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by

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# Amoroso kernel density estimation for nonnegative data and its bias reduction

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**Abstract**

In this paper, we develop nonparametric density estimation for nonnegative data, using Amoroso density as the kernel. It is shown that the resulting Amoroso kernel density estimator is free of boundary bias, having the mean integrated squared error (MISE) of order $O(n^{-4/5})$, as in other boundary-bias-free density estimators from the existing literature, where $n$ is the sample size. Further, we discuss the bias-reduced Amoroso kernel density estimators, having the MISEs of order $O(n^{-8/9})$. We illustrate the finite sample performance of the Amoroso kernel density estimator and its bias-reduced versions, through the simulations.

Keywords: nonparametric density estimator; boundary bias problem; asymmetric kernel; Amoroso kernel;

MSC: 62G07; 62G20

## 1. Introduction

Let $\{X_1, \ldots, X_n\}$ be a random sample from unknown density $f$ with support $\mathbb{R}$. The kernel density estimator (Rosenblatt (1956) and Parzen (1962))

$$\widehat{f}_h^{(K)}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\Big(\frac{x - X_i}{h}\Big) \tag{1}$$

is a popular way to estimate a density nonparametrically, where $K$ is a kernel and $h = h_n > 0$ is a bandwidth. Suppose that $K_{[p]}$ is the $p$th-order kernel ($p \geq 2$ is an integer) such that $\int_{-\infty}^{\infty} K_{[p]}(s)ds = 1$, $\int_{-\infty}^{\infty} s^l K_{[p]}(s)ds = 0$, $l = 1, \ldots, p-1$, and $\int_{-\infty}^{\infty} s^p K_{[p]}(s)ds \neq 0$. It is well known (see, e.g., Wand and Jones (1995)) that, if the support of the underlying density $f$ is the whole real line, using $h \propto n^{-1/5}$, the mean integrated squared error (MISE) of the second-order kernel density estimator $\widehat{f}_h^{(K_{[2]})}$ achieves $O(n^{-4/5})$ when $f$ is twice continuously differentiable with square integrable $f''$. Further, the MISE of the $p$th-order kernel density estimator $\widehat{f}_h^{(K_{[p]})}$ achieves $O(n^{-2p/(2p+1)})$ using $h \propto n^{-1/(2p+1)}$, provided

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The authors preliminarily reported the Amoroso kernel density estimator and its bias-reduced estimators, at the Japanese Joint Statistical Meeting (2016, September).

This version (July 10, 2017) is a revision of the first version (January 13, 2017). The point of the revision is that the present setting in Subsection 3.2 is preferable to the previous one, since the parameter $q$ controls the speed of $r_b \to \infty$.

that $f$ is $p$ times continuously differentiable with a $p$th square integrable derivative $f^{(p)}$. Schucany and Sommers (1977) discussed a fourth-order kernel in the additive form

$$K_{[4],a}(s) = \frac{1}{1-a^2} K_{[2]}(s) - \frac{a^3}{1-a^2} K_{[2]}(as), \quad a \in (0,1). \tag{2}$$

Noting that, by definition, higher-order kernel (hence, the fourth-order kernel $K_{[4]}$) loses nonnegativity, Terrell and Scott (1980), Jones and Foster (1993), and Jones et al. (1995) constructed nonnegative kernel density estimators that achieve the MISEs of order $O(n^{-8/9})$.

However, if the support of the underlying density is a closed interval or semi-infinite interval, then, the standard kernel density estimator (1) has a bias that is $O(1)$ near the boundary. The boundary bias problem is caused by the (location-scale) symmetric kernel that creates a mass outside the support of the underlying density. Various remedies, for example, renormalization, reflection, generalized jackknifing (see, e.g., Jones (1993)), transformation (Marron and Ruppert (1994)), and advanced reflection (Zhang et al. (1999)), were discussed in the literature. On the other hand, there has been a growing interest in an asymmetric kernel density estimation during the last fifteen years, since, by construction, the support of the asymmetric kernel matches the support of the underlying density. To the best of our knowledge, Silverman (1986; page 28) first mentioned the possibility for using gamma or log-normal (LN) density (rather than the location-scale symmetric density) in the nonparametric density estimation for the nonnegative data $X_1, \ldots, X_n$. On the basis of a certain kernel $K_\theta$ with support $[0, \infty)$ and a finite dimensional parameter $\theta$, several estimators in the form of $n^{-1} \sum_{i=1}^n K_{\theta_1(x,b),\theta_2}(X_i)$, $x \geq 0$, have been suggested, where a subcomponent of $\theta$; $\theta_1$ (say) is chosen to be $\theta_1 = \theta_1(x,b)$ as a function of $(x,b)$, and $b = b_n > 0$ is a smoothing parameter. Chen (1999, 2000) developed beta and gamma kernel density estimators for the data from the unit interval and the nonnegative data, respectively (see also Igarashi and Kakizawa (2014b)). Jin and Kawczak (2003) and Scaillet (2004) considered LN, Birnbaum–Saunders (BS), inverse Gaussian (IG), and reciprocal inverse Gaussian (RIG) kernel density estimators. Koul and Song (2013) studied inverse gamma kernel density estimator. Marchant et al. (2013) and Saulo et al. (2013) discussed generalized/skew BS kernel density estimators[1]. Igarashi and Kakizawa (2014b), Igarashi (2016b), and Kakizawa and Igarashi (2017) revisited the IG, RIG, BS, LN, and inverse gamma kernel density estimators, due to the bad parameterization of the respective kernels when $f(0) > 0$ (the previous estimators had an unrealistic

[1]The second author reported symmetrical-based IG/RIG/BS kernel density estimators, that is an extension of Igarashi and Kakizawa (2014b), at the Mathematical Society of Japan (2016 Spring Meeting) and the Japanese Joint Statistical Meeting (2016, September). He also studied log-symmetrical kernel density estimator, including a reformulation of the previous estimators due to Marchant et al. (2013) and Saulo et al. (2013).

problem of "zero value at the origin"). Note that Igarashi and Kakizawa (2014b) applied a generalized inverse Gaussian density (in their paper, it was renamed as "a modified Bessel density"), and then treated the IG, RIG, and BS kernel density estimators in a unified way (the resulting estimator was referred to as a mixture of IG (MIG) kernel density estimator). Igarashi (2016b) discussed a weighted LN kernel density estimator.

In this paper, we apply a family of Amoroso densities to the context of the nonparametric density estimation for nonnegative data. The Amoroso density, with parameters $\alpha, \beta > 0$ and $\gamma \neq 0$,

$$K^{(A)}_{\alpha,\beta,\gamma}(s) = \frac{|\gamma| s^{\alpha\gamma-1} e^{-(s/\beta)^\gamma}}{\beta^{\alpha\gamma}\Gamma(\alpha)},$$

which was proposed by Amoroso (1925) (see also Stacy and Mihram (1965)), contains many densities, for example, gamma ($\gamma = 1$), inverse gamma ($\gamma = -1$), Nakagami ($\gamma = 2$), and Weibull ($\alpha = 1$) densities as special cases (see Crooks (2010))[2]. In the literature, the density $K^{(A)}_{\alpha,\beta,\gamma}$, with parameters $\alpha, \beta, \gamma > 0$, is known as the generalized gamma (GG) density (Stacy (1962)). Motivated by the second gamma kernel density estimator suggested in Chen (2000), Hirukawa and Sakudo (2015) discussed the Nakagami and Weibull kernel density estimators as special cases of the GG kernel density estimator (Hirukawa and Sakudo (2015) additionally restricted $\gamma \geq 1$) under somewhat high-level conditions[3] for the bias and variance approximations; the validity of asymptotic expansion for the MISE, however, seemed not to be discussed rigorously. In this paper, for every constant $\gamma \neq 0$, we consider a new Amoroso kernel density estimator more concretely.

The contribution of this paper is two-fold. First, using an additional parameter $c$ (given in Section 2 below), we optimize the asymptotic MISE (AMISE) of the resulting Amoroso kernel density estimator.

[2]Here, if $\alpha\gamma \geq 1$, then, $K^{(A)}_{\alpha,\beta,\gamma}(0) > 0$; if $\gamma < 0$, then, $K^{(A)}_{\alpha,\beta,\gamma}(0)$ is understood as $\lim_{s\to 0+} K^{(A)}_{\alpha,\beta,\gamma}(s) = 0$. The remaining case $0 < \alpha\gamma < 1$ is not considered here, due to the unboundedness of the density at the origin.

[3]Condition 4 in Hirukawa and Sakudo (2015) would be insufficient for the approximation on $E[f(\varsigma_x)] \approx f(x)$. To be exact, in their Proof of Theorem 1, we must estimate $E[|\varsigma_x - x|] \leq \{E[(\varsigma_x - x)^2]\}^{1/2}$ rather than $E[\varsigma_x - x] = o(1)$. Further, we find that Condition 4 should read

$$E[\varsigma_x] = \frac{\beta\Gamma(\alpha/\gamma)\Gamma(2\alpha/\gamma)}{2^{1/\gamma}\Gamma((\alpha+1)/\gamma)\Gamma((2\alpha-1)/\gamma)} = \begin{cases} x + O(b), & \dfrac{x}{b} \to \infty, \\ O(b), & \dfrac{x}{b} \to \kappa, \end{cases}$$

$$E[\varsigma_x^2] = \frac{\beta^2\Gamma^2(\alpha/\gamma)\Gamma((2\alpha+1)/\gamma)}{2^{2/\gamma}\Gamma^2((\alpha+1)/\gamma)\Gamma((2\alpha-1)/\gamma)} = \begin{cases} x^2 + O(bx), & \dfrac{x}{b} \to \infty, \\ O(b^2), & \dfrac{x}{b} \to \kappa. \end{cases}$$

It follows that

$$E[|\varsigma_x - x|] \leq \{E[(\varsigma_x - x)^2]\}^{1/2} = (E[\varsigma_x^2] - 2xE[\varsigma_x] + x^2)^{1/2} = \begin{cases} O((bx)^{1/2}), & \dfrac{x}{b} \to \infty, \\ O(b), & \dfrac{x}{b} \to \kappa. \end{cases}$$

Our point is that the above estimate on $E[|\varsigma_x - x|]$, multiplied by $(nb^{1/2})^{-1}/\sqrt{x}$ for $x/b \to \infty$ or $(nb)^{-1}$ for $x/b \to \kappa$, contributes to the $O(n^{-1})$-error of the variance of the asymmetric kernel density estimator.

Note that Chen (2000) considered $c = 0, 1$ for the gamma ($\gamma = 1$) kernel density estimator, and that Hirukawa and Sakudo (2015) considered $c = 0$ for the Nakagami ($\gamma = 2$) kernel density estimator. Second, we study bias reductions to improve the convergence rate of the Amoroso kernel density estimator, in the AMISE sense.

The rest of the paper is organized as follows. In Section 2, we introduce a family of Amoroso kernel density estimators and derive its asymptotic bias, variance, and MISE, together with the (pointwise) strong consistency and asymptotic normality. In Section 3, we consider the bias reductions. The asymptotic properties of an additive bias-reduced Amoroso kernel density estimator (Subsection 3.1) and other nonnegative bias-reduced Amoroso kernel density estimators (Subsection 3.2) are shown. Section 4 contains some comments on the additive/nonnegative bias-reduced Amoroso kernel density estimators. In Section 5, we conduct simulation studies to demonstrate the final sample performance of the proposed estimators. All proofs of Theorems are given in Appendix.

**Notation** For the notational simplicity, the dependency on the sample size $n$ is suppressed (e.g., the smoothing parameter is denoted by $b$, instead of $b_n$), but, unless otherwise stated, the limits will be taken as $n$ goes to infinity.

## 2. Amoroso kernel density estimator

We study the problem of nonparametric density estimation for nonnegative data $X_1, \ldots, X_n$. Given constants $\gamma \neq 0$, $c \in \mathbb{R}$, $d > 0$, and a $\rho$-function $\rho_c$ (it is assumed to be positive, continuous, and non-decreasing on $[0, \infty)$), having the form

$$\rho_c(t) = \begin{cases} c + t, & t > d, \\ r_c(t), & t \in [0, d] \end{cases} \quad \text{(we assume } c + d = r_c(d) \geq r_c(0) \geq 1\text{)}, \tag{3}$$

the Amoroso kernel density estimator we consider in this paper is defined as

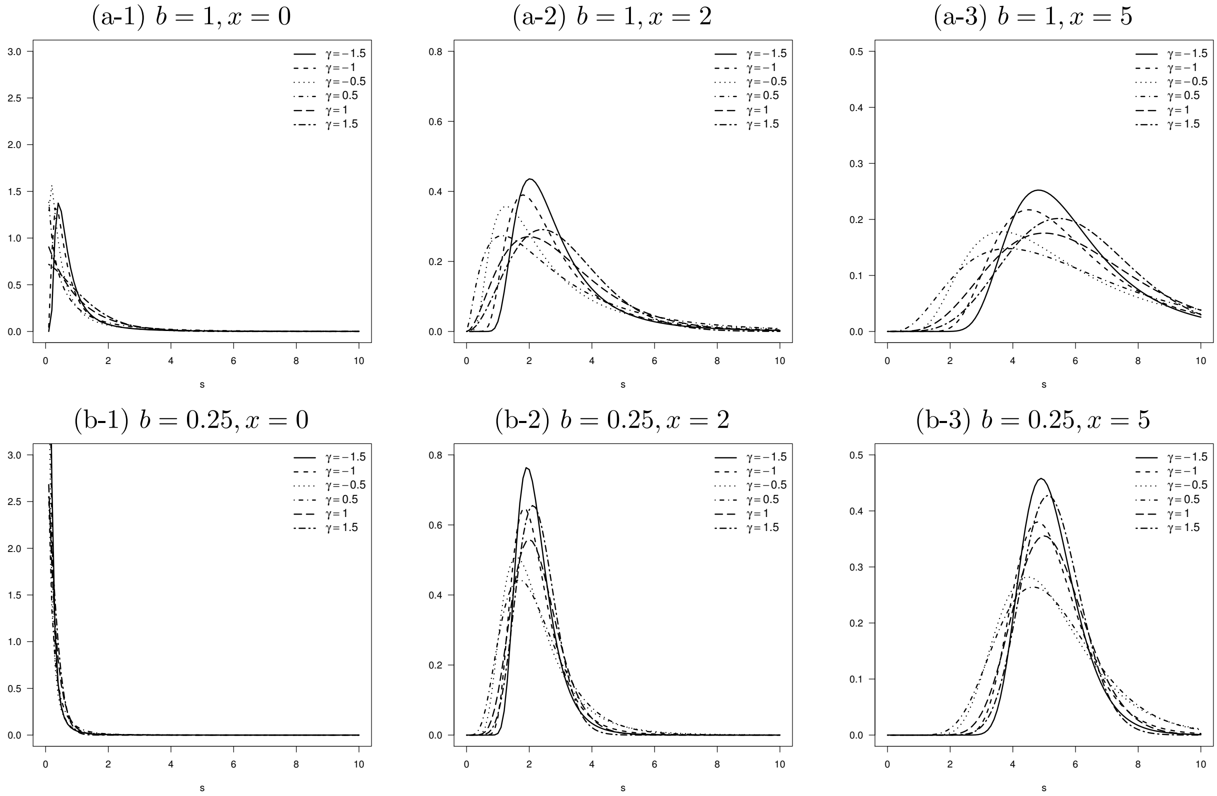
$$\widetilde{f}_{b,c,\gamma}(x) = \frac{1}{n} \sum_{i=1}^{n} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)), b\beta_\gamma(\rho_c(x/b)), \gamma}(X_i), \quad x \geq 0, \tag{4}$$

where $b > 0$ is a smoothing parameter, and $\alpha_\gamma$ and $\beta_\gamma$ are infinitely differentiable functions on $(0, \infty)$, defined by

$$\alpha_\gamma(\rho) = \begin{cases} \dfrac{\rho}{\gamma}, & \gamma > 0, \\ \dfrac{\rho + 1}{|\gamma|}, & \gamma < 0, \end{cases} \quad \beta_\gamma(\rho) = \rho \frac{\Gamma(\alpha_\gamma(\rho))}{\Gamma(\alpha_\gamma(\rho) + 1/\gamma)} = \begin{cases} \rho \dfrac{\Gamma(\rho/\gamma)}{\Gamma((\rho+1)/\gamma)}, & \gamma > 0, \\ \rho \dfrac{\Gamma((\rho+1)/|\gamma|)}{\Gamma(\rho/|\gamma|)}, & \gamma < 0 \end{cases} \tag{5}$$

(both $\alpha_\gamma(\rho)$ and $\alpha_\gamma(\rho) + 1/\gamma$ are positive when $\rho > 0$). In Figure 1, the shapes of the Amoroso kernels $K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c), \gamma}$, $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$, $b = 0.25, 1$, and $c = 1$, are shown for the case

Figure 1: Shapes of the kernels $K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c),\gamma}$, $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$ ($c = 1$).




$x = 0, 2, 5$. We observe that the kernel concentrates at $s = x$, as $b \to 0$, and that its shape varies according to the position $x \geq 0$ where the density estimation is made. Note that the squared kernel is tractable, as follows:

$$\{K^{(A)}_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}(s)\}^2 = b^{-1}|\gamma| v_\gamma(\rho) K^{(A)}_{2\alpha_\gamma(\rho)-1/\gamma, b\beta_\gamma(\rho)/2^{1/\gamma},\gamma}(s),$$

where $v_\gamma$ is an infinitely differentiable function on $(1/2, \infty)$, defined by[4]

$$v_\gamma(\rho) = \frac{\Gamma(2\alpha_\gamma(\rho) - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1/\gamma)}{2^{2\alpha_\gamma(\rho)-1/\gamma}\rho\Gamma^3(\alpha_\gamma(\rho))}.$$

Such an Amoroso kernel density estimator (4) is general, in the following sense. First, Chen (2000) introduced the gamma ($\gamma = 1$) kernel density estimators $\widetilde{f}_{b,1,1}$ ($\rho_1(t) = t + 1$) and $\widetilde{f}_{b,0,1}$ ($d = 2$ and $r_0(t) = t^2/4 + 1$), and then showed that $\widetilde{f}_{b,0,1}$ is superior to $\widetilde{f}_{b,1,1}$ in the AMISE sense. Second,

[4]By definition (see (5)), we see that

$$2\alpha_\gamma(\rho) - 1/\gamma = \begin{cases} \dfrac{2\rho - 1}{\gamma}, & \gamma > 0, \\ \dfrac{2\rho + 3}{|\gamma|}, & \gamma < 0 \end{cases}$$

is positive when $\rho > 1/2$.

Igarashi and Kakizawa (2014b) recommended the gamma kernel density estimator $\widetilde{f}_{b,1/4,1}$ ($d = 2$ and $r_{1/4}(t) = (5/4)(t/2)^{8/5} + 1$), by proving that the AMISE of the gamma kernel density estimator $\widetilde{f}_{b,c,1}$ ($d = 2$ and $r_c(t) = (c+1)(t/2)^{2/(c+1)} + 1$) is minimized at $c = 1/4$. Third, the Nakagami ($\gamma = 2$) kernel density estimator $\widetilde{f}_{b,0,2}$ ($d = 2$ and $r_0(t) = t^2/4 + 1$) was discussed by Hirukawa and Sakudo (2015), as a special case of GG kernel density estimator ($\gamma > 0$). Fourth, Kakizawa and Igarashi (2017) developed the inverse gamma ($\gamma = -1$) kernel density estimator $\widetilde{f}_{b,c,-1}$, and showed that $c = 1/4$ is the best choice in the AMISE sense; interestingly, the estimator $\widetilde{f}_{b,1/4,-1}$ has the same optimal AMISE as the gamma kernel density estimator $\widetilde{f}_{b,1/4,1}$, suggested in Igarashi and Kakizawa (2014b). Fifth, it is possible to adopt a more general form of $r_c(t) = (c+d-c')(t/d)^{d/(c+d-c')} + c'$, where $c+d > c' \geq 1$ (see Igarashi (2016b)). To ensures the smoothness of the resulting density estimator $\widetilde{f}_{b,c,\gamma}$, making use of a smooth $\rho$-function $\rho_c$ is recommended, with the derivative $\rho_c'(t) \geq 0$ for $t \geq 0$.

We make some technical comments on (3) and (5). It may be true that the definition (5), depending on the sign of $\gamma > 0$ or $\gamma < 0$, is possibly inconvenient. But, we emphasize that, by construction,

$$\int_0^\infty sK^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)ds = b\rho \quad \text{for any } \gamma \neq 0 \text{ and } \rho > 0.$$

When $\gamma > 0$, the following moment exists for any $j \in \mathbb{N}$;

$$\int_0^\infty s^j K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)ds = (b\rho)^j \frac{\Gamma^{j-1}(\rho/\gamma)\Gamma((\rho+j)/\gamma)}{\Gamma^j((\rho+1)/\gamma)} \quad \text{if } \rho > 0,$$

but the resulting kernel $K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}$ is bounded if $\rho \geq 1$, i.e.,

$$\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \frac{\gamma\{(\rho-1)/\gamma\}^{(\rho-1)/\gamma}e^{-(\rho-1)/\gamma}\Gamma((\rho+1)/\gamma)}{b\rho\Gamma^2(\rho/\gamma)}$$

($0^0$ is understood to be 1). On the other hand, when $\gamma < 0$, we always have

$$\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \frac{|\gamma|\{(\rho+2)/|\gamma|\}^{(\rho+2)/|\gamma|}e^{-(\rho+2)/|\gamma|}\Gamma(\rho/|\gamma|)}{b\rho\Gamma^2((\rho+1)/|\gamma|)} \quad \text{if } \rho > 0,$$

but we must pay attention to the fact that

$$\int_0^\infty s^j K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)ds = (b\rho)^j \frac{\Gamma^{j-1}((\rho+1)/|\gamma|)\Gamma((\rho+1-j)/|\gamma|)}{\Gamma^j(\rho/|\gamma|)}$$

is well-defined if $j < \rho+1$. We assume the non-decreasingness of the $\rho$-function $\rho_c$ (see (3)), throughout this paper; hence, setting $\rho = \rho_c(x/b)$ with $r_c(0) > 0$, a sufficient condition for the existence of the moment, for each $x \geq 0$, is $\max(0, j-1) < r_c(0) = \inf_{x\geq 0} \rho_c(x/b)$. Since some arguments rely on the case $j = 2, \ldots, \ell$, where $\ell \in \mathbb{N}$, a further restriction $r_c(0) > \ell - 1$ is globally required for the case

$\gamma < 0$. Therefore, the restriction "$r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > \ell - 1$ when $\gamma < 0$" will be often imposed, unless $x/b \to \infty$.

Here, we are interested in studying how the choice of $\gamma \neq 0$ affects on the AMISE performance. In order to derive asymptotic properties of the estimator (4), we make the following assumptions:

A1. $\mathcal{X}^{(n)} = \{X_1, \ldots, X_n\}$ is a random sample from an unknown density $f$ with support $[0, \infty)$.

A2. $b > 0$ is a smoothing parameter satisfying $b \to 0$ and $nb \to \infty$.

A3. (i) $f$ is twice continuously differentiable on $[0, \infty)$. (ii) $f''$ is Hölder continuous, i.e., there exist constants $L_2 > 0$ and $\eta_2 \in (0, 1]$ such that $|f''(s) - f''(t)| \leq L_2 |s - t|^{\eta_2}$ for any $s, t \geq 0$. Also, (iii) $f$, $f'$, and $f''$ are bounded, i.e., $C_0 = \sup_{x \geq 0} f(x)$ and $C_i = \sup_{x \geq 0} |f^{(i)}(x)|$, $i = 1, 2$ are finite.

A4. (i) $\int_0^\infty \{f'(x)\}^2 dx$ and $\int_0^\infty \{x f''(x)\}^2 dx$ are finite. (ii) $\int_0^\infty x^{k_2+1} f(x) dx$ is finite for some constant $k_2 > (\eta_2 + 6)/\eta_2$, where $\eta_2 \in (0, 1]$ is given in A3.

We suppose that Assumption A1 holds without loss of generality. Actually, if the support of the underlying density $f$ is $[\delta, \infty)$ for some known constant $\delta$, then, $x$ and $X_i$ in the definition (4) (see also (14), (21), (22), (23), and (31)) should read as $x - \delta$ and $X_i - \delta$, respectively. It is important to consider the case where $\delta$ is unknown. Probably, the plug-in approach, with $\widehat{\delta} = \min(X_1, \ldots, X_n)$, would be a solution. However, we do not pursue this topic here.

Assumptions A2, A3, and A4 (i) (and additional requirements on the decay of $b \to 0$; see the results presented below) are standard. Especially, Assumption A3 is required for the bias approximation (Theorem 1). Assumption A4 (ii) is imposed to validate the asymptotic expansion for the MISE (see the comment before Theorem 4); the details are included in Appendix A2. As usual, the MISE of the estimator $\widehat{f}$ is defined as $MISE[\widehat{f}] = \int_0^\infty MSE[\widehat{f}(x)] dx$, where $MSE[\widehat{f}(x)] = E[\{\widehat{f}(x) - f(x)\}^2]$ is the mean squared error (MSE) of $\widehat{f}(x)$.

We are ready to describe the asymptotic properties of the Amoroso kernel density estimator (4).

**Theorem 1** *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > 1$ when $\gamma < 0$ (see the comment; $\ell = 2$ at the third paragraph of this section). Suppose that Assumptions* A1–A3 *hold, and define*

$$B_{c|\gamma|}(x) = c|\gamma| f'(x) + x \frac{f''(x)}{2}, \quad V(x) = \frac{f(x)}{2\sqrt{\pi x}}.$$

*Then,*

$$Bias[\widetilde{f}_{b,c,\gamma}(x)] = \begin{cases} b\dfrac{B_{c|\gamma|}(x)}{|\gamma|} + \mathcal{E}_{b,c,\gamma}(x), & \dfrac{x}{b} \to \infty, \\ b\{\rho_c(\kappa) - \kappa\}f'(0) + o(b), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ br_c(0)f'(0) + O(b^2), & x = 0, \end{cases}$$

$$V[\widetilde{f}_{b,c,\gamma}(x)] = \begin{cases} n^{-1}b^{-1/2}|\gamma|^{1/2}V(x)\{1 + O(bx^{-1})\} + O(n^{-1}), & \dfrac{x}{b} \to \infty, \\ n^{-1}b^{-1}|\gamma|f(0)\{v_\gamma(\rho_c(\kappa)) + o(1)\} + O(n^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1}b^{-1}|\gamma|v_\gamma(r_c(0))f(0) + O(n^{-1}), & x = 0, \end{cases}$$

*where* $\mathcal{E}_{b,c,\gamma}(x) = O(b^2 + (bx)^{1+\eta_2/2})$ *for* $x/b \to \infty$. *Here and subsequently,* $\kappa \geq 0$ *is a constant.*

The introduction of the parameter $c$ leaves the derivative $f'$ for the leading term of the bias of the estimator (4), being opposite to Chen (2000). As will be shown at the end of this section, the AMISE of the estimator (4) can be minimized with respect to $c$. That is, Chen (2000)'s choice $c = 0$ (see also Hirukawa and Sakudo (2015)) is not optimal in the AMISE sense.

**Remark 1** Suppose that Assumptions A1, A2, and A3 (i) and (iii) hold.
(i). Given $\gamma \neq 0$, choose $r_c(0) \geq 1$. We have

$$Bias[\widetilde{f}_{b,c,\gamma}(x)] = \begin{cases} O(b + bx), & \dfrac{x}{b} \to \infty, \\ O(b), & \dfrac{x}{b} \to \kappa. \end{cases} \tag{6}$$

Also,

$$\sup_{x\in[0,b^\tau]} |Bias[\widetilde{f}_{b,c,\gamma}(x)]| \leq C_1(2b^\tau + bc) \quad \text{for any } \tau \in (0,1), \tag{7}$$

$$\sup_{x\geq 0} V[\widetilde{f}_{b,c,\gamma}(x)] \leq n^{-1}b^{-1}C_0\widetilde{L}_\gamma. \tag{8}$$

(ii). Given $\gamma \neq 0$, choose $r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > 1$ when $\gamma < 0$ (see the comment; $\ell = 2$ at the third paragraph of this section). We have

$$\sup_{x\in[0,b^\tau]} \left| Bias[\widetilde{f}_{b,c,\gamma}(x)] - b\left\{\rho_c(x/b) - \frac{x}{b}\right\}f'(x)\right| = O(b^{2\tau}) \quad \text{for any } \tau \in (0,1). \tag{9}$$

From Theorem 1, the estimator (4) is (pointwise) weak consistent, i.e.,

$$MSE[\widetilde{f}_{b,c,\gamma}(x)] = \begin{cases} b^2\left\{\dfrac{B_{c|\gamma|}(x)}{|\gamma|}\right\}^2 + n^{-1}b^{-1/2}|\gamma|^{1/2}V(x) + O(b^{2+\eta_2/2} + n^{-1}) & \text{for fixed } x > 0, \\ b^2\{r_c(0)f'(0)\}^2 + n^{-1}b^{-1}|\gamma|v_\gamma(r_c(0))f(0) + O(b^3 + n^{-1}) & \text{for } x = 0 \end{cases}$$

tends to zero (for fixed $x > 0$, assume $b \to 0$ and $nb^{1/2} \to \infty$, instead of Assumption A2). We also prove the (pointwise) strong consistency and asymptotic normality of the estimator (4).

**Theorem 2** *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$. Suppose that Assumptions* A1 *and* A3 (i) *and* (iii) *hold. If $b \to 0$ and $nb/\log n \to \infty$, then, $\widetilde{f}_{b,c,\gamma}(x) \overset{a.s.}{\to} f(x)$ for fixed $x \geq 0$.*

**Theorem 3** *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$. Suppose that Assumptions* A1, A2, *and* A3 (i) *and* (iii) *hold. Then,*

(i). $(nb^{1/2})^{1/2}\{\widetilde{f}_{b,c,\gamma}(x) - E[\widetilde{f}_{b,c,\gamma}(x)]\} \overset{d}{\to} N(0, |\gamma|^{1/2}V(x))$ *for fixed $x > 0$ (here, $b \to 0$ and $nb^{1/2} \to \infty$ are assumed, instead of Assumption* A2*),*

(ii). $(nb)^{1/2}\{\widetilde{f}_{b,c,\gamma}(0) - E[\widetilde{f}_{b,c,\gamma}(0)]\} \overset{d}{\to} N(0, |\gamma|v_\gamma(r_c(0))f(0))$.

**Theorem 3′** *Suppose that Assumptions* A1–A3 *hold.*

(i). *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$. If $nb^{1/2} \to \infty$ and $nb^{5/2+\eta_2} \to 0$, where $\eta_2 \in (0, 1]$ is given in Assumption* A3*, then, for fixed $x > 0$,*

$$(nb^{1/2})^{1/2}\Big\{\widetilde{f}_{b,c,\gamma}(x) - f(x) - b\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\} \overset{d}{\to} N(0, |\gamma|^{1/2}V(x)),$$

*hence, if, in addition, $nb^{5/2} \to 0$, then, $(nb^{1/2})^{1/2}\{\widetilde{f}_{b,c,\gamma}(x) - f(x)\} \overset{d}{\to} N(0, |\gamma|^{1/2}V(x))$.*

(ii). *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > 1$ when $\gamma < 0$ (see the comment; $\ell = 2$ at the third paragraph of this section). If $nb^5 \to 0$, then,*

$$(nb)^{1/2}\{\widetilde{f}_{b,c,\gamma}(0) - f(0) - br_c(0)f'(0)\} \overset{d}{\to} N(0, |\gamma|v_\gamma(r_c(0))f(0)),$$

*hence, if, in addition, $nb^3 \to 0$, then, $(nb)^{1/2}\{\widetilde{f}_{b,c,\gamma}(0) - f(0)\} \overset{d}{\to} N(0, |\gamma|v_\gamma(r_c(0))f(0))$.*

We notice that the convergence rate of the MSE of the estimator (4) near the boundary is slower than that in the interior, i.e.,

$$MSE[\widetilde{f}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-4/5}) & \text{for fixed } x > 0 \text{ (using } b \propto n^{-2/5}), \\ O(n^{-2/3}) & \text{for } x/b \to \kappa \text{ (using } b \propto n^{-1/3}). \end{cases} \tag{10}$$

However, (7) and (8) imply that $\int_0^{b^{\tau_1}} MSE[\widetilde{f}_{b,c,\gamma}(x)]dx = O(b^{3\tau_1} + n^{-1}b^{\tau_1 - 1}) = o(b^2 + n^{-1}b^{-1/2})$ if $\tau_1 \in (2/3, 1)$, and, as will be shown rigorously in Appendix A2, $\int_{b^{-\tau_2}}^{\infty} MSE[\widetilde{f}_{b,c,\gamma}(x)]dx$ is indeed asymptotically negligible, with a suitable choice $\tau_2 \in (0, 1)$ under Assumption A4 (ii); such a different rate phenomenon (10) has negligible impact on the MISE.

**Theorem 4** *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$. Suppose that Assumptions* A1–A4 *hold. Then,*

$$MISE[\widetilde{f}_{b,c,\gamma}] = AMISE[\widetilde{f}_{b,c,\gamma}] + o(b^2 + n^{-1}b^{-1/2}),$$

*where*

$$AMISE[\widetilde{f}_{b,c,\gamma}] = b^2 \int_0^\infty \Big\{\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\}^2 dx + n^{-1}b^{-1/2}\int_0^\infty |\gamma|^{1/2}V(x)dx.$$

The AMISE of the estimator (4) is minimized at

$$b = |\gamma| \left\{ \frac{\int_0^\infty V(x)dx}{4\int_0^\infty B_{c|\gamma|}^2(x)dx} \right\}^{2/5} n^{-2/5}, \tag{11}$$

when $B_{c|\gamma|}(x) \not\equiv 0$; its optimal AMISE

$$\min_{b>0} AMISE[\widetilde{f}_{b,c,\gamma}] = \frac{5}{4^{4/5}} \left\{ \int_0^\infty B_{c|\gamma|}^2(x)dx \right\}^{1/5} \left\{ \int_0^\infty V(x)dx \right\}^{4/5} n^{-4/5} \tag{12}$$

depends only on $c|\gamma|$. This, together with Proposition 3 of Igarashi and Kakizawa (2014b), i.e.,

- If Assumptions A3 (i) and A4 (i) hold, and $x\{f'(x)\}^2 \to 0$ as $x \to \infty$, then, the leading term of the integrated squared bias; $\int_0^\infty B_{c|\gamma|}^2(x)dx$ is minimized at $c|\gamma| = 1/4$,

yields

$$\min_{b>0} AMISE[\widetilde{f}_{b,c,\gamma}] \geq \frac{5}{4^{4/5}} \left\{ \int_0^\infty B_{1/4}^2(x)dx \right\}^{1/5} \left\{ \int_0^\infty V(x)dx \right\}^{4/5} n^{-4/5}, \tag{13}$$

independent of $\gamma \neq 0$. Therefore, using the optimal smoothing parameter (11), $c|\gamma| = 1/4$ is the best choice for the Amoroso kernel density estimator (4). This finding is a substantial extension of Chen (2000), Igarashi and Kakizawa (2014b), Hirukawa and Sakudo (2015), and Kakizawa and Igarashi (2017).

## 3. Bias reductions

This section aims at improving the above-mentioned rates (10) and (12).

### 3.1. Additive bias reduction

For the standard kernel density estimator (1), the additive bias reduction method using the kernel (2) was originally developed by Schucany and Sommers (1977). In principle, the Schucany–Sommers (SS) type bias reduction method is applicable for any density estimator $\widehat{g}_\beta$ with smoothing parameter $\beta \to 0$, such that the bias is given by $Bias[\widehat{g}_\beta(x)] = \beta^q B^{[1]}(x) + o(\beta^q)$ for some constant $q > 0$ and function $B^{[1]}$, independent of $\beta$. Constructing a linear combination of two density estimators with different smoothing parameters $\beta$ and $\beta/a$, for each $a \in (0, 1)$[5], i.e.,

$$\widehat{g}_\beta^{(SS_a)}(x) = \frac{1}{1-a^q}\widehat{g}_\beta(x) - \frac{a^q}{1-a^q}\widehat{g}_{\beta/a}(x), \quad \text{yields} \quad Bias[\widehat{g}_\beta^{(SS_a)}(x)] = o(\beta^q).$$

[5] Igarashi and Kakizawa (2015) formulated the SS type bias-reduced density estimator as

$$\frac{a^q}{a^q-1}\widehat{g}_\beta(x) - \frac{1}{a^q-1}\widehat{g}_{a\beta}(x), \quad a > 1,$$

i.e., the parameter $a$ in Igarashi and Kakizawa (2015) should read as $1/a$.

Further, if $Bias[\widehat{g}_\beta(x)] = \sum_{i=1}^2 \beta^{iq} B^{[i]}(x) + o(\beta^{2q})$, then, $Bias[\widehat{g}_\beta^{(SS_a)}(x)] = -\beta^{2q} B^{[2]}(x)/a^q + o(\beta^{2q})$.

Now, we make the following assumptions (these assumptions are variants of Assumptions A3 and A4 in Section 2):

A3′. (i) $f$ is four times continuously differentiable on $[0, \infty)$. (ii) $f^{(4)}$ is Hölder continuous, i.e., there exist constants $L_4 > 0$ and $\eta_4 \in (0, 1]$ such that $|f^{(4)}(s) - f^{(4)}(t)| \leq L_4 |s - t|^{\eta_4}$ for any $s, t \geq 0$. Also, (iii) $f$, $f'$, $f''$, $f^{(3)}$, and $f^{(4)}$ are bounded, i.e., $C_0 = \sup_{x \geq 0} f(x)$ and $C_i = \sup_{x \geq 0} |f^{(i)}(x)|$, $i = 1, 2, 3, 4$ are finite.

A4′. (i) $\int_0^\infty \{f''(x)\}^2 dx$, $\int_0^\infty \{x f^{(3)}(x)\}^2 dx$, and $\int_0^\infty \{x^2 f^{(4)}(x)\}^2 dx$ are finite. (ii) $\int_0^\infty x^{k_4+1} f(x) dx$ is finite for some constant $k_4 > (3\eta_4 + 20)/\eta_4$, where $\eta_4 \in (0, 1]$ is given in A3′.

The following result is fundamental.

**Theorem 5** (i). *Given $\gamma \neq 0$, choose $r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > 2$ when $\gamma < 0$ (see the comment; $\ell = 3$ at the third paragraph of Section 2). Suppose that Assumptions A1, A2, and A3′ hold. Then,*

$$E[\widetilde{f}_{b,c,\gamma}(x)] = \begin{cases} f(x) + b\dfrac{B_{c|\gamma|}(x)}{|\gamma|} + b^2 \dfrac{B_{c,\gamma}^{[2]}(x)}{\gamma^2} + O(b^3 x^{-1} + \{b(1+x)\}^{2+\eta_4/2}), & \dfrac{x}{b} \to \infty, \\ f(x) + b\Big\{\rho_c(x/b) - \dfrac{x}{b}\Big\} f'(x) + b^2 \eta_\gamma(\kappa, \rho_c(\kappa)) \dfrac{f''(0)}{2} + o(b^2), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ f(0) + b r_c(0) f'(0) + b^2 \eta_\gamma(0, r_c(0)) \dfrac{f''(0)}{2} + O(b^3), & x = 0, \end{cases}$$

*where*

$$B_{c,\gamma}^{[2]}(x) = \delta_{c,\gamma}^{[2]} \frac{f''(x)}{2} + \delta_{c,\gamma}^{[3]} x \frac{f^{(3)}(x)}{6} + 3x^2 \frac{f^{(4)}(x)}{24}$$

*with*

$$\delta_{c,\gamma}^{[2]} = \begin{cases} \dfrac{1}{2}\{(2c^2+1)\gamma^2 + 2(c-1)\gamma + 1\}, & \gamma > 0, \\ \dfrac{1}{2}\{(2c^2+1)\gamma^2 + 2c|\gamma| + 1\}, & \gamma < 0, \end{cases} \qquad \delta_{c,\gamma}^{[3]} = \begin{cases} (3c-1)\gamma + 3, & \gamma > 0, \\ (3c+1)|\gamma| + 3, & \gamma < 0, \end{cases}$$

*and $\eta_\gamma$ is a function on $[0, \infty) \times [r_c(0), \infty)$, defined by*[6]

$$\eta_\gamma(\kappa, \rho) = \rho^2 \frac{\Gamma(\alpha_\gamma(\rho))\Gamma(\alpha_\gamma(\rho) + 2/\gamma)}{\Gamma^2(\alpha_\gamma(\rho) + 1/\gamma)} - 2\kappa\rho + \kappa^2.$$

[6]By definition (see (5)), we see that

$$\alpha_\gamma(\rho) + 2/\gamma = \begin{cases} \dfrac{\rho+2}{\gamma}, & \gamma > 0, \\ \dfrac{\rho-1}{|\gamma|}, & \gamma < 0 \end{cases}$$

is positive when $\rho \geq r_c(0)$, provided that "$r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > 1$ when $\gamma < 0$".

(ii). *Given* $\gamma \neq 0$, *choose* $r_c(0) \geq 1$. *Suppose that Assumptions* A1, A2, *and* A3 (i) *and* (iii) *hold. Then, for* $a_1, a_2 \in (0, 1]$,

$$
\begin{aligned}
&Cov[\widetilde{f}_{b/a_1,c,\gamma}(x), \widetilde{f}_{b/a_2,c,\gamma}(x)] \\
&= \begin{cases} n^{-1}b^{-1/2}\Big(\dfrac{2a_1a_2}{a_1+a_2}\Big)^{1/2}|\gamma|^{1/2}V(x)\{1+O(bx^{-1})\} + O(n^{-1}), & \dfrac{x}{b} \to \infty, \\ n^{-1}b^{-1}|\gamma|f(0)\{\delta_{\gamma,a_1,a_2}(\rho_c(a_1\kappa), \rho_c(a_2\kappa)) + o(1)\} + O(n^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1}b^{-1}|\gamma|f(0)\delta_{\gamma,a_1,a_2}(r_c(0), r_c(0)) + O(n^{-1}), & x = 0, \end{cases}
\end{aligned}
$$

*where* $\delta_{\gamma,a_1,a_2}$ *is an infinitely differentiable function on* $(1/2, \infty)^2$, *defined by*

$$
\delta_{\gamma,a_1,a_2}(\rho_1, \rho_2) = \frac{\{\beta_\gamma(\rho_1)/a_1\}^{\alpha_\gamma(\rho_2)\gamma-1}\{\beta_\gamma(\rho_2)/a_2\}^{\alpha_\gamma(\rho_1)\gamma-1}\Gamma(\alpha_\gamma(\rho_1)+\alpha_\gamma(\rho_2)-1/\gamma)}{[\{\beta_\gamma(\rho_1)/a_1\}^\gamma + \{\beta_\gamma(\rho_2)/a_2\}^\gamma]^{\alpha_\gamma(\rho_1)+\alpha_\gamma(\rho_2)-1/\gamma}\Gamma(\alpha_\gamma(\rho_1))\Gamma(\alpha_\gamma(\rho_2))}.
$$

In Theorems 6, 8′, and 9, we must use the linear $\rho$-function $\rho_c(t) = t + c$ (some critical comments for the use of two-regime $\rho$-function (3), rather than $\rho_c(t) = t + c$, will be given in Section 4). In what follows, we use the notation

$$
\widehat{f}_{b,c,\gamma}(x) = \frac{1}{n}\sum_{i=1}^{n} K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c), \gamma}(X_i), \quad x \geq 0.
$$

We apply the SS type bias reduction method (we set $q = 1$) to the Amoroso kernel density estimator $\widehat{f}_{b,c,\gamma}$, i.e., for each $a \in (0, 1)$, we can construct an estimator

$$
\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) = \frac{1}{1-a}\widehat{f}_{b,c,\gamma}(x) - \frac{a}{1-a}\widehat{f}_{b/a,c,\gamma}(x) = \frac{1}{n}\sum_{i=1}^{n} K^{(SS_a)}_{b,c,\gamma,x}(X_i), \quad x \geq 0, \tag{14}
$$

where

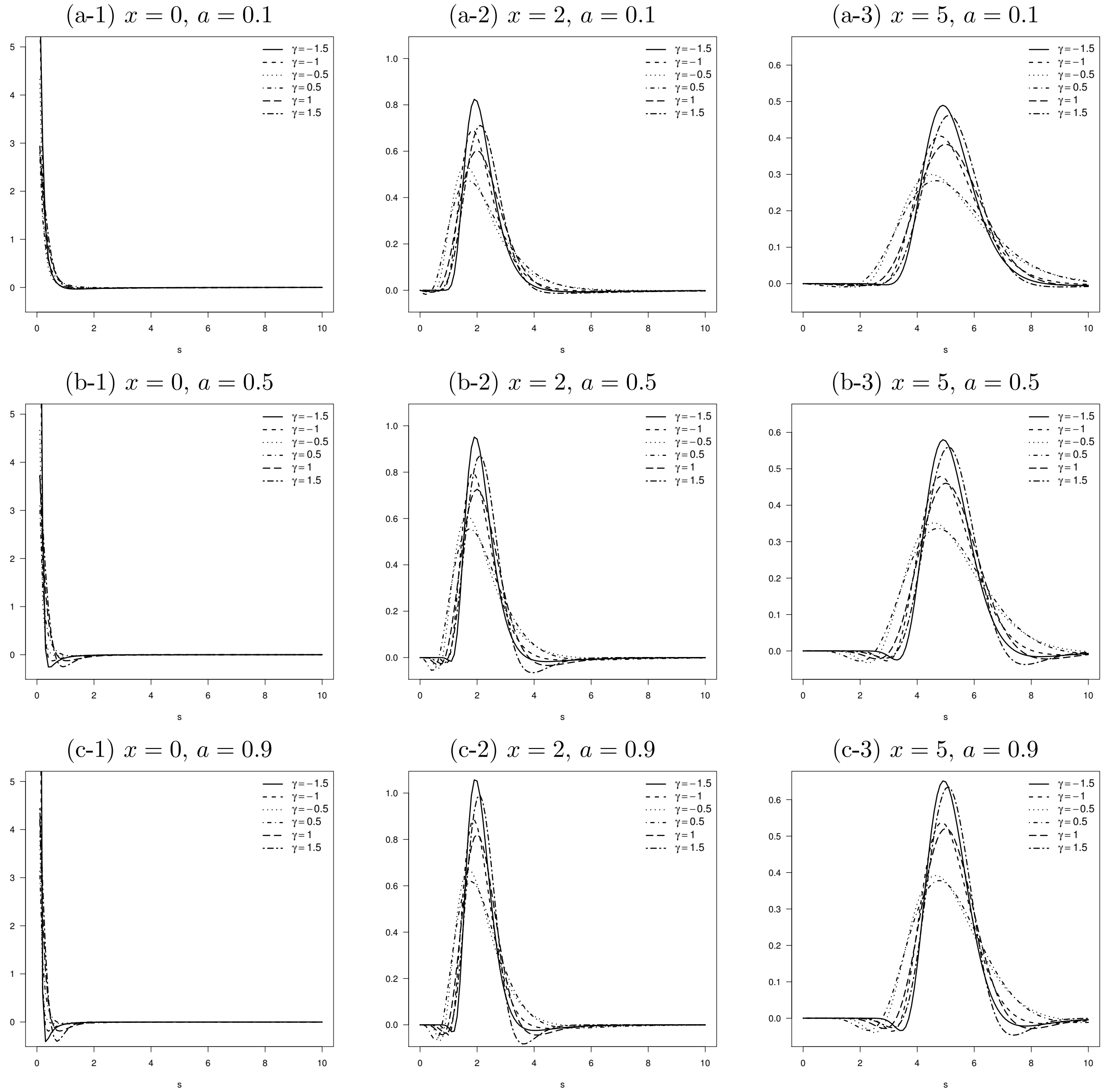
$$
K^{(SS_a)}_{b,c,\gamma,x}(s) = \frac{1}{1-a}\big\{K^{(A)}_{\alpha_\gamma(x/b+c), b\beta_\gamma(x/b+c), \gamma}(s) - aK^{(A)}_{\alpha_\gamma(ax/b+c), (b/a)\beta_\gamma(ax/b+c), \gamma}(s)\big\}.
$$

In Figure 2, the shapes of the SS type kernels $K^{(SS_a)}_{b,c,\gamma,x}$, $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$, $a = 0.1, 0.5, 0.9$, $b = 0.25$, and $c = 1$, are shown for the case $x = 0, 2, 5$. We observe that it becomes a sharp kernel as $a \to 1$, but, instead, it loses the nonnegativity, to a very small extent. Also, its shape varies according to the position $x \geq 0$ where the density estimation is made.

Compared to Theorem 1, the following theorem shows that the bias is reduced from $O(b)$ to $O(b^2)$, whereas the variance remains to be order $n^{-1}b^{-1/2}$ (except for the factor $\lambda(a) \geq 1$) when $x/b \to \infty$, where

$$
\lambda(a) = \frac{1}{(1-a)^2}\Big\{1 + a^{5/2} - 2a\Big(\frac{2a}{a+1}\Big)^{1/2}\Big\}.
$$

Figure 2: Shapes of the kernels $K^{(SS_a)}_{b,c,\gamma,x}$, $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$ ($b = 0.25$ and $c = 1$).



**Theorem 6** *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see the comment; $\ell = 3$ at the third paragraph of Section 2). Suppose that Assumptions A1, A2, and A3$'$ hold, and define*

$$\zeta_{c,\gamma}^{(SS_a)}(\kappa) = -\frac{1}{1-a}\{a\eta_\gamma(\kappa, \kappa + c) - \eta_\gamma(a\kappa, a\kappa + c)\},$$
$$v_{c,\gamma}^{(SS_a)}(\kappa) = \frac{1}{(1-a)^2}\big\{\delta_{\gamma,1,1}(\kappa + c, \kappa + c) + a^2\delta_{\gamma,a,a}(a\kappa + c, a\kappa + c) - 2a\delta_{\gamma,1,a}(\kappa + c, a\kappa + c)\big\}.$$

*Then,*

$$Bias[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] = \begin{cases} -\dfrac{b^2}{a}\dfrac{B_{c,\gamma}^{[2]}(x)}{\gamma^2} + \mathcal{E}_{b,c,\gamma}^{(SS_a)}(x), & \dfrac{x}{b} \to \infty, \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(SS_a)}(\kappa)\dfrac{f''(0)}{2} + o(b^2), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(SS_a)}(0)\dfrac{f''(0)}{2} + O(b^3), & x = 0, \end{cases}$$

$$V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] = \begin{cases} n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x)\{1 + O(bx^{-1})\} + O(n^{-1}), & \dfrac{x}{b} \to \infty, \\ n^{-1}b^{-1}|\gamma|f(0)\{v_{c,\gamma}^{(SS_a)}(\kappa) + o(1)\} + O(n^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1}b^{-1}|\gamma|f(0)v_{c,\gamma}^{(SS_a)}(0) + O(n^{-1}), & x = 0, \end{cases}$$

*where $\mathcal{E}_{b,c,\gamma}^{(SS_a)}(x) = O(b^3x^{-1} + \{b(1+x)\}^{2+\eta_4/2})$ for $x/b \to \infty$.*

**Remark 2** Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 1$ when $\gamma < 0$ (see the comment; $\ell = 2$ at the third paragraph of Section 2). Suppose that Assumptions A1, A2, and A3$'$ (i) and (iii) hold. Then,

$$Bias[\widehat{f}_{b,c,\gamma}(x)] = \begin{cases} b\dfrac{B_{c|\gamma|}(x)}{|\gamma|} + O(b^2 + b^2x^2), & \dfrac{x}{b} \to \infty, \\ bcf'(0) + O(b^2), & \dfrac{x}{b} \to \kappa, \end{cases} \tag{15}$$

hence,

$$Bias[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] = \begin{cases} O(b^2 + b^2x^2), & \dfrac{x}{b} \to \infty, \\ O(b^2), & \dfrac{x}{b} \to \kappa. \end{cases} \tag{16}$$

Also, we have, for any $\tau \in (0, 1)$,

$$\sup_{x\in[0,b^\tau]} |Bias[\widehat{f}_{b,c,\gamma}(x)]| = O(b^{\min(1,2\tau)}) \tag{17}$$

and

$$\begin{aligned} \sup_{x\in[0,b^\tau]} |Bias[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]| &= \frac{1}{1-a}\sup_{x\in[0,b^\tau]}\left|\{Bias[\widehat{f}_{b,c,\gamma}(x)] - bcf'(x)\} - a\Big\{Bias[\widehat{f}_{b/a,c,\gamma}(x)] - \frac{b}{a}cf'(x)\Big\}\right| \\ &= O(b^{2\tau}), \end{aligned} \tag{18}$$

using (9) with $\rho_c(t) = t + c$.

From Theorem 6, the estimator (14) is (pointwise) weak consistent, i.e.,

$$MSE[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] = \begin{cases} \dfrac{b^4}{a^2}\Big\{\dfrac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 + n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x) + O(b^{4+\eta_4/2} + n^{-1}) & \text{for fixed } x>0, \\ \dfrac{b^4}{a^2}\Big\{\zeta^{(SS_a)}_{c,\gamma}(0)\dfrac{f''(0)}{2}\Big\}^2 + n^{-1}b^{-1}|\gamma|v^{(SS_a)}_{c,\gamma}(0)f(0) + O(b^5 + n^{-1}) & \text{for } x=0 \end{cases}$$

tends to zero (for fixed $x>0$, assume $b \to 0$ and $nb^{1/2} \to \infty$, instead of Assumption A2). The (pointwise) strong consistency and asymptotic normality of the estimator (14) can be proved.

**Theorem 7** *Given $\gamma \neq 0$, choose $c \geq 1$. Suppose that Assumptions* A1 *and* A3 (i) *and* (iii) *hold. If $b \to 0$ and $nb/\log n \to \infty$, then, $\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) \xrightarrow{a.s.} f(x)$ for fixed $x \geq 0$.*

**Theorem 8** *Given $\gamma \neq 0$, choose $c \geq 1$. Suppose that Assumptions* A1, A2, *and* A3′ (i) *and* (iii) *hold. Then,*

(i). $(nb^{1/2})^{1/2}\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - E[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x))$ *for fixed $x>0$ (here, $b \to 0$ and $nb^{1/2} \to \infty$ are assumed, instead of Assumption* A2*),*

(ii). $(nb)^{1/2}\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(0) - E[\widehat{f}^{(SS_a)}_{b,c,\gamma}(0)]\} \xrightarrow{d} N(0, |\gamma|v^{(SS_a)}_{c,\gamma}(0)f(0))$.

**Theorem 8′** *Suppose that Assumptions* A1, A2, *and* A3′ *hold.*

(i). *Given $\gamma \neq 0$, choose $c \geq 1$. If $nb^{1/2} \to \infty$ and $nb^{9/2+\eta_4} \to 0$, where $\eta_4 \in (0,1]$ is given in Assumption* A3′*, then, for fixed $x>0$,*

$$(nb^{1/2})^{1/2}\Big\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x) + \frac{b^2}{a}\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x)),$$

*hence, if, in addition, $nb^{9/2} \to 0$, then, $(nb^{1/2})^{1/2}\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x)\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x))$.*

(ii). *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see the comment; $\ell = 3$ at the third paragraph of Section* 2*). If $nb^7 \to 0$, then,*

$$(nb)^{1/2}\Big\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(0) - f(0) - \frac{b^2}{a}\zeta^{(SS_a)}_{c,\gamma}(0)\frac{f''(0)}{2}\Big\} \xrightarrow{d} N(0, |\gamma|v^{(SS_a)}_{c,\gamma}(0)f(0)),$$

*hence, if, in addition, $nb^5 \to 0$, then, $(nb)^{1/2}\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(0) - f(0)\} \xrightarrow{d} N(0, |\gamma|v^{(SS_a)}_{c,\gamma}(0)f(0))$.*

We notice that the convergence rate of the MSE of the estimator (14) near the boundary is slower than that in the interior, i.e.,

$$MSE[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-8/9}) & \text{for fixed } x>0 \text{ (using } b \propto n^{-2/9}), \\ O(n^{-4/5}) & \text{for } x/b \to \kappa \text{ (using } b \propto n^{-1/5}). \end{cases} \tag{19}$$

However, (8) and (18) imply that $\int_0^{\tau_1} MSE[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]dx = O(b^{5\tau_1} + n^{-1}b^{\tau_1-1}) = o(b^4 + n^{-1}b^{-1/2})$ if $\tau_1 \in (4/5, 1)$, and, as will be shown rigorously in Appendix A3, $\int_{b^{-\tau_2}}^{\infty} MSE[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]dx$ is indeed asymptotically negligible, with a suitable choice $\tau_2 \in (0,1)$ under Assumption A4′ (ii); such a different rate phenomenon (19) has negligible impact on the MISE.

**Theorem 9** *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 1$ when $\gamma < 0$ (see Remark 2). Suppose that Assumptions A1, A2, A3′, and A4′ hold. Then,*

$$MISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}] = AMISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}] + o(b^4 + n^{-1}b^{-1/2}),$$

*where*

$$AMISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}] = \frac{b^4}{a^2}\int_0^\infty \Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\lambda(a)\int_0^\infty |\gamma|^{1/2}V(x)dx.$$

The AMISE of the estimator (14) is minimized at

$$b^{(SS_a)} = |\gamma|\{a^2\lambda(a)\}^{2/9}\Bigg[\frac{\int_0^\infty V(x)dx}{8\int_0^\infty \{B^{[2]}_{c,\gamma}(x)\}^2 dx}\Bigg]^{2/9} n^{-2/9},$$

when $B^{[2]}_{c,\gamma}(x) \not\equiv 0$, i.e., the optimal AMISE is given by

$$\min_{b>0} AMISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}] = \frac{9}{8^{8/9}}\Big\{\frac{\lambda^4(a)}{a}\Big\}^{2/9}\Big[\int_0^\infty \{B^{[2]}_{c,\gamma}(x)\}^2 dx\Big]^{1/9}\Big\{\int_0^\infty V(x)dx\Big\}^{8/9} n^{-8/9}, \tag{20}$$

whose convergence rate $n^{-8/9}$ is faster than $n^{-4/5}$ for the optimal AMISE (12). The factor $\{\lambda^4(a)/a\}^{2/9}$ is decreasing for $a \in (0,1)$, with $\lim_{a\to 1}\{\lambda^4(a)/a\}^{2/9} = (27/16)^{8/9}$. Note that, as shown in Section 4, the convergence rate is $n^{-6/7}$ when $\rho_c(t) \neq t + c$.

It should be remarked that the SS type (14) is not well-defined when $a = 1$. However, assuming that $\epsilon$ is independent of $a$, we can consider its limit as $a \to 1$, i.e.,

$$\widehat{f}^{(SS_1)}_{b,c,\gamma}(x) = \lim_{a\to 1}\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) = \widehat{f}_{b,c,\gamma}(x) - b\frac{\partial}{\partial b}\widehat{f}_{b,c,\gamma}(x) = \frac{1}{n}\sum_{i=1}^n K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(X_i)H^{(A)}_{b,c,\gamma,x/b+c}(X_i), \tag{21}$$

where

$$\begin{aligned} H^{(A)}_{b,c,\gamma,\rho}(s) = 1 &+ \frac{1}{|\gamma|}(\rho - c)\Big[\log\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma - \psi(\alpha_\gamma(\rho))\Big] \\ &+\gamma\Big[\Big\{\frac{s}{b\beta_\gamma(\rho)}\Big\}^\gamma - \alpha_\gamma(\rho)\Big]\Big[-\frac{c}{\rho} + \frac{1}{|\gamma|}(\rho - c)\{\psi(\alpha_\gamma(\rho)) - \psi(\alpha_\gamma(\rho) + 1/\gamma)\}\Big] \end{aligned}$$

($\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function). Theorems 6–9 remain valid for the limiting estimator (21); the details will be further discussed as a companion paper (Igarashi and Kakizawa (2017)).

### 3.2. Nonnegative bias reductions

By construction, the SS type bias-reduced estimator loses nonnegativity. Terrell and Scott (1980) and Jones and Foster (1993) proposed nonnegative bias reduction methods for the standard kernel density estimator (1). In recent years, there have been renewed interests on these topics. See Hirukawa (2010; correction 2016), Hirukawa and Sakudo (2014, 2015), Igarashi and Kakizawa (2014a, 2015), Funke and Kawka (2015), and Igarashi (2016a) for the asymmetric kernel density estimation.

We apply the Terrell–Scott (TS) type and Jones–Foster (JF) type bias reduction methods to the Amoroso kernel density estimator $\widehat{f}_{b,c,\gamma}$. In Theorems 10, 12′, and 13–15, we must use $\rho_c(t) = t + c$ (the case of two-regime $\rho$-function (3) will be discussed in Section 4). Following Igarashi (2016a), the TS and JF type bias-reduced Amoroso kernel density estimators, for each $a \in (0, 1)$, are, respectively, defined as

$$\widehat{f}^{(TS_a)}_{b,c,\gamma}(x) = \frac{\{\widehat{f}_{b,c,\gamma}(x) + \epsilon\}^{1/(1-a)}}{\{\widehat{f}_{b/a,c,\gamma}(x) + \epsilon/a\}^{a/(1-a)}}, \tag{22}$$

$$\widehat{f}^{(JF_a)}_{b,c,\gamma}(x) = \{\widehat{f}_{b,c,\gamma}(x) + \epsilon\} \exp\left\{\frac{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)}{\widehat{f}_{b,c,\gamma}(x) + \epsilon} - 1\right\}, \tag{23}$$

where the introduction of $\epsilon = \epsilon_b > 0$ enables us to avoid dividing by zero (in what follows, $\epsilon$ will be assumed to tend to zero at a suitable rate). Note that, assuming $f(x) > 0$, the stochastic expansions of the estimators (22) and (23) are given by

$$\widehat{f}^{(TS_a)}_{b,c,\gamma}(x) = \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) + \frac{\mathcal{Q}(x)}{2af(x)} + \mathcal{R}^{(TS_a)}(x), \tag{24}$$

$$\widehat{f}^{(JF_a)}_{b,c,\gamma}(x) = \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) + \frac{\mathcal{Q}(x)}{2f(x)} + \mathcal{R}^{(JF_a)}(x), \tag{25}$$

respectively, where $\mathcal{Q}(x) = \{\widehat{f}_{b,c,\gamma}(x) - \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) + \epsilon\}^2$,

$$\begin{aligned}
\mathcal{R}^{(TS_a)}(x) = \frac{f(x)}{2} \int_0^1 \sum_{\ell=0}^{3} {}_3C_\ell \left\{\frac{\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)}{f(x)}\right\}^{3-\ell} \left\{\frac{\widehat{f}_{b/a,c,\gamma}(x) + \epsilon/a - f(x)}{f(x)}\right\}^{\ell} \\
\times g^{(TS_a)}_{3-\ell,\ell}\left(\frac{\theta\{\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)\}}{f(x)}, \frac{\theta\{\widehat{f}_{b/a,c,\gamma}(x) + \epsilon/a - f(x)\}}{f(x)}\right)(1-\theta)^2 d\theta, \\
\mathcal{R}^{(JF_a)}(x) = \frac{f(x)}{2} \int_0^1 \sum_{\ell=0}^{3} {}_3C_\ell \left\{\frac{\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)}{f(x)}\right\}^{3-\ell} \left\{\frac{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x)}{f(x)}\right\}^{\ell} \\
\times g^{(JF)}_{3-\ell,\ell}\left(\frac{\theta\{\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)\}}{f(x)}, \frac{\theta\{\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x)\}}{f(x)}\right)(1-\theta)^2 d\theta,
\end{aligned}$$

with

$$g^{(TS_a)}_{i,j}(t, u) = \frac{\partial^{i+j}}{\partial t^i \partial u^j} \frac{(1+t)^{1/(1-a)}}{(1+u)^{a/(1-a)}}, \quad g^{(JF)}_{i,j}(t, v) = \frac{\partial^{i+j}}{\partial t^i \partial v^j} \left\{(1+t)\exp\left(\frac{1+v}{1+t} - 1\right)\right\}.$$

Heuristically, the stochastic expansions (24) and (25) mean that the TS and JF type bias-reduced estimators (22) and (23) have asymptotic properties similar to the SS type bias-reduced estimator (14), except for the additional terms $\mathcal{Q}(x)/\{2af(x)\}$ and $\mathcal{Q}(x)/\{2f(x)\}$. We must show that the remainder term $\mathcal{R}^{(\#_a)}(x)$'s are asymptotically negligible. Here and subsequently, let $\# = TS, JF$, unless otherwise stated.

In order to derive the asymptotic properties of the estimators (22) and (23), we make the following assumptions for some $\iota_1 \in (0,1)$ and $\iota_2 > 1$:

A2$'[\iota_1, \iota_2]$. $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$.

Note that, if $b \propto n^{-\iota_1}$ for some $\iota_1 \in (0,1)$, then, Assumption A2 holds. In what follows, we change the usual unweighted criterion to the weighted criterion $MISE_w[\widehat{f}] = \int_0^\infty w(x) MSE[\widehat{f}(x)]dx$, for a technical reason[7], where we always assume that the weight function $w$ is nonnegative, bounded, and continuous except for a finite number of discontinuities; we write $\overline{w} = \sup_{x \geq 0} w(x)$. For the notational simplicity, we define

$$B_{c,\gamma}^{(TS)}(x) = -\frac{B_{c|\gamma|}^2(x)}{2f(x)} + B_{c,\gamma}^{[2]}(x), \quad \zeta_{c,\gamma}^{(TS_a)}(\kappa) = -\frac{c^2\{f'(0)\}^2}{2f(0)} + \zeta_{c,\gamma}^{(SS_a)}(\kappa)\frac{f''(0)}{2},$$
$$B_{c,\gamma}^{(JF_a)}(x) = -\frac{aB_{c|\gamma|}^2(x)}{2f(x)} + B_{c,\gamma}^{[2]}(x), \quad \zeta_{c,\gamma}^{(JF_a)}(\kappa) = -\frac{ac^2\{f'(0)\}^2}{2f(0)} + \zeta_{c,\gamma}^{(SS_a)}(\kappa)\frac{f''(0)}{2}.$$

**Theorem 10** *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see the comment; $\ell = 3$ at the third paragraph of Section 2). Suppose that Assumptions A1, A2$'[\iota_1, \iota_2]$, and A3$'$ hold for some $\iota_1 \in (0,1)$ and $\iota_2 > 1$. Then, on $\mathcal{I} = \{x \in [0,r] \mid f(x) \geq \varrho\}$ ($r, \varrho > 0$ are constants),*

$$Bias[\widehat{f}_{b,c,\gamma}^{(TS_a)}(x)] = \begin{cases} -\dfrac{b^2}{a}\dfrac{B_{c,\gamma}^{(TS)}(x)}{\gamma^2} + \mathcal{E}_{b,c,\gamma}^{(TS_a)}(x), & \dfrac{x}{b} \to \infty, \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(TS_a)}(\kappa) + o(b^2) + O(n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(TS_a)}(0) + O(b^{\min(3, 1+\iota_2)} + n^{-1}b^{-1}), & x = 0, \end{cases}$$

$$Bias[\widehat{f}_{b,c,\gamma}^{(JF_a)}(x)] = \begin{cases} -\dfrac{b^2}{a}\dfrac{B_{c,\gamma}^{(JF_a)}(x)}{\gamma^2} + \mathcal{E}_{b,c,\gamma}^{(JF_a)}(x), & \dfrac{x}{b} \to \infty, \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(JF_a)}(\kappa) + o(b^2) + O(n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(JF_a)}(0) + O(b^{\min(3, 1+\iota_2)} + n^{-1}b^{-1}), & x = 0, \end{cases}$$

[7] If possible, it will be better for us not to use such a weighted criterion; at present, we do not yet realize whether or not the MISEs after the TS and JF type bias reductions can be treated without a weight function $w$.

*and*

$$V[\widehat{f}^{(\#_a)}_{b,c,\gamma}(x)]$$
$$= \begin{cases} n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x) + O(b^5 + (bx^{-1} + n^{-1/2}b^{-1/2})n^{-1}b^{-1/2}V(x) + n^{-1}), & \frac{x}{b} \to \infty, \\ n^{-1}b^{-1}|\gamma|f(0)v^{(SS_a)}_{c,\gamma}(\kappa) + O(b^5) + o(n^{-1}b^{-1}), & \frac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1}b^{-1}|\gamma|f(0)v^{(SS_a)}_{c,\gamma}(0) + O(b^5 + n^{-1} + n^{-3/2}b^{-3/2}), & x = 0, \end{cases}$$

*where*

$$\mathcal{E}^{(\#_a)}_{b,c,\gamma}(x) = O(b^3x^{-1} + b^{\min(2+\eta_4/2,1+\iota_2)} + n^{-1}\{b^{-1/2}V(x) + 1\}) \quad \textit{for } \frac{x}{b} \to \infty \ (x \in (0,r]).$$

From Theorem 10, the estimators (22) and (23) are (pointwise) weak consistent, i.e.,

$$MSE[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)]$$
$$= \begin{cases} \frac{b^4}{a^2}\Big\{\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 + n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x) + \mathcal{D}^{(TS_a)}_{b,c,\gamma}(x) & \text{for fixed } x \in \mathcal{I}\backslash\{0\}, \\ \frac{b^4}{a^2}\Big[-\frac{c^2\{f'(0)\}^2}{2f(0)} + \zeta^{(SS_a)}_{c,\gamma}(0)\frac{f''(0)}{2}\Big]^2 + n^{-1}b^{-1}|\gamma|v^{(SS_a)}_{c,\gamma}(0)f(0) + \mathcal{D}^{(TS_a)}_{b,c,\gamma}(x) & \text{for } x = 0, \end{cases}$$
$$MSE[\widehat{f}^{(JF_a)}_{b,c,\gamma}(x)]$$
$$= \begin{cases} \frac{b^4}{a^2}\Big\{\frac{B^{(JF_a)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 + n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x) + \mathcal{D}^{(JF_a)}_{b,c,\gamma}(x) & \text{for fixed } x \in \mathcal{I}\backslash\{0\}, \\ \frac{b^4}{a^2}\Big[-\frac{ac^2\{f'(0)\}^2}{2f(0)} + \zeta^{(SS_a)}_{c,\gamma}(0)\frac{f''(0)}{2}\Big]^2 + n^{-1}b^{-1}|\gamma|v^{(SS_a)}_{c,\gamma}(0)f(0) + \mathcal{D}^{(JF_a)}_{b,c,\gamma}(x) & \text{for } x = 0 \end{cases}$$

tend to zero (we suppose that $f(0) > 0$), since $b \propto n^{\iota_1}$ for $\iota_1 \in (0,1)$ implies that $b \to 0$ and $nb \to \infty$ (hence, $nb^{1/2} \to \infty$), where

$$\mathcal{D}^{(\#_a)}_{b,c,\gamma}(x) = \begin{cases} O(b^{\min(4+\eta_4/2,3+\iota_2)} + n^{-1} + n^{-3/2}b^{-1}) & \text{for fixed } x \in \mathcal{I}\backslash\{0\}, \\ O(b^{\min(5,3+\iota_2)} + n^{-1} + n^{-3/2}b^{-3/2}) & \text{for } x = 0. \end{cases}$$

The (pointwise) strong consistency and asymptotic normality of the estimators (22) and (23) can be proved.

**Theorem 11** *Given $\gamma \neq 0$, choose $c \geq 1$. Suppose that Assumptions* A1 *and* A3 (i) *and* (iii) *hold. If $b \to 0$, $nb/\log n \to \infty$, and $\epsilon \to 0$, then, $\widehat{f}^{(\#_a)}_{b,c,\gamma}(x) \xrightarrow{a.s.} f(x)$ for fixed $x \in \mathcal{I}$.*

**Theorem 12** *Suppose that Assumptions* A1, A2$'[\iota_1,\iota_2]$, *and* A3$'$ (i) *and* (iii) *hold for some $\iota_1 \in (0,1)$ and $\iota_2 > 1$.*

(i). *Given $\gamma \neq 0$, choose $c \geq 1$. If $\iota_1 \in (2/13, 1)$, then, for fixed $x \in \mathcal{I}\backslash\{0\}$,*

$$(nb^{1/2})^{1/2}\{\widehat{f}^{(\#_a)}_{b,c,\gamma}(x) - E[\widehat{f}^{(\#_a)}_{b,c,\gamma}(x)]\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x)).$$

(ii). *Given* $\gamma \neq 0$*, choose* $c \geq 1$ *when* $\gamma > 0$ *or* $c > 1$ *when* $\gamma < 0$ *(see the comment;* $\ell = 2$ *at the third paragraph of Section* 2*). Suppose that* $f(0) > 0$*. If* $\iota_1 \in (1/7, 1)$*, then,*

$$(nb)^{1/2}\{\widehat{f}^{(\#_a)}_{b,c,\gamma}(0) - E[\widehat{f}^{(\#_a)}_{b,c,\gamma}(0)]\} \xrightarrow{d} N(0, |\gamma| v^{(SS_a)}_{c,\gamma}(0) f(0)).$$

**Theorem 12′** *Suppose that Assumptions* A1, A2′$[\iota_1, \iota_2]$*, and* A3′ *hold for some* $\iota_1 \in (0,1)$ *and* $\iota_2 > 1$.
(i). *Given* $\gamma \neq 0$*, choose* $c \geq 1$*. If* $2/\min(9 + 2\eta_4, 5 + 4\iota_2) < \iota_1 < 1$*, where* $\eta_4 \in (0,1]$ *is given in Assumption* A3′ (ii) *(i.e., the feasible region of* $(\iota_1, \iota_2)$ *is given by* $2/(9 + 2\eta_4) < \iota_1 < 1$ *and* $\iota_2 > \max\{1, (2\iota_1^{-1} - 5)/4\}$*), then, for fixed* $x \in \mathcal{I}\backslash\{0\}$*,*

$$(nb^{1/2})^{1/2}\Big\{\widehat{f}^{(TS_a)}_{b,c,\gamma}(x) - f(x) + \frac{b^2}{a}\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x)),$$
$$(nb^{1/2})^{1/2}\Big\{\widehat{f}^{(JF_a)}_{b,c,\gamma}(x) - f(x) + \frac{b^2}{a}\frac{B^{(JF_a)}_{c,\gamma}(x)}{\gamma^2}\Big\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x)),$$

*hence, if, in addition,* $\iota_1 \in (2/9, 1)$*, then,* $(nb^{1/2})^{1/2}\{\widehat{f}^{(\#_a)}_{b,c,\gamma}(x) - f(x)\} \xrightarrow{d} N(0, \lambda(a)|\gamma|^{1/2}V(x))$.
(ii). *Given* $\gamma \neq 0$*, choose* $c \geq 1$ *when* $\gamma > 0$ *or* $c > 2$ *when* $\gamma < 0$ *(see the comment;* $\ell = 3$ *at the third paragraph of Section* 2*). Suppose that* $f(0) > 0$*. If* $1/\min(7, 3 + 2\iota_2) < \iota_1 < 1$ *(i.e., the feasible region of* $(\iota_1, \iota_2)$ *is given by* $1/7 < \iota_1 < 1$ *and* $\iota_2 > \max\{1, (\iota_1^{-1} - 3)/2\}$*), then,*

$$(nb)^{1/2}\Big[\widehat{f}^{(TS_a)}_{b,c,\gamma}(0) - f(0) + \frac{b^2}{a}\Big\{-\frac{c^2\{f'(0)\}^2}{2f(0)} + \zeta^{(SS_a)}_{c,\gamma}(0)\frac{f''(0)}{2}\Big\}\Big] \xrightarrow{d} N(0, |\gamma| v^{(SS_a)}_{c,\gamma}(0) f(0)),$$
$$(nb)^{1/2}\Big[\widehat{f}^{(JF_a)}_{b,c,\gamma}(0) - f(0) + \frac{b^2}{a}\Big\{-\frac{ac^2\{f'(0)\}^2}{2f(0)} + \zeta^{(SS_a)}_{c,\gamma}(0)\frac{f''(0)}{2}\Big\}\Big] \xrightarrow{d} N(0, |\gamma| v^{(SS_a)}_{c,\gamma}(0) f(0)),$$

*hence, if, in addition,* $\iota_1 \in (1/5, 1)$*, then,* $(nb)^{1/2}\{\widehat{f}^{(\#_a)}_{b,c,\gamma}(0) - f(0)\} \xrightarrow{d} N(0, |\gamma| v^{(SS_a)}_{c,\gamma}(0) f(0))$.

The following theorem says that the different MSE rate phenomenon

$$MSE[\widehat{f}^{(\#_a)}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-8/9}) & \text{for fixed } x \in \mathcal{I}\backslash\{0\} \text{ (using } b \propto n^{-2/9}), \\ O(n^{-4/5}) & \text{for } x/b \to \kappa \text{ if } f(0) > 0 \text{ (using } b \propto n^{-1/5}) \end{cases} \tag{26}$$

has negligible impact on the weighted MISEs of the estimators (22) and (23).

**Theorem 13** *Given* $\gamma \neq 0$*, choose* $c \geq 1$ *when* $\gamma > 0$ *or* $c > 1$ *when* $\gamma < 0$ *(see Remark* 2*). Suppose that Assumptions* A1, A2′$[\iota_1, \iota_2]$*, and* A3′ *hold for some* $\iota_1 \in (0,1)$ *and* $\iota_2 > 1$*, and that* $\min_{x\in[0,r]} f(x) > 0$*, where* $r > 0$ *is a fixed constant. Let* $w$ *be a truncated weight function, with* $w(y) = 0$ *for any* $y > r$*. Then, the weighted MISEs of the estimators* (22) *and* (23) *over* $[0, r]$ *are given by*

$$MISE_w[\widehat{f}^{(\#_a)}_{b,c,\gamma}] = AMISE_w[\widehat{f}^{(\#_a)}_{b,c,\gamma}] + o(b^4 + n^{-1}b^{-1/2}),$$

*where*

$$AMISE_w[\widehat{f}^{(TS_a)}_{b,c,\gamma}] = \frac{b^4}{a^2}\int_0^r w(x)\Big\{\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\lambda(a)\int_0^r w(x)|\gamma|^{1/2}V(x)dx,$$
$$AMISE_w[\widehat{f}^{(JF_a)}_{b,c,\gamma}] = \frac{b^4}{a^2}\int_0^r w(x)\Big\{\frac{B^{(JF_a)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\lambda(a)\int_0^r w(x)|\gamma|^{1/2}V(x)dx.$$

It is possible to consider the unbounded case $x \in [0, r_b]$, where $r_b$ is divergent, and then derive the weighted MISEs of the estimators (22) and (23) without imposing $w(y) = 0$ for $y > r$. Modifying the argument in Igarashi and Kakizawa (2015), we choose a pair $(q, \iota_0)$, belonging to the subset

$$\widetilde{\mathcal{S}} = \{(0,0)\} \cup \{(q,\iota_0) \mid 0 < q < \eta_4/(4+\eta_4) \text{ and } 0 < \iota_0 < (1-3q)/2\} \tag{27}$$

($\eta_4 \in (0,1]$ is given in Assumption A3′), and consider a set of the points $x$, as follows:

$$\mathcal{I}_{\iota_0}[r_b] = \{x \in [0, r_b] \mid f(x) \geq \varrho b^{\iota_0}\} \tag{28}$$

for some $r_b \equiv r$ or $r_b \to \infty$, with $r_b = O(b^{-q})$, according to $(q,\iota_0) = (0,0)$ or $(q,\iota_0) \in \widetilde{\mathcal{S}}\backslash\{(0,0)\}$[8]. Here and subsequently, $\varrho, r > 0$ are some constants, unless otherwise stated. The present setting is preferable to the previous setting in Igarashi and Kakizawa (2015), since the parameter $q$ controls the speed of $r_b \to \infty$.

In order to re-examine Theorems 10 and 13 from this setting, we make the following assumptions:

A2″. Given $(q,\iota_0) \in \widetilde{\mathcal{S}}$ (we write $p_0 = q + \iota_0$), $b \propto n^{-\iota_1}$ and $\epsilon \propto b^{\iota_2}$ for some $\iota_1 \in (0, 1/(1+2\iota_0))$ and $\iota_2 > 1 + p_0$; note that $(0, 2/9] \subset (0, 1/(1+2\iota_0))$, i.e., $b \propto n^{-2/9}$ ($\iota_1 = 2/9$) is feasible.

A5. Given $r_b \equiv r$ or $r_b \to \infty$, the density $f$ satisfies (i) $\min_{x\in[0,r_b]} f(x) \geq \varrho b^{\iota_0}$ for some constant $\iota_0 \geq 0$ (see (27)), and $w$ is a weight function, independent of $b$, such that (ii) $\int_{r_b}^{\infty} w(x)dx \propto \exp(-b^{-A})$ for some constant $A > 0$, and that (iii) $w(x)\{B^{(TS_a)}_{c,\gamma}(x)\}^2$ and $w(x)\{B^{(JF_a)}_{c,\gamma}(x)\}^2$ are integrable (when $r_b \equiv r$, the requirement (ii) holds iff $w$ is a truncated weight function, with $w(y) = 0$ for any $|y| > r$).

Assumption A4′ does not have to be imposed here for the derivation of the weighted MISE.

**Theorem 14** *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 2$ when $\gamma < 0$ (see the comment; $\ell = 3$ at the third paragraph of Section 2). Suppose that Assumptions A1, A2″, and A3′ hold, and define*

$$\omega_{b,\eta_4,\iota_0,\iota_2}(x) = b^{\eta_4/2}(1+x)^{2+\eta_4/2} + b^{1-2\iota_0}(1+x)^3 + b^{\iota_2-(1+\iota_0)}(1+x),$$
$$\widetilde{\omega}_{b,\iota_0}(x) = b^{1-2\iota_0}(1+x)^3 + n^{-1/2}b^{-(1/2+\iota_0)},$$

[8]If $(q,\iota_0) \in \widetilde{\mathcal{S}}$ (see (27)), then, $r_b = O(b^{-q})$ implies $b^{\eta_4/2}r_b^{2+\eta_4/2} + b^{1-2\iota_0}r_b^3 = o(1)$; note that, in proving Theorem 15, we need $\omega_{b,\eta_4,\iota_0,\iota_2}(r_b) + \widetilde{\omega}_{b,\iota_0}(r_b) = o(1)$.

*where* $(\eta_4, \iota_0, \iota_2)$ *is given in Assumptions* A2$''$ *and* A3$'$. *Then, on* $\mathcal{I}_{\iota_0}[r_b]$ *(see* (28)*),*

$$Bias[\widehat{f}_{b,c,\gamma}^{(TS_a)}(x)] = \begin{cases} -\dfrac{b^2}{a}\dfrac{B_{c,\gamma}^{(TS)}(x)}{\gamma^2} + \mathcal{E}_{b,c,\gamma}^{(TS_a)}(x), & \dfrac{x}{b} \to \infty, \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(TS_a)}(\kappa) + o(b^2) + O(n^{-1}b^{-(1+\iota_0)}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(TS_a)}(0) + O(b^{\min(3-2\iota_0, 1+\iota_2-\iota_0)} + n^{-1}b^{-1}), & x = 0, \end{cases}$$

$$Bias[\widehat{f}_{b,c,\gamma}^{(JF_a)}(x)] = \begin{cases} -\dfrac{b^2}{a}\dfrac{B_{c,\gamma}^{(JF_a)}(x)}{\gamma^2} + \mathcal{E}_{b,c,\gamma}^{(JF_a)}(x), & \dfrac{x}{b} \to \infty, \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(JF_a)}(\kappa) + o(b^2) + O(n^{-1}b^{-(1+\iota_0)}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ -\dfrac{b^2}{a}\zeta_{c,\gamma}^{(JF_a)}(0) + O(b^{\min(3-2\iota_0, 1+\iota_2-\iota_0)} + n^{-1}b^{-1}), & x = 0, \end{cases}$$

*and*

$$V[\widehat{f}_{b,c,\gamma}^{(\#_a)}(x)] = \begin{cases} n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x) + \widetilde{\mathcal{E}}_{b,c,\gamma}^{(\#_a)}(x), & \dfrac{x}{b} \to \infty, \\ n^{-1}b^{-1}|\gamma|f(0)v_{c,\gamma}^{(SS_a)}(\kappa) + O(b^{5-2\iota_0}) + o(n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ n^{-1}b^{-1}|\gamma|f(0)v_{c,\gamma}^{(SS_a)}(0) + O(b^{5-2\iota_0} + (b^{1-2\iota_0} + n^{-1/2}b^{-1/2-\iota_0})n^{-1}b^{-1}), & x = 0, \end{cases}$$

*where, for* $x/b \to \infty$,

$$\mathcal{E}_{b,c,\gamma}^{(\#_a)}(x) = O(b^3x^{-1} + b^2\omega_{b,\eta_4,\iota_0,\iota_2}(x) + n^{-1}b^{-\iota_0}\{b^{-1/2}V(x) + 1\}),$$
$$\widetilde{\mathcal{E}}_{b,c,\gamma}^{(\#_a)}(x) = O(b^{5-2\iota_0}(1+x)^3 + \{bx^{-1} + \widetilde{\omega}_{b,\iota_0}(x)\}n^{-1}b^{-1/2}V(x) + n^{-1}).$$

**Theorem 15** *Given* $\gamma \neq 0$, *choose* $c \geq 1$ *when* $\gamma > 0$ *or* $c > 1$ *when* $\gamma < 0$ *(see Remark* 2*). Suppose that Assumptions* A1, A2$''$, A3$'$, *and* A5 *hold. Then,*

$$MISE_w[\widehat{f}_{b,c,\gamma}^{(\#_a)}] = AMISE_w[\widehat{f}_{b,c,\gamma}^{(\#_a)}] + o(b^4 + n^{-1}b^{-1/2}),$$

*where*

$$AMISE_w[\widehat{f}_{b,c,\gamma}^{(TS_a)}] = \frac{b^4}{a^2}\int_0^\infty w(x)\Big\{\frac{B_{c,\gamma}^{(TS)}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\lambda(a)\int_0^\infty w(x)|\gamma|^{1/2}V(x)dx,$$
$$AMISE_w[\widehat{f}_{b,c,\gamma}^{(JF_a)}] = \frac{b^4}{a^2}\int_0^\infty w(x)\Big\{\frac{B_{c,\gamma}^{(JF_a)}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\lambda(a)\int_0^\infty w(x)|\gamma|^{1/2}V(x)dx.$$

From Theorem 15 (see also Theorem 13), the AMISEs of the estimators (22) and (23) are minimized at

$$b_w^{(TS_a)} = |\gamma|\{a^2\lambda(a)\}^{2/9}\Big[\frac{\int_0^\infty w(x)V(x)dx}{8\int_0^\infty w(x)\{B_{c,\gamma}^{(TS)}(x)\}^2dx}\Big]^{2/9} n^{-2/9},$$
$$b_w^{(JF_a)} = |\gamma|\{a^2\lambda(a)\}^{2/9}\Big[\frac{\int_0^\infty w(x)V(x)dx}{8\int_0^\infty w(x)\{B_{c,\gamma}^{(JF_a)}(x)\}^2dx}\Big]^{2/9} n^{-2/9},$$

when $\sqrt{w(x)}B_{c,\gamma}^{(TS)}(x) \not\equiv 0$ and $\sqrt{w(x)}B_{c,\gamma}^{(JF_a)}(x) \not\equiv 0$, respectively, i.e.,

$$\min_{b>0} AMISE_w[\widehat{f}_{b,c,\gamma}^{(TS_a)}] = \frac{9}{8^{8/9}}\Big\{\frac{\lambda^4(a)}{a}\Big\}^{2/9}\Big[\int_0^\infty w(x)\{B_{c,\gamma}^{(TS)}(x)\}^2 dx\Big]^{1/9}\Big\{\int_0^\infty w(x)V(x)dx\Big\}^{8/9} n^{-8/9}, \tag{29}$$

$$\min_{b>0} AMISE_w[\widehat{f}_{b,c,\gamma}^{(JF_a)}] = \frac{9}{8^{8/9}}\Big\{\frac{\lambda^4(a)}{a}\Big\}^{2/9}\Big[\int_0^\infty w(x)\{B_{c,\gamma}^{(JF_a)}(x)\}^2 dx\Big]^{1/9}\Big\{\int_0^\infty w(x)V(x)dx\Big\}^{8/9} n^{-8/9} \tag{30}$$

have the convergence rate $n^{-8/9}$. We observe that the optimal AMISE (29) is decreasing for $a \in (0,1)$, with $\lim_{a\to 1}\{\lambda^4(a)/a\}^{2/9} = (27/16)^{8/9}$. On the other hand, the dependence of $a$ on the optimal AMISE (30) is greatly different from that on the optimal AMISE (29), since $\int_0^\infty w(x)\{B_{c,\gamma}^{(JF_a)}(x)\}^2 dx$ depends on $a$, $c$, $\gamma$, unknown density $f$, and weight function $w$. Note that, as shown in Section 4, these convergence rates are $n^{-6/7}$ when $\rho_c(t) \neq t + c$.

It should be remarked that the TS and JF type (22) and (23) are not well-defined when $a = 1$. However, we can consider their limits as $a \to 1$ (see Jones and Foster (1993) for the standard kernel density estimator, and Igarashi and Kakizawa (2015) and Igarashi (2016a) for the gamma/MIG/weighted LN and beta kernel density estimators). That is, assuming that $\epsilon$ is independent of $a$, we define

$$\widehat{f}_{b,c,\gamma}^{(\#_1)}(x) = \lim_{a\to 1}\widehat{f}_{b,c,\gamma}^{(TS_a)}(x) = \lim_{a\to 1}\widehat{f}_{b,c,\gamma}^{(JF_a)}(x) = \{\widehat{f}_{b,c,\gamma}(x) + \epsilon\}\exp\Big\{\frac{\widehat{f}_{b,c,\gamma}^{(SS_1)}(x)}{\widehat{f}_{b,c,\gamma}(x) + \epsilon} - 1\Big\}. \tag{31}$$

It turns out that the TS type is linked with the JF type, through the limiting case (31). The asymptotic properties of the limiting estimator (31) will be studied as a companion paper (Igarashi and Kakizawa (2017)).

Here are some examples of $(w, f)$ that we can apply Theorem 15.

(a). For a truncated weight function $w$, with $w(y) = 0$ for any $y > r$, Theorem 15 (i.e., Theorem 13) is applicable, whenever $\min_{x\in[0,r]} f(x) > 0$ (choose $r_b \equiv r$ and $q = \iota_0 = 0$).

(b). Suppose that there exist constants $c_0 \geq 1$[9] and $c_1 > 0$ such that $w(x) \propto x^{c_0-1}\exp\{x^{c_0} - \exp(x^{c_0})\}$ for sufficiently large $x$, and that $\min_{x\geq 0} f(x)\exp(c_1 x) > 0$ (in this case, $w(x)\{B_{c,\gamma}^{(TS)}(x)\}^2$ and $w(x)\{B_{c,\gamma}^{(JF_a)}(x)\}^2$ are integrable). Choosing $r_b = (\iota_0/c_1)\log(1/b)$, Assumption A5 (i) and (ii) can be verified:

- $\min_{x\in[0,r_b]} f(x) \geq \varrho b^{\iota_0}$, where $\varrho = \min_{x\geq 0} f(x)\exp(c_1 x)$,

[9]For the limiting $JF_1$=$TS_1$ type estimator (31), the condition $c_0 > 1$, rather than $c_0 \geq 1$, was required (see Igarashi and Kakizawa (2017)).

- $\int_{r_b}^{\infty} w(x)dx \propto \exp(-b^{-(\iota_0/c_1)^{c_0}\{\log(1/b)\}^{c_0-1}})$; hence, for the case $c_0 = 1$, we can choose $A = \iota_0/c_1$, and, for the case $c_0 > 1$, we can choose any constant $A > 0$ for all sufficiently large $n$, noting that $\lim_{n\to\infty}(\iota_0/c_1)^{c_0}\{\log(1/b)\}^{c_0-1} = \infty$ (we assume $b \to 0$).

(c). Suppose that $w(x) \propto e^{-x}$ (say)[10] for sufficiently large $x$, and that there exists a constant $c_1 > 1$ such that $\min_{x\geq 0} f(x)(1+x)^{c_1} > 0$ (in this case, $w(x)\{B_{c,\gamma}^{(TS)}(x)\}^2$ and $w(x)\{B_{c,\gamma}^{(JF_a)}(x)\}^2$ are integrable). We choose $r_b = b^{-\iota_0/c_1} - 1$ $(= O(b^{-q}))$, where the possible pair $(q, \iota_0)$, depending on $\eta_4 \in (0, 1]$ (see Assumption A3$'$), is pre-determined according to the inequalities $0 < q < \eta_4/(4 + \eta_4)$, $0 < \iota_0 < (1 - 3q)/2$, and $\iota_0 \leq c_1 q$[11]; more precisely,

- if $\eta_4 \in (0, 2/(1 + c_1)]$, then, $(q, \iota_0) \in \widetilde{\mathcal{S}}_1 \subset \widetilde{\mathcal{S}}$, where

$$\widetilde{\mathcal{S}}_1 = \{(q, \iota_0) \mid 0 < q < \eta_4/(4 + \eta_4),\ 0 < \iota_0 \leq c_1 q\},$$

- if $\eta_4 \in (2/(1 + c_1), 1]$, then, $(q, \iota_0) \in \bigcup_{j=2}^{3} \widetilde{\mathcal{S}}_j \subset \widetilde{\mathcal{S}}$, where

$$\begin{aligned}\widetilde{\mathcal{S}}_2 &= \{(q, \iota_0) \mid 0 < q < 1/(3 + 2c_1),\ 0 < \iota_0 \leq c_1 q\},\\ \widetilde{\mathcal{S}}_3 &= \{(q, \iota_0) \mid 1/(3 + 2c_1) \leq q < \eta_4/(4 + \eta_4),\ 0 < \iota_0 < (1 - 3q)/2\}.\end{aligned}$$

Then, Assumption A5 (i) and (ii) can be verified:

- $\min_{x\in[0,r_b]} f(x) \geq \varrho b^{\iota_0}$, where $\varrho = \min_{x\geq 0} f(x)(1 + x)^{c_1}$,
- $\int_{r_b}^{\infty} w(x)dx \propto \exp(-b^{-\iota_0/c_1} + 1)$ (i.e., we can choose $A = \iota_0/c_1$).

## 4. Comment on bias reductions for the case of two-regime $\rho$-function

In this section, we briefly discuss the following estimators for each $a \in (0, 1)$:

$$\widetilde{f}_{b,c,\gamma}^{(SS_a)}(x) = \frac{1}{1-a}\widetilde{f}_{b,c,\gamma}(x) - \frac{a}{1-a}\widetilde{f}_{b/a,c,\gamma}(x), \tag{32}$$

$$\widetilde{f}_{b,c,\gamma}^{(TS_a)}(x) = \frac{\{\widetilde{f}_{b,c,\gamma}(x) + \epsilon\}^{1/(1-a)}}{\{\widetilde{f}_{b/a,c,\gamma}(x) + \epsilon/a\}^{a/(1-a)}}, \tag{33}$$

$$\widetilde{f}_{b,c,\gamma}^{(JF_a)}(x) = \{\widetilde{f}_{b,c,\gamma}(x) + \epsilon\} \exp\left\{\frac{\widetilde{f}_{b,c,\gamma}^{(SS_a)}(x)}{\widetilde{f}_{b,c,\gamma}(x) + \epsilon} - 1\right\}, \tag{34}$$

using two-regime $\rho$-function (3) rather than $\rho_c(t) = t + c$.

[10] For the limiting JF$_1$=TS$_1$ type estimator (31), a smaller weight function $w(x) \propto \exp\{x - \exp(x)\}$ (say) was required (see Igarashi and Kakizawa (2017)).

[11] For the limiting JF$_1$=TS$_1$ type estimator (31), the pair $(q, \iota_0)$ was pre-determined according to the (more restrictive) inequalities $0 < q < \eta_4/(4 + \eta_4)$, $0 < \iota_0 < 1/4 - q$, and $\iota_0 \leq c_1 q$; besides, "$\int_{r_b}^{\infty} w(x)dx \propto \exp(-b^A)$ for some constant $A > 1 + \iota_2$" was made for Assumption A5 (see Igarashi and Kakizawa (2017)).

Given $\gamma \neq 0$, choose $r_c(0) \geq 1$ when $\gamma > 0$ or $r_c(0) > 1$ when $\gamma < 0$ (see the comment; $\ell = 2$[12] at the third paragraph of Section 2). Under Assumptions A1, A2$'[\iota_1, \iota_2]$, and A3$'$, where $\iota_1 \in (0,1)$ and $\iota_2 > 1$, then, the boundary biases of the estimators (32)–(34) (for (33) and (34), assume $f(0) > 0$) are given by

$$Bias[\widetilde{f}^{(SS_a)}_{b,c,\gamma}(x)] = \begin{cases} b\dfrac{f'(0)}{1-a}\Big\{-r_c\Big(\dfrac{ax}{b}\Big) + \dfrac{ax}{b} + c\Big\} + O(b^2), & x \in [db, db/a), \\ b\dfrac{f'(0)}{1-a}\Big\{r_c\Big(\dfrac{x}{b}\Big) - r_c\Big(\dfrac{ax}{b}\Big) - \dfrac{(1-a)x}{b}\Big\} + O(b^2), & x \in (0, db), \\ O(b^2), & x = 0, \end{cases}$$

$$Bias[\widetilde{f}^{(\#_a)}_{b,c,\gamma}(x)] = \begin{cases} b\dfrac{f'(0)}{1-a}\Big\{-r_c\Big(\dfrac{ax}{b}\Big) + \dfrac{ax}{b} + c\Big\} + O(b^2 + n^{-1}b^{-1}), & x \in [db, db/a), \\ b\dfrac{f'(0)}{1-a}\Big\{r_c\Big(\dfrac{x}{b}\Big) - r_c\Big(\dfrac{ax}{b}\Big) - \dfrac{(1-a)x}{b}\Big\} + O(b^2 + n^{-1}b^{-1}), & x \in (0, db), \\ O(b^2 + n^{-1}b^{-1}), & x = 0 \end{cases}$$

(their biases when $x \geq db/a$, as well as their variances, are the same as those in Section 3). Therefore, in order to work the bias reduction methods even near the boundary, we must assume $f'(0) = 0$ or use $r_c(t) = t + c$ (i.e., $\rho_c(t) = t + c$). Otherwise, these AMISEs are given by

$$\begin{aligned} AMISE[\widetilde{f}^{(SS_a)}_{b,c,\gamma}] &= b^3 \frac{\{f'(0)\}^2}{(1-a)^2}\Big[\int_d^{d/a} \{-r_c(at) + at + c\}^2 dt + \int_0^d \{r_c(t) - r_c(at) - (1-a)t\}^2 dt\Big] \\ &\quad + n^{-1}b^{-1/2}\lambda(a)\int_0^\infty |\gamma|^{1/2} V(x)dx, \\ AMISE_w[\widetilde{f}^{(\#_a)}_{b,c,\gamma}] &= b^3 \frac{\{f'(0)\}^2 w(0)}{(1-a)^2}\Big[\int_d^{d/a} \{-r_c(at) + at + c\}^2 dt + \int_0^d \{r_c(t) - r_c(at) - (1-a)t\}^2 dt\Big] \\ &\quad + n^{-1}b^{-1/2}\lambda(a)\int_0^\infty w(x)|\gamma|^{1/2} V(x)dx, \end{aligned}$$

provided that $w(t)$ is continuous at $t = 0$ (for the TS$_a$/JF$_a$ type, we suppose that $\min_{x\in[0,r]} f(x) > 0$). Consequently, in general, the estimators (32)–(34) for the case of two-regime $\rho$-function (3) yield, using $b \propto n^{-2/7}$, the MISEs of order $O(n^{-6/7})$. Since the faster rate $O(n^{-8/9})$ was attained when $\rho_c(t) = t+c$ in Section 3, our finding would be a warning for Hirukawa and Sakudo (2014, 2015) (see also Funke and Kawka (2015)), who discussed the estimators $\widetilde{f}_{b,0,\gamma}$, $\gamma = 1, 2$ ($d = 2$ and $r_0(t) = t^2/4 + 1$).

**Remark 3** Given $\gamma \neq 0$, choose $r_c(0) \geq 1$. Suppose that Assumptions A1 and A3 (i) and (iii) hold. If $b \to 0$ and $nb/\log n \to \infty$, then, Theorem 2 and Slutsky's lemma yield $\widetilde{f}^{(\#_a)}_{b,c,\gamma}(x) \overset{a.s.}{\to} f(x)$ for fixed $x \geq 0$, where $\# = SS, TS, JF$ (we additionally assume $f(x) > 0$ for the TS$_a$/JF$_a$ type).

[12]Unlike Section 3, here, we imposed, for $\gamma < 0$, the restriction $r_c(0) > 1$, rather than $r_c(0) > 2$. Note that $r_c(0) > 1$ is sufficient to show that, in case $\gamma < 0$, the error terms of the biases near the boundary are $O(b^2)$; of course, two-term expansions, with the $O(b^3)$-errors, could be derived under a restriction $r_c(0) > 2$.

## 5. Simulation studies

We conducted the simulation studies to illustrate the finite sample performance of the Amoroso kernel density estimator (4) and the bias-reduced estimators (14), (22), and (23) (and the limiting estimators (21) and (31)). We generated 1000 replicate samples of $n = 100, 300$ from the four densities:

$$\text{A.}\quad f(x) = \frac{1}{2}\Big(\frac{e^{-x/3}}{3} + \frac{xe^{-x/3}}{9}\Big),$$
$$\text{B.}\quad f(x) = \frac{e^{-x/3}}{3},$$
$$\text{C.}\quad f(x) = \frac{1}{2}\Big(\frac{e^{-x/10}}{10} + xe^{-x}\Big),$$
$$\text{D.}\quad f(x) = \frac{1}{2}\Big[\frac{x^{-1}}{\sqrt{2\pi}0.8}\exp\Big\{-\frac{(\log x - 1)^2}{2(0.8)^2}\Big\} + \frac{x^{-1}}{\sqrt{2\pi}0.4}\exp\Big\{-\frac{(\log x - 2)^2}{2(0.4)^2}\Big\}\Big],$$

and calculated the integrated squared error (ISE); $ISE_k = \int_0^\infty \{\widetilde{f}^{[k]}(x) - f(x)\}^2 dx$, where $\widetilde{f}^{[k]}$ is a density estimator using the $k$th sample. Each smoothing parameter $b$ was so selected as to minimize the ISE. In Tables 1–4, we considered the average ISEs; $(1/1000)\sum_{k=1}^{1000} ISE_k$ for the estimator (4), using

$$\rho_c(t) = \begin{cases} t + c, & t \geq 2, \\ (c+1)\Big(\dfrac{t}{2}\Big)^{2/(c+1)} + 1, & t \in [0, 2) \end{cases}$$

($c = 1$ corresponds to $\rho_1(t) = t + 1$), with $c = c'/|\gamma|$, where $c' = 0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2$, and $\gamma = -2, -1.75, -1.5, -1.25, -1, -0.75, -0.5, -0.25, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2$. The results are summarized as follows:

- The average ISEs decreased, as the sample size $n$ increased.
- As expected from the AMISE (13), some results were minimized at $c' = c|\gamma| = 1/4$ when $n = 300$, whereas the corresponding results were not obtained when $n = 100$ (B–D). We believe that the contradiction was caused by small sample size $n$.
- For the case D (A–C), the Amoroso kernel density estimator (4) with $\gamma < 0$ was comparable (inferior) to that with $\gamma > 0$; the average ISEs were minimized at $\gamma = 2$ (A), $\gamma = 1$ (B), $\gamma = 0.5$ (C), and $\gamma = \pm 0.25$ (D).

In Tables 5–8, we also considered the average ISEs for the estimators (4), (14), (22), and (23) (and the limiting estimators (21) and (31)), using $\rho_c(t) = t + c$ ($c = 1$ for $\gamma > 0$ and $c = 1.1$ for $\gamma < 0$[13])

[13] We conducted the simulation using $c = 1$ for $\gamma < 0$ (note that $c = 1$ for $\gamma < 0$ is not allowed in Theorems 9 and 15). We found that there is little effect of $c$.

Table 1: Case A. The average ISEs$\times 10^6$ of $\widetilde{f}_{b,c'/|\gamma|,\gamma}$.

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

$n = 100$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 1283 | **1201** | 1221 | 1325 | 1485 | 1668 | 1856 | 2035 | 2204 |
| | (1356) | (1312) | (1266) | (1229) | (1213) | (1226) | (1260) | (1306) | (1356) |
| 1.75 | 1309 | **1206** | 1228 | 1350 | 1530 | 1729 | 1923 | 2105 | 2272 |
| | (1368) | (1320) | (1272) | (1230) | (1219) | (1241) | (1283) | (1334) | (1385) |
| 1.5 | 1353 | **1221** | 1244 | 1388 | 1589 | 1799 | 1997 | 2177 | 2342 |
| | (1383) | (1336) | (1280) | (1237) | (1231) | (1263) | (1312) | (1364) | (1415) |
| 1.25 | 1428 | **1251** | 1276 | 1445 | 1665 | 1881 | 2076 | 2251 | 2409 |
| | (1405) | (1357) | (1293) | (1247) | (1251) | (1293) | (1345) | (1396) | (1443) |
| 1 | 1561 | **1311** | 1332 | 1527 | 1759 | 1970 | 2156 | 2321 | 2471 |
| | (1438) | (1386) | (1312) | (1264) | (1283) | (1330) | (1380) | (1425) | (1468) |
| 0.75 | 1816 | **1434** | 1435 | 1646 | 1867 | 2059 | 2227 | 2380 | 2520 |
| | (1512) | (1432) | (1344) | (1302) | (1328) | (1368) | (1409) | (1448) | (1485) |
| 0.5 | 2393 | 1714 | **1627** | 1799 | 1971 | 2130 | 2278 | 2419 | 2551 |
| | (1780) | (1522) | (1411) | (1369) | (1375) | (1397) | (1428) | (1462) | (1497) |
| 0.25 | 4673 | 2572 | 1958 | **1921** | 2015 | 2148 | 2288 | 2426 | 2559 |
| | (4954) | (1955) | (1578) | (1429) | (1396) | (1408) | (1439) | (1471) | (1506) |
| $-0.25$ | 4095 | 2594 | 2068 | **2064** | 2160 | 2284 | 2415 | 2544 | 2669 |
| | (2964) | (1843) | (1640) | (1507) | (1464) | (1465) | (1486) | (1514) | (1543) |
| $-0.5$ | 2741 | 2028 | **1891** | 2097 | 2271 | 2416 | 2546 | 2669 | 2785 |
| | (1893) | (1649) | (1568) | (1529) | (1513) | (1513) | (1525) | (1546) | (1572) |
| $-0.75$ | 2350 | 1823 | **1752** | 2027 | 2285 | 2476 | 2629 | 2762 | 2883 |
| | (1710) | (1629) | (1553) | (1516) | (1528) | (1542) | (1558) | (1578) | (1600) |
| $-1$ | 2176 | 1736 | **1681** | 1939 | 2242 | 2481 | 2667 | 2820 | 2951 |
| | (1777) | (1642) | (1560) | (1500) | (1511) | (1547) | (1577) | (1601) | (1624) |
| $-1.25$ | 2053 | 1704 | **1650** | 1871 | 2180 | 2452 | 2668 | 2844 | 2992 |
| | (1734) | (1644) | (1548) | (1498) | (1499) | (1537) | (1577) | (1612) | (1640) |
| $-1.5$ | 1997 | 1689 | **1649** | 1830 | 2118 | 2402 | 2646 | 2844 | 3008 |
| | (1739) | (1624) | (1565) | (1508) | (1497) | (1527) | (1575) | (1616) | (1648) |
| $-1.75$ | 1953 | 1697 | **1663** | 1810 | 2068 | 2349 | 2602 | 2821 | 3004 |
| | (1714) | (1625) | (1568) | (1521) | (1501) | (1522) | (1562) | (1612) | (1651) |
| $-2$ | 1929 | 1723 | **1690** | 1809 | 2031 | 2297 | 2555 | 2785 | 2983 |
| | (1701) | (1636) | (1584) | (1537) | (1506) | (1520) | (1556) | (1603) | (1647) |

Table 1: (continued).

$n = 300$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 538 | **498** | 520 | 588 | 681 | 779 | 872 | 958 | 1036 |
| | (528) | (521) | (508) | (500) | (504) | (522) | (547) | (573) | (596) |
| 1.75 | 551 | **502** | 525 | 603 | 704 | 805 | 899 | 984 | 1061 |
| | (527) | (521) | (509) | (502) | (510) | (532) | (558) | (583) | (606) |
| 1.5 | 573 | **510** | 536 | 624 | 732 | 834 | 927 | 1011 | 1087 |
| | (526) | (522) | (511) | (506) | (519) | (544) | (570) | (593) | (615) |
| 1.25 | 608 | **525** | 554 | 652 | 763 | 865 | 956 | 1037 | 1111 |
| | (527) | (525) | (514) | (512) | (530) | (555) | (580) | (603) | (623) |
| 1 | 666 | **555** | 583 | 688 | 799 | 897 | 983 | 1063 | 1134 |
| | (533) | (529) | (520) | (521) | (544) | (567) | (589) | (611) | (630) |
| 0.75 | 769 | **612** | 630 | 734 | 836 | 928 | 1009 | 1084 | 1153 |
| | (554) | (540) | (533) | (539) | (556) | (577) | (597) | (617) | (636) |
| 0.5 | 991 | 733 | **709** | 788 | 873 | 954 | 1029 | 1099 | 1166 |
| | (654) | (569) | (557) | (556) | (566) | (583) | (603) | (622) | (642) |
| 0.25 | 1847 | 1073 | **828** | 830 | 889 | 961 | 1034 | 1104 | 1170 |
| | (1679) | (760) | (602) | (567) | (572) | (589) | (608) | (628) | (647) |
| −0.25 | 1596 | 1065 | **866** | 884 | 945 | 1013 | 1083 | 1150 | 1213 |
| | (1067) | (710) | (623) | (593) | (595) | (608) | (626) | (643) | (660) |
| −0.5 | 1121 | 851 | **810** | 899 | 985 | 1061 | 1130 | 1194 | 1255 |
| | (709) | (622) | (619) | (612) | (613) | (623) | (637) | (653) | (669) |
| −0.75 | 968 | 767 | **759** | 885 | 996 | 1085 | 1161 | 1228 | 1289 |
| | (629) | (601) | (607) | (616) | (626) | (637) | (650) | (664) | (677) |
| −1 | 905 | 732 | **730** | 864 | 993 | 1095 | 1179 | 1250 | 1314 |
| | (617) | (602) | (603) | (608) | (629) | (647) | (661) | (674) | (687) |
| −1.25 | 871 | 714 | **711** | 840 | 981 | 1095 | 1187 | 1264 | 1332 |
| | (619) | (612) | (604) | (603) | (621) | (646) | (666) | (682) | (696) |
| −1.5 | 848 | 706 | **703** | 816 | 963 | 1086 | 1187 | 1271 | 1343 |
| | (636) | (619) | (608) | (599) | (616) | (639) | (664) | (684) | (700) |
| −1.75 | 826 | 703 | **699** | 801 | 940 | 1072 | 1179 | 1270 | 1347 |
| | (632) | (614) | (606) | (602) | (607) | (634) | (659) | (681) | (700) |
| −2 | 811 | 707 | **700** | 788 | 922 | 1053 | 1167 | 1264 | 1346 |
| | (632) | (620) | (606) | (603) | (610) | (626) | (653) | (677) | (696) |

Table 2: Case B. The average ISEs$\times 10^6$ of $\widetilde{f}_{b,c'/|\gamma|,\gamma}$.

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

$n = 100$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 2933 | 2785 | **2778** | 2890 | 3089 | 3341 | 3617 | 3897 | 4170 |
| | (2643) | (2608) | (2578) | (2546) | (2528) | (2531) | (2555) | (2597) | (2650) |
| 1.75 | 2923 | 2740 | **2732** | 2871 | 3108 | 3394 | 3696 | 3991 | 4274 |
| | (2653) | (2620) | (2581) | (2548) | (2530) | (2537) | (2571) | (2622) | (2683) |
| 1.5 | 2935 | 2697 | **2688** | 2864 | 3146 | 3469 | 3793 | 4102 | 4388 |
| | (2678) | (2636) | (2592) | (2549) | (2532) | (2548) | (2594) | (2655) | (2724) |
| 1.25 | 2990 | 2664 | **2653** | 2877 | 3214 | 3573 | 3913 | 4225 | 4509 |
| | (2715) | (2666) | (2609) | (2554) | (2539) | (2569) | (2628) | (2698) | (2771) |
| 1 | 3133 | 2659 | **2639** | 2931 | 3324 | 3707 | 4050 | 4354 | 4626 |
| | (2770) | (2714) | (2639) | (2566) | (2558) | (2605) | (2676) | (2750) | (2820) |
| 0.75 | 3486 | 2727 | **2679** | 3057 | 3488 | 3865 | 4187 | 4472 | 4727 |
| | (2879) | (2785) | (2686) | (2589) | (2599) | (2659) | (2730) | (2799) | (2863) |
| 0.5 | 4515 | 3034 | **2862** | 3293 | 3685 | 4010 | 4293 | 4551 | 4790 |
| | (3484) | (2912) | (2758) | (2646) | (2666) | (2719) | (2775) | (2838) | (2895) |
| 0.25 | 9362 | 4521 | **3423** | 3550 | 3788 | 4054 | 4316 | 4567 | 4804 |
| | (11306) | (3429) | (2938) | (2757) | (2728) | (2761) | (2812) | (2866) | (2920) |
| $-0.25$ | 7795 | 4392 | **3577** | 3793 | 4040 | 4289 | 4534 | 4769 | 4989 |
| | (5917) | (3292) | (3047) | (2891) | (2844) | (2853) | (2885) | (2927) | (2972) |
| $-0.5$ | 4927 | 3451 | **3275** | 3814 | 4223 | 4524 | 4772 | 4997 | 5204 |
| | (3466) | (3092) | (3016) | (2943) | (2923) | (2927) | (2948) | (2979) | (3018) |
| $-0.75$ | 4322 | 3311 | **3179** | 3687 | 4218 | 4611 | 4911 | 5161 | 5375 |
| | (3504) | (3247) | (3014) | (2922) | (2944) | (2976) | (3005) | (3034) | (3066) |
| $-1$ | 4024 | 3291 | **3202** | 3590 | 4126 | 4590 | 4956 | 5245 | 5490 |
| | (3411) | (3188) | (3010) | (2897) | (2913) | (2966) | (3024) | (3067) | (3108) |
| $-1.25$ | 3927 | 3344 | **3254** | 3564 | 4039 | 4516 | 4927 | 5266 | 5545 |
| | (3433) | (3098) | (2945) | (2877) | (2867) | (2926) | (3005) | (3075) | (3127) |
| $-1.5$ | 3872 | 3454 | **3366** | 3603 | 3980 | 4439 | 4866 | 5232 | 5546 |
| | (3307) | (3168) | (3021) | (2982) | (2884) | (2901) | (2976) | (3049) | (3119) |
| $-1.75$ | 3945 | 3574 | **3513** | 3682 | 4015 | 4403 | 4792 | 5177 | 5514 |
| | (3400) | (3276) | (3182) | (3068) | (3007) | (2963) | (2963) | (3022) | (3097) |
| $-2$ | 3987 | 3647 | **3594** | 3739 | 4031 | 4389 | 4758 | 5115 | 5457 |
| | (3497) | (3263) | (3199) | (3131) | (3089) | (3038) | (3013) | (3019) | (3067) |

Table 2: (continued).

$n = 300$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 1270 | **1203** | 1218 | 1301 | 1428 | 1576 | 1728 | 1876 | 2016 |
| | (1123) | (1115) | (1106) | (1098) | (1096) | (1104) | (1121) | (1144) | (1171) |
| 1.75 | 1261 | **1177** | 1194 | 1294 | 1440 | 1602 | 1763 | 1916 | 2059 |
| | (1124) | (1116) | (1106) | (1097) | (1095) | (1106) | (1127) | (1154) | (1183) |
| 1.5 | 1263 | **1152** | 1172 | 1292 | 1461 | 1637 | 1806 | 1961 | 2103 |
| | (1126) | (1119) | (1108) | (1096) | (1096) | (1111) | (1135) | (1165) | (1195) |
| 1.25 | 1284 | **1132** | 1155 | 1303 | 1494 | 1682 | 1854 | 2008 | 2149 |
| | (1130) | (1126) | (1113) | (1098) | (1099) | (1118) | (1146) | (1176) | (1207) |
| 1 | 1345 | **1124** | 1150 | 1332 | 1543 | 1736 | 1904 | 2055 | 2190 |
| | (1144) | (1138) | (1122) | (1102) | (1106) | (1130) | (1159) | (1188) | (1219) |
| 0.75 | 1494 | **1150** | 1173 | 1389 | 1607 | 1792 | 1953 | 2095 | 2225 |
| | (1194) | (1165) | (1139) | (1112) | (1119) | (1142) | (1170) | (1201) | (1230) |
| 0.5 | 1884 | 1278 | **1257** | 1478 | 1674 | 1840 | 1988 | 2123 | 2246 |
| | (1408) | (1223) | (1170) | (1129) | (1132) | (1154) | (1182) | (1212) | (1242) |
| 0.25 | 3554 | 1840 | **1459** | 1559 | 1706 | 1855 | 1997 | 2129 | 2253 |
| | (3372) | (1423) | (1226) | (1147) | (1145) | (1167) | (1195) | (1226) | (1255) |
| $-0.25$ | 2978 | 1805 | **1515** | 1651 | 1801 | 1945 | 2080 | 2206 | 2324 |
| | (2159) | (1347) | (1242) | (1179) | (1177) | (1197) | (1223) | (1250) | (1277) |
| $-0.5$ | 2086 | 1442 | **1415** | 1673 | 1874 | 2031 | 2166 | 2288 | 2400 |
| | (1405) | (1209) | (1214) | (1196) | (1199) | (1214) | (1236) | (1260) | (1285) |
| $-0.75$ | 1796 | **1358** | 1368 | 1642 | 1885 | 2070 | 2220 | 2349 | 2464 |
| | (1255) | (1213) | (1208) | (1204) | (1217) | (1235) | (1254) | (1275) | (1297) |
| $-1$ | 1681 | **1354** | 1367 | 1608 | 1867 | 2076 | 2245 | 2384 | 2506 |
| | (1277) | (1242) | (1226) | (1214) | (1225) | (1250) | (1272) | (1291) | (1312) |
| $-1.25$ | 1632 | **1391** | 1393 | 1591 | 1839 | 2064 | 2247 | 2400 | 2531 |
| | (1349) | (1305) | (1250) | (1232) | (1227) | (1259) | (1283) | (1305) | (1325) |
| $-1.5$ | 1612 | **1433** | 1437 | 1600 | 1819 | 2039 | 2234 | 2400 | 2541 |
| | (1348) | (1303) | (1274) | (1260) | (1241) | (1253) | (1283) | (1312) | (1335) |
| $-1.75$ | 1633 | **1495** | 1499 | 1626 | 1810 | 2021 | 2216 | 2388 | 2538 |
| | (1365) | (1362) | (1327) | (1288) | (1254) | (1268) | (1287) | (1310) | (1336) |
| $-2$ | 1667 | 1546 | **1545** | 1649 | 1819 | 2013 | 2197 | 2373 | 2527 |
| | (1408) | (1363) | (1332) | (1303) | (1279) | (1285) | (1285) | (1315) | (1334) |

Table 3: Case C. The average ISEs$\times 10^6$ of $\widetilde{f}_{b,c'/|\gamma|,\gamma}$.

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

$n = 100$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 4195 | 4055 | **4012** | 4059 | 4192 | 4368 | 4572 | 4791 | 5025 |
| | (2117) | (2036) | (2044) | (2132) | (2294) | (2480) | (2675) | (2869) | (3061) |
| 1.75 | 4154 | 3987 | **3935** | 3994 | 4142 | 4338 | 4563 | 4807 | 5054 |
| | (2194) | (2095) | (2089) | (2185) | (2352) | (2552) | (2758) | (2959) | (3135) |
| 1.5 | 4107 | 3911 | **3851** | 3923 | 4089 | 4310 | 4561 | 4824 | 5090 |
| | (2284) | (2157) | (2141) | (2239) | (2412) | (2625) | (2841) | (3036) | (3207) |
| 1.25 | 4063 | 3832 | **3760** | 3846 | 4036 | 4288 | 4565 | 4851 | 5134 |
| | (2403) | (2235) | (2200) | (2298) | (2485) | (2706) | (2912) | (3100) | (3264) |
| 1 | 4032 | 3744 | **3656** | 3759 | 3987 | 4277 | 4584 | 4894 | 5195 |
| | (2547) | (2333) | (2268) | (2362) | (2556) | (2771) | (2970) | (3147) | (3310) |
| 0.75 | 4015 | 3660 | **3546** | 3678 | 3954 | 4287 | 4631 | 4964 | 5277 |
| | (2703) | (2459) | (2349) | (2429) | (2613) | (2816) | (3012) | (3191) | (3351) |
| 0.5 | 4066 | 3606 | **3460** | 3632 | 3969 | 4350 | 4716 | 5061 | 5373 |
| | (2869) | (2612) | (2447) | (2488) | (2648) | (2850) | (3050) | (3231) | (3383) |
| 0.25 | 4527 | 3736 | **3477** | 3687 | 4069 | 4463 | 4830 | 5165 | 5470 |
| | (3900) | (2989) | (2549) | (2513) | (2680) | (2894) | (3093) | (3266) | (3414) |
| $-0.25$ | 4611 | 3874 | **3643** | 3846 | 4208 | 4589 | 4944 | 5275 | 5577 |
| | (3201) | (2878) | (2656) | (2652) | (2808) | (2999) | (3176) | (3345) | (3483) |
| $-0.5$ | 4509 | 3979 | **3765** | 3925 | 4252 | 4618 | 4971 | 5300 | 5604 |
| | (3046) | (2792) | (2633) | (2743) | (2906) | (3074) | (3239) | (3391) | (3527) |
| $-0.75$ | 4650 | 4182 | **3972** | 4048 | 4332 | 4671 | 5011 | 5331 | 5634 |
| | (3035) | (2710) | (2615) | (2732) | (2934) | (3126) | (3288) | (3429) | (3562) |
| $-1$ | 4847 | 4432 | **4213** | 4233 | 4441 | 4733 | 5059 | 5372 | 5671 |
| | (2968) | (2644) | (2570) | (2756) | (2952) | (3124) | (3306) | (3458) | (3590) |
| $-1.25$ | 5070 | 4664 | 4443 | **4429** | 4594 | 4839 | 5110 | 5411 | 5703 |
| | (2869) | (2535) | (2501) | (2714) | (2986) | (3197) | (3292) | (3450) | (3594) |
| $-1.5$ | 5193 | 4835 | 4644 | **4627** | 4738 | 4956 | 5196 | 5453 | 5733 |
| | (2696) | (2422) | (2424) | (2663) | (2966) | (3205) | (3365) | (3447) | (3571) |
| $-1.75$ | 5289 | 4996 | **4806** | 4811 | 4867 | 5065 | 5304 | 5549 | 5778 |
| | (2523) | (2340) | (2345) | (2589) | (2881) | (3188) | (3393) | (3549) | (3603) |
| $-2$ | 5412 | 5089 | **4916** | 4920 | 5054 | 5143 | 5369 | 5619 | 5849 |
| | (2454) | (2250) | (2278) | (2502) | (2828) | (3105) | (3366) | (3561) | (3703) |

Table 3: (continued).

$n = 300$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 2023 | **1985** | 1988 | 2010 | 2071 | 2152 | 2241 | 2351 | 2467 |
| | (1100) | (1105) | (1142) | (1173) | (1238) | (1291) | (1320) | (1373) | (1427) |
| 1.75 | 1957 | 1916 | **1912** | 1950 | 2024 | 2109 | 2221 | 2342 | 2467 |
| | (1090) | (1094) | (1122) | (1172) | (1237) | (1266) | (1323) | (1382) | (1434) |
| 1.5 | 1887 | 1839 | **1839** | 1885 | 1966 | 2076 | 2204 | 2338 | 2473 |
| | (1072) | (1079) | (1111) | (1163) | (1206) | (1266) | (1328) | (1388) | (1444) |
| 1.25 | 1823 | **1757** | 1761 | 1819 | 1923 | 2053 | 2196 | 2342 | 2485 |
| | (1067) | (1059) | (1090) | (1137) | (1201) | (1269) | (1335) | (1397) | (1453) |
| 1 | 1763 | **1681** | 1684 | 1761 | 1888 | 2040 | 2200 | 2356 | 2506 |
| | (1068) | (1051) | (1075) | (1128) | (1199) | (1273) | (1343) | (1406) | (1466) |
| 0.75 | 1718 | **1610** | 1612 | 1715 | 1872 | 2045 | 2217 | 2380 | 2534 |
| | (1083) | (1048) | (1061) | (1120) | (1198) | (1279) | (1351) | (1417) | (1473) |
| 0.5 | 1719 | **1564** | 1566 | 1697 | 1881 | 2067 | 2246 | 2413 | 2567 |
| | (1131) | (1063) | (1057) | (1116) | (1203) | (1287) | (1361) | (1426) | (1484) |
| 0.25 | 1951 | 1617 | **1575** | 1718 | 1909 | 2099 | 2279 | 2445 | 2598 |
| | (1722) | (1184) | (1069) | (1122) | (1214) | (1297) | (1371) | (1435) | (1492) |
| $-0.25$ | 1947 | 1649 | **1610** | 1752 | 1944 | 2134 | 2313 | 2479 | 2632 |
| | (1286) | (1119) | (1081) | (1143) | (1231) | (1314) | (1385) | (1448) | (1504) |
| $-0.5$ | 1851 | 1666 | **1644** | 1768 | 1951 | 2140 | 2318 | 2484 | 2638 |
| | (1133) | (1084) | (1092) | (1156) | (1240) | (1320) | (1391) | (1453) | (1509) |
| $-0.75$ | 1906 | 1761 | **1728** | 1814 | 1975 | 2153 | 2327 | 2490 | 2641 |
| | (1148) | (1136) | (1142) | (1173) | (1252) | (1328) | (1397) | (1458) | (1507) |
| $-1$ | 2011 | 1928 | **1893** | 1896 | 2016 | 2178 | 2342 | 2500 | 2651 |
| | (1257) | (1308) | (1339) | (1235) | (1266) | (1338) | (1403) | (1464) | (1517) |
| $-1.25$ | 2164 | 2083 | 2059 | **2039** | 2092 | 2214 | 2364 | 2516 | 2662 |
| | (1396) | (1373) | (1424) | (1427) | (1354) | (1358) | (1407) | (1467) | (1521) |
| $-1.5$ | 2305 | 2307 | 2271 | **2186** | 2215 | 2289 | 2402 | 2535 | 2676 |
| | (1462) | (1505) | (1542) | (1521) | (1529) | (1484) | (1446) | (1468) | (1520) |
| $-1.75$ | 2527 | 2534 | 2479 | 2401 | **2340** | 2376 | 2459 | 2565 | 2690 |
| | (1583) | (1572) | (1609) | (1650) | (1612) | (1586) | (1532) | (1496) | (1520) |
| $-2$ | 2735 | 2699 | 2674 | 2596 | 2520 | **2501** | 2532 | 2627 | 2732 |
| | (1642) | (1609) | (1659) | (1741) | (1739) | (1697) | (1622) | (1629) | (1618) |

Table 4: Case D. The average ISEs$\times 10^6$ of $\widetilde{f}_{b,c'/|\gamma|,\gamma}$.

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

$n = 100$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 2997 | 2940 | **2925** | 2953 | 3023 | 3113 | 3223 | 3355 | 3503 |
| | (1506) | (1491) | (1510) | (1561) | (1650) | (1726) | (1786) | (1855) | (1940) |
| 1.75 | 2889 | 2828 | **2812** | 2852 | 2920 | 3029 | 3164 | 3324 | 3490 |
| | (1493) | (1467) | (1476) | (1544) | (1601) | (1682) | (1763) | (1864) | (1963) |
| 1.5 | 2788 | 2707 | **2692** | 2730 | 2822 | 2954 | 3119 | 3298 | 3485 |
| | (1490) | (1431) | (1436) | (1476) | (1550) | (1643) | (1752) | (1865) | (1974) |
| 1.25 | 2672 | 2578 | **2563** | 2614 | 2732 | 2898 | 3089 | 3292 | 3498 |
| | (1465) | (1386) | (1381) | (1414) | (1508) | (1633) | (1761) | (1882) | (1993) |
| 1 | 2565 | 2451 | **2430** | 2505 | 2657 | 2856 | 3074 | 3302 | 3524 |
| | (1466) | (1352) | (1313) | (1369) | (1489) | (1629) | (1767) | (1899) | (2020) |
| 0.75 | 2466 | 2328 | **2309** | 2410 | 2604 | 2839 | 3087 | 3331 | 3565 |
| | (1465) | (1313) | (1254) | (1326) | (1471) | (1632) | (1786) | (1924) | (2045) |
| 0.5 | 2399 | 2224 | **2211** | 2357 | 2592 | 2858 | 3123 | 3379 | 3617 |
| | (1473) | (1278) | (1197) | (1292) | (1470) | (1649) | (1814) | (1957) | (2084) |
| 0.25 | 2370 | **2163** | 2174 | 2360 | 2622 | 2901 | 3169 | 3425 | 3671 |
| | (1480) | (1254) | (1168) | (1298) | (1498) | (1688) | (1849) | (1987) | (2131) |
| $-0.25$ | 2441 | 2204 | **2186** | 2371 | 2637 | 2920 | 3195 | 3450 | 3700 |
| | (1520) | (1295) | (1208) | (1346) | (1541) | (1722) | (1891) | (2020) | (2151) |
| $-0.5$ | 2524 | 2299 | **2242** | 2377 | 2622 | 2902 | 3177 | 3442 | 3684 |
| | (1547) | (1360) | (1281) | (1379) | (1554) | (1731) | (1885) | (2031) | (2144) |
| $-0.75$ | 2641 | 2433 | **2359** | 2447 | 2649 | 2902 | 3168 | 3427 | 3671 |
| | (1578) | (1417) | (1358) | (1437) | (1582) | (1745) | (1894) | (2027) | (2152) |
| $-1$ | 2779 | 2581 | **2505** | 2558 | 2716 | 2936 | 3179 | 3424 | 3662 |
| | (1620) | (1471) | (1428) | (1489) | (1624) | (1772) | (1903) | (2033) | (2148) |
| $-1.25$ | 2894 | 2728 | **2665** | 2707 | 2823 | 3007 | 3212 | 3441 | 3662 |
| | (1607) | (1505) | (1501) | (1589) | (1695) | (1834) | (1943) | (2073) | (2150) |
| $-1.5$ | 3086 | 2945 | **2876** | 2900 | 3006 | 3146 | 3309 | 3500 | 3688 |
| | (1693) | (1635) | (1641) | (1728) | (1852) | (1974) | (2059) | (2149) | (2198) |
| $-1.75$ | 3264 | 3124 | **3069** | 3086 | 3150 | 3262 | 3429 | 3599 | 3770 |
| | (1747) | (1717) | (1747) | (1811) | (1901) | (2010) | (2156) | (2257) | (2332) |
| $-2$ | 3435 | 3322 | 3273 | **3247** | 3305 | 3416 | 3537 | 3673 | 3846 |
| | (1818) | (1800) | (1844) | (1883) | (1970) | (2092) | (2205) | (2267) | (2380) |

Table 4: (continued).

$n = 300$

| | $c' = 0$ | 0.25 | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma = 2$ | 1329 | **1310** | 1320 | 1352 | 1404 | 1472 | 1549 | 1632 | 1717 |
| | (698) | (679) | (673) | (683) | (704) | (738) | (773) | (810) | (848) |
| 1.75 | 1287 | **1268** | 1278 | 1313 | 1373 | 1448 | 1533 | 1623 | 1712 |
| | (692) | (668) | (660) | (670) | (693) | (726) | (766) | (809) | (851) |
| 1.5 | 1246 | **1224** | 1235 | 1277 | 1345 | 1428 | 1522 | 1618 | 1715 |
| | (685) | (657) | (645) | (654) | (680) | (718) | (763) | (806) | (848) |
| 1.25 | 1211 | **1181** | 1192 | 1243 | 1321 | 1416 | 1516 | 1617 | 1718 |
| | (681) | (645) | (627) | (637) | (667) | (712) | (759) | (806) | (851) |
| 1 | 1175 | **1142** | 1156 | 1214 | 1302 | 1407 | 1514 | 1621 | 1723 |
| | (679) | (633) | (611) | (620) | (657) | (705) | (759) | (808) | (854) |
| 0.75 | 1151 | **1109** | 1125 | 1193 | 1292 | 1404 | 1517 | 1625 | 1729 |
| | (677) | (623) | (594) | (607) | (649) | (705) | (759) | (809) | (854) |
| 0.5 | 1137 | **1088** | 1105 | 1182 | 1289 | 1405 | 1521 | 1631 | 1735 |
| | (678) | (621) | (581) | (598) | (646) | (705) | (761) | (812) | (858) |
| 0.25 | 1138 | **1082** | 1096 | 1179 | 1289 | 1405 | 1523 | 1634 | 1738 |
| | (681) | (618) | (575) | (595) | (646) | (706) | (762) | (811) | (857) |
| $-0.25$ | 1155 | **1082** | 1093 | 1170 | 1283 | 1404 | 1523 | 1634 | 1739 |
| | (688) | (620) | (580) | (592) | (644) | (704) | (760) | (811) | (858) |
| $-0.5$ | 1169 | 1094 | **1093** | 1166 | 1279 | 1399 | 1520 | 1635 | 1741 |
| | (689) | (627) | (581) | (591) | (641) | (701) | (759) | (810) | (856) |
| $-0.75$ | 1195 | 1121 | **1113** | 1172 | 1277 | 1396 | 1517 | 1633 | 1740 |
| | (693) | (636) | (593) | (597) | (642) | (699) | (757) | (808) | (856) |
| $-1$ | 1229 | 1160 | **1146** | 1192 | 1284 | 1395 | 1514 | 1627 | 1736 |
| | (696) | (645) | (608) | (609) | (646) | (700) | (754) | (807) | (853) |
| $-1.25$ | 1272 | 1207 | **1190** | 1223 | 1300 | 1401 | 1514 | 1626 | 1732 |
| | (708) | (657) | (622) | (622) | (654) | (702) | (754) | (804) | (849) |
| $-1.5$ | 1314 | 1254 | **1238** | 1261 | 1325 | 1416 | 1519 | 1625 | 1729 |
| | (708) | (665) | (645) | (637) | (663) | (705) | (754) | (802) | (848) |
| $-1.75$ | 1363 | 1309 | **1292** | 1320 | 1371 | 1444 | 1532 | 1629 | 1728 |
| | (725) | (683) | (672) | (725) | (732) | (749) | (767) | (800) | (843) |
| $-2$ | 1424 | 1379 | **1362** | 1386 | 1430 | 1501 | 1579 | 1648 | 1743 |
| | (775) | (759) | (758) | (808) | (833) | (895) | (921) | (859) | (899) |

and $a = 0.01, 0.2, 0.4, 0.6, 0.8, 1,$ where $\gamma = -1.5, -1, -0.5, 0.5, 1, 1.5$ (we chose $\epsilon = (0.1)^6 b^2$ for the TS/JF type). The results are summarized as follows:

- For the cases A and B, the bias-reduced estimators (14), (22), and (23) with $\gamma = 1$ tended to work well, and outperformed the estimator (4). For the case C (D), the bias-reduced estimators (14), (22), and (23) with $\gamma = 0.5$ ($\gamma = -0.5$) worked well. Note that, for the cases C and D, some bias-reduced estimators with $\gamma = -1.5, 1, 1.5$ underperformed the estimator (4).

- As expected from the decreasingness of the factor $\{\lambda^4(a)/a\}^{2/9}$ in the AMISE of the SS/TS type, for the cases B and C, when $n = 300$, almost all average ISEs of the SS/TS type were minimized at $a = 1$ (when $n = 100$, some corresponding results were not obtained). For the cases A and B (D), it seemed that the TS type (the JF type) outperformed others. Note that, for the case C, the TS type was comparable to the JF type. Although, in view of Figure 3, the graphs of the factor $[\{\lambda^4(a)/a\}^2 \int_0^\infty \{B_{1,\gamma}^{(JF_a)}(x)\}^2 dx]^{1/9}$ in the AMISE of the JF type (we used $w(x) \equiv 1$ as an illustrative purpose here) tell us that, for the cases C and D, the JF type was expected to outperform the TS type, the bad behavior of the JF type for the cases B and C was in conflict with the graphs. We guess that the contradiction was caused by the small sample size $n$.

In summary, the selection of $\gamma \neq 0$ depends on $f$, as expected. We can say that, when $f(0)$ is small or zero, the estimator (4) (the bias-reduced estimators (14), (22), and (23)) using $\gamma < 0$ has better performance.

## 6. Conclusion

In this paper, we have studied the asymptotic properties of the Amoroso kernel density estimator and its bias-reduced estimators, under suitable conditions. We have shown that, using $b \propto n^{-2/5}$, the MISE of the (uncorrected) Amoroso kernel density estimator (4) achieves $O(n^{-4/5})$, whose optimal AMISE (12) is minimized at $c|\gamma| = 1/4$ (see (13)). Further, we have demonstrated that, when $\rho_c(t) = t + c$, the asymptotic MISE (MSE) convergence rates (20), (29), and (30) ((19) and (26)) for the bias-reduced estimators (14), (22), and (23) are faster than that of (12) ((10)) for the estimator (4); especially, the MISEs achieve $O(n^{-8/9})$ using $b \propto n^{-2/9}$. We have illustrated, through the simulations, the finite sample performance of the proposed estimators. It should be remarked that, in practice, we must use data-driven smoothing parameter selectors, such as plug-in, rule-of-thumb, and so on (see Wand and Jones (1995; chapter 3))[14].

[14] We conducted the simulation studies using cross-validation. However, the results are omitted to save space.

Table 5: Case A. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#_a)}_{b,c,\gamma}$ ($\# = SS, TS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\gamma > 0$ or $c = 1.1$ for $\gamma < 0$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

| | $n = 100$ | | | | | | | $n = 300$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | |
| | | | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ | | | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ |
| $\gamma = 1.5$ | 1182 | **1176** | 1186 | 1220 | 1322 | 1886 | 1997 | **534** | 535 | 539 | 550 | 587 | 857 | 927 |
| | (1007) | (998) | (1001) | (1016) | (1060) | (1278) | (1312) | (442) | (443) | (444) | (448) | (460) | (550) | (570) |
| 1 | 1084 | **1079** | 1087 | 1115 | 1196 | 1659 | 1759 | **469** | 469 | 473 | 483 | 515 | 735 | 799 |
| | (981) | (978) | (981) | (997) | (1041) | (1249) | (1283) | (409) | (409) | (411) | (417) | (432) | (523) | (544) |
| 0.5 | 1252 | 1239 | **1233** | 1241 | 1284 | 1560 | 1627 | **500** | 500 | 502 | 508 | 527 | 666 | 709 |
| | (1225) | (1222) | (1224) | (1231) | (1254) | (1382) | (1411) | (465) | (466) | (467) | (470) | (479) | (540) | (557) |
| $-0.5$ | 1585 | 1553 | 1529 | **1520** | 1546 | 1809 | 1885 | 628 | 628 | **627** | 630 | 642 | 762 | 810 |
| | (1410) | (1393) | (1386) | (1393) | (1416) | (1513) | (1542) | (536) | (537) | (538) | (539) | (547) | (597) | (614) |
| $-1$ | 1767 | 1709 | 1640 | **1605** | 1657 | 2189 | 2304 | 717 | 710 | **706** | 711 | 729 | 951 | 1025 |
| | (1513) | (1471) | (1392) | (1336) | (1340) | (1481) | (1512) | (585) | (580) | (570) | (568) | (568) | (612) | (632) |
| $-1.5$ | 2267 | 2162 | 2059 | **1979** | 1989 | 2602 | 2726 | 890 | 881 | 870 | **868** | 881 | 1144 | 1225 |
| | (1875) | (1760) | (1631) | (1533) | (1459) | (1563) | (1590) | (698) | (687) | (667) | (653) | (638) | (653) | (671) |
| | | | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ | | | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | 1158 | 1093 | **1052** | 1053 | 1127 | 1723 | 1997 | 502 | 495 | **494** | 500 | 522 | 772 | 927 |
| | (1057) | (989) | (944) | (933) | (969) | (1219) | (1312) | (452) | (440) | (434) | (435) | (441) | (523) | (570) |
| 1 | 1037 | 1001 | **981** | 989 | 1052 | 1530 | 1759 | 436 | **433** | 433 | 440 | 463 | 670 | 799 |
| | (967) | (935) | (920) | (923) | (961) | (1195) | (1283) | (392) | (389) | (389) | (393) | (406) | (497) | (544) |
| 0.5 | 1314 | 1294 | 1279 | **1275** | 1295 | 1511 | 1627 | 525 | **524** | 525 | 529 | 541 | 644 | 709 |
| | (1227) | (1216) | (1210) | (1211) | (1229) | (1356) | (1411) | (465) | (465) | (465) | (467) | (474) | (528) | (557) |
| $-0.5$ | 1733 | 1697 | 1657 | 1620 | **1587** | 1674 | 1885 | 665 | 663 | 661 | **661** | 664 | 708 | 810 |
| | (1495) | (1454) | (1407) | (1375) | (1359) | (1456) | (1542) | (546) | (543) | (536) | (534) | (535) | (572) | (614) |
| $-1$ | 1954 | 1858 | 1768 | 1646 | **1501** | 1778 | 2304 | 716 | 696 | 673 | 653 | **627** | 749 | 1025 |
| | (1756) | (1687) | (1619) | (1487) | (1336) | (1362) | (1512) | (639) | (624) | (610) | (585) | (535) | (551) | (632) |
| $-1.5$ | 2614 | 2503 | 2339 | 2106 | **1855** | 2092 | 2726 | 902 | 866 | 822 | 765 | **712** | 867 | 1225 |
| | (2224) | (2151) | (2066) | (1838) | (1642) | (1447) | (1590) | (780) | (767) | (743) | (688) | (620) | (594) | (671) |
| | | | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ | | | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | 1158 | **1090** | 1099 | 1161 | 1292 | 1885 | 1997 | 502 | **498** | 509 | 529 | 576 | 856 | 927 |
| | (1057) | (971) | (959) | (988) | (1048) | (1278) | (1312) | (452) | (434) | (435) | (441) | (456) | (550) | (570) |
| 1 | 1037 | **1009** | 1021 | 1069 | 1173 | 1659 | 1759 | **436** | 438 | 449 | 466 | 506 | 735 | 799 |
| | (967) | (941) | (946) | (974) | (1031) | (1248) | (1283) | (392) | (393) | (399) | (409) | (429) | (523) | (544) |
| 0.5 | 1314 | 1277 | 1252 | **1247** | 1282 | 1559 | 1627 | 525 | 517 | **513** | 513 | 529 | 666 | 709 |
| | (1227) | (1217) | (1216) | (1225) | (1251) | (1382) | (1411) | (465) | (463) | (466) | (469) | (479) | (539) | (557) |
| $-0.5$ | 1733 | 1655 | 1584 | **1536** | 1543 | 1809 | 1885 | 665 | 652 | 643 | **638** | 644 | 762 | 810 |
| | (1495) | (1423) | (1383) | (1381) | (1410) | (1513) | (1542) | (546) | (536) | (534) | (536) | (545) | (597) | (614) |
| $-1$ | 1954 | 1797 | 1672 | **1568** | 1622 | 2189 | 2304 | 716 | 692 | **679** | 686 | 717 | 951 | 1025 |
| | (1756) | (1615) | (1493) | (1347) | (1328) | (1481) | (1512) | (639) | (611) | (587) | (565) | (565) | (612) | (632) |
| $-1.5$ | 2614 | 2385 | 2124 | 1948 | **1948** | 2602 | 2726 | 902 | 857 | **832** | 832 | 865 | 1143 | 1225 |
| | (2224) | (2077) | (1796) | (1595) | (1455) | (1563) | (1590) | (780) | (750) | (700) | (651) | (639) | (653) | (671) |

Table 6: Case B. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#_a)}_{b,c,\gamma}$ ($\# = SS, TS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\gamma > 0$ or $c = 1.1$ for $\gamma < 0$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

| | $n = 100$ | | | | | | | $n = 300$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | |
| | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ |
| $\gamma = 1.5$ | 2364 | **2370** | 2394 | 2458 | 2641 | 3602 | 3793 | **1067** | 1069 | 1079 | 1104 | 1182 | 1676 | 1806 |
| | (2026) | (2028) | (2040) | (2070) | (2150) | (2532) | (2594) | (906) | (907) | (911) | (919) | (942) | (1097) | (1135) |
| 1 | **2166** | 2170 | 2188 | 2237 | 2379 | 3154 | 3324 | **933** | 935 | 943 | 964 | 1027 | 1430 | 1543 |
| | (2073) | (2075) | (2084) | (2109) | (2176) | (2500) | (2558) | (898) | (899) | (902) | (910) | (932) | (1071) | (1106) |
| 0.5 | **2184** | 2185 | 2193 | 2217 | 2290 | 2751 | 2862 | **878** | 879 | 884 | 896 | 932 | 1183 | 1257 |
| | (2466) | (2468) | (2472) | (2484) | (2517) | (2711) | (2758) | (1020) | (1021) | (1023) | (1028) | (1043) | (1141) | (1170) |
| $-0.5$ | 2515 | **2514** | 2526 | 2560 | 2660 | 3188 | 3318 | **1028** | 1029 | 1034 | 1049 | 1088 | 1359 | 1444 |
| | (2672) | (2687) | (2704) | (2723) | (2778) | (2937) | (2983) | (1068) | (1068) | (1072) | (1080) | (1094) | (1180) | (1207) |
| $-1$ | 2839 | **2837** | 2863 | 2943 | 3120 | 4054 | 4253 | **1265** | 1267 | 1276 | 1296 | 1353 | 1801 | 1933 |
| | (2528) | (2539) | (2552) | (2604) | (2657) | (2850) | (2908) | (1070) | (1083) | (1090) | (1087) | (1099) | (1201) | (1235) |
| $-1.5$ | 3471 | **3446** | 3453 | 3504 | 3694 | 4798 | 5014 | 1603 | 1607 | **1600** | 1603 | 1650 | 2161 | 2310 |
| | (2739) | (2697) | (2673) | (2699) | (2730) | (2941) | (2997) | (1209) | (1215) | (1202) | (1180) | (1162) | (1259) | (1294) |
| | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | **1988** | 1993 | 2013 | 2069 | 2230 | 3312 | 3793 | **893** | 895 | 903 | 926 | 994 | 1518 | 1806 |
| | (1828) | (1830) | (1841) | (1869) | (1944) | (2419) | (2594) | (808) | (809) | (811) | (822) | (853) | (1043) | (1135) |
| 1 | **1887** | 1891 | 1907 | 1950 | 2078 | 2927 | 3324 | **807** | 809 | 816 | 834 | 890 | 1307 | 1543 |
| | (1936) | (1938) | (1947) | (1970) | (2034) | (2411) | (2558) | (839) | (840) | (843) | (851) | (874) | (1029) | (1106) |
| 0.5 | 2152 | **2148** | 2151 | 2169 | 2228 | 2654 | 2862 | **864** | 864 | 868 | 878 | 908 | 1133 | 1257 |
| | (2416) | (2419) | (2425) | (2437) | (2469) | (2671) | (2758) | (996) | (996) | (998) | (1003) | (1018) | (1120) | (1170) |
| $-0.5$ | 2596 | 2564 | 2541 | **2535** | 2568 | 2950 | 3318 | 1041 | **1040** | 1043 | 1048 | 1067 | 1247 | 1444 |
| | (2558) | (2582) | (2583) | (2599) | (2649) | (2867) | (2983) | (1039) | (1040) | (1043) | (1039) | (1055) | (1145) | (1207) |
| $-1$ | 2778 | 2661 | 2539 | **2421** | 2429 | 3379 | 4253 | 1085 | 1081 | **1077** | 1088 | 1118 | 1478 | 1933 |
| | (2691) | (2638) | (2526) | (2406) | (2307) | (2648) | (2908) | (1027) | (1043) | (1048) | (1061) | (1043) | (1113) | (1235) |
| $-1.5$ | 3667 | 3525 | 3287 | 2994 | **2779** | 3882 | 5014 | 1381 | 1347 | 1319 | **1296** | 1302 | 1708 | 2310 |
| | (3213) | (3182) | (3060) | (2840) | (2454) | (2719) | (2997) | (1147) | (1127) | (1111) | (1097) | (1077) | (1141) | (1294) |
| | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | **1988** | 2094 | 2202 | 2339 | 2586 | 3601 | 3793 | **893** | 942 | 991 | 1050 | 1156 | 1676 | 1806 |
| | (1828) | (1888) | (1941) | (2011) | (2126) | (2532) | (2594) | (808) | (836) | (864) | (893) | (932) | (1097) | (1135) |
| 1 | **1887** | 1962 | 2042 | 2145 | 2336 | 3153 | 3324 | **807** | 841 | 877 | 922 | 1007 | 1430 | 1543 |
| | (1936) | (1976) | (2017) | (2068) | (2158) | (2499) | (2558) | (839) | (857) | (873) | (893) | (924) | (1071) | (1106) |
| 0.5 | 2152 | **2150** | 2161 | 2192 | 2275 | 2750 | 2862 | **864** | 865 | 871 | 886 | 926 | 1183 | 1257 |
| | (2416) | (2433) | (2449) | (2470) | (2511) | (2711) | (2758) | (996) | (1003) | (1010) | (1020) | (1040) | (1141) | (1170) |
| $-0.5$ | 2596 | 2519 | **2498** | 2531 | 2639 | 3187 | 3318 | 1041 | 1032 | **1029** | 1040 | 1082 | 1359 | 1444 |
| | (2558) | (2612) | (2641) | (2691) | (2760) | (2937) | (2983) | (1039) | (1054) | (1053) | (1066) | (1090) | (1179) | (1207) |
| $-1$ | 2778 | 2620 | **2618** | 2781 | 3055 | 4053 | 4253 | **1085** | 1129 | 1186 | 1237 | 1327 | 1800 | 1933 |
| | (2691) | (2586) | (2458) | (2517) | (2638) | (2850) | (2908) | (1027) | (1049) | (1072) | (1078) | (1090) | (1201) | (1235) |
| $-1.5$ | 3667 | 3344 | **3203** | 3331 | 3610 | 4797 | 5014 | **1381** | 1408 | 1485 | 1539 | 1618 | 2160 | 2310 |
| | (3213) | (3018) | (2706) | (2654) | (2694) | (2941) | (2997) | (1147) | (1132) | (1157) | (1170) | (1161) | (1258) | (1294) |

Table 7: Case C. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#_a)}_{b,c,\gamma}$ ($\# = SS, TS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\gamma > 0$ or $c = 1.1$ for $\gamma < 0$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

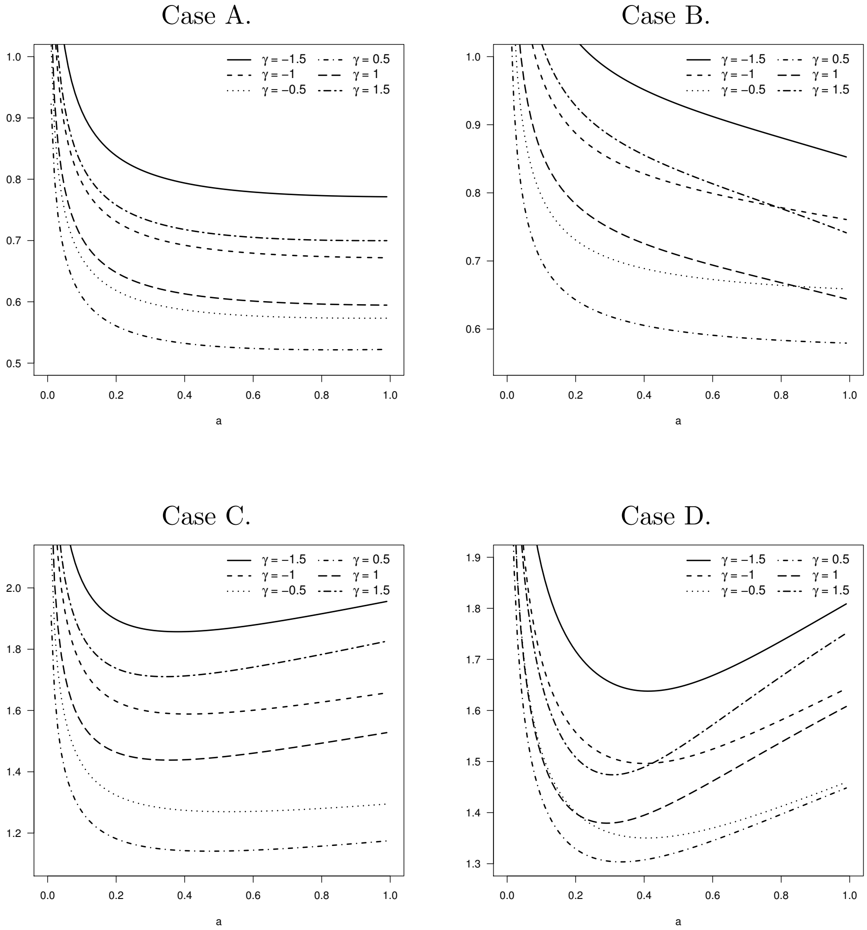
| | $n = 100$ | | | | | | | $n = 300$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | |
| | | | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ | | | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ |
| $\gamma = 1.5$ | 4772 | 4780 | 4757 | 4726 | 4706 | **4553** | 4561 | 2179 | 2181 | **2168** | 2172 | 2195 | 2205 | 2204 |
| | (3003) | (3000) | (2962) | (2922) | (2870) | (2825) | (2841) | (1579) | (1575) | (1525) | (1499) | (1445) | (1337) | (1328) |
| 1 | 3982 | 3984 | 3982 | 3983 | 3983 | **3967** | 3987 | 1770 | **1770** | 1773 | 1784 | 1809 | 1878 | 1888 |
| | (2506) | (2504) | (2497) | (2488) | (2471) | (2526) | (2556) | (1232) | (1229) | (1224) | (1223) | (1208) | (1196) | (1199) |
| 0.5 | **3334** | 3334 | 3335 | 3336 | 3348 | 3431 | 3460 | **1401** | 1402 | 1405 | 1412 | 1433 | 1540 | 1566 |
| | (2342) | (2340) | (2340) | (2341) | (2346) | (2417) | (2447) | (986) | (986) | (987) | (990) | (997) | (1042) | (1057) |
| $-0.5$ | **3734** | 3736 | 3740 | 3771 | 3798 | 3773 | 3792 | **1534** | 1534 | 1534 | 1541 | 1560 | 1646 | 1667 |
| | (2670) | (2660) | (2653) | (2678) | (2670) | (2601) | (2626) | (1081) | (1081) | (1069) | (1073) | (1067) | (1092) | (1104) |
| $-1$ | **4488** | 4585 | 4701 | 4880 | 4837 | 4592 | 4593 | **2028** | 2080 | 2188 | 2182 | 2058 | 2085 | 2093 |
| | (3197) | (3293) | (3422) | (3586) | (3438) | (3058) | (3036) | (1664) | (1780) | (2027) | (2002) | (1611) | (1297) | (1296) |
| $-1.5$ | **5263** | 5467 | 5749 | 6026 | 5832 | 5429 | 5401 | 2663 | 2820 | 3013 | 3063 | 2568 | 2501 | **2497** |
| | (3786) | (3942) | (4182) | (4414) | (4125) | (3527) | (3476) | (2201) | (2500) | (2824) | (2987) | (1962) | (1480) | (1471) |
| | | | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ | | | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | 4988 | 5032 | 5077 | 5050 | 4904 | 4566 | **4561** | 2283 | 2330 | 2411 | 2389 | 2285 | 2216 | **2204** |
| | (3090) | (3113) | (3116) | (3064) | (2965) | (2817) | (2841) | (1599) | (1730) | (1892) | (1812) | (1533) | (1348) | (1328) |
| 1 | 4087 | 4083 | 4090 | 4077 | 4045 | **3956** | 3987 | **1835** | 1850 | 1850 | 1846 | 1853 | 1874 | 1888 |
| | (2544) | (2526) | (2521) | (2506) | (2481) | (2502) | (2556) | (1250) | (1302) | (1294) | (1262) | (1236) | (1198) | (1199) |
| 0.5 | 3321 | 3320 | **3318** | 3318 | 3328 | 3413 | 3460 | **1400** | 1400 | 1402 | 1407 | 1425 | 1528 | 1566 |
| | (2355) | (2350) | (2346) | (2345) | (2347) | (2402) | (2447) | (981) | (981) | (982) | (985) | (993) | (1036) | (1057) |
| $-0.5$ | 3773 | 3768 | **3766** | 3774 | 3801 | 3784 | 3792 | **1532** | 1536 | 1541 | 1549 | 1568 | 1638 | 1667 |
| | (2772) | (2740) | (2716) | (2707) | (2695) | (2582) | (2626) | (1049) | (1067) | (1083) | (1096) | (1103) | (1087) | (1104) |
| $-1$ | **4513** | 4570 | 4677 | 4807 | 5127 | 4648 | 4593 | **2061** | 2084 | 2153 | 2274 | 2383 | 2107 | 2093 |
| | (3246) | (3316) | (3423) | (3487) | (3712) | (3098) | (3036) | (1585) | (1630) | (1768) | (2062) | (2250) | (1338) | (1296) |
| $-1.5$ | **5375** | 5406 | 5541 | 5835 | 6554 | 5570 | 5401 | 2597 | 2729 | 2825 | 3102 | 3612 | 2589 | **2497** |
| | (3976) | (3956) | (4035) | (4209) | (4783) | (3685) | (3476) | (1932) | (2152) | (2311) | (2836) | (3681) | (1664) | (1471) |
| | | | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ | | | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | 4988 | 4960 | 4917 | 4801 | 4726 | **4553** | 4561 | 2283 | 2307 | 2292 | 2211 | 2207 | 2205 | **2204** |
| | (3090) | (3061) | (3023) | (2943) | (2879) | (2824) | (2841) | (1599) | (1705) | (1698) | (1514) | (1447) | (1337) | (1328) |
| 1 | 4087 | 4056 | 4031 | 4003 | 3988 | **3966** | 3987 | 1835 | 1831 | 1813 | **1805** | 1817 | 1878 | 1888 |
| | (2544) | (2511) | (2496) | (2483) | (2471) | (2526) | (2556) | (1250) | (1278) | (1247) | (1227) | (1211) | (1196) | (1199) |
| 0.5 | 3321 | **3318** | 3318 | 3323 | 3339 | 3430 | 3460 | **1400** | 1400 | 1402 | 1409 | 1430 | 1540 | 1566 |
| | (2355) | (2345) | (2340) | (2340) | (2345) | (2417) | (2447) | (981) | (982) | (984) | (988) | (997) | (1042) | (1057) |
| $-0.5$ | 3773 | **3754** | 3756 | 3768 | 3801 | 3774 | 3792 | **1532** | 1538 | 1538 | 1545 | 1559 | 1646 | 1667 |
| | (2772) | (2702) | (2683) | (2676) | (2679) | (2601) | (2626) | (1049) | (1082) | (1083) | (1095) | (1066) | (1092) | (1104) |
| $-1$ | **4513** | 4575 | 4669 | 4884 | 4907 | 4593 | 4593 | **2061** | 2088 | 2202 | 2276 | 2105 | 2085 | 2093 |
| | (3246) | (3287) | (3357) | (3537) | (3491) | (3058) | (3036) | (1585) | (1655) | (1952) | (2154) | (1704) | (1297) | (1296) |
| $-1.5$ | 5375 | **5354** | 5596 | 6086 | 5937 | 5427 | 5401 | 2597 | 2759 | 3058 | 3293 | 2629 | 2503 | **2497** |
| | (3976) | (3877) | (3961) | (4420) | (4249) | (3528) | (3476) | (1932) | (2240) | (2762) | (3218) | (2105) | (1483) | (1471) |

Table 8: Case D. The average ISEs$\times 10^6$ of $\widehat{f}^{(\#_a)}_{b,c,\gamma}$ ($\# = SS, TS, JF$) and $\widehat{f}_{b,c,\gamma}$ ($c = 1$ for $\gamma > 0$ or $c = 1.1$ for $\gamma < 0$).

The bold-faced number indicates the smallest average ISE in each row.
The number in the parentheses stands for the standard deviation$\times 10^6$ of the ISEs.

| | $n = 100$ | | | | | | | $n = 300$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | | $a = 1$ | 0.8 | 0.6 | 0.4 | 0.2 | 0.01 | |
| | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ | $\widehat{f}^{(SS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ |
| $\gamma = 1.5$ | 3225 | 3227 | 3223 | 3219 | 3205 | 3132 | **3119** | 1539 | 1538 | 1534 | 1534 | 1534 | 1527 | **1522** |
| | (1914) | (1904) | (1897) | (1883) | (1875) | (1766) | (1752) | (872) | (867) | (859) | (854) | (829) | (771) | (763) |
| 1 | 2658 | 2656 | 2654 | 2652 | **2649** | 2657 | 2657 | **1259** | 1260 | 1260 | 1263 | 1269 | 1300 | 1302 |
| | (1591) | (1586) | (1573) | (1557) | (1539) | (1496) | (1489) | (701) | (700) | (699) | (695) | (685) | (661) | (657) |
| 0.5 | 2155 | **2154** | 2154 | 2154 | 2158 | 2197 | 2211 | **1048** | 1048 | 1048 | 1051 | 1059 | 1095 | 1105 |
| | (1212) | (1210) | (1206) | (1200) | (1191) | (1191) | (1197) | (602) | (602) | (601) | (600) | (596) | (581) | (581) |
| $-0.5$ | **2169** | 2170 | 2172 | 2176 | 2192 | 2259 | 2270 | **1028** | 1028 | 1030 | 1033 | 1040 | 1095 | 1105 |
| | (1294) | (1293) | (1294) | (1297) | (1304) | (1300) | (1299) | (591) | (592) | (591) | (591) | (587) | (580) | (580) |
| $-1$ | **2651** | 2660 | 2678 | 2721 | 2802 | 2846 | 2838 | **1185** | 1187 | 1189 | 1199 | 1227 | 1330 | 1333 |
| | (1697) | (1699) | (1771) | (1815) | (1886) | (1745) | (1733) | (652) | (659) | (654) | (658) | (666) | (676) | (671) |
| $-1.5$ | **3339** | 3345 | 3379 | 3484 | 3662 | 3576 | 3535 | **1448** | 1457 | 1487 | 1541 | 1581 | 1597 | 1591 |
| | (2183) | (2201) | (2241) | (2357) | (2529) | (2350) | (2313) | (803) | (835) | (916) | (1071) | (1118) | (795) | (785) |
| | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ | $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | 3460 | 3457 | 3446 | 3425 | 3358 | 3166 | **3119** | 1674 | 1672 | 1664 | 1654 | 1629 | 1543 | **1522** |
| | (1972) | (1964) | (1954) | (1945) | (1897) | (1800) | (1752) | (909) | (909) | (899) | (883) | (854) | (780) | (763) |
| 1 | 2823 | 2819 | 2812 | 2796 | 2758 | 2669 | **2657** | 1358 | 1358 | 1357 | 1350 | 1340 | 1307 | **1302** |
| | (1655) | (1644) | (1635) | (1616) | (1575) | (1506) | (1489) | (728) | (727) | (725) | (719) | (704) | (666) | (657) |
| 0.5 | 2231 | 2228 | 2228 | 2222 | 2216 | **2199** | 2211 | 1083 | **1082** | 1083 | 1084 | 1087 | 1098 | 1105 |
| | (1253) | (1252) | (1246) | (1240) | (1221) | (1191) | (1197) | (617) | (616) | (616) | (614) | (607) | (583) | (581) |
| $-0.5$ | **2260** | 2262 | 2261 | 2261 | 2262 | 2268 | 2270 | 1069 | **1069** | 1070 | 1071 | 1077 | 1098 | 1105 |
| | (1332) | (1331) | (1330) | (1330) | (1328) | (1311) | (1299) | (603) | (604) | (604) | (602) | (599) | (581) | (580) |
| $-1$ | 2869 | 2871 | 2876 | 2887 | 2939 | 2940 | **2838** | 1306 | **1306** | 1307 | 1307 | 1315 | 1342 | 1333 |
| | (1748) | (1746) | (1750) | (1778) | (1878) | (1887) | (1733) | (686) | (687) | (686) | (682) | (685) | (684) | (671) |
| $-1.5$ | 3616 | 3610 | 3630 | 3688 | 3782 | 3804 | **3535** | 1599 | 1605 | 1631 | 1678 | 1693 | 1675 | **1591** |
| | (2271) | (2257) | (2296) | (2299) | (2432) | (2561) | (2313) | (830) | (852) | (922) | (1070) | (1109) | (1029) | (785) |
| | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ | $\widehat{f}^{(JF_a)}_{b,c,\gamma}$ | | | | | | $\widehat{f}_{b,c,\gamma}$ |
| 1.5 | 3460 | 3407 | 3352 | 3295 | 3234 | 3132 | **3119** | 1674 | 1646 | 1614 | 1587 | 1558 | 1527 | **1522** |
| | (1972) | (1947) | (1923) | (1894) | (1881) | (1766) | (1752) | (909) | (898) | (877) | (864) | (832) | (771) | (763) |
| 1 | 2823 | 2784 | 2749 | 2711 | 2672 | 2657 | **2657** | 1358 | 1339 | 1322 | 1303 | **1287** | 1299 | 1302 |
| | (1655) | (1630) | (1607) | (1573) | (1541) | (1496) | (1489) | (728) | (721) | (715) | (704) | (689) | (660) | (657) |
| 0.5 | 2231 | 2213 | 2197 | 2179 | **2167** | 2196 | 2211 | 1083 | 1076 | 1069 | **1064** | 1065 | 1095 | 1105 |
| | (1253) | (1242) | (1229) | (1214) | (1197) | (1191) | (1197) | (617) | (613) | (611) | (606) | (599) | (581) | (581) |
| $-0.5$ | 2260 | 2246 | 2228 | 2213 | **2209** | 2260 | 2270 | 1069 | 1063 | 1055 | 1051 | **1051** | 1096 | 1105 |
| | (1332) | (1323) | (1317) | (1311) | (1309) | (1300) | (1299) | (603) | (602) | (600) | (596) | (591) | (580) | (580) |
| $-1$ | 2869 | 2832 | 2806 | **2804** | 2843 | 2846 | 2838 | 1306 | 1285 | 1265 | **1251** | 1253 | 1330 | 1333 |
| | (1748) | (1737) | (1755) | (1843) | (1897) | (1745) | (1733) | (686) | (682) | (672) | (669) | (671) | (676) | (671) |
| $-1.5$ | 3616 | 3550 | **3524** | 3566 | 3681 | 3577 | 3535 | 1599 | **1578** | 1578 | 1606 | 1613 | 1597 | 1591 |
| | (2271) | (2234) | (2256) | (2340) | (2473) | (2350) | (2313) | (830) | (847) | (918) | (1077) | (1116) | (795) | (785) |

Figure 3: Graphs of $[\{\lambda^4(a)/a\}^2 \int_0^\infty \{B_{1,\gamma}^{(JF_a)}(x)\}^2 dx]^{1/9}$.

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## Appendix A

### A1 Technical lemmas

For the notational simplicity, we rewrite $\widetilde{f}_{b,c,\gamma}(x) - E[\widetilde{f}_{b,c,\gamma}(x)]$ as the average of independent zero-mean random variables

$$\Delta^{\dagger}_{b,c,\gamma,i}(x) = K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(X_i) - E[K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(X_i)], \quad i = 1, \ldots, n,$$

i.e.,

$$\widetilde{f}_{b,c,\gamma}(x) - E[\widetilde{f}_{b,c,\gamma}(x)] = \sum_{i=1}^{n} \frac{1}{n} \Delta^{\dagger}_{b,c,\gamma,i}(x) = \overline{\Delta}^{\dagger}_{b,c,\gamma}(x)$$

(for the case $\widehat{f}_{b,c,\gamma}(x) - E[\widehat{f}_{b,c,\gamma}(x)]$, we use the notation $\Delta_{b,c,\gamma,i}(x)$ and $\overline{\Delta}_{b,c,\gamma}(x)$, with $\rho_c(x/b) = x/b + c$).

Throughout this appendix, we denote by $\xi_{\alpha,\beta,\gamma}$ the random variable that is distributed according to the Amoroso density $K^{(A)}_{\alpha,\beta,\gamma}$, where $\alpha, \beta > 0$ and $\gamma \neq 0$. Recalling the definition (5), we have, for $j > 0$,

$$E[\xi^{j}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}] = (b\rho)^j \frac{\Gamma^{j-1}(\alpha_\gamma(\rho))\Gamma(\alpha_\gamma(\rho)+j/\gamma)}{\Gamma^{j}(\alpha_\gamma(\rho)+1/\gamma)} = \begin{cases} (b\rho)^j \dfrac{\Gamma^{j-1}(\rho/\gamma)\Gamma((\rho+j)/\gamma)}{\Gamma^{j}((\rho+1)/\gamma)}, & \gamma > 0, \\ (b\rho)^j \dfrac{\Gamma^{j-1}((\rho+1)/|\gamma|)\Gamma((\rho+1-j)/|\gamma|)}{\Gamma^{j}(\rho/|\gamma|)}, & \gamma < 0 \end{cases} \tag{A1}$$

(this moment, for $\rho > 0$, always exists when $\gamma > 0$, whereas, when $\gamma < 0$, the restriction $\rho > \max(0, j-1)$ is required). Also, it is easy to see that, for $a_1, a_2 \in (0, 1]$,

$$\begin{aligned} &\prod_{i=1}^{2} K^{(A)}_{\alpha_\gamma(\rho_c(a_i x/b)),(b/a_i)\beta_\gamma(\rho_c(a_i x/b)),\gamma}(s) \\ &= b^{-1}|\gamma|\delta_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b)) K^{(A)}_{\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma,bB_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b)),\gamma}(s), \end{aligned} \tag{A2}$$

where $B_{\gamma,a_1,a_2}$ is an infinitely differentiable function on $(0,\infty)^2$, defined by

$$B_{\gamma,a_1,a_2}(\rho_1,\rho_2) = \frac{\{\beta_\gamma(\rho_1)/a_1\}\{\beta_\gamma(\rho_2)/a_2\}}{[\{\beta_\gamma(\rho_1)/a_1\}^\gamma + \{\beta_\gamma(\rho_2)/a_2\}^\gamma]^{1/\gamma}} \quad \text{(note that } B_{\gamma,1,1}(\rho,\rho) = \beta_\gamma(\rho)/2^{1/\gamma}\text{)}.$$

First of all, we compute the $j$th moments (about $x$) of the random variables $\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}$ and $\xi_{\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma,bB_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b)),\gamma}$, and approximate the coefficient in (A2), as follows.

**Lemma A.1** (i). *Given $\gamma \neq 0$ and $r_c(0) > 0$, we have, for $x \geq 0$,*

$$E[\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}] = b\rho_c(x/b).$$

(ii). *Given $\gamma \neq 0$ and $r_c(0) > 0$, we have, $x/b \to \infty$,*

$$E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^j] = \begin{cases} \dfrac{1}{|\gamma|}bx + \dfrac{\delta^{[2]}_{c,\gamma}}{\gamma^2}b^2 + O(b^3x^{-1}), & j = 2, \\ \dfrac{\delta^{[3]}_{c,\gamma}}{\gamma^2}b^2x + O(b^3), & j = 3, \\ \dfrac{3}{\gamma^2}b^2x^2 + O(b^3x), & j = 4, \\ O(b^3x^3), & j = 6. \end{cases}$$

*Also, we have, for $x/b \to \kappa$,*

$$E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^2] = \begin{cases} b^2\eta_\gamma(\kappa,\rho_c(\kappa)) + o(b^2), & \dfrac{x}{b} \to \kappa\ (x \neq 0), \\ b^2\eta_\gamma(0, r_c(0)), & x = 0, \end{cases}$$

*provided that $\rho_c(x/b) > 1$ when $\gamma < 0$ (see* (A1)*).*

(iii). *Given $\gamma \neq 0$ and $r_c(0) > 1/2$, we have, for $a_1, a_2 \in (0,1]$,*

$$E[(\xi_{\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma, bB_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b)),\gamma} - x)^2] = \begin{cases} O(bx), & \dfrac{x}{b} \to \infty, \\ O(b^2), & \dfrac{x}{b} \to \kappa \end{cases}$$

(*see* (A2)). *Specifically, we have, for $a_1 = a_2 = 1$,*

$$E[(\xi_{2\alpha_\gamma(\rho_c(x/b))-1/\gamma, b\beta_\gamma(\rho_c(x/b))/2^{1/\gamma},\gamma} - x)^2] = \begin{cases} O(bx), & \dfrac{x}{b} \to \infty, \\ O(b^2), & \dfrac{x}{b} \to \kappa. \end{cases}$$

(iv). *Given $\gamma \neq 0$ and $r_c(0) > 0$, we have, for $a_1, a_2 \in (0,1]$,*

$$\delta_{\gamma,a_1,a_2}(\rho_c(a_1x/b), \rho_c(a_2x/b)) = \begin{cases} \Big(\dfrac{a_1a_2}{a_1+a_2}\Big)^{1/2} \dfrac{b^{1/2}|\gamma|^{-1/2}}{(2\pi x)^{1/2}}\{1 + O(bx^{-1})\}, & \dfrac{x}{b} \to \infty, \\ \delta_{\gamma,a_1,a_2}(\rho_c(a_1\kappa), \rho_c(a_2\kappa)) + o(1), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ \delta_{\gamma,a_1,a_2}(r_c(0), r_c(0)), & x = 0. \end{cases}$$

*Specifically, we have, for $a_1 = a_2 = 1$,*

$$v_\gamma(\rho_c(x/b)) = \begin{cases} \dfrac{b^{1/2}|\gamma|^{-1/2}}{2(\pi x)^{1/2}}\{1 + O(bx^{-1})\}, & \dfrac{x}{b} \to \infty, \\ v_\gamma(\rho_c(\kappa)) + o(1), & \dfrac{x}{b} \to \kappa \ (x \neq 0), \\ v_\gamma(r_c(0)), & x = 0. \end{cases}$$

**Proof** (i). Use $E[\xi_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}] = b\rho$ (see (A1)).

(ii). Recall (A1). Using

$$\Gamma(z) = \sqrt{2\pi}e^{-z}z^{z-1/2}\Big\{1 + \frac{1}{12z} + \frac{1}{288z^2} + O(z^{-3})\Big\} \quad \text{(see Olver et al. (2010; section 5.11 (ii))),}$$
$$\Big(1 + \frac{1}{z}\Big)^z = e\Big\{1 - \frac{1}{2z} + \frac{11}{24z^2} + O(z^{-3})\Big\}$$

as $z \to \infty$, we can see that, if $\gamma > 0$ and $j > 1$, then, as $\rho \to \infty$,

$$\begin{aligned} E[\xi^j_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}] &= b^j\rho^j \frac{\Gamma^{j-1}(\rho/\gamma)\Gamma((\rho+j)/\gamma)}{\Gamma^j((\rho+1)/\gamma)} \\ &= b^j\rho^j(1 + j\rho^{-1})^{\rho/\gamma+j/\gamma-1/2}(1+\rho^{-1})^{-j(\rho/\gamma+1/\gamma-1/2)}\{1 + O(\rho^{-3})\} \\ &= b^j\rho^j\Big\{1 - \frac{j^2}{2\gamma\rho} + \frac{j^3(3j+8\gamma)}{24\gamma^2\rho^2} + O(\rho^{-3})\Big\}\Big\{1 + \frac{j}{2\gamma\rho} + \frac{j(3j-8\gamma)}{24\gamma^2\rho^2} + O(\rho^{-3})\Big\} \\ &\quad \times\Big\{1 + \frac{j(2j-\gamma)}{2\gamma\rho} + \frac{j^2(2j-\gamma)(2j-3\gamma)}{8\gamma^2\rho^2} + O(\rho^{-3})\Big\} \\ &\quad \times\Big[1 + \frac{j(\gamma-2)}{2\gamma\rho} + \frac{j(\gamma-2)\{(j-2)\gamma-2j\}}{8\gamma^2\rho^2} + O(\rho^{-3})\Big] \\ &= b^j\rho^j + \frac{j(j-1)}{2\gamma}b^j\rho^{j-1} + \frac{j(j-1)\{6\gamma^2 - 4(j+1)\gamma + 3j(j-1)\}}{24\gamma^2}b^j\rho^{j-2} + O(b^j\rho^{j-3}). \end{aligned}$$

Similarly, if $\gamma < 0$ and $1 < j < \rho + 1$, then, as $\rho \to \infty$,

$$\begin{aligned} E[\xi^j_{\alpha_\gamma(\rho), b\beta_\gamma(\rho),\gamma}] &= b^j\rho^j \frac{\Gamma^{j-1}((\rho+1)/|\gamma|)\Gamma((\rho+1-j)/|\gamma|)}{\Gamma^j(\rho/|\gamma|)} \\ &= b^j\rho^j(1+\rho^{-1})^{(j-1)(\rho/|\gamma|+1/|\gamma|-1/2)}\{1 + (1-j)\rho^{-1}\}^{\rho/|\gamma|+(1-j)/|\gamma|-1/2}\{1 + O(\rho^{-3})\} \\ &= b^j\rho^j\Big[1 - \frac{j-1}{2|\gamma|\rho} + \frac{(j-1)\{3(j-1)+8|\gamma|\}}{24\gamma^2\rho^2} + O(\rho^{-3})\Big] \end{aligned}$$

$$
\begin{aligned}
&\times\Big[1-\frac{(1-j)^2}{2|\gamma|\rho}+\frac{(1-j)^3\{3(1-j)+8|\gamma|\}}{24\gamma^2\rho^2}+O(\rho^{-3})\Big]\\
&\times\Big[1+\frac{(j-1)(2-|\gamma|)}{2|\gamma|\rho}+\frac{(j-1)(2-|\gamma|)\{2(j-1)-(j+1)|\gamma|\}}{8\gamma^2\rho^2}+O(\rho^{-3})\Big]\\
&\times\Big[1+\frac{(1-j)\{2(1-j)-|\gamma|\}}{2|\gamma|\rho}+\frac{(1-j)^2\{2(1-j)-|\gamma|\}\{2(1-j)-3|\gamma|\}}{8\gamma^2\rho^2}+O(\rho^{-3})\Big]\\
=\,& b^j\rho^j+\frac{j(j-1)}{2|\gamma|}b^j\rho^{j-1}+\frac{j(j-1)\{6\gamma^2+4(j-2)|\gamma|+3j(j-1)\}}{24\gamma^2}b^j\rho^{j-2}+O(b^j\rho^{j-3}).
\end{aligned}
$$

The result follows by letting $\rho=\rho_c(x/b)=x/b+c$.

(iii). We have

$$
\begin{aligned}
&E[\xi^j_{\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma,bB_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b)),\gamma}]\\
&=b^jB^j_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b))\frac{\Gamma(\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))+(j-1)/\gamma)}{\Gamma(\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma)}\\
&=\begin{cases} x^j\{1+O(bx^{-1})\}, & \dfrac{x}{b}\to\infty,\\ O(b^j), & \dfrac{x}{b}\to\kappa,\end{cases}
\end{aligned}
$$

since, for $x/b\to\infty$,

$$
\begin{aligned}
B_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b))&=\frac{|\gamma|^{1/\gamma}}{(a_1+a_2)^{1/\gamma}}\Big(\frac{x}{b}\Big)^{1-1/\gamma}\{1+O(bx^{-1})\},\\
\frac{\Gamma(\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))+(j-1)/\gamma)}{\Gamma(\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma)}&=\frac{(a_1+a_2)^{j/\gamma}}{|\gamma|^{j/\gamma}}\Big(\frac{x}{b}\Big)^{j/\gamma}\{1+O(bx^{-1})\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&E[(\xi_{\alpha_\gamma(\rho_c(a_1x/b))+\alpha_\gamma(\rho_c(a_2x/b))-1/\gamma,bB_{\gamma,a_1,a_2}(\rho_c(a_1x/b),\rho_c(a_2x/b)),\gamma}-x)^2]\\
&=\begin{cases}\{x^2+O(bx)\}-2x\{x+O(b)\}+x^2, & \dfrac{x}{b}\to\infty,\\ O(b^2), & \dfrac{x}{b}\to\kappa.\end{cases}
\end{aligned}
$$

(iv). Use

$$
\Gamma(z)=\sqrt{2\pi}e^{-z}z^{z-1/2}\{1+O(z^{-1})\}\quad\text{(see Olver et al. (2010; section 5.11 (ii)))},\quad \Big(1+\frac{1}{z}\Big)^z=e\{1+O(z^{-1})\}
$$

as $z\to\infty$. □

Next, we give the uniform/non-uniform bounds of the Amoroso kernel $K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}$, which are useful in this paper.

**Lemma A.2** *Given $\gamma\neq 0$, we have, for any $b>0$,*

$$
\text{(i).}\quad \sup_{\rho\geq 1}\sup_{s\geq 0}K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\leq\frac{\widetilde{L}_\gamma}{b},
$$

$$
\text{(ii).}\quad \sup_{s\geq 0}K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s)\leq\frac{|\gamma|^{1/2}\widetilde{L}_\gamma}{b\sqrt{2\pi}(\rho-1)^{1/2}}\quad \textit{for any } \rho>1,
$$

*where*

$$\widetilde{L}_\gamma = \begin{cases} 1, & \gamma \geq 1, \\ \dfrac{\Gamma(2/\gamma)}{\Gamma(1/\gamma)\Gamma(1/\gamma+1)}, & 0 < \gamma < 1, \\ \dfrac{3\Gamma(1/|\gamma|)\Gamma(3/|\gamma|)}{\Gamma^2(2/|\gamma|)}, & \gamma < 0. \end{cases}$$

**Proof** We know that the mode of $K^{(A)}_{\alpha,\beta,\gamma}$ is given by $\beta(\alpha - 1/\gamma)^{1/\gamma}$ if $\alpha - 1/\gamma \geq 0$, hence, for any $\rho \geq 1$ and $\gamma \neq 0$,

$$\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \frac{|\gamma|(\alpha_\gamma(\rho) - 1/\gamma)^{\alpha_\gamma(\rho)-1/\gamma} e^{-(\alpha_\gamma(\rho)-1/\gamma)}}{b\beta_\gamma(\rho)\Gamma(\alpha_\gamma(\rho))}.$$

Also, since $z^{1-z}e^z\Gamma(z)$ is strictly increasing for $z > 0$ and $R(z) = \sqrt{2\pi}z^{z-1/2}e^{-z}/\Gamma(z)$ is strictly increasing for $z > 0$ with $R(z) \to 1$ as $z \to \infty$ (see Theorem 3.2 of Anderson et al. (1995)), the following inequalities hold for $\rho > 1$ and $\gamma > 0$ (in this case, $\alpha_\gamma(\rho) > 1/\gamma$) or $\rho \geq 1$ and $\gamma < 0$ (in this case, $\alpha_\gamma(\rho) \geq 2/|\gamma| > 1/\gamma$):

$$\text{(i).} \quad \frac{|\gamma|(\alpha_\gamma(\rho) - 1/\gamma)^{\alpha_\gamma(\rho)-1/\gamma} e^{-(\alpha_\gamma(\rho)-1/\gamma)}}{b\beta_\gamma(\rho)\Gamma(\alpha_\gamma(\rho))} \leq \frac{\mathcal{G}_\gamma(\rho)}{b},$$

$$\text{(ii).} \quad \frac{|\gamma|(\alpha_\gamma(\rho) - 1/\gamma)^{\alpha_\gamma(\rho)-1/\gamma} e^{-(\alpha_\gamma(\rho)-1/\gamma)}}{b\beta_\gamma(\rho)\Gamma(\alpha_\gamma(\rho))} \leq \frac{\mathcal{G}_\gamma(\rho)}{b\sqrt{2\pi}\{\alpha_\gamma(\rho) - 1/\gamma\}^{1/2}},$$

where

$$\mathcal{G}_\gamma(\rho) = \frac{|\gamma|(\alpha_\gamma(\rho) - 1/\gamma)\Gamma(\alpha_\gamma(\rho) - 1/\gamma)\Gamma(\alpha_\gamma(\rho) + 1/\gamma)}{\rho\Gamma^2(\alpha_\gamma(\rho))}.$$

Noting that, given $p_1, p_2 > 0$, the function

$$G_{p_1,p_2}(z) = \frac{\Gamma(z)\Gamma(z + p_1 + p_2)}{\Gamma(z + p_1)\Gamma(z + p_2)}$$

is strictly decreasing for $z > 0$, with $G_{p_1,p_2}(z) \geq 1$ (see Alzer (1997; Theorem 10 and page 386)), we have

- for any $\rho > 1$ and $\gamma \geq 1$,

$$\mathcal{G}_\gamma(\rho) = \frac{1}{G_{1-1/\gamma,1/\gamma}(\rho/\gamma)} \leq 1,$$

- for any $\rho > 1$ and $\gamma \in (0, 1)$,

$$\mathcal{G}_\gamma(\rho) = G_{1/\gamma-1,1/\gamma}((\rho - 1)/\gamma + 1) \leq \frac{\Gamma(2/\gamma)}{\Gamma(1/\gamma)\Gamma(1/\gamma+1)},$$

- for any $\rho \geq 1$ and $\gamma < 0$,

$$\mathcal{G}_\gamma(\rho) = \Big(1 + \frac{2}{\rho}\Big)G_{1/|\gamma|,1/|\gamma|}(\rho/|\gamma|) \leq \frac{3\Gamma(1/|\gamma|)\Gamma(3/|\gamma|)}{\Gamma^2(2/|\gamma|)}.$$

On the other hand, we have, for $\rho = 1$ and $\gamma > 0$ (in this case, $\alpha_\gamma(1) = 1/\gamma$),

$$\sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(1),b\beta_\gamma(1),\gamma}(s) = \frac{\gamma}{b\beta_\gamma(1)\Gamma(\alpha_\gamma(1))} = \frac{\Gamma(2/\gamma)}{b\Gamma(1/\gamma)\Gamma(1+1/\gamma)}.$$

Note that

$$\frac{\Gamma(2/\gamma)}{\Gamma(1/\gamma)\Gamma(1+1/\gamma)} = \frac{1}{G_{1-1/\gamma,1/\gamma}(1/\gamma)} \leq 1 \quad \text{for } \gamma \geq 1. \quad \square$$

Using Lemma A.2 (i), we can readily obtain the (nonasymptotic) bounds of the two-sided tail probability and absolute moment of $\overline{\Delta}^{\dagger}_{b,c,\gamma}(x)$.

**Lemma A.3** *Let $\gamma \neq 0$ and $r_c(0) \geq 1$. The following bounds hold for any $n \in \mathbb{N}$, $b, t > 0$, $x \geq 0$, and $j \geq 2$:*

$$\text{(i).}\quad P[|\overline{\Delta}^{\dagger}_{b,c,\gamma}(x)| \geq t] \leq 2\exp\Big\{-\frac{nbt^2}{\widetilde{L}_\gamma(2C_0+t)}\Big\},$$

$$\begin{aligned}\text{(ii).}\quad E[|\overline{\Delta}^{\dagger}_{b,c,\gamma}(x)|^j] &\leq C(j)\{(n^{-1}b^{-1}\widetilde{L}_\gamma)^{j-2} + (n^{-1}b^{-1}C_0\widetilde{L}_\gamma)^{(j-2)/2}\}V[\widehat{f}_{b,c,\gamma}(x)] \\ &\leq 2C(j)(n^{-2}b^{-2}\widetilde{L}_\gamma^2 + n^{-1}b^{-1}C_0\widetilde{L}_\gamma)^{(j-2)/2}V[\widehat{f}_{b,c,\gamma}(x)],\end{aligned}$$

*where the constant $C(j) > 0$ depends only on $j$.*

**Proof** Using Lemma A.2 (i), we have, for $i = 1, \dots, n$,

$$\begin{aligned}|\Delta^{\dagger}_{b,c,\gamma,i}(x)| &\leq \sup_{s\geq 0} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s) \leq b^{-1}\widetilde{L}_\gamma, \\ V[\Delta^{\dagger}_{b,c,\gamma,i}(x)] &\leq \int_0^\infty \{K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)\}^2 f(s)ds \leq b^{-1}C_0\widetilde{L}_\gamma.\end{aligned}$$

Bennett's inequality yields the result (i). Also, Rosenthal's inequality

$$E[|\overline{\Delta}^{\dagger}_{b,c,\gamma}(x)|^j] \leq n^{-j}C(j)\{nE[|\Delta^{\dagger}_{b,c,\gamma,1}(x)|^j] + (nE[\{\Delta^{\dagger}_{b,c,\gamma,1}(x)\}^2])^{j/2}\}$$

yields the result (ii). $\square$

**Remark A.1** Lemma A.2 (i) yields

$$0 \leq \widehat{f}^{(\#_a)}_{b,c,\gamma}(x) \leq M^{(\#_a)}_{b,\epsilon}, \quad |\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)| \leq \frac{1+a^2}{1-a}b^{-1}\widetilde{L}_\gamma, \quad 0 \leq \mathcal{Q}(x) \leq 3\Big\{\frac{a^2+a^4}{(1-a)^2}b^{-2}\widetilde{L}_\gamma^2 + \epsilon^2\Big\},$$

where

$$M^{(\#_a)}_{b,\epsilon} = \begin{cases} \Big(\dfrac{\widetilde{L}_\gamma}{b} + \epsilon\Big)^{1/(1-a)}\epsilon^{-a/(1-a)}, & \# = TS, \\ \Big(\dfrac{\widetilde{L}_\gamma}{b} + \epsilon\Big)e^{1/(1-a)}, & \# = JF. \end{cases}$$

We prove the inequality of the ratio of two gamma functions, which is of independent interest.

**Lemma A.4** *For any $z, p > 0$, we have*

$$\max\Big\{(z+p)^p\Big(1 - \frac{6p^2+6p+1}{12z}\Big), 0\Big\} \leq \frac{\Gamma(z+p)}{\Gamma(z)} \leq (z+p)^p.$$

**Proof** We know (see Olver et al. (2010; section 5.6 (i))) that $R(z) = \sqrt{2\pi}z^{z-1/2}e^{-z}/\Gamma(z)$ is strictly increasing for $z > 0$ and satisfies $e^{-1/(12z)} < R(z) < 1$. Using $e^{1-1/(2z)} \leq (1+z^{-1})^z \leq e$, $1 - 1/(2z) \leq (1+z^{-1})^{-1/2} \leq 1$, and $e^{-1/z} \geq 1 - z^{-1}$, we have

$$\frac{\Gamma(z+p)}{\Gamma(z)} = z^p e^{-p}(1+pz^{-1})^{z+p-1/2}\frac{R(z)}{R(z+p)} \leq z^p(1+pz^{-1})^p$$

and

$$\frac{\Gamma(z+p)}{\Gamma(z)} \geq z^p(1+pz^{-1})^{p-1/2}e^{-p^2/(2z)-1/(12z)} \geq z^p(1+pz^{-1})^p\Big(1 - \frac{6p^2+6p+1}{12z}\Big). \quad \square$$

Finally, we prepare the following lemma, which is the key to establish the validity of the asymptotic expansion of the MISEs of the estimators (4) and (14).

**Lemma A.5** *Let $\gamma \neq 0$ and $c \geq 1-d$ (see (3)). For any $\tau \in (0,1)$, $k > 0$, and sufficiently small $b > 0$, we have*

$$\int_{b^{-\tau}}^{\infty} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)dx = O(b^{\tau(k+1)}s^{k+1}), \quad s > 0.$$

**Proof** It is easy to see that

$$K^{(A)}_{\alpha_\gamma(\rho),b\beta_\gamma(\rho),\gamma}(s) = \begin{cases} \dfrac{b\rho(\rho+1)}{s^2}|\gamma|G_{\{s/(b\beta_\gamma(\rho))\}^\gamma|\gamma|/(\rho+1)}((\rho+1)/|\gamma|), & \gamma > 0, \\ \dfrac{\rho+1}{s}|\gamma|G_{\{s/(b\beta_\gamma(\rho))\}^\gamma|\gamma|/(\rho+1)}((\rho+1)/|\gamma|), & \gamma < 0, \end{cases}$$

where, given $q > 0$,

$$G_q(u) = \frac{(qu)^u e^{-qu}}{u\Gamma(u)} = \frac{e^{u(1-q+\log q)}}{u^{1-u}e^u\Gamma(u)}$$

is strictly decreasing for $u > 0$ (see Theorem 3.2 (2) of Anderson et al. (1995)), and, by definition (see (5)),

$$\frac{1}{\{\beta_\gamma(\rho)\}^\gamma} = \frac{1}{\rho^\gamma}\left\{\frac{\Gamma((\rho+1)/|\gamma|)}{\Gamma(\rho/|\gamma|)}\right\}^{|\gamma|}.$$

Using Lemma A.4 (we set $z = \rho/|\gamma|$ and $p = 1/|\gamma|$), we have

$$\frac{\rho+1}{\rho^\gamma|\gamma|}(1-c_\gamma\rho^{-1})^{|\gamma|} \leq \frac{1}{\{\beta_\gamma(\rho)\}^\gamma} \leq \frac{\rho+1}{\rho^\gamma|\gamma|} \quad (\text{if } \rho > c_\gamma),$$

where $c_\gamma = (6+6|\gamma|+\gamma^2)/(12|\gamma|)$. We can see that, for sufficiently small $b > 0$ (e.g., $b^{-(\tau+1)}/2 \geq d$ (say); see (3)), if $\rho \geq b^{-(\tau+1)}/2$, then,

$$\begin{aligned}
&|\gamma|G_{\{s/(b\beta_\gamma(\rho))\}^\gamma|\gamma|/(\rho+1)}((\rho+1)/|\gamma|) \\
&\leq |\gamma|G_{\{s/(b\beta_\gamma(\rho))\}^\gamma|\gamma|/(\rho+1)}(b^{-(\tau+1)}/(2|\gamma|)) \\
&\leq \left\{\left(\frac{s}{b\rho}\right)^\gamma\frac{b^{-(\tau+1)}}{2|\gamma|}\right\}^{b^{-(\tau+1)}/(2|\gamma|)}\frac{1}{\Gamma(b^{-(\tau+1)}/(2|\gamma|)+1)}\exp\left\{-\left(\frac{s}{b\rho}\right)^\gamma(1-c_\gamma\rho^{-1})^{|\gamma|}\frac{b^{-(\tau+1)}}{2|\gamma|}\right\} \\
&= s^{-\gamma}(1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2}K^{(A)}_{b^{-(\tau+1)}/(2|\gamma|)+1,s^{-\gamma},1}((b\rho)^{-\gamma}(1-2c_\gamma b^{\tau+1})^{|\gamma|}b^{-(\tau+1)}/(2|\gamma|)).
\end{aligned}$$

For sufficiently small $b > 0$, $x \geq b^{-\tau}$ implies $x/b \geq b^{-(\tau+1)} \geq 2d$, hence,

$$\rho_c(x/b) = \frac{x}{b} + c \geq b^{-(\tau+1)} + c \geq \frac{b^{-(\tau+1)}}{2} + (c+d) > \frac{b^{-(\tau+1)}}{2}$$

(see (3)). It follows that, if $\gamma > 0$, then,

$$\begin{aligned}
&\int_{b^{-\tau}}^{\infty} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)dx \\
&\leq b^{-1}s^{-2-\gamma}(1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2}(1+2b^{\tau+1}) \\
&\quad\times\int_{b^{-\tau}}^{\infty}(x+bc)^2K^{(A)}_{b^{-(\tau+1)}/(2\gamma)+1,s^{-\gamma},1}((x+bc)^{-\gamma}(1-2c_\gamma b^{\tau+1})^\gamma b^{-(\tau+1)}/(2\gamma))dx \\
&\leq 2^{-3/\gamma}b^{-3(\tau+1)/\gamma-1}s^{-2-\gamma}\gamma^{-3/\gamma-1}(1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2+3}(1+2b^{\tau+1})
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^{(b^{-\tau}+bc)^{-\gamma}(1-2c_\gamma b^{\tau+1})^{\gamma} b^{-(\tau+1)}/(2\gamma)} y^{-3/\gamma-1} K^{(A)}_{b^{-(\tau+1)}/(2\gamma)+1,s^{-\gamma},1}(y)dy \\
&\le 2^{-(k+3)/\gamma} b^{-(k+3)(\tau+1)/\gamma+k\tau-1} s^{-2-\gamma} \gamma^{-(k+3)/\gamma-1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2+k+3}(1+2b^{\tau+1})(1+b^{\tau+1}c)^{-k} \\
&\quad \times E[(\xi_{b^{-(\tau+1)}/(2\gamma)+1,s^{-\gamma},1})^{-(k+3)/\gamma-1}] \\
&= 2^{-(k+3)/\gamma} b^{-(k+3)(\tau+1)/\gamma+k\tau-1} s^{k+1} \gamma^{-(k+3)/\gamma-1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2+k+3}(1+2b^{\tau+1})(1+b^{\tau+1}c)^{-k} \\
&\quad \times \frac{\Gamma(b^{-(\tau+1)}/(2\gamma)-(k+3)/\gamma)}{\Gamma(b^{-(\tau+1)}/(2\gamma)+1)} \\
&\le 2b^{\tau(k+1)} s^{k+1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2+k+3}(1+2b^{\tau+1})(1+b^{\tau+1}c)^{-k} \Big\{1-\frac{2(k+3)c_{\gamma/(k+3)} b^{\tau+1}}{1-2(k+3)b^{\tau+1}}\Big\}^{-1},
\end{aligned}
$$

where, for the last inequality, we used Lemma A.4 with $z = b^{-(\tau+1)}/(2\gamma) - (k+3)/\gamma$ and $p = (k+3)/\gamma$. Similarly, if $\gamma < 0$, then,

$$
\begin{aligned}
&\int_{b^{-\tau}}^{\infty} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)dx \\
&\le b^{-1} s^{-1+|\gamma|} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2}(1+2b^{\tau+1}) \\
&\quad \times \int_{b^{-\tau}}^{\infty} (x+bc) K^{(A)}_{b^{-(\tau+1)}/(2|\gamma|)+1,s^{|\gamma|},1}((x+bc)^{|\gamma|}(1-2c_\gamma b^{\tau+1})^{|\gamma|} b^{-(\tau+1)}/(2|\gamma|))dx \\
&\le 2^{2/|\gamma|} b^{2(\tau+1)/|\gamma|-1} s^{-1+|\gamma|} |\gamma|^{2/|\gamma|-1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2-2}(1+2b^{\tau+1}) \\
&\quad \times \int_{(b^{-\tau}+bc)^{|\gamma|}(1-2c_\gamma b^{\tau+1})^{|\gamma|} b^{-(\tau+1)}/(2|\gamma|)}^{\infty} y^{2/|\gamma|-1} K^{(A)}_{b^{-(\tau+1)}/(2|\gamma|)+1,s^{|\gamma|},1}(y)dy \\
&\le 2^{(k+2)/|\gamma|} b^{(k+2)(\tau+1)/|\gamma|+k\tau-1} s^{-1+|\gamma|} |\gamma|^{(k+2)/|\gamma|-1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2-k-2}(1+2b^{\tau+1})(1+b^{\tau+1}c)^{-k} \\
&\quad \times E[(\xi_{b^{-(\tau+1)}/(2|\gamma|)+1,s^{|\gamma|},1})^{(k+2)/|\gamma|-1}] \\
&= 2^{(k+2)/|\gamma|} b^{(k+2)(\tau+1)/|\gamma|+k\tau-1} s^{k+1} |\gamma|^{(k+2)/|\gamma|-1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2-k-2}(1+2b^{\tau+1})(1+b^{\tau+1}c)^{-k} \\
&\quad \times \frac{\Gamma(b^{-(\tau+1)}/(2|\gamma|)+(k+2)/|\gamma|)}{\Gamma(b^{-(\tau+1)}/(2|\gamma|)+1)} \\
&\le 2b^{\tau(k+1)} s^{k+1} (1-2c_\gamma b^{\tau+1})^{-b^{-(\tau+1)}/2-k-2}(1+2b^{\tau+1})(1+b^{\tau+1}c)^{-k} \{1+2(k+2)b^{\tau+1}\}^{(k+2)/|\gamma|},
\end{aligned}
$$

where, for the last inequality, we used Lemma A.4 with $z = b^{-(\tau+1)}/(2|\gamma|)$ and $p = (k+2)/|\gamma|$. □

## A2 Proofs of Theorems 1–4 and Remark 1

**Proof of Theorem 1** From Lemma A.1 (i) and (ii), we have, for $x/b \to \infty$,

$$
\begin{aligned}
E[\widetilde{f}_{b,c,\gamma}(x)] &= \int_0^\infty K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s) f(s)ds \\
&= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\} f'(x) + \frac{1}{2} f''(x) E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^2] \\
&\quad + \int_0^\infty (s-x)^2 \int_0^1 \{f''(x+\theta(s-x)) - f''(x)\}(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds \\
&= f(x) + b\frac{B_{c|\gamma|}(x)}{|\gamma|} + O(b^2 + (bx)^{1+\eta_2/2}),
\end{aligned}
$$

since

$$
\Big|\int_0^\infty (s-x)^2 \int_0^1 \{f''(x+\theta(s-x)) - f''(x)\}(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big|
$$

$$\le \frac{L_2}{2} E[|\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x|^{2+\eta_2}]$$
$$\le \frac{L_2}{2}\{E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^4]\}^{(2+\eta_2)/4}$$
$$= O((bx)^{1+\eta_2/2}).$$

Similarly, we have, for $x/b \to \kappa$,

$$\begin{aligned}
E[\widetilde{f}_{b,c,\gamma}(x)] &= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\} f'(x) \\
&\quad + \int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds \\
&= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\} f'(x) + O(b^2),
\end{aligned}$$

since

$$\begin{aligned}
\Big|\int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big| &\le \frac{C_2}{2} E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^2] \\
&= O(b^2).
\end{aligned}$$

On the other hand, we can see that

$$\begin{aligned}
V[\widetilde{f}_{b,c,\gamma}(x)] &= n^{-1}\int_0^\infty \{K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)\}^2 f(s)ds + O(n^{-1}) \\
&= n^{-1}b^{-1}|\gamma| v_\gamma(\rho_c(x/b)) \int_0^\infty K^{(A)}_{2\alpha_\gamma(\rho_c(x/b))-1/\gamma,b\beta_\gamma(\rho_c(x/b))/2^{1/\gamma},\gamma}(s) f(s)ds + O(n^{-1}).
\end{aligned}$$

The variance follows from Lemma A.1 (iv) and

$$\begin{aligned}
&\Big|\int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{2\alpha_\gamma(\rho_c(x/b))-1/\gamma,b\beta_\gamma(\rho_c(x/b))/2^{1/\gamma},\gamma}(s)ds\Big| \\
&\le C_1 E[|\xi_{2\alpha_\gamma(\rho_c(x/b))-1/\gamma,b\beta_\gamma(\rho_c(x/b))/2^{1/\gamma},\gamma} - x|] \\
&\le C_1\{E[(\xi_{2\alpha_\gamma(\rho_c(x/b))-1/\gamma,b\beta_\gamma(\rho_c(x/b))/2^{1/\gamma},\gamma} - x)^2]\}^{1/2} \\
&= \begin{cases} O((bx)^{1/2}), & \frac{x}{b} \to \infty, \\ O(b), & \frac{x}{b} \to \kappa \end{cases} \quad \text{(see Lemma A.1 (iii)).} \quad \square
\end{aligned}$$

**Proof of Remark 1** (i). Lemma A.2 (i) yields

$$V[\widetilde{f}_{b,c,\gamma}(x)] \le n^{-1}\int_0^\infty \{K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)\}^2 f(s)ds \le n^{-1}b^{-1}C_0\widetilde{L}_\gamma.$$

On the other hand, as in Proof of Theorem 1, (6) follows from Lemma A.1 (i) and (ii), since, for $x/b \to \infty$,

$$\begin{aligned}
E[\widetilde{f}_{b,c,\gamma}(x)] &= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\} f'(x) \\
&\quad + \int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds \qquad \text{(A3)} \\
&= f(x) + O(b + bx),
\end{aligned}$$

where

$$\begin{aligned}
\Big|\int_0^\infty (s-x)^2 \int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big| &\le \frac{C_2}{2} E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^2] \\
&= O(bx),
\end{aligned}$$

and, for $x/b \to \kappa$,

$$E[\widetilde{f}_{b,c,\gamma}(x)] = f(x) + \int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds,$$

where

$$\begin{aligned}\Big|\int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big| &\leq C_1 E[\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} + x] \\ &= C_1\{b\rho_c(x/b) + x\} \\ &= O(b).\end{aligned}$$

Similarly, the non-decreasingness of the $\rho$-function $\rho_c$ (i.e., $\sup_{x\in[0,b^\tau]}\rho_c(x/b) = \rho_c(b^{\tau-1}) = b^{\tau-1}+c$) yields

$$\sup_{x\in[0,b^\tau]}\Big|\int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big| \leq C_1(2b^\tau + bc).$$

(ii). (A3) yields, for $x \in [0, b^\tau]$,

$$\begin{aligned}\Big|Bias[\widetilde{f}_{b,c,\gamma}(x)] - b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\}f'(x)\Big| &\leq \frac{C_2}{2}E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^2] \\ &\leq C_2\Big\{b^2\rho_c^2(x/b)\frac{\Gamma(\alpha_\gamma(\rho_c(x/b)))\Gamma(\alpha_\gamma(\rho_c(x/b))+2/\gamma)}{\Gamma^2(\alpha_\gamma(\rho_c(x/b))+1/\gamma)} + x^2\Big\} \\ &\leq C_2\Big\{(b^\tau + bc)^2\frac{\Gamma(\alpha_\gamma(r_c(0)))\Gamma(\alpha_\gamma(r_c(0))+2/\gamma)}{\Gamma^2(\alpha_\gamma(r_c(0))+1/\gamma)} + b^{2\tau}\Big\},\end{aligned}$$

since, given $p > 0$, $\Gamma(z)\Gamma(z+2p)/\Gamma^2(z+p)$ is strictly decreasing for $z > 0$ (see Theorem 6 of Bustoz and Ismail (1986)). □

**Proof of Theorem 2** Lemma A.3 (i) and the Borel-Cantelli lemma immediately yield $\overline{\Delta}^\dagger_{b,c,\gamma}(x) \overset{a.s.}{\to} 0$, provided that $nb/\log n \to \infty$. This, together with (6), yields $\widetilde{f}_{b,c,\gamma}(x) \overset{a.s.}{\to} f(x)$ for fixed $x \geq 0$. □

**Proof of Theorem 3** Recall that

$$\widetilde{f}_{b,c,\gamma}(x) - E[\widetilde{f}_{b,c,\gamma}(x)] = \sum_{i=1}^n \frac{1}{n}\Delta^\dagger_{b,c,\gamma,i}(x), \quad V[\widetilde{f}_{b,c,\gamma}(x)] = \sum_{i=1}^n E\Big[\frac{1}{n^2}\{\Delta^\dagger_{b,c,\gamma,i}(x)\}^2\Big].$$

Using Theorem 1, i.e.,

$$\lim_{n\to\infty} nb^{1/2}V[\widetilde{f}_{b,c,\gamma}(x)] = |\gamma|^{1/2}V(x) \quad \text{for fixed } x > 0, \quad \lim_{n\to\infty} nbV[\widetilde{f}_{b,c,\gamma}(0)] = |\gamma|v_\gamma(r_c(0))f(0) \tag{A4}$$

(for fixed $x > 0$, assume $b \to 0$ and $nb^{1/2} \to \infty$, instead of Assumption A2), and

$$|\Delta^\dagger_{b,c,\gamma,i}(x)| \leq \begin{cases} \dfrac{b^{-1/2}|\gamma|^{1/2}\widetilde{L}_\gamma}{\sqrt{2\pi}\{x+b(c-1)\}^{1/2}} & \text{for fixed } x > 0, \\ b^{-1}\widetilde{L}_\gamma & \text{for } x = 0 \end{cases} \tag{A5}$$

(we used Lemma A.2), it follows that, for any $\delta > 0$ and fixed $x > 0$,

$$\frac{\sum_{i=1}^n E\big[\big|n^{-1}\Delta^\dagger_{b,c,\gamma,i}(x)\big|^{2+\delta}\big]}{\{V[\widetilde{f}_{b,c,\gamma}(x)]\}^{1+\delta/2}} \leq \frac{(nb^{1/2})^{-\delta/2}}{\{nb^{1/2}V[\widehat{f}_{b,c,\gamma}(x)]\}^{\delta/2}}\Big[\frac{|\gamma|^{1/2}\widetilde{L}_\gamma}{\sqrt{2\pi}\{x+b(c-1)\}^{1/2}}\Big]^\delta = O((nb^{1/2})^{-\delta/2}) = o(1),$$

and

$$\frac{\sum_{i=1}^n E\big[\big|n^{-1}\Delta^\dagger_{b,c,\gamma,i}(0)\big|^{2+\delta}\big]}{\{V[\widetilde{f}_{b,c,\gamma}(0)]\}^{1+\delta/2}} \le \frac{(nb)^{-\delta/2}\widetilde{L}_\gamma^\delta}{\{nbV[\widehat{f}_{b,c,\gamma}(0)]\}^{\delta/2}} = O((nb)^{-\delta/2}) = o(1).$$

Therefore, Lyapunov's central limit theorem enables us to see that

$$\frac{\widetilde{f}_{b,c,\gamma}(x) - E[\widetilde{f}_{b,c,\gamma}(x)]}{\{V[\widetilde{f}_{b,c,\gamma}(x)]\}^{1/2}} \xrightarrow{d} N(0,1).$$

The results follow from (A4) and Slutsky's lemma. □

**Proof of Theorem 4** We note that

$$MISE[\widetilde{f}_{b,c,\gamma}] = \Big(\int_0^{b^{\tau_1}} + \int_{b^{\tau_1}}^{b^{-\tau_2}} + \int_{b^{-\tau_2}}^{\infty}\Big) MSE[\widetilde{f}_{b,c,\gamma}(x)]dx,$$

where $\tau_1 \in (2/3, 1)$ and $\tau_2 \in (2/(k_2+1), \eta_2/(\eta_2+3))$ for some $k_2 > (\eta_2+6)/\eta_2$ (see Assumption A4 (ii)). Using (7) and (8), we have

$$\int_0^{b^{\tau_1}} MSE[\widetilde{f}_{b,c,\gamma}(x)]dx \le C_1^2 b^{\tau_1}(2b^{\tau_1} + bc)^2 + n^{-1}b^{\tau_1-1}C_0\widetilde{L}_\gamma = o(b^2 + n^{-1}b^{-1/2}).$$

Lemmas A.2 (i) and A.5 yield

$$\begin{aligned}
\int_{b^{-\tau_2}}^{\infty} MSE[\widetilde{f}_{b,c,\gamma}(x)]dx &\le 2\int_{b^{-\tau_2}}^{\infty}\Big[\Big\{\int_0^\infty K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)f(s)ds\Big\}^2 + f^2(x)\Big]dx \\
&\quad + n^{-1}\int_{b^{-\tau_2}}^{\infty}\int_0^\infty \{K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)\}^2 f(s)dsdx \\
&\le 2C_0\Big\{\int_0^\infty\int_{b^{-\tau_2}}^{\infty} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)dxf(s)ds + b^{\tau_2(k_2+1)}\int_{b^{-\tau_2}}^{\infty} x^{k_2+1}f(x)dx\Big\} \\
&\quad + n^{-1}b^{-1}\widetilde{L}_\gamma\int_0^\infty\int_{b^{-\tau_2}}^{\infty} K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)dxf(s)ds \\
&= O(b^{\tau_2(k_2+1)} + n^{-1}b^{\tau_2(k_2+1)-1}) \qquad \text{(A6)} \\
&= o(b^2).
\end{aligned}$$

Also, in view of Theorem 1 (with $x \ge b^{\tau_1}$), we have

$$\begin{aligned}
\big|V[\widetilde{f}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}|\gamma|^{1/2}V(x)\big| &= o(n^{-1}b^{-1/2}V(x)) + O(n^{-1}), \\
\Big|\{Bias[\widetilde{f}_{b,c,\gamma}(x)]\}^2 - b^2\Big\{\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\}^2\Big| &\le 2b\frac{|B_{c|\gamma|}(x)|}{|\gamma|}|\mathcal{E}_{b,c,\gamma}(x)| + \mathcal{E}^2_{b,c,\gamma}(x),
\end{aligned}$$

where $\mathcal{E}^2_{b,c,\gamma}(x) = O(b^4 + (bx)^{2+\eta_2})$. It follows that

$$\begin{aligned}
&\Big|\int_{b^{\tau_1}}^{b^{-\tau_2}} MSE[\widetilde{f}_{b,c,\gamma}(x)]dx - AMISE[\widetilde{f}_{b,c,\gamma}]\Big| \\
&\le 2b\Big[\int_{b^{\tau_1}}^{b^{-\tau_2}}\Big\{\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\}^2dx\int_{b^{\tau_1}}^{b^{-\tau_2}}\mathcal{E}^2_{b,c,\gamma}(x)dx\Big]^{1/2} + \int_{b^{\tau_1}}^{b^{-\tau_2}}\mathcal{E}^2_{b,c,\gamma}(x)dx + b^2\Big(\int_0^{b^{\tau_1}} + \int_{b^{-\tau_2}}^{\infty}\Big)\Big\{\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\}^2dx \\
&\quad + \int_{b^{\tau_1}}^{b^{-\tau_2}}\big|V[\widetilde{f}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}|\gamma|^{1/2}V(x)\big|dx + n^{-1}b^{-1/2}\Big(\int_0^{b^{\tau_1}} + \int_{b^{-\tau_2}}^{\infty}\Big)|\gamma|^{1/2}V(x)dx \\
&= o(b^2 + n^{-1}b^{-1/2}),
\end{aligned}$$

since $\int_{b^{\tau_1}}^{b^{-\tau_2}}\mathcal{E}^2_{b,c,\gamma}(x)dx = O(b^{4-\tau_2} + b^{2+\eta_2-\tau_2(3+\eta_2)}) = o(b^2)$. □

## A3 Proofs of Theorems 5–9 and Remark 2

**Proof of Theorem 5** (i). From Lemma A.1 (i) and (ii), we have, for $x/b \to \infty$,

$$
\begin{aligned}
E[\widetilde{f}_{b,c,\gamma}(x)] &= \int_0^\infty K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)f(s)ds \\
&= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\}f'(x) + \sum_{j=2}^{4}\frac{1}{j!}f^{(j)}(x)E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^j] \\
&\quad + \frac{1}{6}\int_0^\infty (s-x)^4\int_0^1\{f^{(4)}(x+\theta(s-x)) - f^{(4)}(x)\}(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds \\
&= f(x) + b\frac{B_{c|\gamma|}(x)}{|\gamma|} + b^2\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2} + O(b^3x^{-1} + b^3 + b^3x + (bx)^{2+\eta_4/2}),
\end{aligned}
$$

since

$$
\begin{aligned}
&\Big|\int_0^\infty (s-x)^4\int_0^1\{f^{(4)}(x+\theta(s-x)) - f^{(4)}(x)\}(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big| \\
&\le \frac{L_4}{4}E[|\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x|^{4+\eta_4}] \\
&\le \frac{L_4}{4}\{E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^6]\}^{(4+\eta_4)/6} \\
&= O((bx)^{2+\eta_4/2})
\end{aligned}
$$

(note that $b^3(1+x) \le b^{1-\eta_4/2}\{b(1+x)\}^{2+\eta_4/2} = o(\{b(1+x)\}^{2+\eta_4/2})$). Also, we have, for $x/b \to \kappa$,

$$
\begin{aligned}
E[\widetilde{f}_{b,c,\gamma}(x)] &= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\}f'(x) + \frac{1}{2}f''(x)E[(\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x)^2] \\
&\quad + \frac{1}{2}\int_0^\infty (s-x)^3\int_0^1 f^{(3)}(x+\theta(s-x))(1-\theta)^2 d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds \\
&= f(x) + b\Big\{\rho_c(x/b) - \frac{x}{b}\Big\}f'(x) + b^2\eta_\gamma(x/b,\rho_c(x/b))\frac{f''(x)}{2} + O(b^3),
\end{aligned}
$$

since

$$
\begin{aligned}
&\Big|\int_0^\infty (s-x)^3\int_0^1 f^{(3)}(x+\theta(s-x))(1-\theta)^2 d\theta K^{(A)}_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}(s)ds\Big| \\
&\le \frac{C_3}{3}E[|\xi_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma} - x|^3] \\
&\le \frac{4C_3}{3}(E[\xi^3_{\alpha_\gamma(\rho_c(x/b)),b\beta_\gamma(\rho_c(x/b)),\gamma}] + x^3) \\
&= \frac{4C_3}{3}b^3\Big\{\rho_c^3(x/b)\frac{\Gamma^2(\alpha_\gamma(\rho_c(x/b)))\Gamma(\alpha_\gamma(\rho_c(x/b)) + 3/\gamma)}{\Gamma^3(\alpha_\gamma(\rho_c(x/b)) + 1/\gamma)} + \Big(\frac{x}{b}\Big)^3\Big\} \\
&\le \frac{4C_3}{3}b^3\Big\{\rho_c^3(x/b)\frac{\Gamma^2(\alpha_\gamma(r_c(0)))\Gamma(\alpha_\gamma(r_c(0)) + 3/\gamma)}{\Gamma^3(\alpha_\gamma(r_c(0)) + 1/\gamma)} + \Big(\frac{x}{b}\Big)^3\Big\} \\
&= O(b^3),
\end{aligned}
$$

where, given $p > 0$, $\Gamma^2(z)\Gamma(z+3p)/\Gamma^3(z+p)$ and $\Gamma(z)\Gamma^2(z+3p)/\Gamma^3(z+2p)$ are strictly decreasing for $z > 0$ (see Theorem 10 of Alzer (1997)).

(ii). We have, for $a_1, a_2 \in (0,1]$,

$$
Cov[\widetilde{f}_{b/a_1,c,\gamma}(x), \widetilde{f}_{b/a_2,c,\gamma}(x)] = \frac{1}{n}\int_0^\infty \prod_{i=1}^{2} K^{(A)}_{\alpha_\gamma(\rho_c(a_ix/b)),(b/a_i)\beta_\gamma(\rho_c(a_ix/b)),\gamma}(s)f(s)ds + O(n^{-1}).
$$

The covariance follows from Lemma A.1 (iv) and

$$
\begin{aligned}
&\Big|\int_0^\infty (s-x)\int_0^1 f'(x+\theta(s-x))d\theta \prod_{i=1}^2 K^{(A)}_{\alpha_\gamma(\rho_c(a_ix/b)),(b/a_i)\beta_\gamma(\rho_c(a_ix/b)),\gamma}(s)ds\Big| \\
&\le C_1\prod_{i=1}^2\Big[\int_0^\infty |s-x|\{K^{(A)}_{\alpha_\gamma(\rho_c(a_ix/b)),(b/a_i)\beta_\gamma(\rho_c(a_ix/b)),\gamma}(s)\}^2 ds\Big]^{1/2} \\
&\le C_1\prod_{i=1}^2\Big[b^{-1}|\gamma|v_\gamma(\rho_c(a_ix/b))\int_0^\infty |s-x|K^{(A)}_{2\alpha_\gamma(\rho_c(a_ix/b))-1/\gamma,bB_{\gamma,a_i,a_i}(\rho_c(a_ix/b),\rho_c(a_ix/b)),\gamma}(s)ds\Big]^{1/2} \\
&\le C_1\prod_{i=1}^2[b^{-1}|\gamma|v_\gamma(\rho_c(a_ix/b))\{E[(\xi_{2\alpha_\gamma(\rho_c(a_ix/b))-1/\gamma,(b/a_i)\beta_\gamma(\rho_c(a_ix/b))/2^{1/\gamma},\gamma}-x)^2]\}^{1/2}]^{1/2} \\
&= O(1) \quad \text{for } x/b\to\infty \text{ or } x/b\to\kappa
\end{aligned}
$$

(see Lemma A.1 (iii) and (iv)). □

**Proof of Theorem 6** Theorem 5, with $\rho_c(t) = t + c$, immediately yields the results. □

**Proof of Remark 2** Using Lemma A.1 (i) and (ii), we have (15) (hence (16)), noting that, for $x/b \to \infty$,

$$
\begin{aligned}
E[\widehat{f}_{b,c,\gamma}(x)] &= \int_0^\infty K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)f(s)ds \\
&= f(x) + bcf'(x) + \sum_{j=2}^3 \frac{1}{j!}f^{(j)}(x)E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^j] \\
&\quad + \frac{1}{6}\int_0^\infty (s-x)^4\int_0^1 f^{(4)}(x+\theta(s-x))(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)ds \\
&= f(x) + b\frac{B_{c|\gamma|}(x)}{|\gamma|} + O(b^2 + b^2x^2),
\end{aligned}
$$

where

$$
\begin{aligned}
\Big|\int_0^\infty (s-x)^4\int_0^1 f^{(4)}(x+\theta(s-x))(1-\theta)^3 d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)ds\Big| &\le \frac{C_4}{4}E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^4] \\
&= O((bx)^2).
\end{aligned}
$$

Similarly, we have, for $x/b \to \kappa$,

$$
\begin{aligned}
E[\widehat{f}_{b,c,\gamma}(x)] &= f(x) + bcf'(x) + \int_0^\infty (s-x)^2\int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)ds \\
&= f(x) + bcf'(x) + O(b^2),
\end{aligned}
$$

where

$$
\begin{aligned}
\Big|\int_0^\infty (s-x)^2\int_0^1 f''(x+\theta(s-x))(1-\theta)d\theta K^{(A)}_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}(s)ds\Big| &\le \frac{C_2}{2}E[(\xi_{\alpha_\gamma(x/b+c),b\beta_\gamma(x/b+c),\gamma}-x)^2] \\
&= O(b^2). \quad \square
\end{aligned}
$$

**Proof of Theorem 7** Use Theorem 2 and Slutsky's lemma to get the result. □

**Proof of Theorem 8** We write

$$\Delta_{b,c,\gamma,i}^{(SS_a)}(x) = \frac{1}{1-a}\Delta_{b,c,\gamma,i}(x) - \frac{a}{1-a}\Delta_{b/a,c,\gamma,i}(x), \quad i = 1, \ldots, n.$$

Then,

$$\widehat{f}_{b,c,\gamma}^{(SS_a)}(x) - E[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] = \sum_{i=1}^{n} \frac{1}{n}\Delta_{b,c,\gamma,i}^{(SS_a)}(x), \quad V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] = \sum_{i=1}^{n} E\Big[\frac{1}{n^2}\{\Delta_{b,c,\gamma,i}^{(SS_a)}(x)\}^2\Big].$$

Using Theorem 6, i.e.,

$$\lim_{n\to\infty} nb^{1/2}V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] = |\gamma|^{1/2}\lambda(a)V(x) \quad \text{for fixed } x > 0, \quad \lim_{n\to\infty} nbV[\widehat{f}_{b,c,\gamma}^{(SS_a)}(0)] = |\gamma|v_{c,\gamma}^{(SS_a)}(0)f(0) \tag{A7}$$

(for fixed $x > 0$, assume $b \to 0$ and $nb^{1/2} \to \infty$, instead of Assumption A2), and

$$|\Delta_{b,c,\gamma,i}^{(SS_a)}(x)| \le \begin{cases} \dfrac{b^{-1/2}|\gamma|^{1/2}\widetilde{L}_\gamma}{\sqrt{2\pi}(1-a)}\Big[\dfrac{1}{\{x+b(c-1)\}^{1/2}} + \dfrac{a^{3/2}}{\{x+b(c-1)/a\}^{1/2}}\Big] & \text{for fixed } x > 0, \\ \dfrac{1+a^2}{1-a}b^{-1}\widetilde{L}_\gamma & \text{for } x = 0 \end{cases}$$

(we used (A5)), it follows that, for any $\delta > 0$ and fixed $x > 0$,

$$\begin{aligned} &\frac{\sum_{i=1}^n E\big[\big|n^{-1}\Delta_{b,c,\gamma,i}^{(SS_a)}(x)\big|^{2+\delta}\big]}{\{V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]\}^{1+\delta/2}} \\ &\le \frac{(nb^{1/2})^{-\delta/2}}{\{nb^{1/2}V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]\}^{\delta/2}}\Big\{\frac{|\gamma|^{1/2}\widetilde{L}_\gamma}{\sqrt{2\pi}(1-a)}\Big\}^\delta\Big[\frac{1}{\{x+b(c-1)\}^{1/2}} + \frac{a^{3/2}}{\{x+b(c-1)/a\}^{1/2}}\Big]^\delta \\ &= O((nb^{1/2})^{-\delta/2}) \\ &= o(1), \end{aligned}$$

and

$$\frac{\sum_{i=1}^n E\big[\big|n^{-1}\Delta_{b,c,\gamma,i}^{(SS_a)}(0)\big|^{2+\delta}\big]}{\{V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(0)]\}^{1+\delta/2}} \le \frac{(nb)^{-\delta/2}}{\{nbV[\widehat{f}_{b,c,\gamma}^{(SS_a)}(0)]\}^{\delta/2}}\Big(\frac{1+a^2}{1-a}\widetilde{L}_\gamma\Big)^\delta = O((nb)^{-\delta/2}) = o(1).$$

Therefore, Lyapunov's central limit theorem enables us to see that

$$\frac{\widehat{f}_{b,c,\gamma}^{(SS_a)}(x) - E[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]}{\{V[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]\}^{1/2}} \xrightarrow{d} N(0,1).$$

The results follow from (A7) and Slutsky's lemma. □

**Proof of Theorem 9** We note that

$$MISE[\widehat{f}_{b,c,\gamma}^{(SS_a)}] = \Big(\int_0^{b^{\tau_1}} + \int_{b^{\tau_1}}^{b^{-\tau_2}} + \int_{b^{-\tau_2}}^{\infty}\Big)MSE[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]dx,$$

where $\tau_1 \in (4/5, 1)$ and $\tau_2 \in (4/(k_4+1), \eta_4/(\eta_4+5))$ for some $k_4 > (3\eta_4+20)/\eta_4$ (see Assumption A4′ (ii)). It is easy to see that

$$\begin{aligned} \int_0^{b^{\tau_1}} MSE[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]dx &= O(b^{5\tau_1} + n^{-1}b^{\tau_1-1}) = o(b^4 + n^{-1}b^{-1/2}) \quad \text{(see (8) and (18))}, \\ \int_{b^{-\tau_2}}^{\infty} MSE[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]dx &\le \frac{2}{(1-a)^2}\int_{b^{-\tau_2}}^{\infty}\{MSE[\widehat{f}_{b,c,\gamma}(x)] + a^2MSE[\widehat{f}_{b/a,c,\gamma}(x)]\}dx \\ &= O(b^{\tau_2(k_4+1)} + n^{-1}b^{\tau_2(k_4+1)-1}) \quad \text{(see (A6))} \\ &= o(b^4). \end{aligned}$$

Also, in view of Theorem 6 (with $x \geq b^{\tau_1}$), we have

$$\begin{aligned}
\big|V[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x)\big| &= o(n^{-1}b^{-1/2}V(x)) + O(n^{-1}),\\
\left|\{Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]\}^2 - \frac{b^4}{a^2}\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2\right| &\leq 2\frac{b^2}{a}\frac{|B^{[2]}_{c,\gamma}(x)|}{\gamma^2}|\mathcal{E}^{(SS_a)}_{b,c,\gamma}(x)| + \{\mathcal{E}^{(SS_a)}_{b,c,\gamma}(x)\}^2,
\end{aligned}$$

where $\{\mathcal{E}^{(SS_a)}_{b,c,\gamma}(x)\}^2 = O(b^6x^{-2} + \{b(1+x)\}^{4+\eta_4})$. It follows that

$$\begin{aligned}
&\left|\int_{b^{\tau_1}}^{b^{-\tau_2}} MSE[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]dx - AMISE[\widehat{f}^{(SS_a)}_{b,c,\gamma}]\right|\\
&\leq \frac{2b^2}{a}\left[\int_{b^{\tau_1}}^{b^{-\tau_2}}\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2dx\int_{b^{\tau_1}}^{b^{-\tau_2}}\{\mathcal{E}^{(SS_a)}_{b,c,\gamma}(x)\}^2dx\right]^{1/2} + \int_{b^{\tau_1}}^{b^{-\tau_2}}\{\mathcal{E}^{(SS_a)}_{b,c,\gamma}(x)\}^2dx + \frac{b^4}{a^2}\Big(\int_0^{b^{\tau_1}} + \int_{b^{-\tau_2}}^{\infty}\Big)\Big\{\frac{B^{[2]}_{c,\gamma}(x)}{\gamma^2}\Big\}^2dx\\
&\quad + \int_{b^{\tau_1}}^{b^{-\tau_2}}\big|V[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x)\big|dx + n^{-1}b^{-1/2}\lambda(a)\Big(\int_0^{b^{\tau_1}} + \int_{b^{-\tau_2}}^{\infty}\Big)|\gamma|^{1/2}V(x)dx\\
&= o(b^4 + n^{-1}b^{-1/2}),
\end{aligned}$$

since $\int_{b^{\tau_1}}^{b^{-\tau_2}}\{\mathcal{E}^{(SS_a)}_{b,c,\gamma}(x)\}^2dx = O(b^{6-\tau_1} + b^{4+\eta_4-\tau_2(5+\eta_4)}) = o(b^4)$. $\square$

## A4 Proofs of Theorems 10–15

Assuming $f(x) > 0$, we recall the stochastic expansions (24) and (25), from which we have

$$Bias[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)] = Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + E\Big[\frac{\mathcal{Q}(x)}{2af(x)}\Big] + E[\mathcal{R}^{(TS_a)}(x)], \tag{A8}$$

$$Bias[\widehat{f}^{(JF_a)}_{b,c,\gamma}(x)] = Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + E\Big[\frac{\mathcal{Q}(x)}{2f(x)}\Big] + E[\mathcal{R}^{(JF_a)}(x)], \tag{A9}$$

and, using

$$\begin{aligned}
V\Big[\frac{\mathcal{Q}(x)}{2af(x)} + \mathcal{R}^{(TS_a)}(x)\Big] &\leq 2\Big\{V\Big[\frac{\mathcal{Q}(x)}{2af(x)}\Big] + E[\{\mathcal{R}^{(TS_a)}(x)\}^2]\Big\} = 2\mathcal{J}^{(TS_a)}(x) \quad \text{(say)},\\
V\Big[\frac{\mathcal{Q}(x)}{2f(x)} + \mathcal{R}^{(JF_a)}(x)\Big] &\leq 2\Big\{V\Big[\frac{\mathcal{Q}(x)}{2f(x)}\Big] + E[\{\mathcal{R}^{(JF_a)}(x)\}^2]\Big\} = 2\mathcal{J}^{(JF_a)}(x) \quad \text{(say)},
\end{aligned}$$

we have

$$\big|V[\widehat{f}^{(\#_a)}_{b,c,\gamma}(x)] - V[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]\big| \leq 2\mathcal{J}^{(\#_a)}(x) + 2\{2\mathcal{J}^{(\#_a)}(x)V[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]\}^{1/2}. \tag{A10}$$

Before proving Theorems 10–15, we prepare the following lemma.

**Lemma A.6** *Given $\gamma \neq 0$, choose $c \geq 1$ when $\gamma > 0$ or $c > 1$ when $\gamma < 0$ (see the comment; $\ell = 2$ at the third paragraph of Section 2). Let $\rho_c(t) = t + c$. Given $\iota_0 \in [0,1)$, suppose that Assumptions* A1, A2$'[\iota_1,\iota_2]$, *and* A3$'$ *hold for some $\iota_1 \in (0, 1/(1+2\iota_0))$ and $\iota_2 > 1$. Let $j \geq 2/3$.*

(i). *We have, on* $\mathcal{I}_{\iota_0}[r_b]$ *(see* (28)*),*

$$E\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = \begin{cases} b^2\dfrac{B^2_{c|\gamma|}(x)}{\gamma^2 f(x)} + O(b^{-\iota_0}[b^3(1+x)^3 + b^{1+\iota_2}(1+x) + n^{-1}\{b^{-1/2}V(x)+1\}]), & \dfrac{x}{b}\to\infty, \\ b^2\dfrac{c^2\{f'(0)\}^2}{f(0)} + O(b^{-\iota_0}(b^{3-\iota_0} + b^{1+\iota_2} + n^{-1}b^{-1})), & \dfrac{x}{b}\to\kappa\ (x\neq 0), \\ b^2\dfrac{c^2\{f'(0)\}^2}{f(0)} + O(b^{-\iota_0}(b^3 + b^{1+\iota_2} + n^{-1}b^{-1})), & x=0, \end{cases}$$

$$V\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = \begin{cases} O(\{b^2(1+x)^2 + n^{-1}b^{-1}\}n^{-1}b^{-2\iota_0}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O((b^2+n^{-1}b^{-1})n^{-1}b^{-(1+2\iota_0)}), & \dfrac{x}{b}\to\kappa, \end{cases}$$

*and, assuming* $b^{1-\iota_0}r_b = o(1)$[15]*,*

$$E[|\mathcal{R}^{(\#_a)}(x)|^j] = \begin{cases} O(b^{(3-2\iota_0)j}(1+x)^{3j} + (n^{-1}b^{-1})^{3j/2}b^{1-2\iota_0 j}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O(b^{(3-2\iota_0)j} + (n^{-1}b^{-1})^{3j/2}b^{-2\iota_0 j}), & \dfrac{x}{b}\to\kappa. \end{cases}$$

(ii). *Suppose that* $f(0) > 0$ *(in this case, due to the continuity, there exists a* $\delta > 0$ *such that* $x\in[0,\delta]$ *implies* $f(x) > f(0)/2$ *(say)). For any* $\tau\in[1/2,1)$*, we have*

$$\sup_{x\in[0,b^\tau]} E\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = O(b^2 + n^{-1}b^{-1}),$$
$$\sup_{x\in[0,b^\tau]} V\Big[\frac{\mathcal{Q}(x)}{f(x)}\Big] = O((b^2 + n^{-1}b^{-1})n^{-1}b^{-1}),$$
$$\sup_{x\in[0,b^\tau]} E[|\mathcal{R}^{(\#_a)}(x)|^j] = O(b^{3j} + (n^{-1}b^{-1})^{3j/2}).$$

Proof of Lemma A.6 is postponed to Appendix B.

Now, it is easy to see that, under the same conditions in Lemma A.6,

- we have, on $\mathcal{I}_{\iota_0}[r_b]$,

$$\mathcal{J}^{(\#_a)}(x) = \begin{cases} O(b^{6-4\iota_0}(1+x)^6 + \{b^{2(1-\iota_0)}(1+x)^2 + n^{-1}b^{-(1+2\iota_0)}\}n^{-1}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O(b^{6-4\iota_0} + (b^{2(1-\iota_0)} + n^{-1}b^{-(1+2\iota_0)})n^{-1}b^{-1}), & \dfrac{x}{b}\to\kappa \end{cases} \tag{A11}$$

  (we used Lemma A.6 (i)), and

$$\begin{aligned} &\big\{\mathcal{J}^{(\#_a)}(x)V[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]\big\}^{1/2} \\ &= \begin{cases} O(b^{5-2\iota_0}(1+x)^3 + \{b^{1-2\iota_0}(1+x)^3 + n^{-1/2}b^{-(1/2+\iota_0)}\}n^{-1}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O(b^{5-2\iota_0} + (b^{1-2\iota_0} + n^{-1/2}b^{-(1/2+\iota_0)})n^{-1}b^{-1}), & \dfrac{x}{b}\to\kappa \end{cases} \end{aligned} \tag{A12}$$

  (we used Theorem 6 and (A11)),

[15]In Lemma A5 of Igarashi and Kakizawa (2015), the additional condition "$b^{1-\iota_0}r_b = o(1)$" was required for the proof of $E[|\mathcal{R}^{(\#_a)}(x)|^j]$. Note that the proof of $E[\mathcal{Q}(x)/f(x)]$ relies on "$br_b = b^{\iota_0}(b^{1-\iota_0}r_b) = o(1)$".

- we have, for any $\tau \in [1/2, 1)$ (we assume $f(0) > 0$),

$$\sup_{x\in[0,b^\tau]} MSE[\widehat{f}_{b,c,\gamma}^{(TS_a)}(x)] \le 3 \sup_{x\in[0,b^\tau]} \Big[MSE[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] + \Big\{E\Big[\frac{\mathcal{Q}(x)}{2af(x)}\Big]\Big\}^2 + \mathcal{J}^{(TS_a)}(x)\Big] = O(b^{4\tau} + n^{-1}b^{-1}), \tag{A13}$$

$$\sup_{x\in[0,b^\tau]} MSE[\widehat{f}_{b,c,\gamma}^{(JF_a)}(x)] \le 3 \sup_{x\in[0,b^\tau]} \Big[MSE[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)] + \Big\{E\Big[\frac{\mathcal{Q}(x)}{2f(x)}\Big]\Big\}^2 + \mathcal{J}^{(JF_a)}(x)\Big] = O(b^{4\tau} + n^{-1}b^{-1}) \tag{A14}$$

(we used (8), (18), and Lemma A.6 (ii)).

Also, $r_b = O(b^{-q})$ for some $q \in [0, \eta_4/(4+\eta_4))$, where $\eta_4 \in (0,1]$ is given in Assumption A3$'$, implies that

$$b^{1-\iota_0} r_b = O(b^{1-p_0}) = o(1),$$
$$\omega_{b,\eta_4,\iota_0,\iota_2}(r_b) + \widetilde{\omega}_{b,\iota_0}(r_b) = O(b^{\eta_4/2 - q(2+\eta_4/2)} + b^{1-(3q+2\iota_0)} + b^{\iota_2-(1+p_0)} + n^{-1/2}b^{-(1/2+\iota_0)}) = o(1)$$

for $0 \le \iota_0 < (1-3q)/2$ (hence, $0 \le \iota_0 < 1-q$), $0 < \iota_1 < 1/(1+2\iota_0)$, and $\iota_2 > 1 + p_0$ (we write $p_0 = q + \iota_0$); note that $n^{-1}b^{-(1+2\iota_0)} = o(1)$. These facts play important roles in proving Theorems 10 and 12–15.

**Proof of Theorem 10** Theorem 10 is a special case of Theorem 14 (set $\iota_0 = 0$). $\square$

**Proof of Theorem 11** Use Theorem 2 and Slutsky's lemma to get the result. $\square$

**Proof of Theorem 12** Suppose that $\iota_1 \in (2/13, 1)$ or $\iota_1 \in (1/7, 1)$ according to $x \in \mathcal{I}\backslash\{0\}$ or $x = 0$. Noting that $\mathcal{I} = \mathcal{I}_0[r]$ (set $\iota_0 = 0$), (A11) yields

$$nb^{1/2}\mathcal{J}^{(\#_a)}(x) = O(nb^{13/2} + b^2 + n^{-1}b^{-1}) = o(1) \quad \text{for fixed } x \in \mathcal{I}\backslash\{0\},$$
$$nb\mathcal{J}^{(\#_a)}(0) = O(nb^7 + b^2 + n^{-1}b^{-1}) = o(1) \quad \text{(we suppose } f(0) > 0).$$

It follows from (24) and (25) that

$$(nb^{1/2})^{1/2}\{\widehat{f}_{b,c,\gamma}^{(\#_a)}(x) - E[\widehat{f}_{b,c,\gamma}^{(\#_a)}(x)]\} = (nb^{1/2})^{1/2}\{\widehat{f}_{b,c,\gamma}^{(SS_a)}(x) - E[\widehat{f}_{b,c,\gamma}^{(SS_a)}(x)]\} + o_p(1) \quad \text{for fixed } x \in \mathcal{I}\backslash\{0\},$$
$$(nb)^{1/2}\{\widehat{f}_{b,c,\gamma}^{(\#_a)}(0) - E[\widehat{f}_{b,c,\gamma}^{(\#_a)}(0)]\} = (nb)^{1/2}\{\widehat{f}_{b,c,\gamma}^{(SS_a)}(0) - E[\widehat{f}_{b,c,\gamma}^{(SS_a)}(0)]\} + o_p(1).$$

The result is a consequence of Theorem 8. $\square$

**Proof of Theorem 13** Theorem 13 is a special case of Theorem 15 (set $\iota_0 = 0$). $\square$

**Proof of Theorem 14** Let $x \in \mathcal{I}_{\iota_0}[r_b]$. Recall (A8)–(A10). Note that $n^{-1}b^{-(1+2\iota_0)} = o(1)$, where we assume $\iota_1 \in (0, 1/(1+2\iota_0))$. The bias follows from Theorem 6 and Lemma A.6 (i), whereas the variance follows from Theorem 6, (A11), and (A12), noting that, on $\mathcal{I}_{\iota_0}[r_b]$, $b^{1-2\iota_0}(1+x)^3 \le b^{1-2\iota_0}(1+r_b)^3 = o(1)$ when $x/b \to \infty$. $\square$

**Proof of Theorem 15** We prove the result for $\widehat{f}^{(TS_a)}_{b,c,\gamma}$ (the following arguments remain valid if $TS$ and $B^{(TS)}_{c,\gamma}(x)$ are replaced by $JF$ and $B^{(JF_a)}_{c,\gamma}(x)$, respectively). Now, choosing $\tau_1 \in (4/5, 1)$, we have

$$\big|MISE_w[\widehat{f}^{(TS_a)}_{b,c,\gamma}] - AMISE_w[\widehat{f}^{(TS_a)}_{b,c,\gamma}]\big| \le \sum_{j=1}^{4} I_j,$$

where

$$\begin{aligned}
I_1 &= \overline{w}\int_0^{b^{\tau_1}} MSE[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)]dx, \quad I_2 = \int_{r_b}^{\infty} w(x) MSE[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)]dx,\\
I_3 &= \frac{b^4}{a^2}\Big(\int_0^{b^{\tau_1}} + \int_{r_b}^{\infty}\Big) w(x)\Big\{\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 dx + n^{-1}b^{-1/2}\lambda(a)\overline{w}\Big(\int_0^{b^{\tau_1}} + \int_{r_b}^{\infty}\Big)|\gamma|^{1/2}V(x)dx,\\
I_4 &= \int_{b^{\tau_1}}^{r_b} w(x)\Big|MSE[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] - \frac{b^4}{a^2}\Big\{\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 - n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x)\Big|dx.
\end{aligned}$$

We can see that $I_1 = o(b^4 + n^{-1}b^{-1/2})$, $I_2 \le 2\{(M^{(TS_a)}_{b,\epsilon})^2 + C_0^2\}\int_{r_b}^{\infty} w(x)dx = o(b^4)$, and $I_3 = o(b^4 + n^{-1}b^{-1/2})$; use (A13) ((A14) for the $\text{JF}_a$ type) and Remark A.1 for $I_1$ and $I_2$, respectively. On the other hand, noting that $n^{-1}b^{-(1+2\iota_0)} = o(1)$ and $\omega_{b,\eta_4,\iota_0,\iota_2}(r_b) + \widetilde{\omega}_{b,\iota_0}(r_b) = o(1)$, Theorem 14 (with $x \in [b^{\tau_1}, r_b]$) yields

$$\begin{aligned}
w(x)\big|V[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x)\big| &= o(b^4 w(x) + n^{-1}b^{-1/2}V(x)) + O(n^{-1}w(x)),\\
w(x)\Big|\{Bias[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)]\}^2 - \frac{b^4}{a^2}\Big\{\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2\Big| &\le w(x)\Big[2\frac{b^2}{a}\frac{|B^{(TS)}_{c,\gamma}(x)|}{\gamma^2}|\mathcal{E}^{(TS_a)}_{b,c,\gamma}(x)| + \{\mathcal{E}^{(TS_a)}_{b,c,\gamma}(x)\}^2\Big],
\end{aligned}$$

where $w(x)\{\mathcal{E}^{(TS_a)}_{b,c,\gamma}(x)\}^2 = O(b^6x^{-2}) + o((b^4 + n^{-1})w(x) + n^{-1}b^{-1/2}V(x))$. It follows that

$$\begin{aligned}
I_4 &\le \frac{2b^2}{a}\Big[\int_{b^{\tau_1}}^{r_b} w(x)\Big\{\frac{B^{(TS)}_{c,\gamma}(x)}{\gamma^2}\Big\}^2 dx \int_{b^{\tau_1}}^{r_b} w(x)\{\mathcal{E}^{(TS_a)}_{b,c,\gamma}(x)\}^2 dx\Big]^{1/2} + \int_{b^{\tau_1}}^{r_b} w(x)\{\mathcal{E}^{(TS_a)}_{b,c,\gamma}(x)\}^2 dx\\
&\quad + \int_{b^{\tau_1}}^{r_b} w(x)\Big|V[\widehat{f}^{(TS_a)}_{b,c,\gamma}(x)] - n^{-1}b^{-1/2}\lambda(a)|\gamma|^{1/2}V(x)\Big|dx\\
&= o(b^4 + n^{-1}b^{-1/2}),
\end{aligned}$$

since $\int_{b^{\tau_1}}^{r_b} w(x)\{\mathcal{E}^{(TS_a)}_{b,c,\gamma}(x)\}^2 dx = O(b^{6-\tau_1}) + o(b^4 + n^{-1}b^{-1/2}) = o(b^4 + n^{-1}b^{-1/2})$. □

# Appendix B

**Proof of Lemma A.6** Recall that

$$\mathcal{Q}(x) = \Big[Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon - \frac{a}{1-a}\{\overline{\Delta}_{b,c,\gamma}(x) - \overline{\Delta}_{b/a,c,\gamma}(x)\}\Big]^2.$$

Note that $n^{-2}b^{-2} = o(n^{-1}b^{-1})$. This, together with the assumption $\epsilon \propto b^{\iota_2}$ ($\iota_2 > 1$), will be repeatedly used without mentioning them explicitly, throughout this proof.

Firstly, it is easy to see that

$$E[\mathcal{Q}(x)] = \{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon\}^2 + \frac{a^2}{(1-a)^2}E[\{\overline{\Delta}_{b,c,\gamma}(x) - \overline{\Delta}_{b/a,c,\gamma}(x)\}^2].$$

Here, $E[\{\overline{\Delta}_{b,c,\gamma}(x) - \overline{\Delta}_{b/a,c,\gamma}(x)\}^2] \le 2\{V[\widehat{f}_{b,c,\gamma}(x)] + V[\widehat{f}_{b/a,c,\gamma}(x)]\}$, with

$$V[\widehat{f}_{b,c,\gamma}(x)] + V[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] = \begin{cases} O(n^{-1}\{b^{-1/2}V(x) + 1\}), & \dfrac{x}{b} \to \infty,\\ O(n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa \end{cases} \quad \text{(see Theorems 1 and 6).} \tag{B1}$$

Also,

$$\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon\}^2 = \begin{cases} b^2\dfrac{B^2_{c|\gamma|}(x)}{\gamma^2} + O(b^3(1+x)^3 + b^{1+\iota_2}(1+x)), & \dfrac{x}{b} \to \infty, \\ b^2c^2\{f'(0)\}^2 + O(b^3 + b^{1+\iota_2}), & \dfrac{x}{b} \to \kappa, \end{cases}$$

since (6), (15), and (16) yield

$$\begin{aligned} \{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon\}^2 &= b^2\frac{B^2_{c|\gamma|}(x)}{\gamma^2} + \Big\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon + b\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\} \\ &\quad \times\Big\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon - b\frac{B_{c|\gamma|}(x)}{|\gamma|}\Big\} \\ &= \begin{cases} b^2\dfrac{B^2_{c|\gamma|}(x)}{\gamma^2} + \{\epsilon + O(b+bx)\}\{\epsilon + O(b^2+b^2x^2)\}, & \dfrac{x}{b} \to \infty, \\ b^2c^2\{f'(0)\}^2 + \{\epsilon + O(b)\}\{\epsilon + O(b^2)\}, & \dfrac{x}{b} \to \kappa. \end{cases} \end{aligned}$$

Note that, assuming $\tau \in [1/2, 1)$, (8), (17), and (18) yield $\sup_{x\in[0,b^\tau]} E[\mathcal{Q}(x)] = O(b^2 + n^{-1}b^{-1})$.

Secondly, (6) and Lemma A.3 (ii) (also (B1)) yield

$$\begin{aligned} &V[\mathcal{Q}(x)] \\ &= \frac{a^2}{(1-a)^2}V\Big[2\{\overline{\Delta}_{b,c,\gamma}(x) - \overline{\Delta}_{b/a,c,\gamma}(x)\}\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon\} + \frac{a}{1-a}\{\overline{\Delta}_{b,c,\gamma}(x) - \overline{\Delta}_{b/a,c,\gamma}(x)\}^2\Big] \\ &\le \frac{2^4a^2}{(1-a)^2}E\Big[\{\overline{\Delta}^2_{b,c,\gamma}(x) + \overline{\Delta}^2_{b/a,c,\gamma}(x)\}\{Bias[\widehat{f}_{b,c,\gamma}(x)] - Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)] + \epsilon\}^2 + \frac{a^2}{(1-a)^2}\{\overline{\Delta}^4_{b,c,\gamma}(x) + \overline{\Delta}^4_{b/a,c,\gamma}(x)\}\Big] \\ &= \begin{cases} O(\{b^2(1+x)^2 + n^{-2}b^{-2} + n^{-1}b^{-1}\}n^{-1}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b} \to \infty, \\ O((b^2 + n^{-2}b^{-2} + n^{-1}b^{-1})n^{-1}b^{-1}), & \dfrac{x}{b} \to \kappa. \end{cases} \end{aligned}$$

Note that, assuming $\tau \in [1/2, 1)$, (17), (18), and Lemma A.3 (ii) (also (8)) yield

$$\sup_{x\in[0,b^\tau]} V[\mathcal{Q}(x)] = O((b^2 + n^{-2}b^{-2} + n^{-1}b^{-1})n^{-1}b^{-1}).$$

Thirdly, we estimate $E[|\mathcal{R}^{(\#_a)}(x)|^j]$ for $j \ge 2/3$, in the spirit of Chen et al. (2009) (see also Igarashi and Kakizawa (2014a, 2015)). Consider the event

$$\begin{aligned} S_{x,b} &= \Big\{\mathcal{X}^{(n)} \;\Big|\; \frac{|\overline{\Delta}_{b,c,\gamma}(x)|}{f(x)} \le \frac{1}{4} \text{ and } \frac{|\overline{\Delta}_{b/a,c,\gamma}(x)|}{f(x)} \le \frac{1}{4}\Big\} \quad \text{(say)}, \\ \dot{S}_{x,b} &= \Big\{\mathcal{X}^{(n)} \;\Big|\; \frac{|\overline{\Delta}_{b,c,\gamma}(x)|}{f(x)} \le \frac{1}{4} \text{ and } \frac{|\overline{\Delta}^{(SS_a)}_{b,c,\gamma}(x)|}{f(x)} \le \frac{1}{4}\Big\} \quad \text{(say)}. \end{aligned}$$

It is easy to see that (6), (17), and (18) yield, for $\tau \in [1/2, 1)$,

$$\sup_{x\in\mathcal{I}_{\iota_0}[r_b]} |Bias[\widehat{f}_{b,c,\gamma}(x)]| = \max\Big\{\sup_{x\in[0,b^\tau]} |Bias[\widehat{f}_{b,c,\gamma}(x)]|, \sup_{x\in(b^\tau,r_b]} |Bias[\widehat{f}_{b,c,\gamma}(x)]|\Big\} = O(b + br_b)$$

and

$$\sup_{x\in\mathcal{I}_{\iota_0}[r_b]} |Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]| = \max\Big\{\sup_{x\in[0,b^\tau]} |Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]|, \sup_{x\in(b^\tau,r_b]} |Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]|\Big\} = O(b + br_b)$$

(note that $x \in (b^\tau, r_b]$ implies $x/b \to \infty$). Assuming $b^{1-\iota_0} r_b = o(1)$, it follows that

$$\sup_{x\in\mathcal{I}_{\iota_0}[r_b]}\Big[\frac{1}{f(x)}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}_{b/a,c,\gamma}(x)]+\epsilon/a|\}\Big] = O(b^{1-\iota_0}r_b) = o(1),$$
$$\sup_{x\in\mathcal{I}_{\iota_0}[r_b]}\Big[\frac{1}{f(x)}\{|Bias[\widehat{f}_{b,c,\gamma}(x)]+\epsilon|+|Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]|\}\Big] = O(b^{1-\iota_0}r_b) = o(1).$$

Thus, for all sufficiently large $n$, on $\mathcal{I}_{\iota_0}[r_b]$, we have

$$S_{x,b} \subset \Big\{\mathcal{X}^{(n)} \;\Big|\; \frac{1}{f(x)}|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)| \le \frac{1}{2} \text{ and } \frac{1}{f(x)}|\widehat{f}_{b/a,c,\gamma}(x)+\epsilon/a-f(x)| \le \frac{1}{2}\Big\} \quad \text{(say)},$$
$$\dot{S}_{x,b} \subset \Big\{\mathcal{X}^{(n)} \;\Big|\; \frac{1}{f(x)}|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)| \le \frac{1}{2} \text{ and } \frac{1}{f(x)}|\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)-f(x)| \le \frac{1}{2}\Big\} \quad \text{(say)},$$

hence,

$$E[|\mathcal{R}^{(TS_a)}(x)|^j \chi_{S_{x,b}}] \le \Big\{\frac{2^3 3^{a/(1-a)} a}{(1-a)^3(\varrho b^{\iota_0})^2}\Big\}^j E[\{|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)|+|\widehat{f}_{b/a,c,\gamma}(x)+\epsilon/a-f(x)|\}^{3j}],$$
$$E[|\mathcal{R}^{(JF_a)}(x)|^j \chi_{\dot{S}_{x,b}}] \le \Big\{\frac{2^4 3^2 e^2}{(\varrho b^{\iota_0})^2}\Big\}^j E[\{|\widehat{f}_{b,c,\gamma}(x)+\epsilon-f(x)|+|\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)-f(x)|\}^{3j}]$$

($\chi_S$ denotes the indicator function of a set $S$), using

$$\begin{aligned}
&\max_{|t|\le 1/2, |v|\le 1/2}\Big|\int_0^1 \sum_{\ell=0}^3 {}_3C_\ell t^{3-\ell} v^\ell g^{(TS_a)}_{3-\ell,\ell}(\theta t, \theta v)(1-\theta)^2 d\theta\Big| \\
&\le \Big\{\frac{2^3 3^{1/(1-a)} a}{(1-a)^3}|t|^3 + \frac{2^2 3^{1/(1-a)} a}{(1-a)^3}t^2|v| + \frac{2^2 3^{1/(1-a)} a^2}{(1-a)^3}|t|v^2 + \frac{2^2 3^{a/(1-a)} a}{(1-a)^3}|v|^3\Big\}\int_0^1 (1-\theta)^2 d\theta \\
&\le \frac{2^3 3^{a/(1-a)} a}{(1-a)^3}(|t|+|v|)^3, \qquad \text{(B2)} \\
&\max_{|t|\le 1/2, |v|\le 1/2}\Big|\int_0^1 \sum_{\ell=0}^3 {}_3C_\ell t^{3-\ell} v^\ell g^{(JF)}_{3-\ell,\ell}(\theta t, \theta v)(1-\theta)^2 d\theta\Big| \\
&\le e^2(2^4 3^3|t|^3 + 2^2 3^4 t^2|v| + 2^3 3^2|t|v^2 + 2^2|v|^3)\int_0^1 (1-\theta)^2 d\theta \\
&\le 2^4 3^2 e^2(|t|+|v|)^3. \qquad \text{(B3)}
\end{aligned}$$

We know that, for a random variable $Y$, $E[|Y+C|^{3j}] \le 2^{3j-1}\{|E[Y]+C|^{3j} + E[|Y-E[Y]|^{3j}]\}$, $j \ge 2/3$, $C \in \mathbb{R}$. Combining them with (6) and Lemma A.3 (ii) (also (B1)), it follows that, on $\mathcal{I}_{\iota_0}[r_b]$,

$$E[|\mathcal{R}^{(TS_a)}(x)|^j \chi_{S_{x,b}}] = \begin{cases} O(b^{(3-2\iota_0)j}(1+x)^{3j} + (n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2} n^{-1} b^{-2\iota_0 j}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O(b^{(3-2\iota_0)j} + (n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2} n^{-1} b^{-(1+2\iota_0 j)}), & \dfrac{x}{b}\to\kappa, \end{cases}$$
$$E[|\mathcal{R}^{(JF_a)}(x)|^j \chi_{\dot{S}_{x,b}}] = \begin{cases} O(b^{(3-2\iota_0)j}(1+x)^{3j} + (n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2} n^{-1} b^{-2\iota_0 j}\{b^{-1/2}V(x)+1\}), & \dfrac{x}{b}\to\infty, \\ O(b^{(3-2\iota_0)j} + (n^{-2}b^{-2}+n^{-1}b^{-1})^{(3j-2)/2} n^{-1} b^{-(1+2\iota_0 j)}), & \dfrac{x}{b}\to\kappa. \end{cases}$$

On the other hand, using Lemma A.3 (i), there exist constants $\mathcal{L}, \widetilde{\mathcal{L}} > 0$, independent of $n$, $b$, and $x$, such that

$$E[\chi_{S^c_{x,b}}] \le P\Big[|\overline{\Delta}_{b,c,\gamma}(x)| > \frac{\varrho b^{\iota_0}}{4}\Big] + P\Big[|\overline{\Delta}_{b/a,c,\gamma}(x)| > \frac{\varrho b^{\iota_0}}{4}\Big] \le 4\exp(-nb^{1+2\iota_0}\mathcal{L}),$$
$$E[\chi_{\dot{S}^c_{x,b}}] \le P\Big[|\overline{\Delta}_{b,c,\gamma}(x)| > \frac{\varrho b^{\iota_0}(1-a)}{8}\Big] + P\Big[|\overline{\Delta}_{b/a,c,\gamma}(x)| > \frac{\varrho b^{\iota_0}(1-a)}{8a}\Big] \le 4\exp(-nb^{1+2\iota_0}\widetilde{\mathcal{L}}),$$

where $S^c$ denotes the complement of set $S$. Consequently, we have, on $\mathcal{I}_{\iota_0}[r_b]$,

$$
\begin{aligned}
E[|\mathcal{R}^{(TS_a)}(x)|^j \chi_{S^c_{x,b}}] &= E\Big[\Big|\widehat{f}^{(TS_a)}_{b,c,\gamma}(x) - \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - \frac{\mathcal{Q}(x)}{2af(x)}\Big|^j \chi_{S^c_{x,b}}\Big] \\
&\leq O(b^{-(1+a\iota_2)j/(1-a)} + b^{-j} + b^{-(2+\iota_0)j}) E[\chi_{S^c_{x,b}}] \quad \text{(see Remark A.1)} \\
&= \begin{cases} o(b^{(3-2\iota_0)j}(1+x)^{3j}), & \dfrac{x}{b} \to \infty, \\ o(b^{(3-2\iota_0)j}), & \dfrac{x}{b} \to \kappa, \end{cases} \\
E[|\mathcal{R}^{(JF_a)}(x)|^j \chi_{\dot{S}^c_{x,b}}] &= E\Big[\Big|\widehat{f}^{(JF_a)}_{b,c,\gamma}(x) - \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - \frac{\mathcal{Q}(x)}{2f(x)}\Big|^j \chi_{\dot{S}^c_{x,b}}\Big] \\
&\leq O(b^{-j} + b^{-(2+\iota_0)j}) E[\chi_{\dot{S}^c_{x,b}}] \quad \text{(see Remark A.1)} \\
&= \begin{cases} o(b^{(3-2\iota_0)j}(1+x)^{3j}), & \dfrac{x}{b} \to \infty, \\ o(b^{(3-2\iota_0)j}), & \dfrac{x}{b} \to \kappa, \end{cases}
\end{aligned}
$$

provided that $b \propto n^{-\iota_1}$ ($\iota_1 \in (0, 1/(1+2\iota_0))$). Note that, if $f(0) > 0$ (due to the continuity, there exists a $\delta > 0$ such that $x \in [0, \delta]$ implies $f(x) > f(0)/2$ (say)), then, assuming $\tau \in [1/2, 1)$,

$$
\sup_{x\in[0,b^\tau]} \Big[\frac{1}{f(x)}\{|Bias[\widehat{f}_{b,c,\gamma}(x)] + \epsilon| + |Bias[\widehat{f}_{b/a,c,\gamma}(x)] + \epsilon/a| + |Bias[\widehat{f}^{(SS_a)}_{b,c,\gamma}(x)]|\}\Big] \leq \frac{2O(b)}{f(0)}
$$

(see (17) and (18)); for all sufficiently large $n$, it follows that, on $[0, b^\tau]$,

$$
\begin{aligned}
S_{x,b} &\subset \Big\{\mathcal{X}^{(n)} \;\Big|\; \frac{1}{f(x)}|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| \leq \frac{1}{2} \text{ and } \frac{1}{f(x)}|\widehat{f}_{b/a,c,\gamma}(x) + \epsilon/a - f(x)| \leq \frac{1}{2}\Big\} \quad \text{(say)}, \\
\dot{S}_{x,b} &\subset \Big\{\mathcal{X}^{(n)} \;\Big|\; \frac{1}{f(x)}|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| \leq \frac{1}{2} \text{ and } \frac{1}{f(x)}|\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x)| \leq \frac{1}{2}\Big\} \quad \text{(say)},
\end{aligned}
$$

hence,

$$
\begin{aligned}
E[|\mathcal{R}^{(TS_a)}(x)|^j \chi_{S_{x,b}}] &\leq \Big[\frac{2^3 3^{a/(1-a)} a}{(1-a)^3 \{f(0)/2\}^2}\Big]^j E[\{|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| + |\widehat{f}_{b/a,c,\gamma}(x) + \epsilon/a - f(x)|\}^{3j}], \\
E[|\mathcal{R}^{(JF_a)}(x)|^j \chi_{\dot{S}_{x,b}}] &\leq \Big[\frac{2^4 3^2 e^2}{\{f(0)/2\}^2}\Big]^j E[\{|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| + |\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x)|\}^{3j}]
\end{aligned}
$$

(we used (B2) and (B3)), and consequently, for $j \geq 2/3$,

$$
\begin{aligned}
\sup_{x\in[0,b^\tau]} E[|\mathcal{R}^{(TS_a)}(x)|^j \chi_{S_{x,b}}] &\leq \Big[\frac{2^3 3^{a/(1-a)} a}{(1-a)^3 \{f(0)/2\}^2}\Big]^j \sup_{x\in[0,b^\tau]} E[\{|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| + |\widehat{f}_{b/a,c,\gamma}(x) + \epsilon/a - f(x)|\}^{3j}] \\
&= O(b^{3j} + (n^{-2}b^{-2} + n^{-1}b^{-1})^{(3j-2)/2} n^{-1}b^{-1}), \\
\sup_{x\in[0,b^\tau]} E[|\mathcal{R}^{(JF_a)}(x)|^j \chi_{\dot{S}_{x,b}}] &\leq \Big[\frac{2^4 3^2 e^2}{\{f(0)/2\}^2}\Big]^j \sup_{x\in[0,b^\tau]} E[\{|\widehat{f}_{b,c,\gamma}(x) + \epsilon - f(x)| + |\widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - f(x)|\}^{3j}] \\
&= O(b^{3j} + (n^{-2}b^{-2} + n^{-1}b^{-1})^{(3j-2)/2} n^{-1}b^{-1})
\end{aligned}
$$

(we used (17), (18), and Lemma A.3 (ii) (also (8))). Further, we obtain

$$
\begin{aligned}
\sup_{x\in[0,b^\tau]} E[|\mathcal{R}^{(TS_a)}(x)|^j \chi_{S^c_{x,b}}] &= \sup_{x\in[0,b^\tau]} E\Big[\Big|\widehat{f}^{(TS_a)}_{b,c,\gamma}(x) - \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - \frac{\mathcal{Q}(x)}{2af(x)}\Big|^j \chi_{S^c_{x,b}}\Big] \\
&\leq O(b^{-(1+a\iota_2)j/(1-a)} + b^{-j} + b^{-2j}) \sup_{x\in[0,b^\tau]} E[\chi_{S^c_{x,b}}] \quad \text{(see Remark A.1)} \\
&= o(b^{3j}),
\end{aligned}
$$

$$\begin{aligned}\sup_{x\in[0,b^\tau]} E[|\mathcal{R}^{(JF_a)}(x)|^j \chi_{\dot{S}^c_{x,b}}] &= \sup_{x\in[0,b^\tau]} E\left[\left|\widehat{f}^{(JF_a)}_{b,c,\gamma}(x) - \widehat{f}^{(SS_a)}_{b,c,\gamma}(x) - \frac{\mathcal{Q}(x)}{2f(x)}\right|^j \chi_{\dot{S}^c_{x,b}}\right] \\ &\leq O(b^{-j} + b^{-2j}) \sup_{x\in[0,b^\tau]} E[\chi_{\dot{S}^c_{x,b}}] \quad \text{(see Remark A.1)} \\ &= o(b^{3j}),\end{aligned}$$

provided that $b \propto n^{-\iota_1}$, since, using Lemma A.3 (i), there exist constants $\mathcal{L}', \widetilde{\mathcal{L}}' > 0$, independent of $n$, $b$, and $x$, such that

$$\sup_{x\in[0,b^\tau]} E[\chi_{S^c_{x,b}}] \leq \sup_{x\in[0,b^\tau]} P\left[|\overline{\Delta}_{b,c,\gamma}(x)| > \frac{f(0)}{8}\right] + \sup_{x\in[0,b^\tau]} P\left[|\overline{\Delta}_{b/a,c,\gamma}(x)| > \frac{f(0)}{8}\right] \leq 4\exp(-nb\mathcal{L}'),$$

$$\sup_{x\in[0,b^\tau]} E[\chi_{\dot{S}^c_{x,b}}] \leq \sup_{x\in[0,b^\tau]} P\left[|\overline{\Delta}_{b,c,\gamma}(x)| > \frac{(1-a)f(0)}{16}\right] + \sup_{x\in[0,b^\tau]} P\left[|\overline{\Delta}_{b/a,c,\gamma}(x)| > \frac{(1-a)f(0)}{16a}\right] \leq 4\exp(-nb\widetilde{\mathcal{L}}'). \quad \square$$

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