

On the Unconditional Convergence of Wavelet Expansions for Continuous Functions

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Abstract

In this paper, we study the unconditional convergence of wavelet expansions with Lipschitz wavelets. Especially with the Strömberg wavelet, we shall construct a counter example which shows that uniformly convergent wavelet expansions even for continuous functions do not always converge unconditionally in $L^\infty(\mathbf{R})$.

Keywords Unconditional Convergence, Wavelet Expansion, Strömberg Wavelet.

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1 Introduction

Let Ω be an open set of \mathbf{R} . The space of bounded variations functions is denoted by $BV(\Omega)$ with the norm $\|f\|_{BV} := \|f\|_{L^1} + V(f, \Omega)$, where V is the total variation. The Sobolev space $W^{1,1}(\Omega)$ is a subspace of $BV(\Omega)$ and has the decay property at infinity for the unbounded domain Ω (see e.g., [10]). We remark that $BV(\Omega)$ allows jump-type discontinuities. When Ω is a bounded domain, the similar function spaces $BV(\Omega)$ and $W^{1,1}(\Omega)$ include the space of Lipschitz continuous functions $Lip(\Omega)$, but not the space of Hölder continuous functions $C^\alpha(\Omega)$ with $0 < \alpha < 1$. In case of $\Omega = \mathbf{R}$, after modifying the function on a set of measure zero we find that $Lip(\mathbf{R}) = W_{loc}^{1,\infty}(\mathbf{R})$, and by the Sobolev embedding theorem the following set inclusions hold:

$$Lip(\mathbf{R}) \subset W_{loc}^{1,1}(\mathbf{R}), \quad W^{1,1}(\mathbf{R}) \subset C^0(\mathbf{R}) \cap L^\infty(\mathbf{R}). \quad (1)$$

As for the Fourier expansion $f(t) = \sum_{j \in \mathbf{Z}} c_j e_j(t)$ on Ω , the following results are well-known (see [9] etc.):

(i)_F If $f \in C^\alpha(\Omega)$ for $\alpha > 1/2$, the Fourier series converges uniformly and absolutely, i.e., $\sum_{j \in \mathbf{Z}} |c_j| < \infty$.

(ii)_F If $f \in W^{1,1}(\Omega) \cap C^\alpha(\Omega)$ for $\alpha > 0$, the Fourier series converges uniformly and absolutely, i.e., $\sum_{j \in \mathbf{Z}} |c_j| < \infty$. In fact, $W^{1,1}(\Omega)$ can be relaxed to $BV(\Omega)$.

(iii)_F For the function $f(t) = \sum_{n=1}^{\infty} \frac{\sin nt}{n \log(1+n)} \in W^{1,1}(\Omega)$ with $\Omega = (-\pi, \pi)$ its Fourier series does not converge absolutely.

For a Schauder basis $\{e_j(t)\}$, the ordering of the basis is important in the sense of stable convergence. For a Banach space X , $\sum_{j \in \mathbf{Z}} c_j e_j(t)$ converges to $f(t)$ unconditionally in X if and only if $\sum_{\sigma(j) \in \mathbf{Z}} c_j e_j(t)$ converges to $f(t)$ in X for all permutations σ of \mathbf{Z} . In other words, for the sequence $\{\beta_j\} \subset \{1, -1\}$ the series $\sum_{j \in \mathbf{Z}} \beta_j c_j e_j(t)$ converges in X . Especially for a Hilbert space X with an orthonormal basis $\{e_j(t)\}$, the unconditional convergence holds thanks to the Parseval's identity $\|\sum_{\sigma(j) \in \mathbf{Z}} c_j e_j\|_X^2 = \sum_{\sigma(j) \in \mathbf{Z}} |c_j|^2$. If X is a Banach space, the absolute convergence is stronger than the unconditional one. Noting that

$$\left\| \sum_{j \in \mathbf{Z}} |c_j| e_j \right\|_{L^\infty(\Omega)} = \text{ess. sup}_{t \in \Omega} \sum_{j \in \mathbf{Z}} |c_j e_j(t)| = \sum_{j \in \mathbf{Z}} |c_j| \quad (2)$$

in case of the Banach space $X = L^\infty(\Omega)$ and $|e_j(t)| \equiv 1$, we find that the Fourier series converges to $f(t)$ unconditionally in $L^\infty(\Omega)$ under the assumptions in (i)_F or (ii)_F. Here we pay attention to the fact that the Banach space $X = L^\infty(\Omega)$ for the convergence and the Banach space $\tilde{X} = C^\alpha(\Omega)$ or $W^{1,1}(\Omega)$ for the limit f are different ($\tilde{X} \subset X$) in the Fourier series.

Remark 1.1 *We can take the Hilbert space $X = \tilde{X} = L^2(\Omega)$ for both the convergence and the limit, since the limit f belongs to $L^2(\Omega)$ by the completeness of $L^2(\Omega)$. We need not assume any additional regularity for the limit f unlike the case of $X = L^\infty(\Omega)$.*

Remark 1.2 *If $X = \tilde{X} = L^p(\Omega)$ except the Hilbert space $L^2(\mathbf{R})$, the Fourier series is not an unconditional basis (see [7], [9]).*

Now we shall consider the wavelet expansion $f(t) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}(t)$, where $\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)$. We are concerned with the unconditionality of wavelet expansions except for the particular case of the Hilbert space $L^2(\mathbf{R})$.

There are a lot of various wavelet bases ψ , although the Fourier basis consists of only the analytic function space \mathcal{A} . Therefore, we shall classify wavelets ψ by a belonging function space (or set) Y which restricts the regularity or the decay at infinity. Then the following results about wavelets are known:

(iv)_w If $\psi \in Y = \{y \in C^1(\mathbf{R}); |y(t)| + |y'(t)| \leq g(|t|)\}$ with a decreasing $g \in L^1[0, \infty)$ such that $|g(0)| < \infty$ and $\|tg(\cdot)\|_{L^1[0, \infty)} < \infty$, $\{\psi_{j,k}(t)\}$ is an unconditional basis in $X = \tilde{X} = L^p(\mathbf{R})$ with $1 < p < \infty$ (see [4]).

(v)_w If $\psi \in Y = \{y \in \mathcal{A}(\mathbf{R}); \mathcal{F}[y]$ is characteristic functions of a finite sum of bounded closed intervals (unimodular wavelets) $\}$, $\{\psi_{j,k}(t)\}$ is an unconditional basis in $X = \tilde{X} = L^p(\mathbf{R})$ with $1 < p < \infty$ (see [1], [5]).

Let us choose the Banach space $X = \tilde{X} = W^{1,1}(\mathbf{R})$, and also $\psi \in Y = W^{1,1}(\mathbf{R})$. The choices of $X = \tilde{X} = Y = W^{1,1}(\mathbf{R})$ give that

$$\begin{aligned} \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} \|c_{j,k} \psi_{j,k}\|_{W^{1,1}} &\leq \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} 2^{j/2} |c_{j,k}| \|\psi(2^j \cdot -k)\|_{W^{1,1}} \\ &= \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} 2^{j/2} |c_{j,k}| \int_{\mathbf{R}} \{|\psi(2^j t - k)| + 2^j |\psi'(2^j t - k)|\} dt \\ &\leq \left(\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} 2^{-j/2} |c_{j,k}| \right) \|\psi\|_{L^1} + \left(\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} 2^{j/2} |c_{j,k}| \right) \|\psi'\|_{L^1} \\ &\leq \left(\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} 2^{|j|/2} |c_{j,k}| \right) \|\psi\|_{W^{1,1}}. \end{aligned}$$

Thus, if the scalar series $\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} 2^{|j|/2} |c_{j,k}|$ converges, for $\psi \in W^{1,1}(\mathbf{R})$, the wavelet expansion $\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}$ converges absolutely and also unconditionally in $W^{1,1}(\mathbf{R})$. Thus, we see the following basic observation:

Proposition 1.3 *Assume that $\psi \in W^{1,1}(\mathbf{R})$. Then, the wavelet expansion $\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}(t)$ converges to $f(t)$ unconditionally in $W^{1,1}(\mathbf{R})$ if the coefficients satisfy $\{2^{|j|/2} c_{j,k}\}_{(j,k) \in \mathbf{Z}^2} \in \ell^1$.*

Remark 1.4 *Since the Sobolev embedding theorem gives $W^{1,1}(\mathbf{R}) \subset L^2(\mathbf{R})$, the coefficients $c_{j,k} := (f, \psi_{j,k})_{L^2}$ are well-defined.*

It is not clear whether the condition $\{2^{|j|/2} c_{j,k}\}_{(j,k) \in \mathbf{Z}^2} \in \ell^1$ really restricts the limit $f \in W^{1,1}(\mathbf{R})$. This does not conclude the unconditionality in

$X = \tilde{X} = W^{1,1}(\mathbf{R})$, but indicates the possibility that there exists a counter example (non-unconditionality) if $f \notin W^{1,1}(\mathbf{R})$ and $\{2^{|j|/2}c_{j,k}\}_{(j,k) \in \mathbf{Z}^2} \notin \ell^1$.

From (1), we see that the Sobolev space $W^{1,1}(\mathbf{R})$ can be regarded as a function space between $Lip(\mathbf{R})$ (with a sufficient decay at infinity) and $C^0(\mathbf{R}) \cap L^\infty(\mathbf{R})$. For the non-unconditionality, our strategy is to find a suitable wavelet $\psi \in Y = Lip(\mathbf{R})$ (with an exponential decay at infinity) and a particular function f which belongs to $C^0(\mathbf{R}) \cap L^\infty(\mathbf{R})$ but not $W^{1,1}(\mathbf{R})$, i.e.,

$$f \in \tilde{X} = \{C^0(\mathbf{R}) \cap L^\infty(\mathbf{R})\} \setminus W^{1,1}(\mathbf{R}),$$

because the gap between the regularities of ψ and f possibly makes the unconditionality worse. Indeed, there is a gap between \mathcal{A} and $C^{1/2}(\mathbf{R})$ for the basis and the limit of Fourier series. We shall take $X = L^\infty(\mathbf{R})$ as \tilde{X} which is a little bit weak topology than $W^{1,1}(\mathbf{R})$. Thus, we prove the following result of the wavelet expansion which corresponding to $(iii)_F$ in case of the Fourier expansion:

Theorem 1.5 *There exists $f_0 \in \{C^0(\mathbf{R}) \cap L^\infty(\mathbf{R})\} \setminus W^{1,1}(\mathbf{R})$ satisfying the following:*

- f_0 has the wavelet expansion $f_0(t) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}(t)$ in $L^2(\mathbf{R})$ for some $\psi \in Lip(\mathbf{R})$ and $\{c_{j,k}\}_{(j,k) \in \mathbf{Z}^2} \in \ell^2$ such that $\{2^{|j|/2}c_{j,k}\}_{(j,k) \in \mathbf{Z}^2} \notin \ell^1$.
- $\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}(t)$ converges to $f_0(t)$ uniformly and non-unconditionally in $L^\infty(\mathbf{R})$.

For the proof of Theorem 1.5, it is sufficient to construct a concrete $f_0 \in \{C^0(\mathbf{R}) \cap L^\infty(\mathbf{R})\} \setminus W^{1,1}(\mathbf{R})$. The Strömberg wavelet $\psi^{St} \in Lip(\mathbf{R})$ has an exponential decay at infinity and enables us to construct f_0 simply (see Remark 2.2). Especially for the Franklin wavelet $\psi^{Fr} \in Lip(\mathbf{R})$, the following fact is known:

$(vi)_w$ $\{\psi_{j,k}^{Fr}(t)\}$ is an unconditional basis in $X = \tilde{X} = L^p(\mathbf{R})$ with $1 < p < \infty$ (see Theorem 6.23 in §5 of [4]).

This holds for the spline wavelets of the same order. Therefore, we get the following also for $\psi^{St} \in Lip(\mathbf{R})$:

$(vi)'_w \{\psi_{j,k}^{St}(t)\}$ is an unconditional basis in $X = \tilde{X} = L^p(\mathbf{R})$ with $1 < p < \infty$ (see Theorem 6.14 in §5 of [4]).

These two facts $(vi)_w$ and $(vi)'_w$ are obtained with a small modification the C^1 assumption of $(iv)_w$. We remark that $\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} |c_{j,k}| \psi_{j,k}^{St}(t) \in X = L^p(\mathbf{R})$ with $1 < p < \infty$ and f_0 is not a counter example for $(vi)'_w$. The function space $X = L^\infty(\mathbf{R})$ in Theorem 1.5 is locally stronger than $X = L^p(\mathbf{R})$ with $1 < p < \infty$ in $(vi)'_w$. This causes the non-unconditionality even for the continuous function f_0 .

2 Proof of Theorem 1.5

2.1 Piecewise Linear Spline Wavelets

The B-spline N_2 is defined by

$$N_2(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1, \\ 2 - t & \text{for } 1 \leq t \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Piecewise linear spline wavelets can be given by

$$\psi(t) = \sum_{k \in \mathbf{Z}} b_k N_2(2t - k).$$

Then, we shall give the following function:

$$f_0(t) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}(t) \quad \text{with} \quad c_{j,k} = \begin{cases} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} & \text{for } j \geq 0 \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our purpose is to prove $f_0 \notin W^{1,1}(\mathbf{R})$ and the fact that $\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{j,k} \psi_{j,k}(t)$ converges to $f_0(t)$ uniformly and non-unconditionally in $L^\infty(\mathbf{R})$. Let us put $t_n = 2^{-n}$ ($n \geq 1$). The function f_0 is rewritten as

$$f_0(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi_{j,0}(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \psi(2^j t) = \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k).$$

Furthermore, using the fact that $\text{supp} N_2 \subset [0, 2]$, we shall compute $f_0(t)$ for $t \in [2^{-n}, 2^{-n+1}] = [t_n, t_{n-1}]$ as follows:

- *Case $n = 1$*) For $t \in [2^{-1}, 2^0] = [t_1, t_0]$ we have

$$\begin{aligned}
f_0(t) &= \frac{(-1)^0}{0+1} \{b_0 N_2(2^{0+1}t - 0) + b_1 N_2(2^{0+1}t - 1)\} \\
&\quad + \frac{(-1)^1}{1+1} \{b_1 N_2(2^{1+1}t - 1) + b_2 N_2(2^{1+1}t - 2) + b_3 N_2(2^{1+1}t - 3)\} \\
&\quad + \frac{(-1)^2}{2+1} \{b_3 N_2(2^{2+1}t - 3) + b_4 N_2(2^{2+1}t - 4) + b_5 N_2(2^{2+1}t - 5) \\
&\quad \quad \quad + b_6 N_2(2^{2+1}t - 6) + b_7 N_2(2^{2+1}t - 7)\} \\
&\quad + \dots \\
&= \sum_{j=0}^{\infty} \sum_{k=2^j-1}^{2^{j+1}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k),
\end{aligned}$$

here we used that $2^{0+1}t \in [1, 2]$, $2^{1+1}t \in [2, 4]$, $2^{2+1}t \in [4, 8]$, and $2^{j+1}t \in [2^j, 2^{j+1}]$.

- *Case $n = 2$*) For $t \in [2^{-2}, 2^{-1}] = [t_2, t_1]$ we have

$$\begin{aligned}
f_0(t) &= \frac{(-1)^0}{0+1} \{b_{-1} N_2(2^{0+1}t + 1) + b_0 N_2(2^{0+1}t - 0)\} \\
&\quad + \frac{(-1)^1}{1+1} \{b_0 N_2(2^{1+1}t - 0) + b_1 N_2(2^{1+1}t - 1)\} \\
&\quad + \frac{(-1)^2}{2+1} \{b_1 N_2(2^{2+1}t - 1) + b_2 N_2(2^{2+1}t - 2) + b_3 N_2(2^{2+1}t - 3)\} \\
&\quad + \dots \\
&= \frac{(-1)^0}{0+1} \{b_{-1} N_2(2^{0+1}t + 1) + b_0 N_2(2^{0+1}t - 0)\} \\
&\quad + \sum_{j=1}^{\infty} \sum_{k=2^{j-1}-1}^{2^j-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k),
\end{aligned}$$

here we used that $2^{0+1}t \in [2^{-1}, 1]$, $2^{1+1}t \in [1, 2]$, $2^{2+1}t \in [2, 4]$, and $2^{j+1}t \in [2^{j-1}, 2^j]$.

- *Case $n = 3$*) For $t \in [2^{-3}, 2^{-2}] = [t_3, t_2]$ we have

$$f_0(t) = \frac{(-1)^0}{0+1} \{b_{-1} N_2(2^{0+1}t + 1) + b_0 N_2(2^{0+1}t - 0)\}$$

$$\begin{aligned}
& + \frac{(-1)^1}{1+1} \{b_{-1}N_2(2^{1+1}t+1) + b_0N_2(2^{1+1}t-0)\} \\
& + \frac{(-1)^2}{2+1} \{b_0N_2(2^{2+1}t-0) + b_1N_2(2^{2+1}t-1)\} \\
& + \dots \\
= & \sum_{j=0}^1 \frac{(-1)^j}{j+1} \{b_{-1}N_2(2^{j+1}t+1) + b_0N_2(2^{j+1}t-0)\} \\
& + \sum_{j=2}^{\infty} \sum_{k=2^{j-2}-1}^{2^{j-1}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t-k),
\end{aligned}$$

here we used that $2^{0+1}t \in [2^{-2}, 2^{-1}]$, $2^{1+1}t \in [2^{-1}, 1]$, $2^{2+1}t \in [1, 2]$, and $2^{j+1}t \in [2^{j-2}, 2^{j-1}]$.

Thus, if $n \geq 2$, for $t \in [2^{-n}, 2^{-n+1}] = [t_n, t_{n-1}]$ we recursively have

$$\begin{aligned}
f_0(t) & = \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \{b_{-1}N_2(2^{j+1}t+1) + b_0N_2(2^{j+1}t-0)\} \\
& + \sum_{j=n-1}^{\infty} \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t-k).
\end{aligned}$$

Since $2^{j+1}t \in [2^{j-n+1}, 2^{j-n+2}] \subset [0, 1]$ for $0 \leq j \leq n-2$, we see that

$$N_2(2^{j+1}t+1) = 2 - (2^{j+1}t+1) = 1 - 2^{j+1}t, \quad N_2(2^{j+1}t-0) = 2^{j+1}t$$

and get

$$\begin{aligned}
f_0(t) & = \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \{(1 - 2^{j+1}t)b_{-1} + 2^{j+1}tb_0\} \\
& + \sum_{j=n-1}^{\infty} \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t-k). \tag{3}
\end{aligned}$$

Remark 2.1 In the same way we also get for $t \in [-2^{-n+1}, -2^{-n}] = [-t_{n-1}, -t_n]$

$$\begin{aligned}
f_0(t) & = \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \{(1 - 2^{j+1}t)b_{-1} - 2^{j+1}tb_{-2}\} \\
& + \sum_{j=n-1}^{\infty} \sum_{k=-2^{j-n+2}-1}^{-2^{j-n+1}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t-k). \tag{4}
\end{aligned}$$

2.2 Unbounded Variation

It is sufficient to show that $f_0 \notin BV(\mathbf{R})$ instead of $f_0 \notin W^{1,1}(\mathbf{R})$, since $W^{1,1}(\mathbf{R}) \subset BV(\mathbf{R})$. Especially when $t = t_n$, noting that $N_2(2^{j+1}t_n - k) = \delta_{1,2^{j+1}t_n - k}$, that is, the summation with respect to k runs over only $k = 2^{j+1}t_n - 1 = 2^{j-n+1} - 1$, (3) can be changed into

$$f_0(t_n) = \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ (1 - 2^{j-n+1})b_{-1} + 2^{j-n+1}b_0 \right\} + \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+1}-1}.$$

Hence it follows that for $n \geq 3$

$$\begin{aligned} f_0(t_{n-1}) - f_0(t_n) &= \sum_{j=0}^{n-3} \frac{(-1)^j}{j+1} \left\{ (1 - 2^{j-n+2})b_{-1} + 2^{j-n+2}b_0 \right\} + \sum_{j=n-2}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+2}-1} \\ &\quad - \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ (1 - 2^{j-n+1})b_{-1} + 2^{j-n+1}b_0 \right\} - \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+1}-1} \\ &= \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ (1 - 2^{j-n+2})b_{-1} + 2^{j-n+2}b_0 \right\} - \frac{(-1)^{n-2}}{n-1} b_0 \\ &\quad + \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+2}-1} + \frac{(-1)^{n-2}}{n-1} b_0 \\ &\quad - \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} \left\{ (1 - 2^{j-n+1})b_{-1} + 2^{j-n+1}b_0 \right\} - \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} b_{2^{j-n+1}-1} \\ &= \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} 2^{j-n+1} (b_0 - b_{-1}) + \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} (b_{2^{j-n+2}-1} - b_{2^{j-n+1}-1}). \end{aligned}$$

Our next task is to find a suitable wavelet whose coefficients b_k satisfy $\sum_{n=1}^{\infty} |f_0(t_{n-1}) - f_0(t_n)| = \infty$.

In particular, for convenience, we shall choose the Strömberg wavelet $\psi^{St} \in Lip(\mathbf{R})$ given by

$$\psi^{St}(t) = \sum_{k \in \mathbf{Z}} b_k N_2(2t - k),$$

where

$$b_k = \begin{cases} -4(\sqrt{3} - 2)^k & \text{if } k \geq 1, \\ -\frac{5}{2} + \frac{\sqrt{3}}{2} & \text{if } k = 0, \\ -(2 - \sqrt{3})^{-\frac{k}{2}} \left(\cos \frac{k\pi}{2} + \sqrt{2} \sin \frac{k\pi}{2} \right) & \text{if } k \leq -1, \end{cases}$$

(see [3]). Noting that

$$\sum_{j=0}^{n-2} \frac{(-2)^j}{j+1} = 2^{-1} \int_0^2 \frac{1 - (-y)^{n-1}}{1+y} dy = 2^{-1} \log 3 - 2^{n-1} \int_0^1 \frac{(-z)^{n-1}}{1+2z} dz,$$

we can rewrite

$$\begin{aligned} f_0(t_{n-1}) - f_0(t_n) &= (b_0 - b_{-1}) 2^{1-n} \sum_{j=0}^{n-2} \frac{(-2)^j}{j+1} + \frac{(-1)^{n-1}}{n} (b_1 - b_0) \\ &\quad + \sum_{j=n}^{\infty} \frac{(-1)^j}{j+1} (b_{2^{j-n+2}-1} - b_{2^{j-n+1}-1}) \\ &= (b_{-1} - b_0) \left(\int_0^1 \frac{(-z)^{n-1}}{1+2z} dz - 2^{-n} \log 3 \right) + \frac{(-1)^{n-1}}{n} (b_1 - b_0) \\ &\quad + \sum_{j=n}^{\infty} \frac{(-1)^j}{j+1} (b_{2^{j-n+2}-1} - b_{2^{j-n+1}-1}). \end{aligned}$$

Especially, for $n = 2m + 1$ ($m \geq 1$) we obtain

$$\begin{aligned} t_{2m} - t_{2m+1} &= 2^{-2m-1}, \\ f_0(t_{2m}) - f_0(t_{2m+1}) &= (b_{-1} - b_0) \int_0^1 \frac{z^{2m}}{1+2z} dz + \frac{b_1 - b_0}{2m+1} \\ &\quad - (b_{-1} - b_0) 2^{-2m-1} \log 3 + \sum_{j=2m+1}^{\infty} \frac{(-1)^j}{j+1} \{b_{2^{j-2m+1}-1} - b_{2^{j-2m}-1}\} \\ &=: I + II - III + IV. \end{aligned}$$

Using $b_{-1} - b_0 = \frac{1}{2}(3 + \sqrt{3})$ and $b_1 - b_0 = \frac{3}{2}(7 - 3\sqrt{3})$, we get

$$\begin{aligned} I + II &\geq (b_{-1} - b_0) \int_0^1 \frac{z^{2m}}{1+2z} dz + \frac{b_1 - b_0}{2m+1} > \frac{13(2 - \sqrt{3})}{6(m+1)}, \\ |IV| &\leq \frac{1}{2m+2} \sum_{j=2m+1}^{\infty} |b_{2^{j-2m+1}-1} - b_{2^{j-2m}-1}| \\ &= \frac{2}{(m+1)(2 - \sqrt{3})} \sum_{j=2m+1}^{\infty} \left\{ (2 - \sqrt{3})^{2^{j-2m}} - (2 - \sqrt{3})^{2^{j-2m+1}} \right\} \\ &\leq \frac{2}{(m+1)(2 - \sqrt{3})} (2 - \sqrt{3})^{2^{(2m+1)-2m}} = \frac{2(2 - \sqrt{3})}{m+1}. \end{aligned}$$

Since $|III| < \frac{2-\sqrt{3}}{7(m+1)}$ for $m \geq 2$, there exists $c > 0$ such that for $m \geq 2$

$$\begin{aligned} |f_0(t_{2m}) - f_0(t_{2m+1})| &\geq |I + II + IV| - |III| \geq |I + II - |IV|| - |III| \\ &\geq \frac{c}{m+1}. \end{aligned} \quad (5)$$

Since $\sum_{n=1}^{\infty} |f_0(t_{n-1}) - f_0(t_n)| \geq \sum_{m=1}^{\infty} |f_0(t_{2m}) - f_0(t_{2m+1})| = \infty$, we find that $f_0 \notin BV(\mathbf{R})$. Therefore we can conclude that $f_0 \notin W^{1,1}(\mathbf{R})$.

Remark 2.2 *Thanks to the Strömberg wavelet $\psi^{St} \in Lip(\mathbf{R})$, we can know that $|I+II+IV| \neq 0$. The information of the exact values of b_k is required to find (5). Therefore, it would be difficult to get (5) for general piecewise linear spline wavelets $\psi \in Lip(\mathbf{R})$ or even for the Franklin wavelet $\psi^{Fr} \in Lip(\mathbf{R})$ whose values of b_k are very complicated (see [3]).*

2.3 Continuity and Uniform Convergence

As for the continuity (at $t = 0$), with

$$f_0(0) = \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}0 - k) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} b_{-1},$$

by (3) we get

$$\begin{aligned} |f_0(t) - f_0(0)| &\leq \left| \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} \right| |b_{-1}| + \left| \sum_{j=0}^{n-2} \frac{(-1)^j}{j+1} 2^{j+1} \right| |b_0 - b_{-1}| t \\ &\quad + \left| \sum_{j=n-1}^{\infty} \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t - k) \right| \\ &=: I' + II' + III'. \end{aligned}$$

We can estimate I' , II' and III' as follows:

$$\begin{aligned} I' &= \left| \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j+1} \right| (\sqrt{3} - 1) = \left| \int_0^1 \frac{(-z)^{n-1}}{1+z} dz \right| (\sqrt{3} - 1) \\ &\leq \int_0^1 \frac{z^{n-1}}{1+0} dz (\sqrt{3} - 1) \leq \frac{C}{n}, \end{aligned}$$

$$\begin{aligned}
II' &\leq \left| \sum_{j=0}^{n-2} \frac{(-2)^j}{j+1} \right| \cdot 2|b_0 - b_{-1}|t_{n-1} \\
&= \left| 2^{-1} \log 3 - 2^{n-1} \int_0^1 \frac{(-z)^{n-1}}{1+2z} dz \right| \cdot 2 \left| -\frac{1}{2}(3 + \sqrt{3}) \right| 2^{-n+1} \\
&\leq \left(2^{-1} \log 3 + 2^{n-1} \int_0^1 \frac{z^{n-1}}{1+2 \cdot 0} dz \right) \cdot (3 + \sqrt{3}) 2^{-n+1} \leq \frac{C}{n}, \\
III' &\leq \sum_{j=n-1}^{\infty} \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} \frac{|b_k|}{j+1} \\
&\leq \frac{1}{n} \sum_{j=n-1}^{\infty} \left(\sup_{2^{j-n+1}-1 \leq k \leq 2^{j-n+2}-1} |b_k| \right) \sum_{k=2^{j-n+1}-1}^{2^{j-n+2}-1} 1 \\
&\leq \frac{1}{n} \sum_{j=n-1}^{\infty} 4(2 - \sqrt{3})^{2^{j-n+1}-1} (2^{j-n+1} + 1) \leq \frac{C}{n}.
\end{aligned}$$

Thus we find that $|f_0(t) - f_0(0)| \rightarrow 0$ as $n \rightarrow \infty$ for $t \in [2^{-n}, 2^{-n+1}] = [t_n, t_{n-1}]$. This means the right continuity of f_0 . Similarly, the left continuity of f_0 follows from (4) instead of (3), and we get $f_0 \in C^0(\mathbf{R})$.

Remark 2.3 *We remark that f_0 is not only continuous but also uniformly continuous. More precisely, f_0 satisfies $|f_0(t) - f_0(s)| \leq C/\log|t - s|^{-1}$ for $0 < |t - s| < 1/2$, that is log-Hölder continuous.*

Remark 2.4 *In general, f_0 is continuous if $f_J \rightarrow f_0$ uniformly, but the converse does not hold. It is known that $f_J \rightarrow f_0$ uniformly if $\{f_J(t)\}_J$ is uniformly equicontinuous and $f_J(t) \rightarrow f_0(t)$ pointwise. Let us take the sequence of partial sums*

$$f_J(t) = \sum_{j=0}^J \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi_{j,0}^{St}(t).$$

Similarly, we also find that $|f_J(t) - f_J(0)| \rightarrow 0$ as $n \rightarrow \infty$ for $t \in [t_n, t_{n-1}]$. We remark that the corresponding I'_J , II'_J and III'_J tend to 0 independently of J . This means that $\{f_J(t)\}_J$ is uniformly equicontinuous. We see that $f_J(t_0) \rightarrow f_0(t_0)$ for a fixed $t_0 \neq 0$, since $|b_k|$ is rapidly decreasing at $\pm\infty$ and

$$f_J(t_0) = \sum_{j=0}^J \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi_{j,0}^{St}(t_0) = \sum_{j=0}^J \sum_{k \in \mathbf{Z}} \frac{(-1)^j}{j+1} b_k N_2(2^{j+1}t_0 - k) \sim \sum_{j=0}^J \frac{(-1)^j}{j+1} b_{[2^{j+1}t_0]},$$

where $[\alpha]$ is the largest integer not greater than α . Meanwhile, we immediately see that $f_J(0) \rightarrow f_0(0)$ as an alternating series. Thus, $f_J(t) \rightarrow f_0(t)$ pointwise and we can conclude that $f_J \rightarrow f_0$ uniformly.

2.4 Non-unconditional Convergence

The non-unconditional convergence implies that there exists a sequence $\beta_j \in \{1, -1\}$ such that the series $\sum_{j \in \mathbf{Z}} \beta_j c_j e_j(t)$ does not converge. In order to know the non-unconditional convergence of $f_0(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi_{j,0}^{St}(t)$, with $\beta_j = \bar{c}_j/|c_j|$ especially for $\sum_{j \in \mathbf{Z}} \beta_j c_j e_j(t)$ we shall consider the divergence of

$$\tilde{f}_0(t) := \sum_{j=0}^{\infty} \left| \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \right| \psi_{j,0}^{St}(t).$$

We remark that we can not deal with $\sum_{j=0}^{\infty} \left| \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \right| |\psi_{j,0}^{St}(t)|$ instead of $\tilde{f}_0(t)$. Let us define the interval $I_J = (2^{-J-3}, 2^{-J-3} + 2^{-J-4})$ for $J \geq 1$. Taking $L^\infty(\mathbf{R})$ -norm, we have

$$\begin{aligned} \|\tilde{f}_0\|_{L^\infty(\mathbf{R})} &\geq \|\tilde{f}_0\|_{L^\infty(I_J)} = \left\| \sum_{j=0}^{\infty} \frac{\psi_{j,0}^{St}(t)}{(j+1)2^{\frac{j}{2}}} \right\|_{L^\infty(I_J)} \\ &\geq \left\| \sum_{j=0}^{J-1} \frac{\psi_{j,0}^{St}(t)}{(j+1)2^{\frac{j}{2}}} \right\|_{L^\infty(I_J)} - \left\| \sum_{j=J}^{\infty} \frac{\psi_{j,0}^{St}(t)}{(j+1)2^{\frac{j}{2}}} \right\|_{L^\infty(I_J)} \\ &=: L_J - M_J. \end{aligned}$$

We note that $0 < 2^{j+1}t - k < 2$ if $N_2(2^{j+1}t - k) \neq 0$, i.e., $2^{j+1}t - 2 < k < 2^{j+1}t$. Therefore we may consider $[2^{j+1}t] - 1 \leq k \leq [2^{j+1}t]$. As for the 1st term, we get

$$\begin{aligned} L_J &\geq \operatorname{ess. sup}_{t \in I_J} \sum_{j=0}^{J-1} \frac{\psi_{j,0}^{St}(t)}{(j+1)2^{\frac{j}{2}}} = \operatorname{ess. sup}_{t \in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \sum_{k \in \mathbf{Z}} b_k N_2(2^{j+1}t - k) \\ &= \operatorname{ess. sup}_{t \in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \left\{ b_{-1} N_2(2^{j+1}t + 1) + b_0 N_2(2^{j+1}t - 0) \right\} \\ &= \operatorname{ess. sup}_{t \in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \left\{ b_{-1}(1 - 2^{j+1}t) + b_0 2^{j+1}t \right\} \end{aligned}$$

$$= \operatorname{ess. sup}_{t \in I_J} \sum_{j=0}^{J-1} \frac{1}{j+1} \{b_{-1} - 2^{j+1}t(b_{-1} - b_0)\},$$

here we used that $0 < 2^{j+1}t \leq \frac{3}{8}$ for $0 \leq j \leq J$ and $t \in I_J$. Since $b_{-1} = \sqrt{4 - 2\sqrt{3}} = \sqrt{3} - 1$ and $b_{-1} - b_0 = \frac{\sqrt{3}+3}{2}$, we see that

$$\begin{aligned} L_J &\geq \sum_{j=0}^{J-1} \frac{1}{j+1} \{b_{-1} - 2^{j+1} \cdot (2^{-J-3} + 2^{-J-4}) \cdot (b_{-1} - b_0)\} \\ &\geq \sum_{j=0}^{J-1} \frac{1}{j+1} \{b_{-1} - 2^{(J-1)+1} \cdot (2^{-J-3} + 2^{-J-4}) \cdot (b_{-1} - b_0)\} \\ &\geq \sum_{j=0}^{J-1} \frac{c}{j+1} \quad (c > 0). \end{aligned}$$

As for the 2nd term, noting that $|b_k|$ is decreasing for $k \geq 0$, we get

$$\begin{aligned} M_J &\leq \operatorname{ess. sup}_{t \in I_J} \sum_{j=J}^{\infty} \frac{|\psi_{j,0}^{St}(t)|}{(j+1)2^{\frac{j}{2}}} = \operatorname{ess. sup}_{t \in I_J} \sum_{j=J}^{\infty} \frac{1}{j+1} \left| \sum_{k \in \mathbf{Z}} b_k N_2(2^{j+1}t - k) \right| \\ &= \operatorname{ess. sup}_{t \in I_J} \sum_{j=J}^{\infty} \frac{1}{j+1} \left| b_{[2^{j+1}t]-1} N_2(2^{j+1}t - [2^{j+1}t] + 1) \right. \\ &\quad \left. + b_{[2^{j+1}t]} N_2(2^{j+1}t - [2^{j+1}t]) \right| \\ &\leq \operatorname{ess. sup}_{t \in I_J} \sum_{j=J}^{\infty} \frac{|b_{[2^{j+1}t]-1}| + |b_{[2^{j+1}t]}|}{j+1} \\ &\leq \frac{C}{J} + \operatorname{ess. sup}_{t \in I_J} \sum_{j=J+2}^{\infty} \frac{2|b_{[2^{j+1}t]-1}|}{j+1} \leq \frac{C}{J} + \sum_{j=J+2}^{\infty} \frac{2|b_{2^{j-J-2}-1}|}{j+1}, \end{aligned}$$

here we used that $[2^{j+1}t] - 1 \geq [2^{j-J-2}] - 1 = 2^{j-J-2} - 1 (\geq 0)$ for $j \geq J+2$ and $t \in I_J$. Moreover, we easily see that

$$M_J \leq \frac{C}{J} + c_1 + \sum_{h=2}^{\infty} \frac{8(2 - \sqrt{3})^{2^h-1}}{J+h+2} \leq C' + \sum_{h=2}^{\infty} \frac{8(2 - \sqrt{3})^h}{1+1+2} \leq C'',$$

where C' is independent of J . Thus, it follows that

$$\|\tilde{f}_0\|_{L^\infty(\mathbf{R})} \geq L_J - M_J \geq \sum_{j=0}^{J-1} \frac{c}{j+1} - C''.$$

This holds for all $J \geq 0$, that is, $\|\tilde{f}_0\|_{L^\infty(\mathbf{R})} = \infty$.

Remark 2.5 *If we take only $B^0(\mathbf{R})$ -norm (sup-norm) instead of $L^\infty(\mathbf{R})$ -norm (ess.sup-norm), by substituting $t = 0$ we immediately find that*

$$\|\tilde{f}_0\|_{B^0(\mathbf{R})} \geq \left| \sum_{j=0}^{\infty} \left| \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \right| \psi_{j,0}^{St}(0) \right| = \left| \sum_{j=0}^{\infty} \frac{\sum_{k \in \mathbf{Z}} b_k N_2(-k)}{j+1} \right| = \sum_{j=0}^{\infty} \frac{b_{-1}}{j+1} = \infty.$$

In the above estimate of $\|\tilde{f}_0\|_{L^\infty(\mathbf{R})}$, we used the sequence of the interval $\{I_J\}$ since the essential supremum excludes the measure zero set $\{t = 0\}$. We also remark that $\sum_{j=0}^{\infty} \left| \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \right| \psi_{j,0}^{St}(t)$ does not converge to $\tilde{f}_0(t)$ uniformly (while $\sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)2^{\frac{j}{2}}} \psi_{j,0}^{St}(t)$ converges to $f_0(t)$ uniformly).

Concluding Remarks

- We have considered a continuous wavelet with sufficient decay. In general, such a wavelet is difficult to be exactly represented in the time domain. We paid attention to a little-known fact that the **Strömberg wavelet** has simple exact values at nodes. The sequence of those simple exact values decays rapidly and plays an important role to derive some estimates in the proof. So, we have utilized the **Strömberg wavelet** effectively in this paper.
- In our construction, the unbounded variation of a function causes the non-unconditional convergence of a wavelet expansion. We remark that the proof would become simpler if one considered a discontinuous function with the unbounded variation. But we can not expect the uniform convergence, if discontinuous functions are expanded by continuous wavelets. So, we have succeeded to construct a **continuous function** whose wavelet expansion converges **uniformly and non-unconditionally** in L^∞ .

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