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Rotational beta expansion: ergodicity and soficness

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Abstract. We study a family of piecewise expanding maps on the plane, generated by composition of a rotation and an expansive similitude of expansion constant β. We give two constants B_1 and B_2 depending only on the fundamental domain that if $\beta > B_1$ then the expanding map has a unique absolutely continuous invariant probability measure, and if $\beta > B_2$ then it is equivalent to 2-dimensional Lebesgue measure. Restricting to a rotation generated by q-th root of unity ζ with all parameters in $\mathbb{Q}(\zeta, \beta)$, the map gives rise to a sofic system when $\cos(2\pi/q) \in \mathbb{Q}(\beta)$ and β is a Pisot number. It is also shown that the condition $\cos(2\pi/q) \in \mathbb{Q}(\beta)$ is necessary by giving a family of non-sofic systems for $q = 5$.

1. Introduction.

Let $1 < \beta \in \mathbb{R}$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$. Fix $\xi, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1/\eta_2 \notin \mathbb{R}$. Then $\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0,1), y \in [0,1)\}$ is a fundamental domain of the lattice $\mathcal{L}$ generated by η_1 and η_2 in $\mathbb{C}$, i.e.,

$$\mathbb{C} = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d)$$

is a disjoint partition of $\mathbb{C}$. Define a map $T : \mathcal{X} \to \mathcal{X}$ by $T(z) = \beta\zeta z - d$ where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta\zeta z \in \mathcal{X} + d$. Given a point z in $\mathcal{X}$, we obtain an expansion

$$\begin{aligned} z &= \frac{d_1}{\beta\zeta} + \frac{T(z)}{\beta\zeta} \\ &= \frac{d_1}{\beta\zeta} + \frac{d_2}{(\beta\zeta)^2} + \frac{T^2(z)}{(\beta\zeta)^2} \\ &= \sum_{n=1}^{\infty} \frac{d_n}{(\beta\zeta)^n} \end{aligned}$$

with $d_n = d(T^{n-1}(z))$. We call T the *rotational beta transformation* and $d_1 d_2 \dots$ the *expansion* of z with respect to T. We note that the map T generalizes the notions of beta expansion [**8**], [**18**], [**19**] and negative beta expansion [**7**], [**9**], [**16**] in a natural dynamical manner to the complex plane $\mathbb{C}$. More number theoretical generalizations had been studied by means of numeration system in complex bases, e.g., [**2**], [**5**], [**11**], [**14**]. Since

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T is a piecewise expanding map, by a general theory developed in [**3**], [**6**], [**12**], [**13**], [**20**], [**21**], [**22**], there exists an invariant probability measure μ which is absolutely continuous to the two-dimensional Lebesgue measure[1]. The number of ergodic components is known to be finite [**6**], [**12**], [**20**]. An explicit upper bound in terms of the constants in a Lasota–Yorke type inequality was given by Saussol [**20**]. However this bound may be large[2]. By using the special shape of the map T, we can show that the number is one if β is sufficiently large. Define the *width* $w(\mathcal{X})$ of $\mathcal{X}$ as

$$w(\mathcal{X}) := \min\{|\eta_1|, |\eta_2|\} \sin(\theta(\mathcal{X})),$$

where $\theta(\mathcal{X}) \in (0, \pi)$ is the angle between η_1 and η_2. Then $w(\mathcal{X})$ is the minimum height of the parallelogram formed by $\mathcal{X}$. Let $r(P)$ be the *covering radius* of a point set $P \subseteq \mathbb{C}$, i.e., $r(P)$ is the infimum of the positive real numbers R such that every point in $\mathbb{C}$ is within distance R of at least one point in P. Let us define

$$B_n = \max\left\{\nu_n(\theta(\mathcal{X})), \frac{2r(\mathcal{L})}{w(\mathcal{X})}\right\} \quad (n = 1, 2)$$

with

$$\nu_1(\theta(\mathcal{X})) := \begin{cases} 2 & \text{if } \dfrac{1}{2} < \tan\left(\dfrac{\theta(\mathcal{X})}{2}\right) < 2 \\ \dfrac{1 + |\cos\theta(\mathcal{X})|}{2(\sin\theta(\mathcal{X}) + |\cos\theta(\mathcal{X})| - 1)} & \text{otherwise} \end{cases}$$

and

$$\nu_2(\theta(\mathcal{X})) := 1 + \frac{\sqrt{2}}{\sin\theta(\mathcal{X})\sqrt{1 + |\cos\theta(\mathcal{X})|}} = 1 + \frac{1}{\sin\theta(\mathcal{X}) \max\left\{\sin\frac{\theta(\mathcal{X})}{2}, \cos\frac{\theta(\mathcal{X})}{2}\right\}}.$$

Note that B_1 and B_2 do not depend on ξ and are determined only by η_1 and η_2.

Theorem 1.1. *If $\beta > B_1$ then $(\mathcal{X}, T)$ has a unique absolutely continuous invariant probability measure μ. Moreover, if $\beta > B_2$ then μ is equivalent to the 2-dimensional Lebesgue measure restricted to $\mathcal{X}$.*

One can confirm the inequality $B_1 \leq B_2$ in Figure 1. The uniqueness implies that $(\mathcal{X}, T)$ is ergodic with respect to μ. In the last section, we give a rotational beta transformation where the number of ergodic components exceeds one, when β is small (see Example 6.1). It is an intriguing problem to improve the above bounds B_1 and B_2, which may not be optimal, see Examples 6.3, 6.4 and 6.5. Hereafter, ACIM stands for absolutely continuous invariant probability measure.

Remark 1.2. The covering radius $r(\mathcal{L})$ is computed from the successive minima

[1] For example, we can see this fact by Lemma 2.1 of [**20**] for some iterate of T.

[2] Saussol [**20**] did not aim at giving a good bound of it, but was interested in showing the finiteness of the number of components. Indeed, when we apply Lasota–Yorke type inequality, these two objectives (finiteness proof and minimizing the upper bound) are in confrontation.

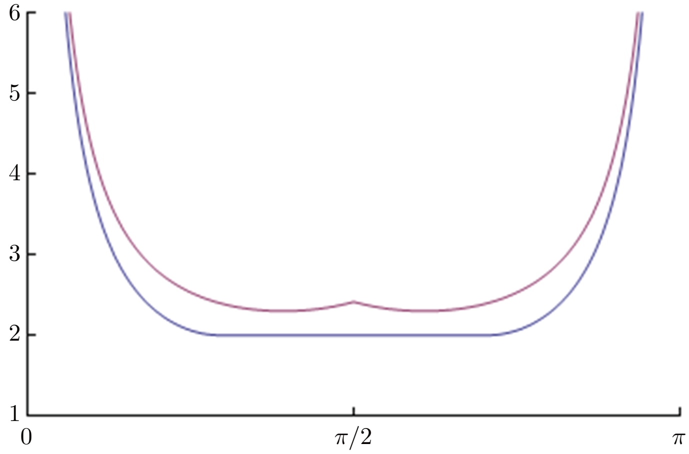

Figure 1. Comparison of $\nu_1(\theta(\mathcal{X}))$ and $\nu_2(\theta(\mathcal{X}))$.

of $\mathcal{L}$, which are derived by the 'homogeneous' continued fraction algorithm due to Gauss. The term $2r(\mathcal{L})/w(\mathcal{X})$ in Theorem 1.1 is expected to be replaced by a smaller one, since we may substitute $r(\mathcal{L})$ with $r(\mathcal{L}+T^{-n}(z))$ for a non negative integer n and a point z in $\mathcal{X}$ to obtain the same conclusion. See the proof in Section 2.

Remark 1.3. The beta and negative beta transformations could be understood in a similar framework in 1-dimension by choosing $\zeta = \pm 1$ and $\mathcal{X} = [\xi, \xi+\eta)$ with $\mathcal{L} = \eta\mathbb{Z}$. In this case, $(\mathcal{X}, T)$ has a unique ACIM with respect to the 1-dimensional Lebesgue measure. This result follows from Li–Yorke [**15**] which reads that every support of an ACIM contains at least one discontinuity in its interior, and the fact that a neighborhood of each discontinuity of T is mapped similarly to neighborhoods of two end points of $\mathcal{X}$. The problem of discontinuities becomes harder in dimension > 1.

Later on, we are interested in the associated symbolic dynamical system over the alphabet $\mathcal{A} := \{d(z) \mid z \in \mathcal{X}\}$. Let $\mathcal{A}^{\mathbb{Z}}$ (resp. $\mathcal{A}^*$) be the set of all bi-infinite (resp. finite) words over $\mathcal{A}$. We say $w \in \mathcal{A}^*$ is admissible if w appears in the expansion $d_1 d_2 \dots$ for some $z \in \mathcal{X} \setminus \bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$[3]. Here $\partial(\mathcal{X})$ denotes the boundary of $\mathcal{X}$. Let

$$\mathcal{X}_T := \left\{ w = (w_j) \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, w_j w_{j+1} \dots w_k \text{ is admissible } \forall (j,k) \in \mathbb{Z}^2 \text{ with } j \le k \right\}$$

which is compact by the product topology of $\mathcal{A}^{\mathbb{Z}}$. The symbolic dynamical system associated to T is the topological dynamics $(\mathcal{X}_T, s)$ given by the shift operator $s((w_j)) = (w_{j+1})$. We say $(\mathcal{X}_T, s)$ (or simply, $(\mathcal{X}, T)$) is sofic if there is a finite directed graph G labeled by $\mathcal{A}$ such that for each $w \in \mathcal{X}_T$, there exists a bi-infinite path in G labeled w and vice versa. Here is a characterization of sofic systems using the forward orbits of the discontinuities:

[3]We exclude the null set $\bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$, i.e., the set of forward/backward discontinuities to concentrate on the essential part of the dynamics.

Lemma 1.4. *The system $(\mathcal{X}, T)$ is sofic if and only if $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of segments.*

Note that the two open segments in $\partial(\mathcal{X})$, one from $\xi + \eta_1$ to $\xi + \eta_1 + \eta_2$ and the other from $\xi + \eta_2$ to $\xi + \eta_1 + \eta_2$, are outside of $\mathcal{X}$. For these segments, the images by T are defined by an infinitesimal small perturbation, e.g., we take the image of the segment connecting $\xi + \eta_1(1 - \varepsilon)$ and $\xi + \eta_1(1 - \varepsilon) + \eta_2$ for a small positive ε. We prove this lemma in Section 3.

From the above lemma, we see that for $(\mathcal{X}, T)$ to be sofic, the set of slopes of the discontinuous segments consisting $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$ must be finite. This means that ζ must be a root of unity. Hereafter, we assume that ζ is a q-th root of unity with $q > 2$ and $\xi, \eta_1, \eta_2 \in \mathbb{Q}(\zeta, \beta)$ with $\eta_1/\eta_2 \notin \mathbb{R}$. We let $\kappa(\xi + \eta_1 x + \eta_2 y) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ be a bijection from $\mathcal{X}$ to $[0,1)^2$ and consider the analog of T on $[0,1)^2$.

Since $\mathbb{Q}(\zeta, \beta)$ is quadratic over $\mathbb{Q}(\zeta + \zeta^{-1}, \beta)$, every element of $\mathbb{Q}(\zeta, \beta)$ is uniquely expressed as a linear combination of η_1 and η_2 over $\mathbb{Q}(\zeta + \zeta^{-1}, \beta)$. We find $a_{jk}, b_j \in \mathbb{Q}(\zeta + \zeta^{-1}, \beta)$ such that

$$\zeta \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

and

$$(\beta\zeta - 1)\xi = b_1\eta_1 + b_2\eta_2.$$

Let U be the map from $[0,1)^2$ to itself, which satisfies $U \circ \kappa = \kappa \circ T$. We can write

$$U\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \beta(a_{11}x + a_{12}y) + b_1 - \lfloor \beta(a_{11}x + a_{12}y) + b_1 \rfloor \\ \beta(a_{21}x + a_{22}y) + b_2 - \lfloor \beta(a_{21}x + a_{22}y) + b_2 \rfloor \end{pmatrix}.$$

This expression suggests an important role of the field $\mathbb{Q}(\zeta + \zeta^{-1})$ in our problem. In the following, we give a sufficient condition so that $(\mathcal{X}, T)$ is a sofic system.

Theorem 1.5. *Let ζ be a q-th root of unity ($q > 2$) and β be a Pisot number. Let $\eta_1, \eta_2, \xi \in \mathbb{Q}(\zeta, \beta)$. If $\zeta + \zeta^{-1} \in \mathbb{Q}(\beta)$, then the system $(\mathcal{X}, T)$ is sofic.*

In proving this theorem, we give an upper bound on the number of the *intercepts* of the segments in $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$. The details will be given in Section 4. For $q = 3, 4, 6$, since $2\cos(2\pi p/q)$ is an integer, we have the following result.

Corollary 1.6. *If ζ is a 3rd, 4th or 6th root of unity, then the system $(\mathcal{X}, T)$ is sofic for any Pisot number β.*

On the other hand, we can give a family of non-sofic systems when $\zeta + \zeta^{-1} \notin \mathbb{Q}(\beta)$. From here on, i denotes $\sqrt{-1}$.

Theorem 1.7. *Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi i/5)$. If $\beta > 2.90332$ such that $\sqrt{5} \notin \mathbb{Q}(\beta)$, then $(\mathcal{X}, T)$ is not a sofic system.*

Most of the large Pisot numbers satisfy the conditions of Theorem 1.7, e.g., any integer greater than 2. The proof of Theorem 1.7 suggests that $(\mathcal{X}, T)$ rarely becomes sofic for general β and ζ. Meanwhile, Example 6.3 shows that there are sofic rotational beta expansions beyond Theorem 1.5. It is of interest to characterize such quintuples $(\beta, \zeta, \eta_1, \eta_2, \xi)$, giving an analogy of Parry numbers in 1-dimensional beta expansion (cf. [**1**], [**8**], [**18**]).

2. Proof of Theorem 1.1.

Let t be a positive real number. We denote by $B_{-t}(A)$ the set of points of A which have distance at least t from $\partial(A)$. We shall study the n-th inverse image $T^{-n}(z) = \{z' \in \mathcal{X} | T^n(z') = z\}$ for $n \in \mathbb{N}$ and $z \in \mathcal{X}$. Put $r = r(\mathcal{L}), w = w(\mathcal{X})$ and $\theta = \theta(\mathcal{X})$. For $j = 1, 2$, set $\nu_j = \nu_j(\theta(\mathcal{X}))$. First we claim that if $\beta > B_2$, then for all $z \in \mathcal{X}$, $\bigcup_{n=1}^{\infty} T^{-n}(z)$ is dense in $\mathcal{X}$. Note that $T^{-1}(z) = ((z + \mathcal{L}) \cap \beta\zeta\mathcal{X}) / \beta\zeta$.

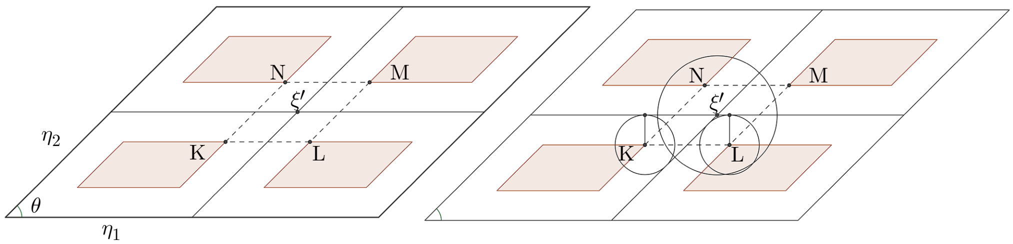

Figure 2. (Left) $B_{-r/\beta}(\mathcal{X}) + \mathcal{L}$; (Right) Balls $B(\xi', 2r/\beta)$, $B(K, r/\beta)$ and $B(L, r/\beta)$.

Consider the region $B_{-r}(\beta\zeta\mathcal{X})$. If $\beta w > 2r$, then $B_{-r}(\beta\zeta\mathcal{X}) \neq \emptyset$. Moreover, since this region $B_{-r}(\beta\zeta\mathcal{X})$ has no intersection with any ball $B(x, r)$ centered at $x \in \mathbb{C} \setminus \beta\zeta\mathcal{X}$ of radius r, the set $(z + \mathcal{L}) \cap \beta\zeta\mathcal{X}$ can not be empty and gives an r-covering of $B_{-r}(\beta\zeta\mathcal{X})$. That is, for each $z' \in B_{-r}(\beta\zeta\mathcal{X})$, there exists $d \in \mathcal{L}$ such that $z + d \in \beta\zeta\mathcal{X}$ and the ball $B(z + d, r)$ contains z'. As such, we see that $B_{-r/\beta}(\mathcal{X})$ is r/β-covered by $T^{-1}(z)$. Consequently, $B_{-r/\beta}(\mathcal{X}) + \mathcal{L}$ (see Figure 2 (left)) is r/β-covered by $T^{-1}(z) + \mathcal{L}$. Now, we enlarge the radius r/β to form a covering of the entire space $\mathbb{C}$. To this end, we claim that extending the radius by a factor of ν_2 suffices. From the inequality $\nu_2 > 1 + 1/\sin\theta$, we only have to check that a rhombus KLMN in Figure 2 determined by adjacent translates of $B_{-r/\beta}(\mathcal{X})$ is covered. Since ν_2 is invariant under $\theta \leftrightarrow \pi - \theta$, we prove the statement for $\theta \in (0, \pi/2]$. Consider the Voronoï diagram of its four vertices K, L, M and N. Then it can be seen easily that the minimum length required to achieve the goal is given by the circumradius of the triangles $\triangle$KLN and $\triangle$LMN, which are the acute triangles determined by the smaller diagonal of the rhombus. This gives the constant ν_2 and proves the claim. For an obtuse θ, we have to switch to the other angle $\pi - \theta$. Refer to Figure 3 below to compare the Voronoï diagrams of two particular rhombuses.

Let $\beta > B_2$. We show by induction that for all $n \in \mathbb{N}$, $T^{-n}(z)$ provides an r_n-covering of $B_{-r_n}(\mathcal{X})$, where $r_n = r\nu_2^{n-1}/\beta^n$. Suppose this is the case for all $k \leq n$ for some $n \in \mathbb{N}$. We note that $T^{-(n+1)}(z) = ((T^{-n}(z) + \mathcal{L}) \cap \beta\zeta\mathcal{X}) / \beta\zeta$. From $\beta > B_2$, we have $r_n < r$. Thus $\beta w > 2r_n$, implying that $B_{-r_n}(\beta\zeta\mathcal{X}) \neq \emptyset$. As $T^{-n}(z) + \mathcal{L}$ gives an r_n-

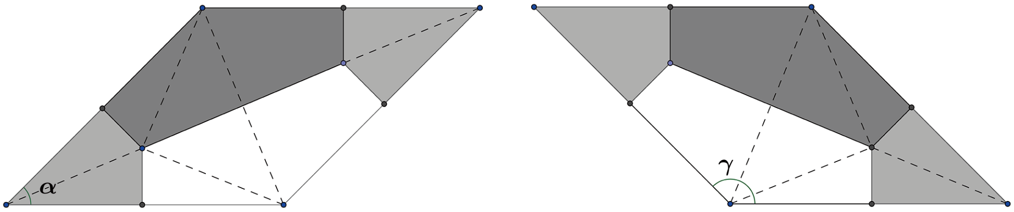

Figure 3. Voronoï diagrams where $\alpha \in (0, \pi/2]$ and $\gamma \in (\pi/2, \pi)$.

covering of $B_{-r_n}(\mathcal{X})+\mathcal{L}$, we can enlarge r_n by a factor of ν_2 to obtain a covering of $\mathbb{C}$, and consequently, of $\beta\zeta\mathcal{X}$. Now, for all $c \in \mathbb{C} \setminus \beta\zeta\mathcal{X}$, we have $B(c, r_n\nu_2) \cap B_{-r_n\nu_2}(\beta\zeta\mathcal{X}) = \emptyset$. This implies that $(T^{-n}(z) + \mathcal{L}) \cap \beta\zeta\mathcal{X}$ is an $r_n\nu_2$-covering of $B_{-r_n\nu_2}(\beta\zeta\mathcal{X})$. From this, it follows that $T^{-(n+1)}(z)$ is an r_{n+1}-covering of $B_{-r_{n+1}}(\mathcal{X})$. This finishes the induction which completes the proof of the claim.

We continue to use the symmetry $\theta \leftrightarrow \pi - \theta$ and assume that $\theta \in (0, \pi/2]$. In the course of the above proof, if we choose $z = \xi$, we can come up with a considerably finer covering of $\mathbb{C}$. Observe that inside the parallelogram KLMN, there is a point $\xi' \in \xi + \mathcal{L}$ (see Figure 2). The ball centered at ξ' already covers a significant portion of the parallelogram. To proceed, we first note that some rectangular strips along the perimeter of the translates of $\mathcal{X}$ can be covered by balls $B(x, 2r/\beta)$ where $x \in T^{-1}(\xi)+\mathcal{L}$ as shown in Figure 4 (left). Therefore, around ξ', we need to cover a region comprising of four kite-shaped areas given in Figure 4 (right).

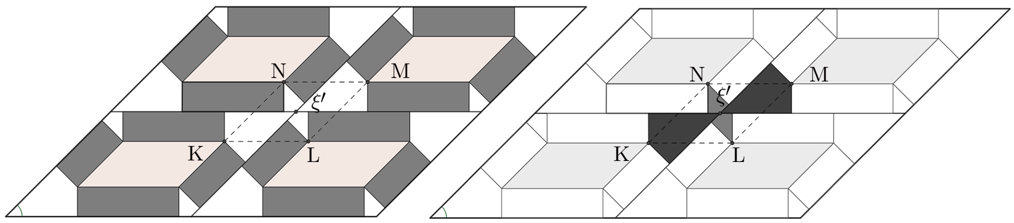

Figure 4. (Left) Rectangular strips ; (Right) Kites.

Now, if $1/2 < \tan(\theta/2) < 2$, the ball $B(\xi', 2r/\beta)$ contains the bases of the two perpendiculars emanating from M to the lines ℓ_1 and ℓ_2, where ℓ_j is the line parallel to η_j and passing through ξ' for $j = 1, 2$. This means that the kite containing M is covered by the balls $B(\xi', 2r/\beta)$ and $B(\mathrm{M}, r/\beta)$. A similar argument shows that remaining kites are also covered. Hence, we see that $(\xi \cup T^{-1}(\xi)) + \mathcal{L}$ gives a $2r/\beta$-covering of $\mathbb{C}$.

For the other cases, we have to enlarge the radius a little more. Figure 2 (right) shows such a case where there is a small remaining region yet to be covered. In Figure 5, we take a minimum $\rho > 1$ such that the balls $B(\xi', (\rho+1)r/\beta)$ and $B(\mathrm{M}, \rho r/\beta)$ intersect on the boundary of the kite.

A small computation yields that if $\rho = (1 + \cos\theta)/(2(-1 + \sin\theta + \cos\theta))$, $\mathbb{C}$ can be covered by balls centered at the elements of $(\xi \cup T^{-1}(\xi)) + \mathcal{L}$ of radius $\rho r/\beta$.

We can proceed with the same induction to see that if $\beta > B_1$, $\bigcup_{j=0}^{n} T^{-j}(\xi)$ gives

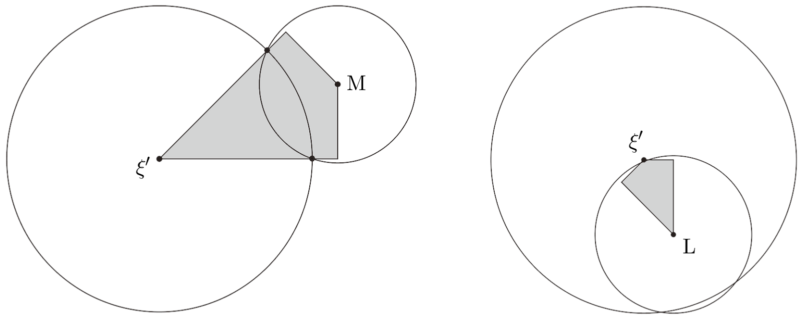

Figure 5. Covering the kites.

an $r\nu_1^{n-1}/\beta^n$-covering of $B_{-r\nu_1^{n-1}/\beta^n}(\mathcal{X})$.

We saw that the choice $z = \xi$ makes the radius of the covering smaller. However ξ is unfortunately on the boundary of $\mathcal{X}$, which is not suitable for the later use. So we select an appropriate $z \in \mathcal{X}$ which is very close to ξ. Since all inequalities in the above proof are open, one may find an $\varepsilon_0 > 0$ that the first inductive step works for every point $z \in B(\xi, \varepsilon_0)$. Let $\mathcal{Y} := \mathcal{X} \setminus \bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$ and select $z \in \mathcal{Y}$ such that $z \in B(\xi, \varepsilon_0 \nu_1^{n-2}/\beta^{n-1})$ for some integer $n \geq 2$. This choice of z is possible because $\bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$ is a null set. Then the induction similarly works at least n steps and we obtain the following statement.

If $\beta > B_1$, then for any $\varepsilon > 0$ there exist $z \in \mathcal{Y}$ and a positive integer n such that $\bigcup_{j=0}^{n} T^{-j}(z)$ is an ε-covering of $B_{-\varepsilon}(\mathcal{X})$.

We are ready to prove the first part of the theorem. Suppose $\beta > B_1$. The proof of Theorem 5.2 in [**20**] implies that the support of each ACIM contains an open ball where the associated Radon-Nikodym density has a positive lower bound. Any such balls belonging to different ergodic ACIM's must be disjoint. Let us assume that μ_j $(j = 1, 2)$ are two different ergodic ACIM's of $(\mathcal{X}, T)$ with corresponding densities h_j. Note that $h_j(z) > 0$ implies $h_j(T(z)) > 0$ for almost all z since h_j is a fixed point of the Perron–Frobenius operator, whose associated Jacobian is positive and constant. Choose open balls $B(x_j, s)$ such that $\operatorname{essinf}_{B(x_j,s)} h_j > 0$ for $j = 1, 2$. From the above result, we can find $z \in \mathcal{Y}$ and positive integers m_j $(j = 1, 2)$ such that $T^{-m_j}(z)$ and $B(x_j, s)$ have a nontrivial intersection. For $j = 1, 2$, let $u_j \in B(x_j, s) \cap T^{-m_j}(z)$. Then $T^{m_j}(u_j) = z$. Moreover, for some small balls $B(u_j, \delta_j)$ inside $B(x_j, s)$, we have

$$\begin{aligned} T^{m_j}\left(B(u_j, \delta_j)\right) &= B\left(T^{m_j}(u_j), \delta'_j\right) \\ &= B(z, \delta'_j), \end{aligned}$$

where $\delta'_j > 0$ is some small radius for $j = 1, 2$. Therefore, $\operatorname{essinf}_{B(z,\delta'_j)} h_j > 0$, which is a contradiction. Thus, we see that the number of ergodic components is one, showing the first statement.

The second statement is subtler than the first one. Let $\beta > B_2$. For $\varepsilon > 0$, let

$$N_\varepsilon = \{x \in \mathcal{X} \mid \operatorname{essinf}_{B(x,\varepsilon)} h = 0\},$$

where h is the density of the ACIM μ and put $N = \bigcap_{\varepsilon} N_{\varepsilon}$. According to Proposition 5.1 in [**20**], we know $\mu(N) = 0$. We claim that N is contained in $\bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$. Assume that $z \notin \bigcup_{j=-\infty}^{\infty} T^j(\partial(\mathcal{X}))$. Choose $B(x,s) \subset \operatorname{supp}\mu$ with $x \in \mathcal{X}$ and $s > 0$ such that $\operatorname{essinf}_{B(x,s)} h > 0$. Then there is a positive integer n such that $T^{-n}(z) \cap B(x,s) \neq \emptyset$. This means that there is a small ball $B(w,\varepsilon) \subset B(x,s)$ that $T^n(w) = z$ and $T^n(B(w,\varepsilon)) = B(z,\varepsilon')$. However, $\operatorname{essinf}_{B(z,\varepsilon')} h > 0$ shows that $z \notin N$, which shows the claim. The claim implies $m(N) = 0$ where m is the 2-dimensional Lebesgue measure[4]. Now we assume that $S \subset \mathcal{X}$ is measurable with $m(S) > 0$. Since $m(S \setminus N) = m(S) > 0$, take a Lebesgue density point $z \in S \setminus N$, i.e.,

$$\lim_{t \to 0} \frac{m(B(z,t) \cap (S \setminus N))}{m(B(z,t))} = 1.$$

Since $z \notin N$, there are positive c and ε_0 such that $\operatorname{essinf}_{B(z,\varepsilon_0)} h > c$. Thus

$$\mu(S) = \mu(S \setminus N) = \int_{S \setminus N} h dm > \int_{B(z,\varepsilon) \cap (S \setminus N)} c dm > 0,$$

for a small $\varepsilon \leq \varepsilon_0$ which shows that m is absolutely continuous to μ. $\square$

3. Proof of Lemma 1.4.

Recall that $\mathcal{Y} = \mathcal{X} \setminus \bigcup_{n=-\infty}^{\infty} T^n(\partial(\mathcal{X}))$. Define the set of *predecessors* associated to a point $z \in \mathcal{Y}$ by

$$P(z) = \bigcup_{n=1}^{\infty} \left\{ d(z')d(T(z')) \cdots d(T^{n-1}(z')) \in \mathcal{A}^* \mid z' \in T^{-n}(z) \right\},$$

that is, the set of codings of all trajectories into $z \in \mathcal{Y}$ of the inverse images of the point z. Introduce an equivalence relation $z_1 \sim z_2$ by $P(z_1) = P(z_2)$. It is clear that the cardinality of equivalence classes in $\mathcal{Y}/\sim$ is finite if and only if the system is sofic (cf. [**17**, Theorem 3.2.10]). By the definition of the map T, it is plain to see by induction on K, that $\mathcal{X} \setminus \bigcup_{n=1}^{K} T^n(\partial(\mathcal{X}))$ consists of finite number of open polygons and each end point of a discontinuity segment must be on *another segment of a different slope*[5]. An open polygon may be cut into two or more pieces by a broken line of $T^{K+1}(\partial(\mathcal{X}))$. We see that any points x and y separated by the broken line are inequivalent, as one of $P(x)$ and $P(y)$ has at least one more predecessor than the other. Suppose that $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$ is an infinite union of segments. Then as we increase K by 1, at least one open polygon of $\mathcal{X} \setminus \bigcup_{n=1}^{K} T^n(\partial(\mathcal{X}))$ is separated by a broken line coming from $T^{K+1}(\partial(\mathcal{X}))$. In fact, if not then $T^{K+1}(\partial(\mathcal{X}))$ must be totally contained in $Q := \partial(\mathcal{X}) \cup \bigcup_{n=1}^{K} T^n(\partial(\mathcal{X}))$ and we have $T^m(\partial(\mathcal{X})) \subset Q$ for $m \geq K+1$. However there are only finitely many segments whose end points lie on other segments of different slopes in Q, which shows that the sequence

[4] The proof of Proposition 5.1 in [**20**] guarantees $\mu(N) = 0$. The author wrote that this implies that N is a null set (with respect to μ) but it does not necessarily mean $m(N \cap \operatorname{supp}(\mu)) = 0$.

[5] If a segment of $T^n(\partial(\mathcal{X}))$ falls into $\partial(\mathcal{X})$, then we discard the segment, because the soficness is defined over $\mathcal{Y}$.

$(T^m(\partial(\mathcal{X})))$ $(m > K)$ is eventually periodic, giving a contradiction. Consequently we always find an additional equivalent class through $K \to K+1$. This shows that the system can not be sofic.

For the reverse implication, we consider the partition of $\mathcal{X}$ into finitely many disjoint polygons induced by $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$. Taking discontinuities into account, such polygons may not be open nor closed. Let $P_1, \dots, P_r$ be the polygons in the partition. It is clear that for $j \in \{1, \dots, r\}$, $T(P_j) = \bigcup_{k \in \mathcal{I}} P_k$ for some $\mathcal{I} \subseteq \{1, \dots, r\}$. This follows from the fact that the set $\bigcup_{n=0}^{\infty} T^n(\partial(\mathcal{X}))$ is T-invariant. For $d \in \mathcal{A}$, let

$$[d] := \{z \in \mathcal{X} | d_1(z) = d\}.$$

Suppose $P_j \cap [d] \neq \emptyset$. Since $\beta\zeta[d] = \beta\zeta\mathcal{X} \cap (\mathcal{X} + d)$, then the boundary of $T([d])$ lies in $\bigcup_{n=0}^{1} T^n(\partial(\mathcal{X}))$. Note that

$$T(P_j \cap [d]) \subseteq T(P_j) = \bigcup_{k \in \mathcal{I}} P_k$$

and

$$\begin{aligned}\beta\zeta(P_j \cap [d]) &= \beta\zeta P_j \cap \beta\zeta[d] \\ &= \beta\zeta P_j \cap (\mathcal{X} + d).\end{aligned}$$

Thus, $T(P_j \cap [d]) = \bigcup_{k \in \mathcal{I}^*} P_k$ where $\mathcal{I}^* \subseteq \mathcal{I}$. From the partition, we define a labeled directed graph G. Let

$$V(G) := \{P_1, \dots, P_r\}$$

be the vertex set of G. We build the edge set and define the labeling as follows. For $j, k \in \{1, \dots, r\}$ and $d \in \mathcal{A}$, there is an edge labeled d from P_j to P_k if P_k is contained in $T(P_j \cap [d])$. It is clear that G is a sofic graph describing $(\mathcal{X}, T)$. $\square$

Remark 3.1. The sofic shift obtained in the latter part of the above proof is irreducible if $(\mathcal{X}, T)$ admits the ACIM equivalent to the Lebesgue measure. By construction, the resulting labeled graph is the minimum left resolving presentation of the irreducible sofic shift. Therefore it is easy to check whether the system is a shift of finite type or not by checking synchronizing words through backward reading of the graph (see [**17**, Theorem 3.4.17]).

4. Proof of Theorem 1.5.

We have to study the growth of $\bigcup_{n=1}^{K} U^n\big(\partial\big([0,1)^2\big)\big)$ as K increases. Our idea is to record only the information of the set of *lines* which include this finite union of segments. Thus, we are interested in studying the union of the lines containing the segments whose defining equations are of the form $f(X,Y) = (A,B)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) + C$, where $(0,0) \neq (A,B) \in \mathbb{R}^2$. We often identify the line and its defining equation. Then the image under U of the line is given by the defining equation $f(X', Y') = 0$ with

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \beta \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} - \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \Delta := \left\{ \begin{pmatrix} \lfloor \beta(a_{11}x + a_{12}y) + b_1 \rfloor - b_1 \\ \lfloor \beta(a_{21}x + a_{22}y) + b_2 \rfloor - b_2 \end{pmatrix} \middle| \, 0 \le x, y < 1 \right\}.$$

Since Δ is a bounded set of lattice points, it is a finite set. As multiplication by ζ acts as q-fold rotation on $\mathbb{C}$, we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

Therefore the image of the line under U is

$$\frac{1}{\beta}(A, B) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} X + c_1 \\ Y + c_2 \end{pmatrix} + C = 0.$$

Multiplying by β, we obtain a correspondence of the coefficient vectors of the defining equations:

$$\left(A^{(n)}, B^{(n)}, C^{(n)}\right) \to \left(A^{(n+1)}, B^{(n+1)}, C^{(n+1)}\right) \tag{4.2}$$

where

$$\left(A^{(n+1)}, B^{(n+1)}\right) = \left(A^{(n)}, B^{(n)}\right) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \tag{4.3}$$

$$C^{(n+1)} = \beta C^{(n)} + \left(A^{(n)}, B^{(n)}\right) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{4.4}$$

with $\left(A^{(0)}, B^{(0)}, C^{(0)}\right) = (A, B, C)$. Note that (4.2) is not one-to-one, since we have many choices for $\binom{c_1}{c_2}$ from Δ. Here we introduce an obvious restriction on $C^{(n)}$ that four values

$$\left\{ A^{(n)}s + B^{(n)}t + C^{(n)} \;\middle|\; (s,t) \in \{(0,0), (1,0), (0,1), (1,1)\} \right\}$$

are not simultaneously positive nor negative, to ensure that the resulting lines intersect the closure of $\mathcal{X}$. All the same we have to note that the resulting lines may contain irrelevant ones[6] which do not actually contain a segment of $\bigcup_{j=0}^{\infty} U^j(\partial([0,1)^2))$. From (4.1), $(A^{(n)}, B^{(n)})$ is clearly periodic with period q, and our task is to prove that the set of all $C^{(n)}$ given by this iteration is finite. We call the set $\overline{U} \subset \mathbb{Q}(\beta)$ of all the $C^{(n)}$'s

[6] Therefore the resulting lines are *potential* discontinuities. In the actual algorithm to obtain the associated graph of the sofic shift, it is simpler to abandon such irrelevant lines at each step. However in doing so, we have to record the position of end points of discontinuity segments, which makes the process involved.

arising from $\partial([0,1)^2)$, together with 0 and -1, the set of *intercepts* of U.

Let $\beta_1=\beta,\beta_2,\dots,\beta_d$ be the conjugates of β. For $k=1,\dots,d$, define $\sigma_k:\mathbb{Q}(\beta)\to\mathbb{Q}(\beta_k)$ to be the conjugate map that sends β to β_k. To demonstrate the finiteness of $\overline{U}$, we show that $\sigma_k\left(C^{(n)}\right)$ is bounded for $k=1,\dots,d$. From (4.3), (4.4) and (4.1), we have $C^{(n+1)}=\beta C^{(n)}+m$ where m is an element of

$$M:=\left\{(A,B)\begin{pmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}^n\begin{pmatrix}c_1\\ c_2\end{pmatrix}\;\middle|\;(A,B)\in\{(0,1),(1,0)\},n=0,1,\dots,q-1,\begin{pmatrix}c_1\\ c_2\end{pmatrix}\in\Delta\right\}.$$

Here we use the fact that $(A,B,C)\in\{(1,0,0),(1,0,-1),(0,1,0),(0,1,-1)\}$ gives $\partial([0,1)^2)$. By the finiteness of Δ, $M\subset\mathbb{Q}(\beta)$ is also a finite set. Taking a common denominator, there is a fixed $N\in\mathbb{N}$ that $C^{(n)}\in(1/N)\mathbb{Z}[\beta]$. Let $\omega_k:=\max\{1,\max_{m\in M}\{|\sigma_k(m)|\}\}$. Then, if $k=2,\dots,d$, we have

$$\left|\sigma_k\left(C^{(n)}\right)\right|\le|(\beta_k)^n|+\omega_k\sum_{j=0}^{n-1}|\beta_k|^j\le\frac{\omega_k}{1-|\beta_k|}.$$

For $k=1$, since the line $A^{(n)}X+B^{(n)}Y+C^{(n)}=0$ passes through $[0,1]^2$, it follows that

$$\left|\sigma_1\left(C^{(n)}\right)\right|=\left|C^{(n)}\right|\le\max_{l=0,1,\dots q-1}\left(\left|A^{(l)}\right|+\left|B^{(l)}\right|\right).$$

by the periodicity of $A^{(n)}$ and $B^{(n)}$. □

5. Proof of Theorem 1.7.

Put $\omega=(1+\sqrt{5})/2$. From $\xi=0$, $\eta_1=1$, $\eta_2=\zeta=\exp(2\pi i/5)$ and a trivial relation $\zeta^2=(\zeta+\zeta^{-1})\zeta-1$, we have $b_1=b_2=0$ and $a_{11}=0,a_{12}=-1,a_{21}=1,a_{22}=1/\omega$. Therefore, we have

$$U\left(\begin{pmatrix}x\\ y\end{pmatrix}\right)=\begin{pmatrix}-\beta y-\lfloor-\beta y\rfloor\\ \beta(x+y/\omega)-\lfloor\beta(x+y/\omega)\rfloor\end{pmatrix}.$$

Clearly, $\sqrt{5}\notin\mathbb{Q}(\beta)$ is equivalent to $\mathbb{Q}(\beta)\cap\mathbb{Q}(\omega)=\mathbb{Q}$. Since $\mathbb{Q}(\omega)$ is a Galois extension over $\mathbb{Q}$, this implies that $\mathbb{Q}(\omega)$ and $\mathbb{Q}(\beta)$ are linearly disjoint and there exists a conjugate map $\sigma\in\mathrm{Gal}(\mathbb{Q}(\beta,\omega)/\mathbb{Q}(\beta))$ with $\sigma(\beta)=\beta$ and $\sigma(\omega)=-1/\omega$.

From (4.3) and (4.4) we see,

$$C^{(n+1)}=\beta C^{(n)}+\left(Ac_{11}^{(n+1)}+Bc_{21}^{(n+1)}\right)c_1+\left(Ac_{12}^{(n+1)}+Bc_{22}^{(n+1)}\right)c_2\in\overline{U}$$

for some $\binom{c_1}{c_2}\in\Delta$ and

$$\begin{pmatrix}c_{11}^{(n)} & c_{12}^{(n)}\\ c_{21}^{(n)} & c_{22}^{(n)}\end{pmatrix}=\begin{pmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}^n.$$

Consider the case where $(A,B,C)=(1,0,-1)$. Then,

$$C^{(n+1)} = \beta C^{(n)} + c_{11}^{(n+1)} c_1 + c_{12}^{(n+1)} c_2.$$

Applying σ, we get $\sigma\left(C^{(n+1)}\right) = \beta\sigma\left(C^{(n)}\right) + \sigma\left(c_{11}^{(n+1)} c_1 + c_{12}^{(n+1)} c_2\right)$. It follows that

$$\begin{aligned}\left|\sigma\left(C^{(n+1)}\right)\right| &\geq \beta\left|\sigma\left(C^{(n)}\right)\right| - \left|\sigma\left(c_{11}^{(n+1)} c_1 + c_{12}^{(n+1)} c_2\right)\right| \\ &= \beta\left|\sigma\left(C^{(n)}\right)\right| - \left|\sigma\left(c_{11}^{(n+1)}\right) c_1 + \sigma\left(c_{12}^{(n+1)}\right) c_2\right| \\ &\geq \beta\left|\sigma\left(C^{(n)}\right)\right| - D,\end{aligned}$$

where

$$\begin{aligned}D &:= \max_{n\in\mathbb{N}} \max_{\Delta} \left\{\left|\sigma\left(c_{11}^{(n)}\right) c_1 + \sigma\left(c_{12}^{(n)}\right) c_2\right|\right\} \\ &\leq \max_{n\in\mathbb{N}} \max_{\Delta} \left\{\left|\sigma\left(c_{11}^{(n)}\right)\right| |c_1| + \left|\sigma\left(c_{12}^{(n)}\right)\right| |c_2|\right\}.\end{aligned}$$

Direct computation yields

$$\left(\sigma\left(c_{11}^{(n)}\right), \sigma\left(c_{12}^{(n)}\right)\right) = \begin{cases}(1,0) & n \equiv 0 \pmod 5 \\ (-\omega, 1) & n \equiv 1 \pmod 5 \\ (\omega, -\omega) & n \equiv 2 \pmod 5 \\ (-1, \omega) & n \equiv 3 \pmod 5 \\ (0, -1) & n \equiv 4 \pmod 5.\end{cases}$$

Hence, $D \leq \omega \max_{\Delta} \{|c_1| + |c_2|\} = \omega\left(\lfloor\beta\omega\rfloor + \lceil\beta\rceil\right)$. Accordingly, for all $n \in \mathbb{N}$,

$$\left|\sigma\left(C^{(n+1)}\right)\right| \geq \beta\left|\sigma\left(C^{(n)}\right)\right| - \omega\left(\lfloor\beta\omega\rfloor + \lceil\beta\rceil\right).$$

Therefore, if

$$\left|\sigma\left(C^{(n)}\right)\right| > \frac{\omega\left(\lfloor\beta\omega\rfloor + \lceil\beta\rceil\right)}{\beta - 1}$$

for some $n \in \mathbb{N}$, then $\left\{\sigma\left(C^{(n)}\right) \mid n \in \mathbb{N}\right\}$ diverges. Now, it is easy to check that $\left(A^{(1)}, B^{(1)}, C^{(1)}\right) = (\omega - 1, 1, (\omega-1)\lfloor-\beta\rfloor + \lfloor\beta\omega\rfloor - \beta)$ gives a line which actually includes a discontinuity segment. Under the assumption $\beta > \left(13 + 3\sqrt{5} - \sqrt{70 - 2\sqrt{5}}\right)/4 \approx 2.90332$, we have

$$\begin{aligned}\left|\sigma\left((\omega-1)\lfloor-\beta\rfloor + \lfloor\beta\omega\rfloor - \beta\right)\right| &= \sigma\left((\omega-1)\lfloor-\beta\rfloor + \lfloor\beta\omega\rfloor - \beta\right) \\ &= -\omega\lfloor-\beta\rfloor + \lfloor\beta\omega\rfloor - \beta,\end{aligned}$$

and

$$-\omega\lfloor-\beta\rfloor + \lfloor\beta\omega\rfloor - \beta > \frac{\omega\left(\lfloor\beta\omega\rfloor + \lceil\beta\rceil\right)}{\beta - 1}.$$

We therefore conclude that $\{\sigma\left(C^{(n)}\right) \mid n \in \mathbb{N}\}$ is unbounded. Now we have shown that once we had chosen $C^{(1)}$ as above, for *every* possible sequence $\left(C^{(n)}\right)$, its conjugate sequence $\left(\sigma\left(C^{(n)}\right)\right)$ $(n = 1, 2, 3, \dots)$ diverges. This implies that the set of discontinuities can not be finite. $\square$

6. Examples.

Taking β small, we can find a family of systems $(\mathcal{X}, T)$ with more than one ACIM.

Example 6.1. Let $\zeta = i$ and $\beta = 1.039$. Set $\eta_1 = 2.92$, $\eta_2 = \exp(\pi i/3)$ and $\xi = 0$. From the distribution of eventual orbits of T of randomly chosen points, it is not difficult to make explicit the polygons bounded by horizontal and vertical segments within which restrictions of T are well-defined. This leads us to a rigorous proof of the existence of two distinct ergodic components. A system is generated by a rectangle E and an octagon F composed of two rectangles. The ratio of two sides of the rectangle E is $1 : \beta$. By successive applications of T, the four vertices of E are easily computed:

$$x + \frac{\sqrt{3}i}{2},\ x + yi,\ x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + yi,\ x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + \frac{\sqrt{3}i}{2}$$

with $x = \eta_1 - (\sqrt{3}/2)\beta - 1/2$ and $y = \beta x - \sqrt{3}/2$ and the vertices of F in counter-clockwise ordering are

$$x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + yi,\ \gamma + yi,\ \gamma + vi,\ u + vi,$$
$$u + v'i,\ \gamma + v'i,\ \gamma + \frac{\sqrt{3}i}{2},\ x + \frac{1}{\beta}\left(\frac{\sqrt{3}}{2} - y\right) + \frac{\sqrt{3}i}{2}$$

with $\gamma = -\beta y + \eta_1 - 1/2$, $u + vi = T^3(\gamma + \sqrt{3}i/2)$ and $u + v'i = T^2(x - 1/2)$. The images $T(F)$ and $T^2(F)$ are similar to F with the ratio β and β^2. We readily confirm the set equation $E \cup F = T(E) \cup T^3(F)$. Hence the restriction of T is well defined on the set

$$Y := E \cup F \cup T(F) \cup T^2(F)$$

and defines a piecewise expanding map. Thus there is at least one ACIM whose support is contained in Y. The same discussion can be done for a different system which is disjoint from Y. The resulting supports of ACIM's are clearly disjoint. The same situation happens when β and η_1 satisfy

$$\frac{\sqrt{3}}{2}\beta + 1 + \frac{\sqrt{3}}{\beta} - \frac{\sqrt{3}}{2\beta^3} \le \eta_1 \le \frac{1}{2} + \frac{\sqrt{3}}{\beta} + \frac{\sqrt{3}}{2\beta^3}$$

while other parameters are fixed. This example gives an uncountable family of systems with at least two ACIM's.

In the following we give some examples of sofic systems.

Example 6.2. Let $\zeta = \exp(2\pi i/3)$ and $\beta = 1 + \sqrt{2}$. Set $\eta_1 = 1$, $\eta_2 = \zeta^2$ and $(\beta\zeta - 1)\xi = 3 - \beta$. From $r(\mathcal{L}) = 1/\sqrt{3}$ and $w(\mathcal{X}) = \sqrt{3}/2$ we have $\beta > B_2 = 7/3$ and there is a unique ACIM equivalent to Lebesgue measure by Theorem 1.1. We consider the symbolic dynamical system associated to the map T. The set $\mathcal{A}$ is given by

$$\big\{a = -1 - \zeta^2, b = -\zeta^2, c = 1 - \zeta^2, d = 2 - \zeta^2, e = -2 - 2\zeta^2, f = -1 - 2\zeta^2,$$
$$g = -2\zeta^2, h = 1 - 2\zeta^2, j = -2 - 3\zeta^2, k = -1 - 3\zeta^2, l = -3\zeta^2\big\}.$$

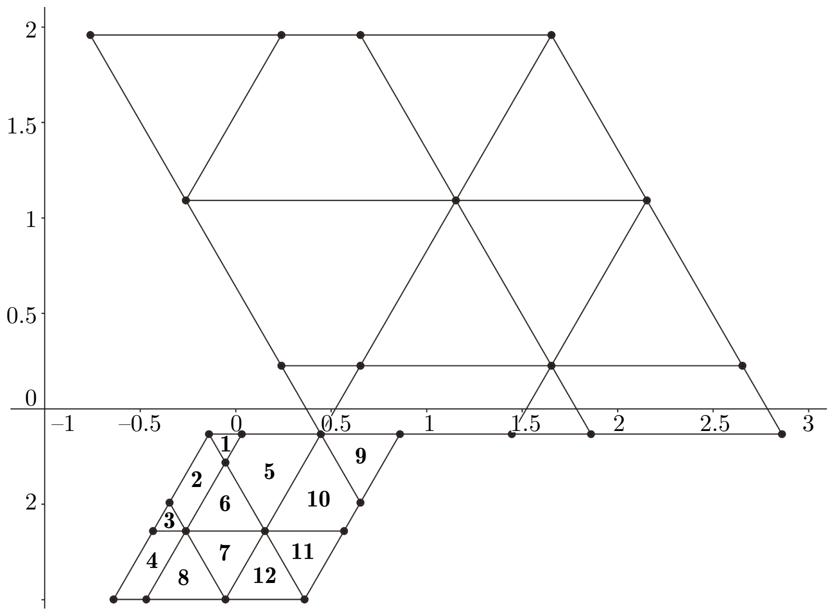

Figure 6. $\mathcal{X}$ and $\beta\zeta\mathcal{X}$.

In Figure 6, we see that the discontinuity lines are finite and partition the fundamental domain $\mathcal{X}$ into disjoint components P_n, $n = 1, \ldots, 12$. We also see in the figure the expanded fundamental region $\beta\zeta\mathcal{X}$.

It is easy to confirm that the image of P_n under T is given by Table 1. From this table, we construct the sofic graph (see Figure 7) as described in Section 3.

Example 6.3. This example is a kind of a *square root system* of the negative beta expansion introduced by Ito–Sadahiro [**7**]. Let $\zeta = i$ and set $\eta_1 = 1$, $\eta_2 = \beta i$ and $\xi = -1 - \beta i$. We have

$$T(x + yi) = -\beta y - \lfloor -\beta y + 1 \rfloor + \beta x i.$$

By taking its square, we can separate the variables:

$$T^2(x + yi) = -\beta^2 x - \lfloor -\beta^2 x + 1 \rfloor + \left(-\beta^2 y - \beta \lfloor -\beta y + 1 \rfloor\right) i.$$

Thus we can study this map Gaussian coordinate-wise by defining

Table 1.

P_n	$T(P_n)$	δ	P_n	$T(P_n)$	δ	P_n	$T(P_n)$	δ
1	7	b	6	9, 10	b	9	7, 11, 12	j
2	11, 12	b		2, 3, 6	c		4	k
	4, 7, 8	c		12	g	10	5, 6, 9, 10	f
3	12	c	7	1, 5	c		2, 3, 6	g
4	11	c		11	g		7, 8, 12	k
	4, 7, 8, 12	d		4, 7, 8	h	11	1, 5	g
5	9	a	8	9, 10	c		11	k
	1, 2, 5, 6	b		2, 3, 6	d		4, 7, 8	l
	7, 11, 12	f		12	h	12	9, 10	g
	4, 7, 8	g	9	9	e		2, 3, 6	h
				1, 2	f		12	l

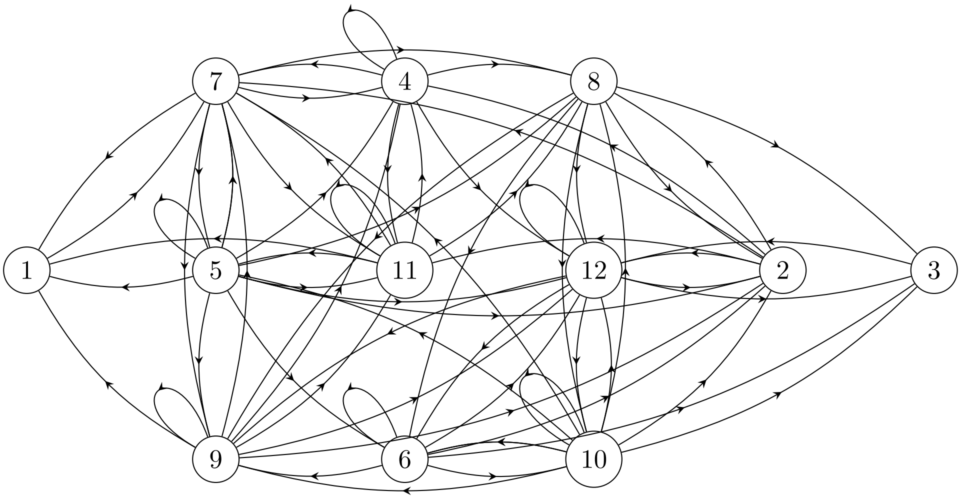

Figure 7. Sofic graph for 3-fold rotation.

$$f(x) = -\beta^2 x - \lfloor -\beta^2 x + 1 \rfloor ,$$

a 1-dimensional piecewise expansive map from $[-1, 0)$ to itself and

$$g(y) = -\beta^2 y - \beta \lfloor -\beta y + 1 \rfloor$$

defined on $[-\beta, 0)$. We easily see that f and g give isomorphic systems through the relation $g(\beta x) = \beta f(x)$. Liao–Steiner [**16**] showed that the unique ACIM of f is equivalent to the 1-dimensional Lebesgue measure if and only if $\beta^2 \geq (1 + \sqrt{5})/2$. Thus the ACIM of T is equivalent to the 2-dimensional Lebesgue measure if and only if $\beta \geq \sqrt{(1 + \sqrt{5})/2}$. In view of the shape of f, one see that if β^2 is a Pisot number, then the system $(\mathcal{X}, T)$ is sofic (cf. Theorem 3.3 in [**10**]). This give examples of sofic rotational beta expansion beyond the scope of Theorem 1.5. One can also show that when β is the Salem number whose minimum polynomial is $x^4 - x^3 - x^2 - x + 1$, the system becomes sofic.

This example is essentially 1-dimensional. We do not yet succeed in giving a 'genuine' 2-dimensional sofic rotational beta expansion beyond Theorem 1.5.

Example 6.4. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi i/5)$. Let $\beta = (1+\sqrt{5})/2$. We describe the symbolic dynamical system associated to given rotation beta transformation through its sofic graph. Here, we use the map U instead of T. The alphabet $\mathcal{A} = \Delta + \binom{b_1}{b_2} = \Delta$ is given by

$$\left\{ a = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, c = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, d = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, e = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, f = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}.$$

The partition of the fundamental region $[0,1)^2$ is given in Figure 8. The sofic graph is described in Table 2. Since the incidence matrix of this graph is primitive, we can determine the ACIM whose density is positive and constant on each partition. Therefore the ACIM is equivalent to the Lebesgue measure, although we can not apply Theorem 1.1 for $\beta < 2$.

Example 6.5. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi i/7)$. Let $\beta = 1 +$

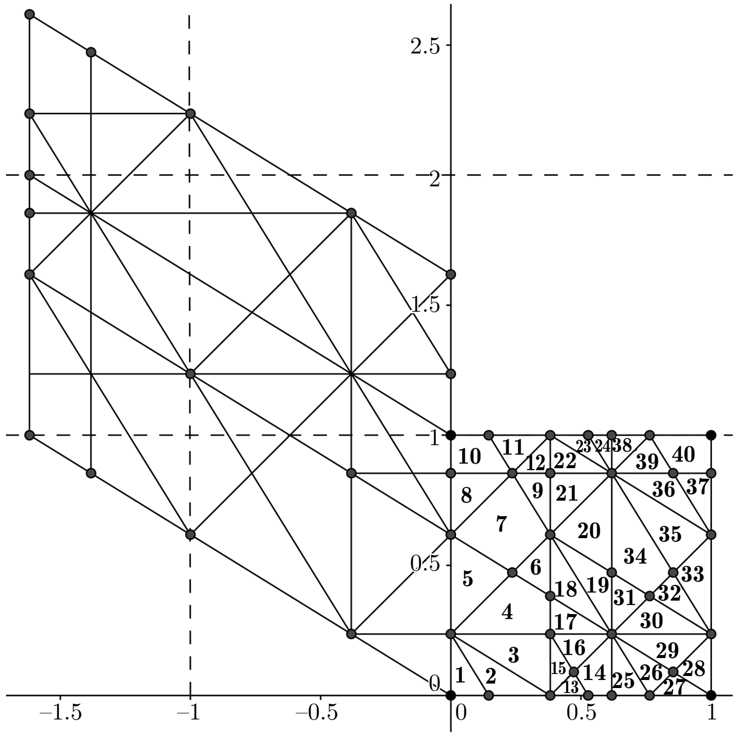

Figure 8. 5-fold sofic case.

Table 2.

P_n	$U(P_n)$	δ
1	$28, 29$	b
2	$30, 32, 33$	b
3	$31, 34, 35$	b
4	$9, 12, 19, 20, 21, 22$	b
5	$6, 7, 18$	b
6	11	b
	2	d
7	$37, 40$	a
	$8, 10$	b
	$26, 27, 28, 29$	c
	1	d
8	$36, 38, 39$	a
	25	c
9	$30, 31$	c
10	$23, 24$	a
	$13, 14, 15, 16$	c
11	$17, 18$	c
12	19	c
13	37	b

P_n	$U(P_n)$	δ
14	40	b
	$26, 27$	d
15	36	b
16	$38, 39$	b
	25	d
17	$23, 24$	b
	$13, 14$	d
18	$3, 15, 16$	d
19	$4, 17$	d
20	$33, 35$	c
	5	d
21	$32, 34$	c
22	20	c
23	21	c
24	22	c
25	$28, 29$	d
26	$30, 32$	d
27	33	d
28	35	d

P_n	$U(P_n)$	δ
29	$31, 34$	d
30	$19, 20$	d
31	$6, 18$	d
32	$9, 21$	d
33	$11, 12, 22$	d
	2	f
34	$36, 37$	c
	$7, 8$	d
35	$39, 40$	c
	10	d
	$27, 28$	e
	1	f
36	38	c
	$25, 26, 29$	e
37	$30, 31$	e
38	23	c
	$13, 15$	e
39	24	c
	$14, 16$	e
40	$17, 18, 19$	e

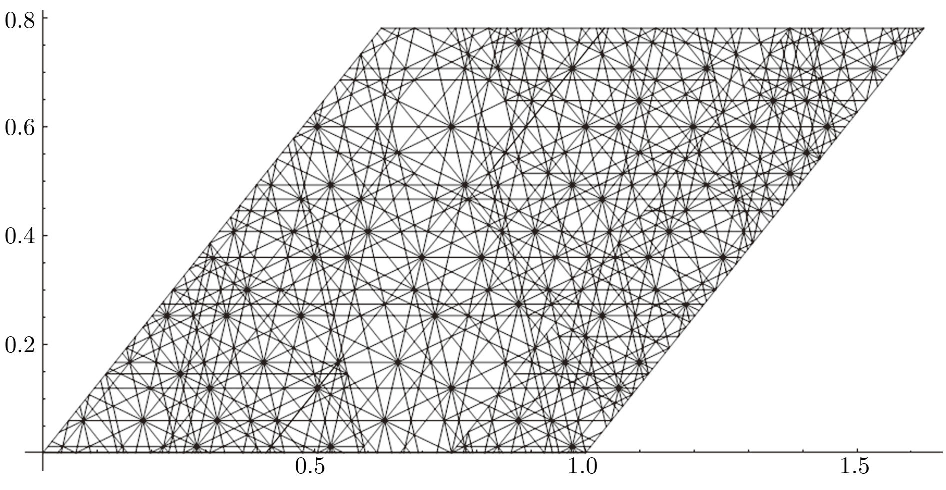

Figure 9. Sofic 7-fold rotation.

$2\cos(2\pi/7) \approx 2.24698$, the cubic Pisot number whose minimum polynomial is $x^3 - 2x^2 - x + 1$. From $r(\mathcal{L}) = 1/(2\cos(\pi/7))$ and $w(\mathcal{X}) = \sin(2\pi/7)$ we have $\beta > B_1 \approx 2.00272$ and there is a unique ACIM by Theorem 1.1, but $\beta < B_2 \approx 2.41964$. From Theorem 1.5, we know that the corresponding dynamical system is sofic. Figure 9 shows the sofic dissection of $\mathcal{X}$ by 224 discontinuity segments. The number of states of the sofic graph is 3292 (!), computed by Euler's formula. It is possible to show that the corresponding incidence matrix of the sofic graph is primitive, and consequently the ACIM is equivalent to the Lebesgue measure.

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