A FAMILY OF NON-SOFIC BETA EXPANSIONS

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Abstract. Let $\beta_n > 1$ be a root of $x^n - x - 1$ for $n = 4, 5, \dots$. We will prove that β_n is not a Parry number, i.e., the associated beta transformation does not correspond a sofic symbolic system. A generalization is shown in the last section.

1. Beta expansions

Fix a real number $\beta > 1$. The map from $[0, 1)$ to itself defined by $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ is called the *beta transformation.* Putting $a_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor$, we obtain an expansion:

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots$$

with $a_i \in \mathcal{A} := \mathbb{Z} \cap [0, \beta)$, which gives a generalization of the decimal expansion to the real base β. Let $\mathcal{A}^{\mathbb{N}}$ (resp. $\mathcal{A}^{\mathbb{Z}}$) be the set of right infinite (resp. bi-infinite) words over $\mathcal{A}$ which is compact by the product topology of $\mathcal{A}$. Define $d_\beta : [0, 1) \to \mathcal{A}^{\mathbb{N}}$ by $d_\beta(x) = a_1 a_2 \dots$. The *expansion of one* of β is the infinite word $c_1 c_2 \dots \in \mathcal{A}^{\mathbb{N}}$ obtained as a limit of the expansion $1 - \epsilon$ when $\epsilon > 0$ tends to zero, which is denoted by $d_\beta(1 - 0)$. The map d_β is not surjective and the image $d_\beta([0, 1))$ is characterized as

$$\{\xi = (\xi_n) \in \mathcal{A}^{\mathbb{N}} \mid s^n(\xi) \ll d_\beta(1 - 0) \quad (n = 0, 1, \dots)\}$$

where s is a shift operator $s((\xi_n)) = (\xi_{n+1})$, and $\ll$ is the natural lexicographic order on $\mathcal{A}^{\mathbb{N}}$. We say that $\xi \in \mathcal{A}^{\mathbb{N}}$ is *admissible* if it satisfies the *Parry condition*

$$s^n(\xi) \ll d_\beta(1 - 0) \quad (n = 0, 1, \dots),$$

see [12, 8]. Let $\mathcal{A}^*$ be the set of finite words over $\mathcal{A}$. An element $w \in \mathcal{A}^*$ is admissible if $w0^\infty = w00\dots$ is admissible. Define a compact subset of $\mathcal{A}^{\mathbb{Z}}$ by

$$X_\beta = \{(\xi_n) \in \mathcal{A}^{\mathbb{Z}} \mid \xi_n \xi_{n+1} \dots \xi_m \text{ is admissible for all } n \text{ and } m \text{ with } n < m\}.$$

The symbolic dynamical system (X_β, s) is called *beta shift.* We see that (X_β, s) is a subshift of finite type if and only if $d_\beta(1 - 0)$ is purely periodic. Further (X_β, s) is sofic if and only if $d_\beta(1 - 0)$ is eventually periodic. We say that β is a *simple Parry number* if (X_β, s) is a shift of finite type, and a *Parry number*[1] if (X_β, s) is sofic. It is well known that (X_β, s) is sofic if β is a Pisot number, that is, a real algebraic integer greater than one whose all conjugates lie within the open unit disk. In fact, this follows from a general fact that beta expansions of elements of $\mathbb{Q}(\beta) \cap [0, 1)$ are eventually periodic provided β is a Pisot number [2, 16]. According to [1], let U

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[1] Parry coined it *beta number* but it is confusing to say β is a beta number. Recent articles use this name.

be the set of real algebraic numbers greater than one whose remaining conjugates lie in the closed unit disk. Pisot numbers are contained in U. A non-Pisot element of U is called a Salem number. We can show that a Salem number β is a root of a reciprocal polynomial having exactly two real conjugates: β and $1/\beta$ and all other conjugates have modulus one. Boyd [4, 5] showed that Salem numbers of degree 4 are Parry numbers, and gave some heuristic discussion on the existence of non-Parry Salem number of higher degree. However until now, we have no idea how to prove that $d_\beta(1-0)$ is not eventually periodic when β is a Salem number. In this note, we will show the following

Theorem 1.1. *Let $\beta_n > 1$ be the root of $x^n - x - 1$ for $n = 2, 3, \dots$. Then β_n is a Parry number if and only if $n = 2, 3$.*

The result may be compared with Boyd [3] in which it is shown that $\log \beta_n$ can not be a Mahler measure, which appears as an entoropy of a toral automorphism. According to [12], we know that if β is a Parry number, then it must be a Perron number whose other conjugates have modulus less than 2. Here a Perron number is an algebraic integer greater than one, all of whose other conjugates have modulus strictly less than the number itself. Solomyak [19] further studied distribution of conjugates of Parry numbers, describing the intriguing region Φ where the conjugates densely lie. This improves the modulus bound to $(1+\sqrt{5})/2$. He also gave an example of a Perron number $(1+\sqrt{13})/2 \notin U$ whose conjugate lie in the interior of Φ. Theorem 1.1 seems to be the first result on a family of non-Parry Perron numbers whose conjugates lie in the interior of Φ in [19], see Appendix.

The key to the proof is the *Lagrange inversion formula* which gives the inverse of Taylor expansion of a holomorphic function defined in some region. As Theorem 1.1 covers all n, we must rely on numerical computation. The dependencies on computation are sketched within the proofs. If we permit a finite number of exceptions, then the proof becomes computer independent and we can treat more general cases. A generalization of Theorem 1.1 in this sense is given in the last section.

Hereafter the Landau O symbol : $f(x) = O(g(x))$ will be used to mean that there exists a constant C that $|f(x)| \le C|g(x)|$ for all x in an appropriate ball (possibly centered at ∞) which is clear from the context. Vinogradoff symbols are not used. We write $n \gg 1$ only to mean that n is sufficiently large.

2. Proof

Let $\beta \notin U$ be a Perron number. Then one can select a conjugate $\gamma \neq \beta$ of β with $|\gamma| > 1$. Let x' be the image of x by the conjugate map from $\mathbb{Q}(\beta)$ to $\mathbb{Q}(\gamma)$ and $d_\beta(1-0) = c_0 c_1 \dots$. Put

$$T_\beta^k(1-0) = \beta^k \left(1 - \sum_{m=1}^{k} \frac{c_m}{\beta^m}\right) \in \mathbb{Z}[\beta].$$

Note that $T_\beta^0(1-0) = 1$ and we have

$$T_\beta^k(1-0) = \sum_{m=1}^{\infty} \frac{c_{m+k}}{\beta^m}.$$

Lemma 2.1. *If there is $k \in \mathbb{N}$ with $|(T_\beta^k(1-0))'| > \frac{\lfloor \beta \rfloor}{|\gamma|-1}$, then β is not a Parry number.*

Proof. Putting $x_m = T_\beta^m(1-0)$, we have $x_{m+1} = \beta x_m - c_{m+1}$. Since $|x'_k| > \lfloor \beta \rfloor/(|\gamma|-1)$, we have

$$|x'_{m+1}| = |\gamma x'_m - c_{m+1}| > |x'_m|$$

for $m \geq k$. Therefore the sequence $(|x'_m|)_{m=1,2,\dots}$ diverges, which is impossible if c_i is eventually periodic. □

This lemma gives a computational way to show that β_n is not a Parry number for a fixed n.

For $n = 2$, $\beta_2 = (1+\sqrt{5})/2$ is the best known Pisot number, the golden mean. It is also well known that β_3 the smallest Pisot number [18, 1]. We will show that β_n for $n \geq 4$ is not a Parry number.

Lemma 2.2. *β_n $(n \geq 4)$ is a Perron number and not contained in U.*

Proof. Let $V = \{1, \dots, n\}$ and define the directed edge E by

$$i \to i+1 \quad (i = 1, 2, \dots, n-1), n \to 1, n \to 2.$$

The adjacency matrix of this graph is clearly primitive and its Perron-Frobenius root is β_n, which shows that β_n is a Perron number. From $(\beta_n)^{n+1} - \beta_n - 1 = \beta_n^2 - 1 > 0$, we see

$$\beta_2 > \beta_3 > \beta_4 > \cdots > 1$$

Since β_3 is the smallest Pisot number, β_n for $n \geq 4$ is not a Pisot number. It is neither a Salem number, since it does not have a positive real conjugate. □

Lemma 2.3. *The polynomial $x^n - x - 1$ is irreducible over $\mathbb{Q}$ for $n \geq 2$.*

Proof. This result is due to Selmer [17]. □

The Bürmann-Lagrange formula is discussed in Part I-Chap. 7 of [7]. We briefly review it in a special form, to obtain an explicit truncation error bound. Denote by $B(x, r)$ the ball of radius r centered at x. Let $g(z)$ be a holomorphic function with $g(0) = 0$ and $g'(z) \neq 0$ in $z \in B(0, r)$. Then g is locally univalent and admits a holomorphic inverse which is to be made explicit. Define a function

$$h(w) = \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{\zeta g'(\zeta)}{g(\zeta) - w} d\zeta$$

where C is the counter-clockwise contour which circumscribes $B(0, r)$. Since $g'(z)$ does not vanish, by the residue theorem we have $h(g(z)) = z$ in a neighborhood of the origin, and hence in $B(0, r)$ by the identity theorem for holomorphic functions. Using

$$\frac{1}{1-z} = \sum_{k=0}^{m} z^k + \frac{z^{m+1}}{1-z}$$

we have

$$h(w) = c_1 w + \cdots + c_m w^m + \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{\zeta g'(\zeta) w^{m+1}}{g(\zeta)^{m+1}(g(\zeta) - w)} d\zeta \tag{2.1}$$

with

$$c_k = \frac{1}{2\pi k\sqrt{-1}} \oint_C \frac{d\zeta}{g(\zeta)^k}.$$

This (2.1) is the *Lagrange inversion formula* in a complex analytic form. A different formulation is found in pp.131-133 of [20]. It has many interesting applications in combinatorics.

Proposition 2.4. *Fix $m \in \mathbb{Z}$. For an integer $n \geq 12|m|$, there is a root of $x^n - x - 1$ which satisfies the asymptotic formula:*

$$\exp\left(\frac{2\pi m\sqrt{-1}}{n}\right) + \frac{\log 2}{n} + \frac{(1+\log 2)\log 2 + 2\sqrt{-1}\pi m(1+\log 4)}{2n^2} + C(n)$$

with $C(n) = O\left(\frac{1}{n^3}\right)$.

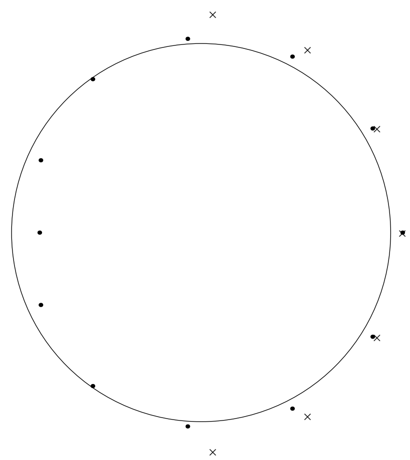


FIGURE 1. Roots of $x^{12} - x - 1$ (black dots) and approximations ($\times$ dots) by the formula of Proposition 2.4.

Proof. Consider a root γ of $x^n - x - 1$ lying in a ball $B(1, 1/2)$. Since $|\arg \gamma| < \pi/6$, we have

$$\frac{1}{n} = \frac{\log(\gamma)}{\log(1+\gamma) + 2\pi m\sqrt{-1}}$$

where log denotes the principal branch of the logarithm, $m \in \mathbb{Z}$ and $|m| < n/12$. We fix m and study the asymptotic behavior of γ when n tends to ∞. Introduce a complex variable $z = \gamma - 1$ to define

$$g(z) = \frac{\log(z+1)}{\log(z+2) + 2\pi m\sqrt{-1}}.$$

Then $g(z)$ is holomorphic, $g(0) = 0$ and $g'(z) \neq 0$ in $B(0, 1/2)$. Lagrange inversion (2.1) gives

$$h(w) = (\log 2 + 2\pi m\sqrt{-1})w + \left(\frac{(1+\log 2)\log 2}{2} + \sqrt{-1}\pi m(1+\log 4) - 2\pi^2 m^2\right) w^2 + E(w)$$

with

$$E(w) = \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{\zeta g'(\zeta) w^3}{g(\zeta)^3 (g(\zeta) - w)} d\zeta = O(w^3)$$

where C is the contour for $B(0, 1/2)$.

Putting $w = 1/n$, the Taylor expansion of $\exp(2\pi m\sqrt{-1}/n)$ leads to the required asymptotic formula. □

We see that $x^n - x - 1$ has a unique root greater than 1. Denote this root by β_n. Let γ_n be the complex root of $x^n - x - 1$ closest to β in $\mathbb{C}$ with $\Im \gamma_n > 0$.

Corollary 2.5.

$$\left| \beta_n - \left(1 + \frac{\log 2}{n}\right) \right| \le \frac{2}{3n^2} \qquad (n \ge 8) \tag{2.2}$$

$$\left| \gamma_n - \left(1 + \frac{\log 2 + 2\pi\sqrt{-1}}{n}\right) \right| \le \frac{24}{n^2} \qquad (n \ge 6). \tag{2.3}$$

Note that Selmer [17] obtained a weaker form of (2.2). In the course of the later proofs, we shall use numerical values of β_n and γ_n for small n's. However they are not literally small. In particular, we will use γ_n with $n \le 3605$ which is computed by the complex Newton method with the initial value $1 + (\log 2 + 2\pi\sqrt{-1})/n$.

Proof. We use g, E_n in the proof of Proposition 2.4. For $m = 0$, we use the numerical estimates $\min\{|g(\zeta)| \mid |\zeta| = 1/2\} \ge 0.44$ and $\max\{|g'(\zeta)| \mid |\zeta| = 1/2\} \le 8$. Assuming $n \ge 100$, it suffices to have

$$\frac{(1 + \log 2)\log 2}{2n^2} + \frac{8 \cdot 0.5^2}{0.44^3 n^3 \cdot (0.44 - 1/100)} < \frac{2}{3n^2}.$$

This is valid for $n \ge 684$. We can check the statement for $6 \le n \le 683$ by numerical computation. For $m = 1$, we use $\min\{|g(\zeta)| \mid |\zeta| = 1/2\} \ge 0.0636$ and $\max\{|g'(\zeta)| \mid |\zeta| = 1/2\} \le 0.32$. Then the similar inequality

$$\frac{|(1 + \log 2)\log 2 + 2(1 + \log 4)\sqrt{-1}\pi - 4\pi^2|}{2n^2} + \frac{0.32 \cdot 0.5^2}{0.0636^3 n^3 \cdot (0.0636 - 1/1400)} < \frac{24}{n^2}.$$

holds for $n \ge 1441$. The remaining cases $8 \le n \le 1440$ are confirmed by direct computation. □

We derive three lemmas 2.6, 2.7 and 2.8 which are used in the proof of Theorem 1.1. Similarly to the proof of Corollary 2.5, their proofs are finished for large n's by (2.2) and (2.3), while the remaining small n's have to be checked by numerical computation.

Since $\beta_n < 2$ for all $n \ge 2$, we have $\mathcal{A} = \{0, 1\}$ and $c_1 = 1$. Let $m_0 \ge 2$ the smallest index that $c_{m_0} = 1$. First we have

Lemma 2.6.

$$m_0 \ge \frac{n \log n}{\log 2}$$

for $n \ge 8$.

Proof. By the definition of $d_\beta(1-0)$, we have $m_0 = \left\lfloor \frac{\log(1-1/\beta_n)}{\log(1/\beta_n)} \right\rfloor$. By (2.2), it suffices to show

$$-\frac{\log\left(\frac{\log 2}{n}+\frac{2}{3n^2}\right)}{\log\left(1+\frac{\log 2}{n}+\frac{2}{3n^2}\right)} > \frac{n\log n}{\log 2}$$

for $n \geq 8$. □

More precise computation gives

$$m_0 = \frac{n\log n - n\log\log 2}{\log 2} - \frac{\log n}{2\log 2} + O(1),$$

but we do not need this precision for the later use.

Lemma 2.7. *For $n \geq 6$ and $m_1 \geq \frac{n\log n}{\log 2}$, we have*

$$|\gamma_n^{m_1}(1-1/\gamma_n)| > 4$$

and

$$\left|\gamma_n^{m_1-2}\right| > \frac{n}{2}.$$

Proof. Let C be the counter-clockwise path around $B(0,1/2)$. The Taylor expansion

$$\log(1+z) = \sum_{i=1}^{m} \frac{(-1)^{i-1}z^m}{i} + \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{\log(1+\zeta)z^{m+1}}{\zeta^{m+1}(\zeta - z)} d\zeta$$

gives an estimate

$$|\log(1+z) - z| \leq \frac{2\log 2}{1/2 - |z|}|z^2|$$

for $|z| < 1/2$. Since $|\gamma_n| > 1$, we have

$$|\gamma_n^{m_1}(1-1/\gamma_n)| \geq \left|\gamma_n^{n\log n/\log 2 - 1}(\gamma_n - 1)\right|.$$

As

$$\log(\gamma_n) = \log\left(1 + \frac{\log 2 + 2\pi\sqrt{-1}}{n} + \frac{A}{n^2}\right)$$

for $|A| \leq 24$, we have

$$\log(\gamma_n) = \frac{\log 2 + 2\pi\sqrt{-1}}{n} + \frac{A}{n^2} + \frac{B}{n^2}$$

with $|B| \leq \frac{2\log 2}{1/2 - 7/2000} \cdot 6.4^2 \leq 115$ for $n \geq 2000$. Here we used an estimate

$$\left|\frac{\log 2 + 2\pi\sqrt{-1}}{n} + \frac{A}{n^2}\right| \leq \frac{6.4}{n}$$

valid for $n \geq 305$. Therefore we have

$$\log(\gamma_n) = \frac{\log 2 + 2\pi\sqrt{-1}}{n} + \frac{C}{n^2} \tag{2.4}$$

with $|C| \le 139$. Consequently

$$\begin{aligned}
&\left(\frac{n\log n}{\log 2}-1\right)\log(\gamma_n)\\
=&\quad \log n+\frac{2\pi\sqrt{-1}\log n}{\log 2}+\frac{C\log n}{n\log 2}-\frac{\log 2+2\pi\sqrt{-1}}{n}-\frac{C}{n^2}\\
=&\quad \log n+\frac{2\pi\sqrt{-1}\log n}{\log 2}+\frac{D\log n}{n}
\end{aligned}$$

with $|D| \le 201$. On the other hand, we have

$$\begin{aligned}
\log(\gamma_n-1) &= \log\left(\frac{\log 2+2\pi\sqrt{-1}}{n}+\frac{A}{n^2}\right)\\
&= \log(\log 2+2\pi\sqrt{-1})-\log n+\log\left(1+\frac{A}{n(\log 2+2\pi\sqrt{-1})}\right)\\
&= \log(\log 2+2\pi\sqrt{-1})-\log n+\frac{A}{n(\log 2+2\pi\sqrt{-1})}+\frac{E}{n^2}
\end{aligned}$$

where

$$|E| \le \frac{2\cdot 3.8^2\log 2}{1/2-3.8/2000} \le 41.$$

Here we used $|A/(\log 2+2\pi\sqrt{-1})| \le 3.8$. Summing up, we have

$$|\gamma_n^{n\log n/\log 2-1}(\gamma_n-1)| = |\log 2+2\pi\sqrt{-1}|\exp\left(\frac{D\log n}{n}+\frac{F}{n}+\frac{E}{n^2}\right)$$

with $|D| < 201$, $|E| < 41$, $|F| \le 3.8$ and $n \ge 2000$. For $n \ge 3606$, the last value exceeds 4 and we obtain the first estimate of Lemma 2.7. For $6 \le n < 3605$, we have to rely on numerical computation. For the second estimate, using (2.4),

$$\begin{aligned}
&\Re\left(\left(\frac{n\log n}{\log 2}-2\right)\log(\gamma_n)\right)\\
&=\Re\left(\left(\frac{n\log n}{\log 2}-2\right)\left(\frac{\log 2+2\pi\sqrt{-1}}{n}+\frac{C}{n^2}\right)\right)\\
&=\log n+\Re(C)\left(\frac{\log n}{n\log 2}-\frac{2}{n^2}\right)-\frac{2\log 2}{n}\\
&=\log n+G\frac{\log n}{n}
\end{aligned}$$

with $|G| \le 201$ and $n \ge 2000$. So we have

$$|\gamma_n^{m_1-2}| \ge n\exp\left(G\frac{\log n}{n}\right) > \frac{n}{2}$$

for $n \ge 2237$. The remaining $6 \le n < 2236$ are confirmed by numerical computation. ☐

Lemma 2.8. *For $n \ge 8$, we have*

$$\frac{1}{|\gamma_n|-1} \le \frac{3n}{2}$$

Proof. Using (2.3), we have

$$|\gamma_n\overline{\gamma_n}| = 1 + \frac{2\log 2}{n} + \frac{2\Re A}{n^2} + \frac{|\log 2 + 2\pi\sqrt{-1}|^2}{n^2} = 1 + \frac{2\log 2}{n} + \frac{H}{n^2}$$

with $|H| \le 90$. We see

$$\left|\sqrt{1+z} - \left(1+\frac{z}{2}\right)\right| \le \frac{\sqrt{6}|z|^2}{1/2-|z|},$$

in a similar manner. Thus we obtain

$$|\gamma_n| - 1 = \frac{\log 2}{n} + \frac{H}{2n^2} + \frac{J}{n^2} \tag{2.5}$$

with $|J| \le \frac{1.5^2\sqrt{6}}{1/2-1.5/2000} \le 12$ for $n \ge 2000$. Here we used an estimate

$$\frac{2\log 2}{n} + \frac{H}{n^2} \le \frac{1.5}{n}$$

for $n \ge 800$. Using (2.5), we see that the statement is true for $n > 2153$. The remaining $8 \le n \le 2152$ are checked by direct computation. □

Proof of the Theorem 1.1.
Since every finite subword of $d_\beta(1-0)$ is admissible, by the Parry condition, $10^t1 \in \mathcal{A}^*$ is not admissible for $t < m_0 - 2$. From the definition of m_0, we have $c_{m_0+i} = 0$ for $1 \le i \le m_0 - 2$. By Lemma 2.1, our goal is to prove

$$\left|(T_\beta^{2m_0-2}(1-0))'\right| > \frac{1}{|\gamma_n| - 1}. \tag{2.6}$$

From Lemma 2.7 and $T_{\beta_n}^{2m_0-2}(1-0) = \beta_n^{2m_0-2}(1-\beta_n^{-1}-\beta_n^{-m_0})$, we have

$$\begin{aligned}\left|(T_\beta^{2m_0-2}(1-0))'\right| &= \left|\gamma_n^{2m_0-2}(1-\gamma_n^{-1}-\gamma_n^{-m_0})\right| \\ &\ge \left|\gamma_n^{2m_0-2}(1-\gamma_n^{-1})\right| - \left|\gamma_n^{m_0-2}\right| \\ &\ge 3\left|\gamma_n^{m_0-2}\right| > 3n/2.\end{aligned}$$

which proves the theorem for $n \ge 8$ with the help of Lemma 2.8. For $n = 6, 7$, we can check (2.6) directly. For $n = 4$, we have

$$d_\beta(1-0) = 10000000100000000000001000000000100000\ldots$$

and

$$\left|(T_\beta^m(1-0))'\right| > \frac{1}{|\gamma_n| - 1}$$

for $m = 35$. For $n = 5$, we get

$$d_\beta(1-0) = 10000000000010000000000000\ldots$$

and one can take $m = 26$. □

3. A generalization

There may be several ways to generalize Theorem 1.1. Here we present a straight forward one.

Theorem 3.1. *Let G be a polynomial with non negative integer coefficients such that $G(1) > 1$, $G(0) \neq 0$ and it is not a power of another polynomial. Let $\alpha_n > 1$ be the real root of $x^n - G(x)$. Then there is a positive integer n_0 that α_n is a non-Parry Perron number for $n \geq n_0$.*

Proof. Put $F(x) = x^n - G(x)$. Since $x > 1$ implies $F'(x) > 0$ for $n \gg 1$, $F(1) < 0$ shows that there is a unique root $\alpha_n > 1$ of F. Fixing $r > 1$, from the non-negativity of the coefficients of G, we see that $G(r)$ is the maximum of $|G(r\zeta)|$ for all ζ with $|\zeta| = 1$. It is unique in the sense that $|G(r\zeta)| = G(r)$ implies $\zeta = 1$. We know that α_n is a Perron number by virtue of Rouché's theorem for a counter-clockwise circular path of radius α_n centered at 0 avoiding outward the real root α_n by a small perturbation. Let $K(F)$ be the factor of F whose leading coefficient is equal to that of F, having properties that every root of $K(F)$ is not a root of unity and $F/K(F)$ is a product of cyclotomic polynomials. Theorem 5 of Schinzel [13] reads that there exists a positive integer n_1 that $K(F)$ is irreducible for $n \gg 1$ and $(n, n_1) = 1$. Reviewing its proof, n_1 must be greater than one only when $x^n - G(y)$ is reducible as a polynomial of $\mathbb{Q}(y)[x]$, which happens when $G(y) = h(y)^k$ with $k \geq 2$ or $G(y) = -4h(y)^4$ for some $h \in \mathbb{Q}(y)$ by the theorem of Capelli (Theorem 9.1 in [9]). Thus under our assumption, we can take $n_1 = 1$. The remainder of the proof proceeds similarly to Theorem 1.1. Applying the Lagrange inversion formula to

$$g(z) = \frac{\log(z+1)}{\log G(z+1) + 2\pi m\sqrt{-1}},$$

we obtain the asymptotic expansion

$$\alpha_n = 1 + \frac{\log G(1)}{n} + O\left(\frac{1}{n^2}\right)$$

and find a conjugate

$$\eta_n = 1 + \frac{\log G(1) + 2\pi m\sqrt{-1}}{n} + O\left(\frac{1}{n^2}\right)$$

for $n \gg 1$. We select $m \in \mathbb{N}$ with $\exp(2\pi m/\sqrt{3}) > G(1)$. Clearly α_n and η_n are the roots of $K(F)$ for $n \gg 1$. We obtain asymptotic expansions:

$$m_0 := \left\lfloor \frac{\log(1 - 1/\alpha_n)}{\log(1/\alpha_n)} \right\rfloor = \frac{n \log n}{\log G(1)} - \frac{n \log\log G(1)}{\log G(1)} + O(\log n),$$

$$|\eta_n^{2m_0-2}(1 - 1/\eta_n)| = \frac{|\log G(1) + 2\pi m\sqrt{-1}|}{(\log G(1))^2} n + O(\log n),$$

$$|\eta_n^{m_0-2}| = \frac{n}{\log G(1)} + O(\log n)$$

and

$$\frac{1}{|\eta_n| - 1} = \frac{n}{\log G(1)} + O(1).$$

Therefore

$$\begin{aligned}|T^{2m_0-2}(1-0)'| &\geq |\eta_n^{2m_0-2}(1-1/\eta_n)| - |\eta_n^{m_0-2}| \\ &= \frac{|\log G(1) + 2\pi m\sqrt{-1}|}{(\log G(1))^2} n - \frac{n}{\log G(1)} + O(\log n) \\ &> \frac{1}{|\eta_n| - 1}\end{aligned}$$

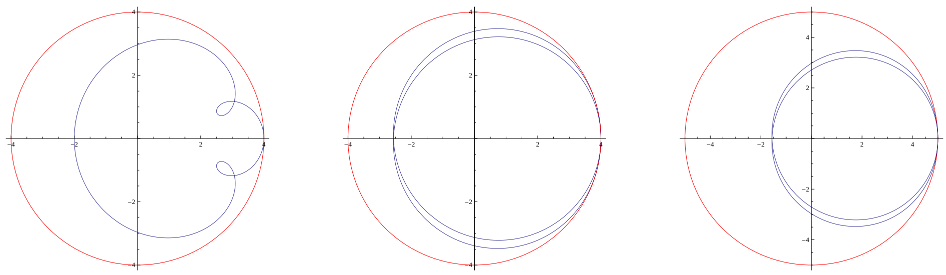
The last inequality holds for $n \gg 1$ by the choice of m. □

We may expect some generalization of Theorem 3.1 for polynomials of the form $x^n f(x) - g(x)$ for fixed f and g, as Lagrange inversion formula likewise applies.

Without any change of the proof, the non-negativity condition of coefficients of G can be relaxed to:

$$\exists r_0 > 1, \ 1 < \forall r < r_0, \ \forall \zeta \neq 1 \text{ with } |\zeta| = 1 \qquad |G(r\zeta)| < G(r).$$

This is a geometric condition on a surface $G(r\exp(t\sqrt{-1}))$ parametrized by r and t, which seems hard to check, but fulfilled by $G(x) = x^3 - x^2 + 2x + 2$, for example. This is confirmed by checking the condition in the limit case $r = 1$ (see Figure 2), and the fact that the surface is non singular at $(r,t) = (1,0)$ and the curvature of the curve $G(\exp(t\sqrt{-1}))$ at $t = 0$ is larger than $1/G(1)$. In general, we can not judge only by the section at $r = 1$. Indeed $x^3 + 3x^2 - x + 1$ fulfills the condition but $-x^3 + 3x^2 + x + 2$ does not. They require a detailed study around $(r,t) = (1,\pi)$.



(a) $x^3 - x^2 + 2x + 2$ (b) $x^3 + 3x^2 - x + 1$ (c) $-x^3 + 3x^2 + x + 2$

FIGURE 2. Curves for $G(\exp(\sqrt{-1}t))$ and a circle of radius $G(1)$

Irreducibility of lacunary polynomials is a classical subject and many related works are found in literature, see for e.g. [10, 11, 15]. To make explicit the constants n_0 in Theorem 3.1, the reader may consult [14, 6].

The set of simple Parry numbers is dense in $[1,\infty)$. We know little about the topology of the set of non-Parry Perron numbers in $\mathbb{R}$, nor on the set of their conjugates in $\mathbb{C}$.

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Appendix.

In this Appendix, we exclusively quote the results in [19]. We claim that all conjugates of β_n of $x^n - x - 1$ with $4 \le n \le 500$ are in the interior of Φ. Negative conjugates for even n have modulus less than one, and there is nothing to prove. For any complex conjugate g with $|g| > 1$ for all $4 \le n \le 500$, we can confirm that

0 lies in the interior of the convex polygon:

$$P = \left\{ 1 + \sum_{i=1}^{6} \frac{y_i}{g^{ki}} \;\middle|\; y_i \in [0,1] \right\} \tag{3.1}$$

in $\mathbb{C}$ where k is the minimum positive integer such that $|\arg g^k| \geq \pi/3$. For fixed g and k, $(y_1, \dots, y_6) \mapsto 1 + \sum_{i=1}^{6} y_i/g^{-ki}$ defines a $\mathbb{R}$-linear function from $\mathbb{R}^6$ to $\mathbb{C}$ and the interior of P is the image of the open cube $(0,1)^6$. In view of Lemma 3.4, since $0 = 1 + \sum_{i=1}^{6} y_i/g^{-ki}$ with $y_i \neq 1$, we have $|1/g| > \rho_\phi$ for $\phi = \arg g$. By using Lemma 4.2, $\phi \mapsto \rho_\phi$ is continuous for $0 < \phi < \pi$, we see that g is in the interior of Φ.

Computation suggests that for every complex number g with $1 < |g| \leq 1.15$ and $|\arg g| \geq \pi/3$, the polygon (3.1) contains 0 as an inner point, and consequently g lies in the interior of Φ. However this may be laborious to prove. We have to study how the shape of the polygon (3.1) varies as $|g|$ and $\arg(g)$ change. Since all conjugates of $x^n - x - 1$ approaches to the unit circle, it is likely that all conjugates are in the interior of Φ for all $n \geq 4$.

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