

Geometric Classifier for Multiclass, High-Dimensional Data

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Abstract: In this paper, we consider a geometric classifier which is applicable to multiclass classification for high-dimensional data. We show the consistency property and the asymptotic normality of the geometric classifier under certain mild conditions. We discuss sample size determination so that the geometric classifier can ensure its misclassification rates are less than prespecified thresholds. We give a two-stage procedure to estimate the sample sizes required in such the geometric classifier and propose a *misclassification rate adjusted classifier (MRAC)* based on the geometric classifier. We evaluate the performance of the MRAC theoretically and numerically. Finally, we demonstrate the MRAC in actual data analyses by using a microarray data set.

Keywords: Asymptotic normality; Geometric classifier; HDLSS; Sample size determination; Two-stage procedure.

Subject Classifications: 62H30; 62H10; 62L10.

1. INTRODUCTION

High-dimensional data situations occur in many areas of modern science such as genetic microarrays, medical imaging, text recognition, finance, chemometrics, and so on. A common feature of high-dimensional data is that the data dimension is high, however, the sample size is relatively low. This is the so-called “HDLSS” or “large p , small n ” situation where $p/n \rightarrow \infty$; here p is the data dimension and n is the sample size. Aoshima and Yata (2011a,b) provided a variety of statistical inference for high-dimensional data such as given-bandwidth confidence region, two-sample test, classification, variable selection, regression, pathway analysis and so on. They considered sample size determination to ensure prespecified high accuracy for high-dimensional, non-Gaussian inference and developed the theory of Stein (1945, 1949)’s two-stage procedure which was originally given for inference on the univariate Gaussian mean. Aoshima and Yata (2015) verified the asymptotic normality of statistics appearing in inference on high-dimensional mean vectors under certain mild conditions. In this paper, we focus on high-dimensional classification and make an attempt to give a multiclass classifier to hold misclassification rates less than prespecified thresholds.

Suppose we have independent and p -variate populations, π_i , $i = 1, \dots, k$, having an unknown mean vector $\boldsymbol{\mu}_i$ and unknown covariance matrix $\boldsymbol{\Sigma}_i (> \mathbf{O})$ for each i . We assume that $\limsup_{p \rightarrow \infty} \|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2/p < \infty$ for all $i \neq j$, where $\|\cdot\|$ denotes the Euclidean norm. Also, we assume that $\text{tr}(\boldsymbol{\Sigma}_i)/p \in (0, \infty)$ as $p \rightarrow \infty$ for $i = 1, \dots, k$. Here, for a function, $f(\cdot)$, “ $f(p) \in (0, \infty)$ as $p \rightarrow \infty$ ” implies $\liminf_{p \rightarrow \infty} f(p) > 0$ and $\limsup_{p \rightarrow \infty} f(p) < \infty$. We do not assume that $\boldsymbol{\Sigma}_1 = \dots = \boldsymbol{\Sigma}_k$. The eigen-decomposition of $\boldsymbol{\Sigma}_i$ is given by $\boldsymbol{\Sigma}_i = \mathbf{H}_i \boldsymbol{\Lambda}_i \mathbf{H}_i^T$, where $\boldsymbol{\Lambda}_i$ is a diagonal matrix of eigenvalues, $\lambda_{i1} \geq \dots \geq \lambda_{ip} > 0$, and \mathbf{H}_i is an orthogonal matrix of the corresponding eigenvectors. We have independent and identically distributed (i.i.d.) observations, $\mathbf{x}_{i1}, \dots, \mathbf{x}_{in_i}$, from each π_i . Let $\mathbf{x}_{ij} = \mathbf{H}_i \boldsymbol{\Lambda}_i^{1/2} \mathbf{z}_{ij} + \boldsymbol{\mu}_i$, where \mathbf{z}_{ij} is

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considered as a sphered data vector from a distribution with the zero mean vector and the identity covariance matrix. We assume $n_i \geq 2$, $i = 1, \dots, k$. We estimate $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ by $\bar{\boldsymbol{x}}_{in_i} = \sum_{j=1}^{n_i} \boldsymbol{x}_{ij}/n_i$ and $\boldsymbol{S}_{in_i} = \sum_{j=1}^{n_i} (\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_{in_i})(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_{in_i})^T / (n_i - 1)$.

As for population π_i , $i = 1, \dots, k$, we make the following assumption:

(A-i) Let \boldsymbol{y}_{ij} , $j = 1, \dots, n_i$, be i.i.d. random q_i -vectors having $E(\boldsymbol{y}_{ij}) = \mathbf{0}$ and $\text{Var}(\boldsymbol{y}_{ij}) = \boldsymbol{I}_{q_i}$ for each i ($= 1, \dots, k$), where $q_i \geq p$. Let $\boldsymbol{y}_{ij} = (y_{i1j}, \dots, y_{iq_i j})^T$ in which $\limsup_{p \rightarrow \infty} E(y_{irj}^4) < \infty$ for all r ,

$$E(y_{irj}^2 y_{isj}^2) = E(y_{irj}^2) E(y_{isj}^2) = 1 \quad \text{and} \quad E(y_{irj} y_{isj} y_{itj} y_{iu j}) = 0$$

for all $r \neq s, t, u$. Then, the observations, \boldsymbol{x}_{ijs} , from each π_i ($i = 1, \dots, k$) are given by

$$\boldsymbol{x}_{ij} = \boldsymbol{\Gamma}_i \boldsymbol{y}_{ij} + \boldsymbol{\mu}_i, \quad j = 1, \dots, n_i, \quad (1.1)$$

where $\boldsymbol{\Gamma}_i$ is a $p \times q_i$ matrix such that $\boldsymbol{\Gamma}_i \boldsymbol{\Gamma}_i^T = \boldsymbol{\Sigma}_i$.

Here, \boldsymbol{I}_{q_i} denotes the identity matrix of dimension q_i . Note that (1.1) includes the case that $\boldsymbol{\Gamma}_i = \boldsymbol{H}_i \boldsymbol{\Lambda}_i^{1/2}$ and $\boldsymbol{y}_{ij} = \boldsymbol{z}_{ij}$. Also, note that (A-i) is met when π_i s have $N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ for $i = 1, \dots, k$. In addition, we assume the following assumptions for $\boldsymbol{\Sigma}_i$ s as necessary:

(A-ii) $\frac{\text{tr}(\boldsymbol{\Sigma}_i^4)}{\text{tr}(\boldsymbol{\Sigma}_i^2)^2} \rightarrow 0$ and $\frac{\text{tr}(\boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_l)}{\text{tr}(\boldsymbol{\Sigma}_j^2)} \in (0, \infty)$ as $p \rightarrow \infty$ for $i, j, l = 1, \dots, k$.

Note that “ $\text{tr}(\boldsymbol{\Sigma}_i^4)/\text{tr}(\boldsymbol{\Sigma}_i^2)^2 \rightarrow 0$ as $p \rightarrow \infty$ ” is equivalent to the condition that “ $\lambda_{i1}/\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2} \rightarrow 0$ as $p \rightarrow \infty$ ”. Also, the sphericity condition such as “ $\text{tr}(\boldsymbol{\Sigma}_i^2)/\text{tr}(\boldsymbol{\Sigma}_i)^2 \rightarrow 0$ as $p \rightarrow \infty$ for $i = 1, \dots, k$ ” holds under (A-ii).

Remark 1.1. If all λ_{ij} s are bounded such as $\lambda_{ij} \in (0, \infty)$ as $p \rightarrow \infty$, (A-ii) trivially holds. For a spiked model such as $\lambda_{ij} = a_{ij} p^{\alpha_{ij}}$ ($j = 1, \dots, t_i$) and $\lambda_{ij} = c_{ij}$ ($j = t_i + 1, \dots, p$) with positive constants, a_{ij} s, c_{ij} s and α_{ij} s, and positive integers t_i s, (A-ii) holds under the condition that $\alpha_{ij} < 1/2$ for $j = 1, \dots, t_i$ ($< \infty$); $i = 1, \dots, k$.

Let \boldsymbol{x}_0 be an observation vector of an individual belonging to one of the k populations. When $k = 2$, a typical classification rule is that one classifies the individual into π_1 if

$$(\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_{1n_1})^T \boldsymbol{S}_{1n_1}^{-1} (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_{1n_1}) - \log \left\{ \frac{\det(\boldsymbol{S}_{2n_2})}{\det(\boldsymbol{S}_{1n_1})} \right\} < (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_{2n_2})^T \boldsymbol{S}_{2n_2}^{-1} (\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_{2n_2}),$$

and into π_2 otherwise. However, the inverse matrix of \boldsymbol{S}_{in_i} does not exist in the HDLSS context ($p > n_i$). Dudoit et al. (2002) considered substituting the inverse matrix defined by only diagonal elements of \boldsymbol{S}_{in_i} . Chan and Hall (2009) and Aoshima and Yata (2014a) considered distance-based classifiers. Particularly, Aoshima and Yata (2014a) gave a distance-based classifier for multiclass, non-Gaussian high dimensional data and considered sample size determination to hold misclassification rates less than prespecified thresholds. When $k = 2$, the distance-based classifier is simplified as follows: One classifies the individual into π_1 if

$$\left(\boldsymbol{x}_0 - \frac{\bar{\boldsymbol{x}}_{1n_1} + \bar{\boldsymbol{x}}_{2n_2}}{2} \right)^T (\bar{\boldsymbol{x}}_{2n_2} - \bar{\boldsymbol{x}}_{1n_1}) - \frac{\text{tr}(\boldsymbol{S}_{1n_1})}{2n_1} + \frac{\text{tr}(\boldsymbol{S}_{2n_2})}{2n_2} < 0 \quad (1.2)$$

and into π_2 otherwise. Here, $-\text{tr}(\boldsymbol{S}_{1n_1})/(2n_1) + \text{tr}(\boldsymbol{S}_{2n_2})/(2n_2)$ is a bias-correction term. Aoshima and Yata (2014a) showed that the classifier holds a consistency property in which misclassification rates go to zero as $p \rightarrow \infty$ even when (A-i) is not met. In that sense, the classifier is quite robust and applicable

to actual high-dimensional data. On the other hand, Aoshima and Yata (2011a) considered substituting $\{\text{tr}(\mathbf{S}_{in_i})/p\}\mathbf{I}_p$ for \mathbf{S}_{in_i} in order to use a geometric representation of HDLSS data from each π_i and gave a two-class quadratic classifier called the *geometric classifier* as follows: One classifies the individual into π_1 if

$$\frac{p\|\mathbf{x}_0 - \bar{\mathbf{x}}_{1n_1}\|^2}{\text{tr}(\mathbf{S}_{1n_1})} - \frac{p\|\mathbf{x}_0 - \bar{\mathbf{x}}_{2n_2}\|^2}{\text{tr}(\mathbf{S}_{2n_2})} - p \log \left\{ \frac{\text{tr}(\mathbf{S}_{2n_2})}{\text{tr}(\mathbf{S}_{1n_1})} \right\} - \frac{p}{n_1} + \frac{p}{n_2} < 0 \quad (1.3)$$

and into π_2 otherwise. Here, $-p/n_1 + p/n_2$ is a bias-correction term. Aoshima and Yata (2014a,b) showed that the classifier holds the consistency property even when $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$. Recently, Aoshima and Yata (2014b) provided a general theory of quadratic classifiers for high-dimensional data in non-sparse settings.

In this paper, we develop the geometric classifier by (1.3) to multiclass classification when $k (\geq 2)$. In Section 2, we show the consistency property and the asymptotic normality of the geometric classifier for multiclass high-dimensional data. In Section 3, we discuss sample size determination so that the geometric classifier can ensure its misclassification rates are less than prespecified thresholds. We give a two-stage procedure to estimate the sample sizes required in such the geometric classifier and propose a *misclassification rate adjusted classifier (MRAC)* based on the geometric classifier. In Section 4, we evaluate the performance of the MRAC numerically as well. Finally, in Section 5, we demonstrate the MRAC in actual data analyses by using a microarray data set.

2. ASYMPTOTIC PROPERTIES OF THE GEOMETRIC CLASSIFIER

Let

$$W_i(\mathbf{x}_0|n_i) = \frac{p\|\mathbf{x}_0 - \bar{\mathbf{x}}_{in_i}\|^2}{\text{tr}(\mathbf{S}_{in_i})} - \frac{p}{n_i} + p \log \{\text{tr}(\mathbf{S}_{in_i})\} \quad (2.1)$$

for $i = 1, \dots, k$. We consider the geometric classifier when $k (\geq 2)$ as follows: One classifies the individual into π_i if

$$\max \left\{ \underset{j=1, \dots, k}{\text{argmin}} W_j(\mathbf{x}_0|n_j) \right\} = i. \quad (2.2)$$

When $\underset{j=1, \dots, k}{\text{argmin}} W_j(\mathbf{x}_0|n_j) = \{i_1, \dots, i_l\}$ with integers $l \in [2, k]$ and $i_1 < \dots < i_l$, we have $\max\{\underset{j=1, \dots, k}{\text{argmin}} W_j(\mathbf{x}_0|n_j)\} = i_l$. Note that the difference, $W_1(\mathbf{x}_0|n_1) - W_2(\mathbf{x}_0|n_2)$, is equivalent to (1.3).

2.1. Consistency Property

Let $\Delta_{ij(1)} = \|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2$ and $\Delta_{ij(2)} = \text{tr}(\boldsymbol{\Sigma}_i) - \text{tr}(\boldsymbol{\Sigma}_j) + \text{tr}(\boldsymbol{\Sigma}_j) \log\{\text{tr}(\boldsymbol{\Sigma}_j)/\text{tr}(\boldsymbol{\Sigma}_i)\}$ for all $i \neq j$. Note that $\Delta_{ij(2)} \geq 0$ ($i \neq j$) with equality if and only if $\text{tr}(\boldsymbol{\Sigma}_i) = \text{tr}(\boldsymbol{\Sigma}_j)$. Let

$$\Delta_{ij} = \frac{p}{\text{tr}(\boldsymbol{\Sigma}_j)} (\Delta_{ij(1)} + \Delta_{ij(2)})$$

for all $i \neq j$. We assume the followings as $p \rightarrow \infty$ either when n_i is fixed or $n_i \rightarrow \infty$ for $i = 1, \dots, k$:

$$\text{(A-iii)} \quad \frac{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \boldsymbol{\Sigma}_i (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}{\Delta_{ij}^2} = o(1) \quad \text{and} \quad \frac{\text{tr}(\boldsymbol{\Sigma}_i^2) \text{tr}(\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_j)^2}{\text{tr}(\boldsymbol{\Sigma}_i)^2 \Delta_{ij}^2} = o(1) \quad \text{for all } i \neq j;$$

$$\text{(A-iv)} \quad \frac{\max\{\text{tr}(\boldsymbol{\Sigma}_i^2), \text{tr}(\boldsymbol{\Sigma}_j^2)\}}{\min\{n_i, n_j\} \Delta_{ij}^2} = o(1) \quad \text{for all } i \neq j.$$

We denote the error rate of misclassifying an individual from π_i (into another class) by $e(i)$. Then, we have the following result.

Theorem 2.1. *Under (A-i), (A-iii) and (A-iv), it holds that as $p \rightarrow \infty$*

$$\frac{W_j(\mathbf{x}_0|n_i) - W_i(\mathbf{x}_0|n_j)}{\Delta_{ij}} = 1 + o_P(1) \text{ when } \mathbf{x}_0 \in \pi_i \text{ for all } i \neq j;$$

$$e(i) \rightarrow 0 \text{ for } i = 1, \dots, k.$$

Remark 2.1. When $k = 2$, Aoshima and Yata (2014a) gave partial results of Theorem 2.1 under different conditions.

Remark 2.2. If $\max_{i' \neq i, \dots, k} \{\text{tr}(\boldsymbol{\Sigma}_{i'}^2)\} / \Delta_{ij}^2 \rightarrow 0$ as $p \rightarrow \infty$ for all $i \neq j$, (A-iii) and (A-iv) naturally hold. Then, one can claim Theorem 2.1 even when n_i is fixed for $i = 1, \dots, k$.

2.2. Asymptotic Normality

Let

$$\widetilde{W}(\mathbf{x}_0|n_i, n_j) = \frac{p\|\mathbf{x}_0 - \bar{\mathbf{x}}_{in_i}\|^2}{\text{tr}(\boldsymbol{\Sigma}_i)} - \frac{p\|\mathbf{x}_0 - \bar{\mathbf{x}}_{jn_j}\|^2}{\text{tr}(\boldsymbol{\Sigma}_j)} - p \log \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_j)}{\text{tr}(\boldsymbol{\Sigma}_i)} \right\} - \frac{p \text{tr}(\mathbf{S}_{in_i})}{\text{tr}(\boldsymbol{\Sigma}_i)n_i} + \frac{p \text{tr}(\mathbf{S}_{jn_j})}{\text{tr}(\boldsymbol{\Sigma}_j)n_j}$$

for all $i \neq j$. Note that $W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j)$ is equivalent to $\widetilde{W}(\mathbf{x}_0|n_i, n_j)$ with $\boldsymbol{\Sigma}_i = \mathbf{S}_{in_i}$ and $\boldsymbol{\Sigma}_j = \mathbf{S}_{jn_j}$ for all $i \neq j$. We have that $E\{\widetilde{W}(\mathbf{x}_0|n_i, n_j)\} = -\Delta_{ij}$ when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. Under (A-i), it holds that

$$\begin{aligned} \frac{\text{tr}(\boldsymbol{\Sigma}_j)^2}{4p^2} \text{Var}\{\widetilde{W}(\mathbf{x}_0|n_i, n_j)\} &= \frac{\text{tr}(\boldsymbol{\Sigma}_j)^2}{\text{tr}(\boldsymbol{\Sigma}_i)^2} \left(\frac{\text{tr}(\boldsymbol{\Sigma}_i^2)}{n_i} + \frac{\text{tr}(\boldsymbol{\Sigma}_i^2)}{2n_i(n_i - 1)} \right) + \frac{\text{tr}(\boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)}{n_j} + \frac{\text{tr}(\boldsymbol{\Sigma}_j^2)}{2n_j(n_j - 1)} \\ &\quad + (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T (\boldsymbol{\Sigma}_i + \boldsymbol{\Sigma}_j/n_j) (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) + O\left(\frac{\text{tr}(\boldsymbol{\Sigma}_i^2) \text{tr}(\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_j)^2}{\text{tr}(\boldsymbol{\Sigma}_i)^2} \right) \end{aligned}$$

when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. Let

$$\delta_{ij} = \frac{2p}{\text{tr}(\boldsymbol{\Sigma}_j)} \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_j)^2}{\text{tr}(\boldsymbol{\Sigma}_i)^2} \left(\frac{\text{tr}(\boldsymbol{\Sigma}_i^2)}{n_i} + \frac{\text{tr}(\boldsymbol{\Sigma}_i^2)}{2n_i(n_i - 1)} \right) + \frac{\text{tr}(\boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)}{n_j} + \frac{\text{tr}(\boldsymbol{\Sigma}_j^2)}{2n_j(n_j - 1)} \right\}^{1/2}$$

for all $i \neq j$. We assume extra assumptions as $p \rightarrow \infty$ and $n_i \rightarrow \infty$, $i = 1, \dots, k$:

$$\text{(A-v)} \quad \frac{(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \boldsymbol{\Sigma}_i (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)}{\delta_{ij}^2} = o(1) \quad \text{and} \quad \frac{\text{tr}(\boldsymbol{\Sigma}_i^2) \text{tr}(\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_j)^2}{\text{tr}(\boldsymbol{\Sigma}_i)^2 \delta_{ij}^2} = o(1) \quad \text{for all } i \neq j.$$

Note that under (A-ii) it holds $\delta_{ij} = O\{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}\}$ for all $i \neq j$, so that $\text{tr}(\boldsymbol{\Sigma}_i)/\text{tr}(\boldsymbol{\Sigma}_j) \rightarrow 1$ as $p \rightarrow \infty$ for all $i \neq j$ under (A-ii) and (A-v). Then, we have the following results.

Theorem 2.2. *Assume that $\Delta_{ij(1)}/\text{tr}(\boldsymbol{\Sigma}_j) \rightarrow 0$ as $p \rightarrow \infty$ for all $i \neq j$. Under (A-i), (A-ii) and (A-v), it holds that as $p \rightarrow \infty$ and $n_i \rightarrow \infty$, $i = 1, \dots, k$*

$$\frac{W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j) + \Delta_{ij}}{\delta_{ij}} \Rightarrow Y_{ij} \quad \text{when } \mathbf{x}_0 \in \pi_i \text{ for all } i \neq j,$$

where “ \Rightarrow ” denotes the convergence in distribution and Y_{ij} denotes a random variable distributed as the standard normal distribution.

Remark 2.3. When $k = 2$, Aoshima and Yata (2011a) gave the asymptotic normality under some stronger conditions.

Corollary 2.1. Assume that $\Delta_{ij(1)}/\text{tr}(\Sigma_j) \rightarrow 0$ as $p \rightarrow \infty$ for all $i \neq j$. Under (A-i), (A-ii) and (A-v), the classification rule by (2.2) has that as $p \rightarrow \infty$ and $n_i \rightarrow \infty$, $i = 1, \dots, k$

$$e(i) \leq \sum_{j(\neq i)=1}^k \Phi(-\Delta_{ij}/\delta_{ij}) + o(1) \quad \text{for } i = 1, \dots, k,$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

Remark 2.4. When $k = 2$, the above result is given as

$$e(1) = \Phi(-\Delta_{12}/\delta_{12}) + o(1) \quad \text{and} \quad e(2) = \Phi(-\Delta_{21}/\delta_{21}) + o(1).$$

3. SAMPLE SIZE DETERMINATION TO CONTROL MISCLASSIFICATION RATES

Let $\Delta_{ij*} = \{\text{tr}(\Sigma_j)/p\}\Delta_{ij} = \Delta_{ij(1)} + \Delta_{ij(2)}$ for all $i \neq j$. Let $\Delta_{i*} = \min_{j(\neq i)=1, \dots, k} \min\{\Delta_{ij*}, \Delta_{ji*}\}$ for $i = 1, \dots, k$. We are interested in determining the sample size for (2.2) to ensure the requirement:

$$e(i) \leq \alpha_i \quad \text{whenever } \Delta_{i*} \geq \Delta_{i*L} \quad \text{for } i = 1, \dots, k,$$

where $\alpha_i \in (0, 1/2)$ and $\Delta_{i*L} (> 0)$ $i = 1, \dots, k$, are prespecified constants. We assume $\Delta_{i*L} = o\{\text{tr}(\Sigma_i^2)^{1/2}\}$, $i = 1, \dots, k$.

3.1. Sample Size Determination

Let z_α be the upper α point of the standard normal distribution. We consider n_i s satisfying

$$\delta_{ij} \leq \frac{2\Delta_{(ij)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \quad (3.1)$$

for all $i \neq j$, where $\Delta_{(ij)} = p \max\{\Delta_{i*L}, \Delta_{j*L}\} / \max\{\text{tr}(\Sigma_i), \text{tr}(\Sigma_j)\}$ ($i \neq j$). Note that $\Delta_{(ij)} = \Delta_{(ji)}$ and $\Delta_{(ij)} \leq \min\{\Delta_{ij}, \Delta_{ji}\}$ for all $i \neq j$. Under (3.1), we have that

$$\begin{aligned} \Delta_{ij} + \Delta_{(ij)} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} &\geq \Delta_{(ij)} \left(1 + \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \right) = \frac{2z_{\alpha_i/(k-1)}\Delta_{(ij)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \\ &\geq z_{\alpha_i/(k-1)}\delta_{ij}; \\ \Delta_{ji} - \Delta_{(ij)} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} &\geq \frac{2z_{\alpha_j/(k-1)}\Delta_{(ij)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \geq z_{\alpha_j/(k-1)}\delta_{ji}, \end{aligned}$$

so that from Theorem 2.2 it follows that for $i = 1, \dots, k$

$$\sum_{i(\neq j)=1}^k P\left(W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j) \geq \Delta_{(ij)} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}}\right) \leq \alpha_i + o(1) \quad \text{when } \mathbf{x}_0 \in \pi_i$$

under (3.1) and the assumptions of Theorem 2.2. First, we consider the case when $\liminf_{p \rightarrow \infty} |\text{tr}(\Sigma_i)/\text{tr}(\Sigma_j) - 1| > 0$ for $i \neq j$. In the case, it holds $\liminf_{p \rightarrow \infty} \Delta_{ij}/p > 0$. Under (A-i) and (A-ii), from Theorem 2.1

we have that

$$\begin{aligned} & P\left(W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j) \geq \Delta_{(ij)} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}}\right) \\ &= P\left(-1 + o_P(1) \geq \frac{\Delta_{(ij)} z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{\Delta_{ij} z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}}\right) = o(1) \quad \text{when } \mathbf{x}_0 \in \pi_i \end{aligned}$$

even if n_i s are fixed for $i \neq j$. Next, we consider the case when $\text{tr}(\mathbf{\Sigma}_1) = \dots = \text{tr}(\mathbf{\Sigma}_k)$. Let $\sigma_i = \text{tr}(\mathbf{\Sigma}_i^2)^{1/2}$ for $i = 1, \dots, k$. From the fact that $\text{tr}(\mathbf{\Sigma}_i \mathbf{\Sigma}_j) \leq \{\text{tr}(\mathbf{\Sigma}_i^2) \text{tr}(\mathbf{\Sigma}_j^2)\}^{1/2}$ ($i \neq j$), it holds that for $i \neq j$

$$\delta_{ij} \leq \frac{2p}{\text{tr}(\mathbf{\Sigma}_j)} \text{tr}\left(\frac{\text{tr}(\mathbf{\Sigma}_i^2)}{n_i - 1} + \frac{\text{tr}(\mathbf{\Sigma}_j^2)^{1/2} \max_{l=i,j} \text{tr}(\mathbf{\Sigma}_l^2)^{1/2}}{n_j - 1}\right)^{1/2} \leq \frac{2p}{\text{tr}(\mathbf{\Sigma}_j)} \max_{l=i,j} \sigma_l^{1/2} \left(\frac{\sigma_i}{n_i - 1} + \frac{\sigma_j}{n_j - 1}\right)^{1/2}.$$

Let us write $\sigma_{(i)} = \max_{j(\neq i)=1, \dots, k} \sigma_j$ and $\alpha_{(i)} = \min_{j(\neq i)=1, \dots, k} \alpha_j$ for $i = 1, \dots, k$. From the above arguments, we can find n_i , $i = 1, \dots, k$, to satisfy (3.1) by

$$n_i \geq \frac{(z_{\alpha_i/(k-1)} + z_{\alpha_{(i)}/(k-1)})^2 \max_{l=1, \dots, k} \sigma_l}{\Delta_{i^*L}^2} \sigma_i^{1/2} (\sigma_i^{1/2} + \sigma_{(i)}^{1/2}) + 1 \quad (\text{hereafter called } C_i). \quad (3.2)$$

Note that $n_i \rightarrow \infty$, $i = 1, \dots, k$, as $p \rightarrow \infty$ from the fact that $\Delta_{i^*L} = o\{\text{tr}(\mathbf{\Sigma}_i^2)^{1/2}\}$, $i = 1, \dots, k$. For example, when $k = 2$, $\text{tr}(\mathbf{\Sigma}_1) = \text{tr}(\mathbf{\Sigma}_2)$ and $\Delta_{1^*L} = \Delta_{2^*L}$, the smallest integer (n_1, n_2) satisfying (3.2) holds the following optimality:

$$\min \sum_{i=1}^2 n_i \quad \text{subject to} \quad \frac{2p}{\text{tr}(\mathbf{\Sigma}_j)} \max_{l=1,2} \sigma_l^{1/2} \left(\frac{\sigma_1}{n_1 - 1} + \frac{\sigma_2}{n_2 - 1}\right)^{1/2} \leq \frac{2\Delta_{(ij)}}{z_{\alpha_i} + z_{\alpha_j}} \quad \text{for } i \neq j.$$

According to (3.2), we take samples from each π_i and calculate $W_i(\mathbf{x}_0|n_i)$, $i = 1, \dots, k$, in (2.1). We consider the following classification procedure based on the *misclassification rate adjusted classifier* by Aoshima and Yata (2014a):

Misclassification rate adjusted classifier (MRAC)

Step 1: Set $i = 0$.

Step 2: Put $i = i + 1$. If $i = k$, go to Step 4; otherwise go to Step 3.

Step 3: If it holds that

$$W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j) < p \frac{\max\{\Delta_{i^*L}, \Delta_{j^*L}\}}{\max\{\text{tr}(\mathbf{S}_{in_i}), \text{tr}(\mathbf{S}_{jn_j})\}} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}}$$

for all $j = i + 1, \dots, k$, go to Step 4; otherwise go to Step 2.

Step 4: Classify \mathbf{x}_0 into π_i .

We have the following result.

Theorem 3.1. Under (A-i) to (A-iii), for the MRAC with (3.2), it holds that as $p \rightarrow \infty$

$$\limsup e(i) \leq \alpha_i \quad \text{whenever } \Delta_{i^*} \geq \Delta_{i^*L} \text{ for } i = 1, \dots, k. \quad (3.3)$$

3.2. Designing a Lower Bound, Δ_{i^*L}

First, we consider a lower bound of $\Delta_{ij(1)}$. Let $\widehat{\Delta}_{ij(1)} = \|\bar{\mathbf{x}}_{in_i} - \bar{\mathbf{x}}_{jn_j}\|^2 - \text{tr}(\mathbf{S}_{in_i})/n_i - \text{tr}(\mathbf{S}_{jn_j})/n_j$. By using the two sample test by Aoshima and Yata (2015), under certain regularity conditions, it holds that as $p \rightarrow \infty$ and $n_i \rightarrow \infty$, $i = 1, \dots, k$

$$\frac{\widehat{\Delta}_{ij(1)} - \Delta_{ij(1)}}{\kappa_{ij}} \Rightarrow Y_{ij} \quad \text{for all } i \neq j,$$

where Y_{ij} denotes a random variable distributed as the standard normal distribution and

$$\kappa_{ij} = \left\{ 2 \frac{W_{in_i}}{n_i(n_i - 1)} + 2 \frac{W_{jn_j}}{n_j(n_j - 1)} + 4 \frac{\text{tr}(\mathbf{S}_{in_i} \mathbf{S}_{jn_j})}{n_i n_j} \right\}^{1/2}$$

having W_{in_i} s defined by (9) in Yata and Aoshima (2013). Here, W_{in_i} is an unbiased estimator of $\text{tr}(\boldsymbol{\Sigma}_i^2)$ and $\text{Var}\{W_{in_i}/\text{tr}(\boldsymbol{\Sigma}_i^2)\} \rightarrow 0$ as $p \rightarrow \infty$ and $n_i \rightarrow \infty$ under (A-i). See Aoshima and Yata (2014a) for the details. It follows that $P(\widehat{\Delta}_{ij(1)} - \kappa_{ij} z_{\alpha'} \leq \Delta_{ij(1)}) \rightarrow 1 - \alpha'$ for given $\alpha' \in (0, 1/2)$. Thus, one may design a lower bound of $\Delta_{ij(1)}$ by

$$\Delta_{ij(1)L} = \widehat{\Delta}_{ij(1)} - \kappa_{ij} z_{\alpha'} \quad (3.4)$$

for sufficiently small α' . Next, we consider a lower bound of $\Delta_{ij(2)}$. For $i \neq j$ it holds that

$$\Delta_{ij(2)} \geq \frac{\text{tr}(\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_j)^2}{2 \max\{\text{tr}(\boldsymbol{\Sigma}_i), \text{tr}(\boldsymbol{\Sigma}_j)\}}$$

with equality if and only if $\text{tr}(\boldsymbol{\Sigma}_i) = \text{tr}(\boldsymbol{\Sigma}_j)$. We note that as $p \rightarrow \infty$ and $n_i \rightarrow \infty$, $i = 1, \dots, k$

$$\frac{\text{tr}(\mathbf{S}_{in_i})}{\text{tr}(\boldsymbol{\Sigma}_i)} = 1 + O_P\left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}}{n_i^{1/2} \text{tr}(\boldsymbol{\Sigma}_i)} \right\} = 1 + o_P(1)$$

under (A-i). Thus, one may design a lower bound of $\Delta_{ij(2)}$ by

$$\Delta_{ij(2)L} = \frac{\text{tr}(\mathbf{S}_{in_i} - \mathbf{S}_{jn_j})^2}{2 \max\{\text{tr}(\mathbf{S}_{in_i}), \text{tr}(\mathbf{S}_{jn_j})\}}$$

for $i \neq j$. Let $\Delta_{ij^*L} = \Delta_{ij(1)L} + \Delta_{ij(2)L}$ for all $i \neq j$. Note that $\Delta_{ij^*L} = \Delta_{ji^*L}$ for $i \neq j$. Finally, we choose a lower bound, Δ_{i^*L} , by $\Delta_{i^*L} = \min_{j(\neq i)=1, \dots, k} \Delta_{ij^*L}$ for sufficiently small α' .

3.3. Two-Stage Procedure

In order to estimate C_i s in (3.2), we proceed with the following two steps:

1. Choose $m_i (\geq 4)$ satisfying

$$\frac{m_i}{C_i} \leq 1, \quad \frac{C_i}{m_i^2} \rightarrow 0 \quad \text{and} \quad \frac{C_i}{m_i} \frac{\text{tr}(\boldsymbol{\Sigma}_i^4)}{\text{tr}(\boldsymbol{\Sigma}_i^2)^2} \rightarrow 0 \quad \text{as } p \rightarrow \infty \quad \text{under (A-ii)} \quad (3.5)$$

for $i = 1, \dots, k$. Note that (3.5) holds when $m_i/C_i \in (0, 1)$ as $p \rightarrow \infty$. Take pilot samples, \mathbf{x}_{ij} , $j = 1, \dots, m_i$, of size m_i from each π_i . Then, calculate W_{im_i} for each π_i according to (9) in Yata and Aoshima

(2013). Let $\hat{\sigma}_i = W_{im_i}^{1/2}$ and $\hat{\sigma}_{(i)} = \max_{j(\neq i)=1, \dots, k} \hat{\sigma}_j$ for $i = 1, \dots, k$. Define the total sample size for each π_i by

$$N_i = \max \left\{ m_i, \left\lceil \frac{(z_{\alpha_i/(k-1)} + z_{\alpha_{(i)}/(k-1)})^2 \max_{l=1, \dots, k} \hat{\sigma}_l}{\Delta_{i^*L}^2} \hat{\sigma}_i^{1/2} (\hat{\sigma}_i^{1/2} + \hat{\sigma}_{(i)}^{1/2}) + 1 \right\rceil \right\}, \quad (3.6)$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

2. For each i , if $N_i = m_i$, do not take any additional samples from π_i and otherwise, that is if $N_i > m_i$, take additional samples, \mathbf{x}_{ij} , $j = m_i + 1, \dots, N_i$, of size $N_i - m_i$ from π_i . By combining the initial samples and the additional samples, calculate $\bar{\mathbf{x}}_{iN_i}$ and \mathbf{S}_{iN_i} , $i = 1, \dots, k$. Then, follow MRAC by using $W_i(\mathbf{x}_0|N_i)$ and $\text{tr}(\mathbf{S}_{iN_i})$ instead of $W_i(\mathbf{x}_0|n_i)$ and $\text{tr}(\mathbf{S}_{in_i})$.

Theorem 3.2. Under (A-i) to (A-iii), (3.3) holds for the MRAC with (3.5) and (3.6).

Remark 3.1. When $k = 2$, Aoshima and Yata (2011a) gave a two-stage classification rule based on the geometric classifier. See Theorem 4.3 in Aoshima and Yata (2011a) for the details. We emphasize that the MRAC can claim (3.3) for $k \geq 2$ even under milder conditions than the original one by Aoshima and Yata (2011a).

Remark 3.2. Under (A-i), (A-ii) and (3.5), it holds $N_i/C_i = 1 + o_P(1)$ as $p \rightarrow \infty$, which is in the HDLSS situation in the sense that $N_i/p = o_P(1)$ under the condition that $\max_{j=1, \dots, k} \{\text{tr}(\boldsymbol{\Sigma}_j^2)\}/\Delta_{i^*L}^2 = o(p)$.

Remark 3.3. Even when $m_i/C_i > 1$ for some i , the assertion in Theorem 3.2 is still claimed. However, it may cause oversampling in the sense that $N_i/C_i > 1$ w.p.1.

4. SIMULATION

In order to examine the performance of the MRAC with (3.5) and (3.6), we used computer simulations. First, we considered 2 classes having Gaussian distributions. Independent pseudo random observations were generated from $\pi_i : N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, $i = 1, 2$. We considered $\boldsymbol{\Sigma}_1 = \mathbf{B}\{(-1)^{|i-j|} 0.3^{|i-j|^{1/3}}\} \mathbf{B}$ and $\boldsymbol{\Sigma}_2 = c\{(-1)^{|i-j|} 0.4^{|i-j|^{1/3}}\}$, where $\mathbf{B} = \text{diag}[\{0.5 + 1/(p+1)\}^{1/2}, \dots, \{0.5 + p/(p+1)\}^{1/2}]$. Note that $\text{tr}(\boldsymbol{\Sigma}_1) = p$ and $\text{tr}(\boldsymbol{\Sigma}_2) = cp$. We set $\boldsymbol{\mu}_1 = (1, \dots, 1, 0, \dots, 0)^T$ whose the first 30 elements are 1 and $\boldsymbol{\mu}_2 = (0, \dots, 0)^T$, so that $\Delta_{ij(1)} = \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2 = 30$. We prespecified $\Delta_{1^*L} = \Delta_{2^*L} = \Delta_{12(1)} = 30$. We set $(\alpha_1, \alpha_2) = (0.05, 0.15)$ and $m_i = \lceil 0.5 \times (C_i - 1) \rceil + 1$, $i = 1, 2$, where C_i is defined by (3.2). We considered four cases: (a) $p = 500$ when $c = 1$, (b) $p = 1000$ when $c = 1$, (c) $p = 500$ when $c = 1.2$, and (d) $p = 1000$ when $c = 1.2$. By averaging the outcomes from 2000 ($= R$, say) replications, the findings were summarized in Table 1. Under a fixed scenario, suppose that the r th replication ends with $N_i = n_{ir}$ ($i = 1, 2$) observations for $r = 1, \dots, R$. Let $\bar{n}_i = R^{-1} \sum_{r=1}^R n_{ir}$ and $V(n_i) = (R-1)^{-1} \sum_{r=1}^R (n_{ir} - \bar{n}_i)^2$. In the end of the r th replication, we checked whether the classifier does (or does not) classify \mathbf{x}_0 from π_i correctly and defined $P_{ir} = 0$ (or 1) accordingly for each i . We calculated $\bar{e}(i) = R^{-1} \sum_{r=1}^R P_{ir}$ for each i as an estimate of $e(i)$. Their estimated standard errors were given by $s\{\bar{e}(i)\}$ for each i , where $s^2\{\bar{e}(i)\} = R^{-1} \bar{e}(i) \{1 - \bar{e}(i)\}$. As observed in Table 1, the two-class MRAC with (3.5) and (3.6) gave adequate performances for all the cases when considered those standard errors. Especially, when $\text{tr}(\boldsymbol{\Sigma}_1) \neq \text{tr}(\boldsymbol{\Sigma}_2)$ such as in (c) and (d), the MRAC gave good performances because $\Delta_{i^*} > \Delta_{i^*L}$, $i = 1, 2$.

Next, we considered 3 classes having non-Gaussian distributions generated by $y_{ijl} = (8/10)^{1/2} w_{ijl}$, where w_{ijl} , $j = 1, \dots, p$ ($l = 1, 2, \dots$) are independently distributed as t -distribution with 10 degrees of freedom for each π_i ($i = 1, 2, 3$). Note that $E(y_{ijl}) = 0$, $E(y_{ijl}^2) = 1$, and y_{ijl} , $j = 1, \dots, p$ ($i = 1, 2, 3$; $l = 1, 2, \dots$) are independent. Let $\mathbf{x}_{il} = \mathbf{H}_i \boldsymbol{\Lambda}_i^{1/2} (y_{i1l}, \dots, y_{ipl})^T + \boldsymbol{\mu}_i$ ($i = 1, 2, 3$; $l = 1, 2, \dots$), where $\boldsymbol{\Lambda}_i = \mathbf{H}_i^T \boldsymbol{\Sigma}_i \mathbf{H}_i$. Then, the distribution of \mathbf{x}_{il} satisfies (A-i) for each π_i . We considered $\boldsymbol{\Sigma}_1 = \mathbf{B}\{(-1)^{|i-j|} 0.3^{|i-j|^{1/3}}\} \mathbf{B}$, $\boldsymbol{\Sigma}_2 = \mathbf{B}\{(-1)^{|i-j|} 0.4^{|i-j|^{1/3}}\} \mathbf{B}$ and $\boldsymbol{\Sigma}_3 = 1.2\{(-1)^{|i-j|} 0.4^{|i-j|^{1/3}}\}$. We set

Table 1. Accuracy of the two-class MRAC with (3.5) and (3.6)

	C_i	\bar{n}_i	$\bar{n}_i - C_i$	$V(n_i)$	$\bar{e}(i)$	$s\{\bar{e}(i)\}$
When $\text{tr}(\Sigma_1) = \text{tr}(\Sigma_2)$ ($c = 1$)						
$p = 500: (m_1, m_2) = (10, 11)$						
π_1	18.55	19.15	0.6	16.16	0.047	0.00471
π_2	20.37	21.19	0.81	27.85	0.151	0.00801
$p = 1000: (m_1, m_2) = (19, 21)$						
π_1	36.29	36.87	0.58	14.1	0.038	0.00428
π_2	40.01	40.74	0.73	26.86	0.17	0.00839
When $\text{tr}(\Sigma_1) \neq \text{tr}(\Sigma_2)$ ($c = 1.2$)						
$p = 500: (m_1, m_2) = (13, 15)$						
π_1	23.11	23.28	0.17	13.86	0.027	0.00362
π_2	27.74	27.95	0.21	30.8	0.072	0.00576
$p = 1000: (m_1, m_2) = (24, 28)$						
π_1	45.47	45.91	0.44	16.38	0.013	0.00253
π_2	54.85	55.46	0.61	36.86	0.048	0.00476

$\mu_1 = (1, \dots, 1, 0, \dots, 0)^T$ whose the first 40 elements are 1, $\mu_2 = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0)^T$ whose the 21st to the 60th elements are 1, and $\mu_3 = (0, \dots, 0)^T$. Then, we had $\Delta_{i*} \geq 40$ for $i = 1, 2, 3$. We prespecified $\Delta_{i*L} = 40$, $i = 1, 2, 3$. We set $m_i = \lceil 0.5 \times (C_i - 1) \rceil + 1$ for each π_i . We considered four cases: (a) $p = 500$ when $(\alpha_1, \alpha_2, \alpha_3) = (0.1, 0.1, 0.1)$, (b) $p = 1000$ when $(\alpha_1, \alpha_2, \alpha_3) = (0.1, 0.1, 0.1)$, (c) $p = 500$ when $(\alpha_1, \alpha_2, \alpha_3) = (0.05, 0.1, 0.15)$, and (d) $p = 1000$ when $(\alpha_1, \alpha_2, \alpha_3) = (0.05, 0.1, 0.15)$. By averaging the outcomes from 2000 ($= R$, say) replications, the findings were summarized in Table 2. Throughout, the three-class MRAC with (3.5) and (3.6) gave adequate performances for all the cases when considered those standard errors.

5. EXAMPLE

We analyzed gene expression data by Armstrong et al. (2002) in which the data set consisted of 12582 ($= p$) genes. We had 3 classes of leukemia subtypes, that is, π_1 : acute lymphoblastic leukemia (24 samples), π_2 : mixed-lineage leukemia (20 samples), and π_3 : acute myeloid leukemia (28 samples). We used the MRAC and compared the geometric classifier by (3.5) and (3.6) with the distance-based classifier by Aoshima and Yata (2014a). The total sample size of the distance-based classifier is defined by

$$N_{i*} = \max \left\{ m_i, \left\lceil \frac{(z_{\alpha_i/(k-1)} + z_{\alpha_{(i)/(k-1)}})^2 \max_{l=1, \dots, k} \hat{\sigma}_l}{\Delta_{i(1)L}^2} \hat{\sigma}_i^{1/2} (\hat{\sigma}_i^{1/2} + \hat{\sigma}_{(i)}^{1/2}) + 1 \right\rceil \right\}$$

for each π_i , where $\Delta_{i(1)} = \min_{j(\neq i)=1, \dots, k} \Delta_{ij(1)}$ for $i = 1, \dots, k$, and $\Delta_{i(1)L}$ is a lower bound of $\Delta_{i(1)}$ such as $\Delta_{i(1)} \geq \Delta_{i(1)L}$. Since $\Delta_{i*} \geq \Delta_{i(1)}$, N_{i*} s are larger than N_i s in (3.6) w.p.1 when $\Delta_{i*} > \Delta_{i(1)L}$.

We prespecified $(\alpha_1, \alpha_2, \alpha_3) = (0.05, 0.15, 0.1)$, so that $\alpha_{(1)} = 0.1$, $\alpha_{(2)} = 0.05$ and $\alpha_{(3)} = 0.05$. We set $m_1 = m_2 = m_3 = 10$. According to Section 3.2, by setting $\alpha' = 0.05$ and $n_i = m_i (= 10)$, $i = 1, 2, 3$, we had $\Delta_{12*L} = 6.11 \times 10^9$, $\Delta_{13*L} = 2.45 \times 10^{10}$ and $\Delta_{23*L} = 8.09 \times 10^9$. Thus, we prespecified $\Delta_{1L*} = \min(\Delta_{12L*}, \Delta_{13L*}) = 6.11 \times 10^9$, $\Delta_{2L*} = \min(\Delta_{12L*}, \Delta_{23L*}) = 6.11 \times 10^9$ and $\Delta_{3L*} = \min(\Delta_{13L*}, \Delta_{23L*}) = 8.09 \times 10^9$. Also, we had $\Delta_{12(1)L} = 5.96 \times 10^9$, $\Delta_{13(1)L} = 2.37 \times 10^{10}$ and

Table 2. Accuracy of the three-class MRAC with (3.5) and (3.6)

	C_i	\bar{n}_i	$\bar{n}_i - C_i$	$V(n_i)$	$\bar{e}(i)$	$s\{\bar{e}(i)\}$
When $(\alpha_1, \alpha_2, \alpha_3) = (0.1, 0.1, 0.1)$						
$p = 500: (m_1, m_2, m_3) = (11, 13, 13)$						
π_1	19.72	20.63	0.91	14.96	0.05	0.00485
π_2	23.28	24.15	0.88	28.75	0.075	0.00587
π_3	24.93	26.03	1.09	38.36	0.106	0.00688
$p = 1000: (m_1, m_2, m_3) = (20, 24, 26)$						
π_1	38.66	39.24	0.59	15.42	0.038	0.00428
π_2	45.9	46.56	0.66	26.17	0.067	0.00559
π_3	49.23	50.0	0.77	39.49	0.107	0.0069
When $(\alpha_1, \alpha_2, \alpha_3) = (0.05, 0.1, 0.15)$						
$p = 500: (m_1, m_2, m_3) = (13, 15, 14)$						
π_1	23.48	24.47	0.99	20.58	0.02	0.00313
π_2	27.75	28.65	0.9	34.94	0.074	0.00585
π_3	26.56	27.65	1.1	43.55	0.105	0.00685
$p = 1000: (m_1, m_2, m_3) = (24, 28, 27)$						
π_1	46.21	46.86	0.65	19.26	0.016	0.00281
π_2	54.92	55.54	0.62	31.61	0.057	0.00516
π_3	52.5	53.29	0.79	39.49	0.126	0.00742

$\Delta_{23(1)L} = 7.81 \times 10^9$ according to (3.4). Thus, we prespecified $\Delta_{1(1)L} = 5.96 \times 10^9$, $\Delta_{2(1)L} = 5.96 \times 10^9$ and $\Delta_{3(1)L} = 7.81 \times 10^9$.

By using pilot samples of size $m_1 = m_2 = m_3 = 10$, we calculated $W_{1m_1} = 2.59 \times 10^{19}$, $W_{2m_2} = 2.16 \times 10^{19}$ and $W_{3m_3} = 2.51 \times 10^{19}$. From (3.6), the total sample size for π_1 was calculated by

$$N_1 = \max \left\{ 10, \left\lceil \frac{(z_{\alpha_1/2} + z_{\alpha_{(1)}/2})^2 \max_{l=1,2,3} \hat{\sigma}_l}{\Delta_{1*L}^2} \hat{\sigma}_1^{1/2} (\hat{\sigma}_1^{1/2} + \hat{\sigma}_{(1)}^{1/2}) + 1 \right\rceil \right\} = 19.$$

Similarly, we had $N_2 = 16$ and $N_3 = 12$. We considered constructing the geometric classifier, $W_i(\mathbf{x}_0|N_i)$, $i = 1, 2, 3$, by $(N_1, N_2, N_3) = (19, 16, 12)$ samples and checking the accuracy of the MRAC by using remaining $(24 - N_1, 20 - N_2, 28 - N_3) = (5, 4, 16)$ samples. We randomly split the data set from each π_i into training sets of sizes $(N_1, N_2, N_3) = (19, 16, 12)$ and test sets of sizes $(5, 4, 16)$. We constructed $W_i(\mathbf{x}_0|N_i)$, $i = 1, 2, 3$, by the training sets and checked the accuracy of the MRAC by using the test sets. We repeated this procedure 100 times. Then, we had the average of misclassification rates as $\bar{e}(1) = 0.044$, $\bar{e}(2) = 0.09$ and $\bar{e}(3) = 0.064$. Also, for the distance-based classifier by Aoshima and Yata (2014a), we calculated the total sample sizes as $(N_{1*}, N_{2*}, N_{3*}) = (20, 17, 12)$ and had the average of misclassification rates as $\bar{e}(1) = 0.023$, $\bar{e}(2) = 0.08$ and $\bar{e}(3) = 0.066$. Similarly, for various settings of α_i s, we investigated the performances of the geometric classifier and the distance-based classifier in the MRAC. Throughout, we used the same settings as $m_1 = m_2 = m_3 = 10$ and $(\Delta_{1*L}, \Delta_{2*L}, \Delta_{3*L}) = (6.11 \times 10^9, 6.11 \times 10^9, 8.09 \times 10^9)$ or $(\Delta_{1(1)L}, \Delta_{2(1)L}, \Delta_{3(1)L}) = (5.96 \times 10^9, 5.96 \times 10^9, 7.81 \times 10^9)$. We summarized the results in Table 3. Both the classifiers seem to give adequate performances in such a HDLSS situation. The geometric classifier would save more observations compared to the distance-based classifier specially in small sample size settings. On the other hand, the distance-based classifier is very versatile and it holds (3.3) under milder conditions than the geometric classifier. See Sections 3 and 4 in Aoshima and Yata (2014a) for the details.

Table 3. Average misclassification rates of the MRAC by the geometric classifier with (3.5) and (3.6) and by the distance-based classifier by Aoshima and Yata (2014a). We set $m_1 = m_2 = m_3 = 10$ and $(\Delta_{1*L}, \Delta_{2*L}, \Delta_{3*L}) = (6.11 \times 10^9, 6.11 \times 10^9, 8.09 \times 10^9)$ or $(\Delta_{1(1)L}, \Delta_{2(1)L}, \Delta_{3(1)L}) = (5.96 \times 10^9, 5.96 \times 10^9, 7.81 \times 10^9)$. When $\alpha_i \leq 0.05$ at least for two π_i s, the result was not available within the data sets

$(\alpha_1, \alpha_2, \alpha_3)$	Geometric classifier				Distance-based classifier			
	$\bar{e}(1)$	$\bar{e}(2)$	$\bar{e}(3)$	(N_1, N_2, N_3)	$\bar{e}(1)$	$\bar{e}(2)$	$\bar{e}(3)$	(N_{1*}, N_{2*}, N_{3*})
(0.15, 0.15, 0.15)	0.097	0.135	0.071	(13, 12, 10)	0.058	0.113	0.069	(14, 13, 10)
(0.1, 0.15, 0.15)	0.081	0.1	0.071	(15, 14, 10)	0.047	0.095	0.071	(15, 14, 10)
(0.1, 0.1, 0.15)	0.08	0.08	0.088	(16, 16, 10)	0.054	0.055	0.084	(17, 16, 10)
(0.1, 0.1, 0.1)	0.074	0.085	0.084	(16, 16, 10)	0.04	0.088	0.071	(17, 16, 11)
(0.05, 0.15, 0.1)	0.044	0.09	0.064	(19, 16, 12)	0.023	0.08	0.066	(20, 17, 12)
(0.05, 0.1, 0.1)	0.064	0.065	0.071	(19, 18, 12)	0.035	0.08	0.067	(20, 19, 12)
(0.1, 0.05, 0.1)	0.104	0.06	0.086	(19, 18, 12)	0.07	0.05	0.086	(20, 19, 12)
(0.1, 0.1, 0.05)	0.066	0.075	0.06	(19, 18, 12)	0.038	0.07	0.062	(20, 19, 12)

A. APPENDIX

Proof of Theorem 2.1. Under (A-iv), it holds that $\text{Var}\{\|\bar{\mathbf{x}}_{in_i} - \boldsymbol{\mu}_i\|^2 - \text{tr}(\mathbf{S}_{in_i})/n_i\} = O\{\text{tr}(\boldsymbol{\Sigma}_i^2)/n_i^2\} = o(\Delta_{ij}^2)$ and $\text{Var}\{(\bar{\mathbf{x}}_{jn_j} - \boldsymbol{\mu}_j)^T(\mathbf{x}_0 - \boldsymbol{\mu}_i) | \mathbf{x}_0 \in \pi_i\} = O\{\text{tr}(\boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)/n_j\} = O\{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2} \text{tr}(\boldsymbol{\Sigma}_j^2)^{1/2}/n_j\} = o(\Delta_{ij}^2)$ for all i, j . Note that $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T \boldsymbol{\Sigma}_j (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)/n_j \leq \|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2 \lambda_{j1}/n_j \leq \|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2 \text{tr}(\boldsymbol{\Sigma}_j^2)^{1/2}/n_j = o(\Delta_{ij}^2)$ for all $i \neq j$, under (A-iv). Then, it holds that $\text{Var}\{(\mathbf{x}_0 - \boldsymbol{\mu}_i) - (\bar{\mathbf{x}}_{jn_j} - \boldsymbol{\mu}_j)\}^T (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) | \mathbf{x}_0 \in \pi_i\} = (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T (\boldsymbol{\Sigma}_i + \boldsymbol{\Sigma}_j/n_j) (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) = o(\Delta_{ij}^2)$ for all $i \neq j$, under (A-iii) and (A-iv). Thus by using Chebyshev's inequality, under (A-iii) and (A-iv) we obtain that

$$\begin{aligned} \|\mathbf{x}_0 - \boldsymbol{\mu}_i - (\bar{\mathbf{x}}_{in_i} - \boldsymbol{\mu}_i)\|^2 - \text{tr}(\mathbf{S}_{in_i})/n_i &= \|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 + o_P(\Delta_{ij}); \\ \|\mathbf{x}_0 - \boldsymbol{\mu}_i - (\bar{\mathbf{x}}_{jn_j} - \boldsymbol{\mu}_j) + \boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2 - \text{tr}(\mathbf{S}_{jn_j})/n_j &= \|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 + \Delta_{ij(1)} + o_P(\Delta_{ij}) \end{aligned}$$

when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. Under (A-i) and (A-iv) we have that $\text{Var}\{\text{tr}(\mathbf{S}_{in_i})\} = O\{\text{tr}(\boldsymbol{\Sigma}_i^2)/n_i\} = o(\Delta_{ij}^2)$ and $\text{Var}(\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 | \mathbf{x}_0 \in \pi_i) = O\{\text{tr}(\boldsymbol{\Sigma}_i^2)\}$ for all $i \neq j$, so that $\text{tr}(\mathbf{S}_{in_i}) = \text{tr}(\boldsymbol{\Sigma}_i) + o_P(\Delta_{ij})$ and $\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 = \text{tr}(\boldsymbol{\Sigma}_i) + O_P\{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}\}$ when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. Note that $\text{tr}(\boldsymbol{\Sigma}_i)/p \in (0, \infty)$ as $p \rightarrow \infty$ for $i = 1, \dots, k$. Then, under (A-i), (A-iii) and (A-iv), we have that

$$\begin{aligned} \frac{W_j(\mathbf{x}_0 | n_j) - W_i(\mathbf{x}_0 | n_i)}{\Delta_{ij}} &= p \frac{\|\mathbf{x}_0 - \boldsymbol{\mu}_i - (\bar{\mathbf{x}}_{jn_j} - \boldsymbol{\mu}_j) + \boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|^2 - \text{tr}(\mathbf{S}_{jn_j})/n_j}{\text{tr}(\mathbf{S}_{jn_j}) \Delta_{ij}} \\ &\quad - p \frac{\|\mathbf{x}_0 - \boldsymbol{\mu}_i - (\bar{\mathbf{x}}_{in_i} - \boldsymbol{\mu}_i)\|^2 - \text{tr}(\mathbf{S}_{in_i})/n_i}{\text{tr}(\mathbf{S}_{in_i}) \Delta_{ij}} + p \log \left\{ \frac{\text{tr}(\mathbf{S}_{jn_j})}{\text{tr}(\mathbf{S}_{in_i})} \right\} / \Delta_{ij} \\ &= p \frac{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 + \Delta_{ij(1)}}{\text{tr}(\boldsymbol{\Sigma}_j) \Delta_{ij}} - p \frac{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2}{\text{tr}(\boldsymbol{\Sigma}_i) \Delta_{ij}} + p \log \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_j)}{\text{tr}(\boldsymbol{\Sigma}_i)} \right\} / \Delta_{ij} + o_P(1) \\ &= p \frac{\{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 - \text{tr}(\text{tr}(\boldsymbol{\Sigma}_i))\} \{\text{tr}(\boldsymbol{\Sigma}_i) - \text{tr}(\boldsymbol{\Sigma}_j)\}}{\text{tr}(\boldsymbol{\Sigma}_i) \text{tr}(\boldsymbol{\Sigma}_j) \Delta_{ij}} + 1 + o_P(1) \\ &= 1 + o_P(1) \end{aligned} \tag{A.1}$$

when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. Hence, we conclude the results. \square

Proof of Theorem 2.2. We note that $\max\{\text{tr}(\boldsymbol{\Sigma}_i^2)/n_i^2, \text{tr}(\boldsymbol{\Sigma}_j^2)/n_j^2\} = o(\delta_{ij}^2)$ for all $i \neq j$ under (A-ii). Also, note that $(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T (\boldsymbol{\Sigma}_i + \boldsymbol{\Sigma}_j/n_j) (\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) = o(\delta_{ij}^2)$ for all $i \neq j$ under (A-ii) and (A-v) since $\delta_{ji}/(n_j \delta_{ij}) =$

$o(1)$ for all $i \neq j$ under (A-ii). Let $\omega(\mathbf{x}_0|n_i, n_j) = 2\{p/\text{tr}(\boldsymbol{\Sigma}_j)\}(\mathbf{x}_0 - \boldsymbol{\mu}_i)^T\{(\bar{\mathbf{x}}_{jn_j} - \boldsymbol{\mu}_j) - \text{tr}(\boldsymbol{\Sigma}_j)(\bar{\mathbf{x}}_{in_i} - \boldsymbol{\mu}_i)/\text{tr}(\boldsymbol{\Sigma}_i)\}$ for $i \neq j$. Then, similar to (A.1), under (A-i), (A-ii), (A-v) and $\Delta_{ij(1)}/\text{tr}(\boldsymbol{\Sigma}_j) = o(1)$ for all $i \neq j$, we have that

$$\begin{aligned}
& W_j(\mathbf{x}_0|n_j) - W_i(\mathbf{x}_0|n_i) \\
&= p \frac{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 + \Delta_{ij(1)} - 2(\mathbf{x}_0 - \boldsymbol{\mu}_i)^T(\bar{\mathbf{x}}_{jn_j} - \boldsymbol{\mu}_j)}{\text{tr}(\mathbf{S}_{jn_j})} - p \frac{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 - 2(\mathbf{x}_0 - \boldsymbol{\mu}_i)^T(\bar{\mathbf{x}}_{in_i} - \boldsymbol{\mu}_i)}{\text{tr}(\mathbf{S}_{in_i})} \\
&\quad + p \log \left\{ \frac{\text{tr}(\mathbf{S}_{jn_j})}{\text{tr}(\mathbf{S}_{in_i})} \right\} + o_P(\delta_{ij}) \\
&= p \frac{\{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 - \text{tr}(\boldsymbol{\Sigma}_i)\}\{\text{tr}(\mathbf{S}_{in_i} - \mathbf{S}_{jn_j})\}}{\text{tr}(\mathbf{S}_{in_i})\text{tr}(\mathbf{S}_{jn_j})} + p \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\mathbf{S}_{jn_j})} - \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\mathbf{S}_{in_i})} \right\} - p \log \left\{ \frac{\text{tr}(\mathbf{S}_{in_i})}{\text{tr}(\mathbf{S}_{jn_j})} \right\} \\
&\quad - \omega(\mathbf{x}_0|n_i, n_j) + p \frac{\Delta_{ij(1)}}{\text{tr}(\boldsymbol{\Sigma}_j)} + o_P(\delta_{ij}) \tag{A.2}
\end{aligned}$$

when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$ since $\text{tr}(\mathbf{S}_{in_i})/\text{tr}(\boldsymbol{\Sigma}_i) - 1 = O_P(\delta_{ij}/p) = o_P(1)$. Here, we note that $\text{tr}(\boldsymbol{\Sigma}_i^2)/p = o\{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}\}$, $i = 1, \dots, k$, under (A-ii) from the fact that $\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}/\text{tr}(\boldsymbol{\Sigma}_i) = o(1)$ under (A-ii). It holds that $\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 = \text{tr}(\boldsymbol{\Sigma}_i) + O_P\{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}\}$ when $\mathbf{x}_0 \in \pi_i$ and $\text{tr}(\mathbf{S}_{in_i}) = \text{tr}(\boldsymbol{\Sigma}_i) + O_P[\{\text{tr}(\boldsymbol{\Sigma}_i^2)/n_i\}^{1/2}]$, $i = 1, \dots, k$. Then, under (A-ii) and (A-v), we have that

$$\begin{aligned}
p \frac{\{\|\mathbf{x}_0 - \boldsymbol{\mu}_i\|^2 - \text{tr}(\boldsymbol{\Sigma}_i)\}\{\text{tr}(\mathbf{S}_{in_i} - \mathbf{S}_{jn_j})\}}{\text{tr}(\mathbf{S}_{in_i})\text{tr}(\mathbf{S}_{jn_j})} &= O_P \left\{ \frac{\max_{l=i,j} \text{tr}(\boldsymbol{\Sigma}_l^2)}{\min\{n_i, n_j\}^{1/2}p} + \frac{\text{tr}(\boldsymbol{\Sigma}_i^2)^{1/2}|\text{tr}(\boldsymbol{\Sigma}_i - \boldsymbol{\Sigma}_j)|}{p} \right\} \\
&= o_P(\delta_{ij}) \tag{A.3}
\end{aligned}$$

when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. On the other hand, under (A-i) and (A-ii), it holds that

$$\begin{aligned}
p \log \left\{ \frac{\text{tr}(\mathbf{S}_{in_i})}{\text{tr}(\mathbf{S}_{jn_j})} \right\} - p \log \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\boldsymbol{\Sigma}_j)} \right\} &= p \log \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\mathbf{S}_{jn_j})} \right\} - p \log \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\mathbf{S}_{in_i})} \right\} - p \log \left\{ \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\boldsymbol{\Sigma}_j)} \right\} \\
&= p \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\mathbf{S}_{jn_j})} + p - p \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\mathbf{S}_{in_i})} - p \frac{\text{tr}(\boldsymbol{\Sigma}_i)}{\text{tr}(\boldsymbol{\Sigma}_j)} + o_P(\delta_{ij}) \tag{A.4}
\end{aligned}$$

for all $i \neq j$. Then, by combining (A.2) with (A.3) and (A.4), under the assumptions of Theorem 2.2 we have that

$$W_j(\mathbf{x}_0|n_j) - W_i(\mathbf{x}_0|n_i) = \omega(\mathbf{x}_0|n_i, n_j) + \Delta_{ij} + o_P(\delta_{ij})$$

when $\mathbf{x}_0 \in \pi_i$. Note that $\text{Var}\{\omega(\mathbf{x}_0|n_i, n_j)\}/\delta_{ij}^2 = 1 + o(1)$ for all $i \neq j$ under (A-ii). Then, in a way similar to the proof of Theorem 3 in Aoshima and Yata (2014a), under (A-i) and (A-ii) we can claim that $\omega(\mathbf{x}_0|n_i, n_j)/\delta_{ij} \Rightarrow Y_{ij}$ for all $i \neq j$. Thus it concludes the result. \square

Proof of Corollary 2.1. By using Theorem 2.2 and Bonferroni's inequality, we have that $1 - e(i) \geq 1 - \sum_{j(\neq i)=1}^k \Phi\{-\Delta_{ij}/\delta_{ij}\} + o(1)$ when $\mathbf{x}_0 \in \pi_i$. This concludes the proof. \square

Proof of Theorem 3.1. From (3.2), it holds that $\delta_{ij} \leq 2\Delta_{(ij)}\{1 + o(1)\}/(z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)})$ when $\text{tr}(\boldsymbol{\Sigma}_i)/\text{tr}(\boldsymbol{\Sigma}_j) = 1 + o(1)$ for all $i \neq j$. We denote the error of misclassifying an individual from π_i into π_j by $e(j|i)$ for $i \neq j$. Then, under (3.2) and the assumptions of Theorem 2.2, we have that

$$\begin{aligned}
e(j|i) &= P \left\{ \frac{W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j)}{\delta_{ij}} \geq \frac{p \max\{\Delta_{i^*L}/\text{tr}(\mathbf{S}_{jn_j}), \Delta_{j^*L}/\text{tr}(\mathbf{S}_{in_i})\}}{\delta_{ij}} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \right\} \\
&\leq P \left\{ Y_{ij} \geq \frac{\Delta_{(ij)}}{\delta_{ij}} \frac{2z_{\alpha_i/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \right\} + o(1) \leq P(Y_{ij} \geq z_{\alpha_i/(k-1)}) + o(1) = \frac{\alpha_i}{k-1} + o(1)
\end{aligned}$$

when $\mathbf{x}_0 \in \pi_i$ for $i \neq j$, where Y_{ij} denotes a random variable distributed as the standard normal distribution. We note that (A-v) holds under (A-iii) when $\liminf \delta_{ij}/\Delta_{ij} > 0$ for all $i \neq j$. On the other hand, when $\delta_{ij}/\Delta_{ij} = o(1)$ for $i \neq j$, from Theorem 2.1 it holds that for $\mathbf{x}_0 \in \pi_i$

$$\begin{aligned} e(j|i) &= P\left\{ \frac{W_i(\mathbf{x}_0|n_i) - W_j(\mathbf{x}_0|n_j)}{\Delta_{ij}} \geq \frac{p \max\{\Delta_{i^*L}/\text{tr}(\mathbf{S}_{jn_j}), \Delta_{j^*L}/\text{tr}(\mathbf{S}_{in_i})\}}{\Delta_{ij}} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \right\} \\ &= P\left\{ -1 + o_P(1) \geq \frac{\Delta_{(ij)}\{1 + o_P(1)\}}{\Delta_{ij}} \frac{z_{\alpha_i/(k-1)} - z_{\alpha_j/(k-1)}}{z_{\alpha_i/(k-1)} + z_{\alpha_j/(k-1)}} \right\} = o(1) \end{aligned}$$

under (A-i) to (A-iii) without (A-v). We note that $\delta_{ij}/\Delta_{ij} = o(1)$ for $i \neq j$ under (A-ii) when it holds that $\liminf_{p \rightarrow \infty} \Delta_{ij(1)}/\text{tr}(\mathbf{\Sigma}_j) > 0$ or $\liminf_{p \rightarrow \infty} |\text{tr}(\mathbf{\Sigma}_i)/\text{tr}(\mathbf{\Sigma}_j) - 1| > 0$. Thus one can claim $e(j|i) \leq \alpha_i/(k-1) + o(1)$ for all $i \neq j$ under (3.2) and (A-i) to (A-iii). Then, from Bonferroni's inequality, we have that $1 - e(i) \geq 1 - \sum_{j(\neq i)=1}^k e(j|i) \geq 1 - \alpha_i + o(1)$ when $\mathbf{x}_0 \in \pi_i$. This concludes the proof. \square

Proof of Theorem 3.2. Let $C_{iL} = \lfloor C_i - (\omega C_i)^{1/2} \rfloor$, $i = 1, \dots, k$, where $\omega (> 0)$ is a variable such that $\omega \rightarrow 0$ as $p \rightarrow \infty$. Then, from the proof of Theorem 5 in Aoshima and Yata (2014a), it holds that $\max\{m_i, C_{iL}\} \leq N_i < C_i + (\omega C_i)^{1/2}$ as $p \rightarrow \infty$ w.p.1. Then, in a way similar to the proofs of Theorems 2.4 and 2.5 in Aoshima and Yata (2011a), under (A-i) to (A-iii) we have that for all $i \neq j$

$$\begin{aligned} \text{tr}(\mathbf{S}_{iN_i}) &= \text{tr}(\mathbf{\Sigma}_i) + O_P[\{\text{tr}(\mathbf{\Sigma}_i^2)/C_{iL}\}^{1/2}]; \\ \|\bar{\mathbf{x}}_{iN_i} - \boldsymbol{\mu}_i\|^2 - \text{tr}(\mathbf{S}_{iN_i})/N_i &= o_P(\Delta_{ij}); \\ (\bar{\mathbf{x}}_{jN_j} - \boldsymbol{\mu}_j)^T(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j) &= o_P(\Delta_{ij}); \text{ and} \\ \omega(\mathbf{x}_0|N_i, N_j) &= \omega(\mathbf{x}_0|C_{iL}, C_{jL}) + o_P(\Delta_{ij}) \text{ when } \mathbf{x}_0 \in \pi_i, \end{aligned}$$

where $\omega(\mathbf{x}_0|N_i, N_j)$ is given in the proof of Theorem 2.2. Similar to the proof of Theorem 2.2, under (A-i) to (A-iii) we have that

$$W_j(\mathbf{x}_0|N_i) - W_i(\mathbf{x}_0|N_j) = \omega(\mathbf{x}_0|C_{iL}, C_{jL}) + \Delta_{ij} + o_P(\Delta_{ij})$$

when $\mathbf{x}_0 \in \pi_i$ for all $i \neq j$. Then, in a way similar to the proof of Theorem 3.1, we can conclude the results. \square

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REFERENCES

- Aoshima, M. and Yata, K. (2011a). Two-Stage Procedures for High-Dimensional Data, *Editor's special invited paper, Sequential Analysis* 30: 356–399.
- Aoshima, M. and Yata, K. (2011b). Authors' Response, *Sequential Analysis* 30: 432–440.
- Aoshima, M. and Yata, K. (2014a). A Distance-Based, Misclassification Rate Adjusted Classifier for Multi-class, High-Dimensional Data, *Annals of Institute of Statistical Mathematics* 66: 983–1010.

- Aoshima, M. and Yata, K. (2014b). High-Dimensional Quadratic Classifiers in Non-Sparse Settings, submitted.
- Aoshima, M. and Yata, K. (2015). Asymptotic Normality for Inference on Multisample, High-Dimensional Mean Vectors under Mild Conditions, *Methodology and Computing in Applied Probability* 17: 419–439 .
- Armstrong, S. A., Staunton, J. E., Silverman, L. B., Pieters, R., den Boer, M. L., Minden, M. D., Sallan, S. E., Lander, E. S., Golub, T. R., and Korsmeyer, S. J. (2002). MLL Translocations Specify a Distinct Gene Expression Profile That Distinguishes a Unique Leukemia, *Nature Genetics* 30: 41–47.
- Chan, Y. -B. and Hall, P. (2009). Scale Adjustments for Classifiers in High-Dimensional, Low Sample Size Settings, *Biometrika* 96: 469–478.
- Dudoit, S., Fridlyand, J., and Speed, T. P. (2002). Comparison of Discrimination Methods for the Classification of Tumors Using Gene Expression Data, *Journal of American Statistical Association* 97: 77–87.
- Stein, C. (1945). A Two-Sample Test for a Linear Hypothesis Whose Power is Independent of the Variance, *Annals of Mathematical Statistics* 16: 243–258.
- Stein, C. (1949). Some Problems in Sequential Estimation (abstract), *Econometrica* 17: 77–78.
- Yata, K. and Aoshima, M. (2013). Correlation Tests for High-Dimensional Data Using Extended Cross-Data-Matrix Methodology, *Journal of Multivariate Analysis* 117: 313–331.