# Overlap coincidence to strong coincidence in substitution tiling dynamics

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**Abstract**

Overlap coincidence is an equivalent criterion to pure discrete spectrum of the dynamics of self-affine tilings in $\mathbb{R}^d$. In the case of $d = 1$, strong coincidence on $m$-letter irreducible substitution has been introduced in [10, 8] which implies that the system is metrically conjugate to a domain exchange in $\mathbb{R}^{m-1}$. However being a domain exchange does not imply the property of pure discrete spectrum of the tiling dynamics. The relation between two coincidences has not been established completely. In this paper we generalize strong coincidence to higher dimensions and show the implication from overlap coincidence to the new strong coincidence when the associated height group is trivial. Furthermore we introduce a new criterion 'simultaneous coincidence' and show the implication from overlap coincidence to the simultaneous coincidence. The triviality of height group is shown in [6, 18] for 1-dimension irreducible Pisot substitutions.

*Keywords:* Overlap coincidence, Strong coincidence, Simultaneous coincidence, Pure discrete spectrum, Pisot substitution.

*2008 MSC:* Primary: 52C23

## 1. Introduction

The principal aim of this paper is to give a better understanding of pure discrete spectrum of self-affine tiling dynamical systems which have zero-entropy and whose spectral type varies from weakly-mixing to pure discrete. The study is strongly motivated by atomic configurations of quasicrystals, which show pure point diffraction. Indeed, equivalence of pure point diffraction of quasicrystal structure and pure

discrete spectrum of its associated dynamical system is known in quite a general setting. If we restrict ourselves to 1-dimension substitutive systems, the problem of pure discrete spectrum can be reformulated using notions of 'coincidence' on word combinatorics. Many notions of coincidences have been introduced in the study of the dynamical spectrum of self-affine tilings. A lot of coincidences among these are proved to be equivalent. However relation between strong coincidence of 1-dimension Pisot substitution and other coincidences is not completely understood.

Overlap coincidence is defined in self-affine tiling and characterizes pure discrete spectrum of the tiling dynamics [19, 14, 13]. Basically what it means is that every two tiles in the tiling, which overlap after translating by a return vector, have at least one tile in common after some iterations (see subsection 2.3 for the detail). This coincidence has been proved to be equivalent with algebraic coincidence [13] and super coincidence [17]. For an irreducible Pisot substitution in 1-dimension, all of these coincidences are equivalent to the fact that balanced pair algorithm terminates and each balanced pair leads to a coincidence [11, 20]. This is a combinatorial condition that we can quickly check for a given substitution.

Strong coincidence has been introduced by [10] in constant length substitution sequences, generalized in unimodular Pisot substitutions by [8], and extended in the case of non-unimodular Pisot substitutions by [17]. This combinatorial condition guarantees that there is a geometric realization of substitutions which is metrically conjugate to a domain exchange. In view of the balanced pair algorithm, strong coincidence only implies that every balanced pair leads to a coincidence, i.e., termination of the balanced pair algorithm seems to be necessary to assume. In [11] Hollander and Solomyak proved the termination of the balanced pair algorithm in two letter case, establishing the equivalence between strong coincidence and pure discrete spectrum. From this equivalence with the result of Barge and Diamond [5] which guarantees the strong coincidence in two-letter case, we obtained that two-letter irreducible Pisot substitution dynamical systems have pure discrete spectrum.

However apart from two-letter irreducible Pisot substitution sequences, the relation between the strong coincidence and pure discrete spectrum is not clearly understood. This relation is important for an approach towards 'Pisot substitution conjecture' [2]. Recently Nakaishi [16] claimed that the dynamics of irreducible unimodular Pisot substitution sequence satisfying strong coincidence has pure discrete spectrum through domain exchange flow. Though we do not yet have a full account of this claim, it indicates that strong coincidence in irreducible unimodular Pisot

substitution sequences implies overlap coincidence in irreducible unimodular Pisot substitution tilings in $\mathbb{R}$. This gives us a motivation to look at the other direction. As mentioned above, strong coincidence and overlap coincidence are both combinatorial objects which are defined in 1-dimension substitution sequence. However, it seems that there is no reason to restrict it to 1-dimension substitutions to see the relationship between these coincidences and seems to be better to transfer the problem into geometric setting. So we first generalize strong coincidence in substitution sequences into substitution tiling in $\mathbb{R}^d$, and give a stronger version 'simultaneous coincidence' in §2 and show that overlap coincidence implies simultaneous coincidence in Pisot family substitution tiling in $\mathbb{R}^d$ provided that the associated substitution Delone multi-color set is 'admissible' (Def. 3.2) and the corresponding 'height group' (Def. 3.10) is trivial. Since it is shown in [6, 18] that this group is trivial for irreducible Pisot substitutions, we know that every irreducible Pisot substitution having pure discrete spectrum must admit the simultaneous coincidence.

Here we point out that our higher dimensional generalization of strong coincidence is somewhat too weak condition by itself. It must be fulfilled in conjunction with some constraints to make a reasonable connection to other coincidences. Indeed, without the constraints of admissibility and trivial height group, for a given substitution tiling, it is always possible to build an associated substitution Delone multi-color set satisfying our extended notion of strong coincidence. The first author [1] obtained a converse statement from strong coincidence to overlap coincidence using the extended notion of strong coincidence defined in this paper. But in [1], strong coincidence is required for many choices of admissible *control points* having trivial height group to deduce overlap coincidence. It is of interest to minimize the constraints which are sufficient to derive converse direction from strong coincidence to overlap coincidence.

## 2. Preliminary

### *2.1. Tiles and tilings*

A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of $T$) is a compact set in $\mathbb{R}^d$ which is the closure of its interior, and $i = l(T) \in \{1, \ldots, m\}$ is the color of $T$. We let $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. We say that a set $P$ of tiles is a *patch* if the number of tiles in $P$ is finite and the tiles of $P$ have mutually disjoint interiors. A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \bigcup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors.

**Definition 2.1.** Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_\mathcal{A}$ the set of non-empty patches. We say that $\Omega : \mathcal{A} \to \mathcal{P}_\mathcal{A}$ is a *tile-substitution* (or simply *substitution*) with a $d \times d$ expansive matrix $Q$ if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$ such that

$$\Omega(T_j) = \{u + T_i : \ u \in \mathcal{D}_{ij}, \ i = 1, \dots, m\} \tag{2.1}$$

with

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le m. \tag{2.2}$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

The substitution (2.1) is extended to all translates of prototiles by $\Omega(x + T_j) = Qx + \Omega(T_j)$ and to patches and tilings by $\Omega(P) = \bigcup\{\Omega(T) : \ T \in P\}$. The substitution $\Omega$ can be iterated, producing larger and larger patches $\Omega^k(P)$. We say that $\mathcal{T}$ is a *substitution tiling* if $\mathcal{T}$ is a tiling and $\Omega(\mathcal{T}) = \mathcal{T}$ with some substitution $\Omega$. In this case, we also say that $\mathcal{T}$ is a *fixed point* of $\Omega$. We say that a substitution tiling is *primitive* if the corresponding substitution matrix $S$, with $S_{ij} = \sharp(\mathcal{D}_{ij})$, is primitive, and *irreducible* if the characteristic polynomial of $S$ is irreducible. We say that $\mathcal{T}$ has *finite local complexity* (FLC) if $\forall \ R > 0$, $\exists$ finitely many translational classes of patches whose support lies in some ball of radius $R$. A tiling $\mathcal{T}$ is *repetitive* if for any compact set $K \subset \mathbb{R}^d$, $\{t \in \mathbb{R}^d : \mathcal{T} \cap K = (t + \mathcal{T}) \cap K\}$ is relatively dense. A repetitive fixed point of a primitive tile-substitution with FLC is called a *self-affine tiling*. Let $\lambda > 1$ be the Perron-Frobenius eigenvalue of the substitution matrix $S$. Let $D = \{\lambda_1, \dots, \lambda_d\}$ be the set of (real and complex) eigenvalues of $Q$. We say that $Q$ (or the substitution $\Omega$) fulfills a *Pisot family* if for every $\lambda \in D$ and every Galois conjugate $\lambda'$ of $\lambda$, if $\lambda' \notin D$, then $|\lambda'| < 1$.

Given a tiling $\mathcal{T}$ in $\mathbb{R}^d$, we define the *tiling space* as the orbit closure of $\mathcal{T}$ under the translation action: $X_\mathcal{T} = \overline{\{-g + \mathcal{T} : \ g \in \mathbb{R}^d\}}$, in the well-known "local topology": for a small $\epsilon > 0$ two tilings $\mathcal{S}_1, \mathcal{S}_2$ are $\epsilon$-close if $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on the ball of radius $\epsilon^{-1}$ around the origin, after a translation of size less than $\epsilon$. Then $X_\mathcal{T}$ is compact and we get a topological dynamical system $(X_\mathcal{T}, \mathbb{R}^d)$ where $\mathbb{R}^d$ acts by translations. This system is minimal (i.e. every orbit is dense) whenever $\mathcal{T}$ is repetitive.

### *2.2. Point sets*

Recall that a Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone multi-color set* in $\mathbb{R}^d$ if each $\Lambda_i$ is Delone and

$\mathrm{supp}(\mathbf{\Lambda}) := \cup_{i=1}^{m}\Lambda_i \subset \mathbb{R}^d$ is Delone. A *cluster* of $\mathbf{\Lambda}$ is a family of points $\mathbf{P} = (P_i)_{i\le m}$ where $P_i \subset \Lambda_i$ is finite for all $i \le m$. We say that $\Lambda \subset \mathbb{R}^d$ is a *Meyer set* if it is a Delone set and $\Lambda - \Lambda$ is uniformly discrete in $\mathbb{R}^d$ [12]. The colors of points in the Delone multi-color set have the same meaning as the colors of tiles on tilings. Various notions such as primitivity, FLC and repetitivity in point sets are defined in similar way as in tilings.

**Definition 2.2.** $\mathbf{\Lambda} = (\Lambda_i)_{i\le m}$ is called a *substitution Delone multi-color set* if $\mathbf{\Lambda}$ is a Delone multi-color set and there exist an expansive matrix $Q$ and finite sets $\mathcal{D}_{ij}$ for $i, j \le m$ such that

$$\Lambda_i = \bigcup_{j=1}^{m}(Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m, \tag{2.3}$$

where the unions on the right-hand side are disjoint.

For any given substitution Delone multi-colour set $\mathbf{\Lambda} = (\Lambda_i)_{i\le m}$, we define $\Phi_{ij} = \{f : x \mapsto Qx + a \ : \ a \in \mathcal{D}_{ij}\}$. Then $\Phi_{ij}(\Lambda_j) = Q\Lambda_j + \mathcal{D}_{ij}$, where $i \le m$. We define $\Phi$ an $m \times m$ array for which each entry is $\Phi_{ij}$. For any $k \in \mathbb{Z}_+$ and $x \in \Lambda_j$ with $j \le m$, we let $\Phi^k(x) = \Phi^{k-1}((\Phi_{ij}(x))_{i\le m})$. For any $k \in \mathbb{Z}_+$, $\Phi^k(\mathbf{\Lambda}) = \mathbf{\Lambda}$ and $\Phi^k(\Lambda_j) = \bigcup_{i\le m}(Q^k\Lambda_j + (\mathcal{D}^k)_{ij})$ where

$$(\mathcal{D}^k)_{ij} = \bigcup_{n_1,n_2,\ldots,n_{(k-1)}\le m} (\mathcal{D}_{in_1} + Q\mathcal{D}_{n_1n_2} + \cdots + Q^{k-1}\mathcal{D}_{n_{(k-1)}j}).$$

We say that a cluster $\mathbf{P}$ is *legal* if it is a translate of a subcluster of a cluster generated from one point of $\mathbf{\Lambda}$, that is to say, $a + \mathbf{P} \subset \Phi^k(x)$ (i.e. $a + \mathbf{P} \subset \bigcup_{i=1}^{m}(Q^kx + (\mathcal{D}^k)_{ij})$) for some $k \in \mathbb{Z}_+$, $a \in \mathbb{R}^d$, $x \in \Lambda_j$, and $j \le m$.

**Definition 2.3.** Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-color set in $\mathbb{R}^d$ with an expansive matrix $Q$. We say that $\mathbf{\Lambda}$ admits an *algebraic coincidence* if there exist $M \in \mathbb{Z}_+$ and $\xi \in \Lambda_i$ for some $i \le m$ such that $Q^M\Xi(\mathbf{\Lambda}) \subset \Lambda_i - \xi$, where $\Xi(\mathbf{\Lambda}) = \bigcup_{i\le m}(\Lambda_i - \Lambda_i)$.

Since $\mathbf{\Lambda}$ is primitive, there exists $N \in \mathbb{N}$ such that $Q^N\Xi(\mathbf{\Lambda}) \subset \Lambda_i - \Lambda_i$ for any $i \le m$. So $\mathbf{\Lambda}$ admits an algebraic coincidence if and only if there exists $M \in \mathbb{N}$ and $\xi \in \Lambda_i$ such that $Q^M(\Lambda_i - \Lambda_i) \subset \Lambda_i - \xi$. We say that $\mathbf{\Lambda}$ is a *Pisot family substitution Delone multi-color set* if $Q$ fulfills the condition of Pisot family.

If a self-affine tiling $\mathcal{T}$ is given, we can get an associated substitution Delone multi-color set $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i\le m}$ of $\mathcal{T}$ taking representative points of tiles in the relatively same positions for the same color tiles in the tiling (see [13, Lemma 5.4]). On the other

hand, if $\mathbf{\Lambda}$ is a primitive substitution Delone multi-color set for which every $\mathbf{\Lambda}$-cluster is legal, then $\mathbf{\Lambda} + \mathcal{A} := \{x + T_i : x \in \Lambda_i, i \leq m\}$ is a tiling of $\mathbb{R}^d$, where $\mathcal{A}$ is the set of prototiles from the associated tile equations (see [14]). This bijection establishes a topological conjugacy of $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ and $(X_{\mathcal{T}}, \mathbb{R}^d)$.

Note that if we translate representative points of tiles of $\mathcal{T}$ by $\Lambda'_i = \Lambda_i - g_i$, then the set equation will be

$$\Lambda'_i = \bigcup_{j=1}^{m} Q\Lambda'_j + \mathcal{D}'_{ij}$$

with $\mathcal{D}'_{ij} = \{d_{ij} - Qg_j + g_i \ : \ d_{ij} \in \mathcal{D}_{ij}, 1 \leq i, j \leq m\}$, or in short, $\mathcal{D}'_{ij} = \mathcal{D}_{ij} - Qg_j + g_i$. The corresponding tile equation becomes

$$QA'_j = \bigcup A'_i + \mathcal{D}'_{ij}$$

which is satisfied by $A'_j = A_j + g_j$. So we set $\mathrm{supp}(T'_j) = A'_j$ and the color of $T'_j$ to be the one of $T_j$. Such modification only translates supports of tiles and corresponding point sets, which causes non essential changes of the description of $\mathcal{T}$. Hereafter, we do not distinguish such changes of reference points and use the same symbols $\Lambda_i$ and $T_i$.

Since the representative points of tiles in a tiling $\mathcal{T}$ are taken in the relatively same position for the same type of tiles, for each $i \leq m$ we define a *reference point* $c_i \in \mathbb{R}^d$ for the prototile $T_i$ and can take an associated substitution Delone multi-color set $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \leq m}$ satisfying $\mathcal{T} = \{T_i - c_i + u \mid u \in \Lambda_i, i \leq m\}$.

The topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ has a unique invariant Borel probability measure $\mu$ when the tiling $\mathcal{T}$ is a self-affine tiling (see [19, 14]). The dynamical spectrum of the system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ refers to the spectrum of the unitary operator $U_x$ arising from the translational action on the space of $L^2$-functions on $X_{\mathcal{T}}$. We say that the tiling $\mathcal{T}$ has *pure discrete dynamical spectrum* if the eigenfunctions for the $\mathbb{R}^d$-action span a dense subspace of $L^2(X_{\mathcal{T}}, \mu)$ [19].

*2.3. Overlap coincidence*

Let $\Xi(\mathcal{T}) = \{y \in \mathbb{R}^d \ : \ T = y + S, \text{ where } T, S \in \mathcal{T}\}$. A triple $(u, y, v)$, with $u + T_i, v + T_j \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* if

$$(u + A_i - y)^\circ \cap (v + A_j)^\circ \neq \emptyset,$$

where $A_i = \mathrm{supp}(T_i)$ and $A_j = \mathrm{supp}(T_j)$. An overlap $(u, y, v)$ is a *coincidence* if

$$u - y = v \text{ and } u + T_i, v + T_i \in \mathcal{T} \text{ for some } i \leq m.$$

Let $\mathcal{O} = (u, y, v)$ be an overlap in $\mathcal{T}$, we define *$\ell$-th inflated overlap*

$$\Omega^\ell \mathcal{O} = \{(u', Q^\ell y, v') \, : u' + T_k \in \Omega^\ell(u + T_i), v' + T_r \in \Omega^\ell(v + T_j), \text{ and } (u', Q^\ell y, v') \text{ is an overlap}\}.$$

**Definition 2.4.** We say that a self-affine tiling $\mathcal{T}$ admits *overlap coincidence* if there exists $\ell \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $\Omega^\ell \mathcal{O}$ contains a coincidence.

**Theorem 2.5.** *[14, 13] Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ for which $\Xi(\mathcal{T})$ is a Meyer set. Then $(X_\mathcal{T}, \mathbb{R}^d, \mu)$ has pure discrete dynamical spectrum if and only if $\mathcal{T}$ admits overlap coincidence.*

**Theorem 2.6.** *[13] Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ for which $\Xi(\mathcal{T})$ is a Meyer set and $\mathbf{\Lambda} = \mathbf{\Lambda}_\mathcal{T} = (\Lambda_i)_{i \le m}$ be an associated substitution Delone multi-color set. Then $\mathcal{T}$ admits overlap coincidence if and only if $\mathbf{\Lambda}$ admits algebraic coincidence.*

## 3. Strong coincidence and Simultaneous coincidence in $\mathbb{R}^d$

A substitution $\sigma$ over $m$ letters $\{1, 2, \ldots, m\}$ is called primitive, when its substitution matrix $M_\sigma$ is primitive. We say that $\sigma$ is *irreducible* if the characteristic polynomial of $M_\sigma$ is irreducible, and it is *Pisot*, if the Perron Frobenius root $\beta$ of $M_\sigma$ is a Pisot number. Then we can construct a bi-infinite sequence which is generated by $\sigma$ from some fixed letters around the origin and define the natural *suspension tiling* $\mathcal{T}$ in $\mathbb{R}$ with an expansion factor $\beta$ arose from $\sigma$ by associating to the letters of the bi-infinite sequence the intervals whose lengths are given by a left eigenvector of $M_\sigma$ corresponding to $\beta$. In this case, $\beta$ is identified with the $1 \times 1$ expansive matrix $Q = (\beta)$. Let $\Omega$ be the corresponding tile-substitution and let $\mathcal{A} = \{T_1, \ldots, T_m\}$ be the corresponding prototiles of the intervals whose left end points are all at the origin. Taking the left end points of the intervals of tiles in the tiling, we can get an associated substitution Delone multi-color set. We can interpret "strong coincidence" of $\sigma$ into the tiling setting in the following way:

**Definition 3.1.** Let $\mathcal{T}$ be the suspension tiling in $\mathbb{R}$ of a substitution $\sigma$. We say that $\mathcal{T}$ admits *prefix strong coincidence* if for any pair of prototiles $\{T_i, T_j\} \subset \mathcal{A}$, there exists $L \in \mathbb{N}$ *such that two supertiles $\Omega^L T_i$ and $\Omega^L T_j$ have at least one common tile where $\Omega$ is the tile-substitution.*

We generalize this definition of strong coincidence to $\mathbb{R}^d$ by using the reference points.

**Definition 3.2.** Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$. Let $c_i$ be the reference point of $T_i$ for $i \le m$. We say that the set of the reference points is *admissible* if $\cap_{i \le m}(\mathrm{supp}(T_i) - c_i)$ has non-empty interior.

Clearly, the left end points of the 1-dimensional suspension tiling form a set of admissible reference points.

**Definition 3.3.** Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ with an expansion $Q$. Let $\mathcal{A} = \{T_1, \cdots, T_m\}$ be the prototile set of $\mathcal{T}$. Let $c_i$ be the reference point of $T_i$ for $i \leq m$ whose set is admissible. Let $\mathbf{\Lambda}$ be an associated substitution Delone multi-color set for which $\mathcal{T} = \{u_i + (T_i - c_i) \mid u_i \in \Lambda_i, i \leq m\}$. If for any $1 \leq i, j \leq m$, there is a positive integer $L$ that

$$\Omega^L(T_i - c_i) \cap \Omega^L(T_j - c_j) \neq \emptyset, \tag{3.1}$$

i.e. the left hand side contains at least one tile, then we say that $\mathbf{\Lambda}$ admits *strong coincidence*.

Set

$$\mathcal{G} := \bigcup_{k=0}^{\infty} Q^{-k}(\Lambda_i - \Lambda_i), \quad \text{for some } i \leq m.$$

This set is independent of the choice of $i$. Indeed by the primitivity of $\mathbf{\Lambda}$, for any $i, j \leq m$, there exists $n \in \mathbb{N}$ that

$$Q^n(\Lambda_i - \Lambda_i) \subset \Lambda_j - \Lambda_j \,.$$

In plain words, $\mathcal{G}$ is the set of eventual return vectors, i.e., vectors $v \in \mathbb{R}^d$ such that there is an $n \in \mathbb{N}$ that $Q^n v$ is a return vector of $\mathcal{T}$.

**Remark 3.4.** If $\mathbf{\Lambda}$ admits strong coincidence, then notice that for any $i, j \leq m$, there is a common tile $T_k - c_k + \eta \in \Omega^L(T_i - c_i) \cap \Omega^L(T_j - c_j)$ for some $\eta \in \mathbb{R}^d$, $L \in \mathbb{N}$ and $k \leq m$. Thus $Q^L\Lambda_i + \eta \subset \Lambda_k$ and $Q^L\Lambda_j + \eta \subset \Lambda_k$. So

$$Q^L(\Lambda_i \cup \Lambda_j) \subset \Lambda_k - \eta \,. \tag{3.2}$$

Thus for any $i, j \leq m$,

$$Q^L(\Lambda_i - \Lambda_j) \subset \Lambda_k - \Lambda_k \tag{3.3}$$

and we obtain

$$\Lambda - \Lambda \subset \mathcal{G}, \quad \text{where } \Lambda = \cup_{i \leq m} \Lambda_i \,.$$

**Remark 3.5.** Assume that $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \leq m}$ admits algebraic coincidence. There is $L \in \mathbb{N}$ and $\eta \in \mathbb{R}^d$ that $Q^L(\Lambda_i - \Lambda_i) \subset \Lambda_i - \eta$ for some $i$. This implies

$$Q^L(\Lambda_i - \Lambda_i) - Q^L(\Lambda_i - \Lambda_i) \subset \Lambda_i - \Lambda_i.$$

This is equivalent to the fact that the set of eventual return vectors $\mathcal{G}$ forms an additive group, i.e., $\mathcal{G} = \langle \mathcal{G} \rangle_{\mathbb{Z}}$, where $\langle \mathcal{G} \rangle_{\mathbb{Z}}$ is the additive group generated by the elements of $\mathcal{G}$. Among self-affine tilings, it is interesting to characterize when $\mathcal{G} = \langle \mathcal{G} \rangle_{\mathbb{Z}}$ holds. For e.g., consider the suspension tiling of Thue-Morse substitution $0 \to 01,\ 1 \to 10$

which does not admit overlap coincidence. Taking the left end points of the tiles in the tiling, we have

$$\Lambda_k \cap [0,\infty) = \left\{ \sum_{i=0}^{\ell} b_i 2^i \ : \ b_i \in \{0,1\},\ \sum_{i=0}^{\ell} b_i \equiv k \pmod 2 \text{ and } \ell \in \mathbb{Z}_{\geq 0} \right\}$$

with $k \in \{0,1\}$. One can easily express each element of $\Lambda_1 \cap [0,\infty)$ as a difference of $\Lambda_0 \cap [0,\infty)$. So we can show that $\mathcal{G} = \mathbb{Z}[1/2]$, which forms a group. This example also shows that the converse of Remark 3.4 does not hold: $\Lambda - \Lambda \subset \mathcal{G}$ does not imply strong coincidence. A similar idea using beta-integer with golden mean base works in Example 4.6, and we can prove that $\mathcal{G} = \mathbb{Z}[(1+\sqrt{5})/2]$. We do not know yet any example of a self-affine tiling for which $\mathcal{G}$ is not a group.

Given a self-affine tiling $\mathcal{T}$, there are many ways to associate a substitution Delone multi-color set. Indeed, Definition 3.1 used the left end points but one may choose other reference points. Arnoux-Ito [8] gave the strong coincidence with respect to right end points as well, that is, *suffix strong coincidence*[1]. There is a standard way to associate Delone multi-color set to a given self-affine tiling. A tile map $\gamma : \mathcal{T} \to \mathcal{T}$ sends a tile $T$ to the one in $\Omega(T)$ such that $\gamma(T_1)$ and $\gamma(T_2)$ are located in the same relative position in $\Omega(T_1)$ and $\Omega(T_2)$ whenever $T_1$ and $T_2$ have the same color. A control point $c(T)$ of $T \in \mathcal{T}$ is defined by

$$c(T) = \bigcap_{n=1}^{\infty} Q^{-n}(\gamma^n T).$$

The color of the control point $c(T)$ is given by the color of the tile $(T)$. By definition control points of the same color tiles are located in the same relative position and the set of control points $\mathcal{C}$ is invariant under the expansion by $Q$, that is, $Q\mathcal{C} \subset \mathcal{C}$. We can choose control points as representative points. Let $\Lambda_i$ be chosen to be the control points of tiles in $\mathcal{T}$ of color $i$. Then $\mathcal{C} = (\Lambda_i)_{i \leq m}$. The set of control points $\mathcal{C}$ forms a substitution Delone multi-color set.

**Example 3.6.** The substitution $\sigma$ defined by $a \to aba, b \to bab$ satisfies neither prefix nor suffix strong coincidence. However define the tile map $\gamma$ by choosing $T_b$ in $\sigma(T_a)$ and the left most $T_b$ in $\sigma(T_b)$, then $\mathbf{\Lambda}$ admits strong coincidence by the choice of the control points.

Now we introduce a stronger notion than Definition 3.3:

[1] In [8], prefix (resp. suffix) strong coincidence is called positive (resp. negative) strong coincidence.

**Definition 3.7.** Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ with expansion $Q$. Let $\mathcal{A} = \{T_1, \cdots, T_m\}$ be the prototile set of $\mathcal{T}$. Let $c_i$ be the reference point of $T_i$ for $i \le m$ whose set is admissible. Let $\mathbf{\Lambda}$ be an associated substitution Delone multi-color set for which $\mathcal{T} = \{u_i + (T_i - c_i) \mid u_i \in \Lambda_i, i \le m\}$. If there is a positive integer $L$ that

$$\bigcap_{i=1}^{m} \Omega^L(T_i - c_i) \neq \emptyset, \tag{3.4}$$

i.e. the left hand side contains at least one tile, then we say that $\mathbf{\Lambda}$ admits *simultaneous coincidence*.

**Remark 3.8.** By the similar argument as Remark 3.4, if $\mathbf{\Lambda}$ admits simultaneous coincidence then there exists $L \in \mathbb{N}$, $\eta \in \mathbb{R}^d$ and $k \le m$ such that

$$Q^L(\bigcup_{i=1}^{m} \Lambda_i) \subset \Lambda_k - \eta\,. \tag{3.5}$$

**Remark 3.9.** In $d = 1$ in Arnoux-Ito's framework of [8], they have chosen the left (or right) end points as control points. It is plausible that their definition of geometric substitution acting on broken segments would work by other choices of control points by shifting the origin to the control points located on the 1-dimensional suspension tiling in the expanding line of $M_\sigma$. In this case, a unit segment will grow in prefix and suffix directions at a time, which fits better with the prefix-suffix construction by Canterini and Siegel [9]. The definition of strong coincidence would naturally be extended in the form to assure that some iterates of every pair of two unit segments share a common segment.

For a Pisot substitution tiling in $\mathbb{R}$, one can consider a substitution Delone multi-color set $\mathbf{\Lambda}$ taking reference points from the left end points of tile intervals. The height group of $\mathbf{\Lambda}$ is defined by Sing [18] which generalizes an idea of Dekking [10] for constant length substitution. We extend this definition to general substitution Delone multi-color sets in $\mathbb{R}^d$.

**Definition 3.10.** *The height group* of $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ in $\mathbb{R}^d$ is the quotient group

$$\langle \Lambda - \Lambda \rangle_{\mathbb{Z}} \,/\, \langle \Lambda_i - \Lambda_i \mid i \le m \rangle_{\mathbb{Z}}, \quad \text{where } \Lambda = \cup_{i \le m} \Lambda_i\,.$$

Given a tiling $\mathcal{T}$ in $\mathbb{R}^d$, the height group also **depends** on the choice of $\mathbf{\Lambda}$.

**Example 3.11.** Consider again the substitution in Example 3.6. Taking the left end points of $T_a$ and $T_b$, we have $\Lambda = \Lambda_a \cup \Lambda_b = \mathbb{Z}$, $\langle \Lambda - \Lambda \rangle_{\mathbb{Z}} = \mathbb{Z}$, and $\langle \Lambda_i - \Lambda_i \mid i \in \{a, b\} \rangle_{\mathbb{Z}} = 2\mathbb{Z}$. Thus the height group is $\mathbb{Z}/2\mathbb{Z}$. The same is true for the right end points. However the third choice in Example 3.6 gives the height group $(2/3)\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/3\mathbb{Z}$. We note that the height group is not trivial by any choice of admissible reference points. In fact, let $c_a$ and $c_b$ be reference points of the prototiles $T_a = ([0, 1], a)$ and $T_b = ([0, 1], b)$ in $\mathbb{R}$. To be admissible, $|c_b - c_a| < 1$. But if the height group were trivial, then we have $c_b - c_a \in 2\mathbb{Z}$, which shows $c_a = c_b$ and $\langle \Lambda - \Lambda \rangle_{\mathbb{Z}} = \mathbb{Z}$.

## 4. From overlap coincidence to strong coincidence

A *van Hove sequence* for $\mathbb{R}^d$ is a sequence $\mathcal{F} = \{F_n\}_{n\geq 1}$ of bounded measurable subsets of $\mathbb{R}^d$ satisfying

$$\lim_{n\to\infty} \mathrm{Vol}((\partial F_n)^{+r})/\mathrm{Vol}(F_n) = 0, \text{ for all } r > 0, \qquad (4.1)$$
$$\text{where } (\partial F_n)^{+r} = \{x \in \mathbb{R}^d : \mathrm{dist}(x, F_n) \leq r\}.$$

For any $\mathcal{S} \subset \mathcal{T}$ and van Hove sequence $\{F_n\}_{n\geq 1}$, we define

$$\mathrm{dens}(\mathcal{S}) := \lim_{n\to\infty} \frac{\mathrm{vol}(\mathcal{S} \cap F_n)}{\mathrm{vol}(F_n)}$$

if the limit exists. Here the density limit is dependent on the van Hove sequence. In the case of a self-affine tiling $\mathcal{T}$, if $\mathcal{S}$ is a set of all the translates of a patch in $\mathcal{T}$, the density exists uniformly [14].

The following theorem shows that overlap coincidence of $\mathcal{T}$ implies strong coincidence of $\mathbf{\Lambda}_\mathcal{T}$ as long as the reference points for the tiling can be chosen to satisfy (4.2). This theorem can be proved using various results for pure discrete spectrum such as [19, Thm. 6.2] and [7, Prop. 1].

**Theorem 4.1.** *Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ with expansion $Q$ for which $\Xi(\mathcal{T})$ is a Meyer set. Let $\mathbf{\Lambda} = \mathbf{\Lambda}_\mathcal{T} = (\Lambda_i)_{i\leq m}$ be an associated admissible substitution Delone multi-color set. Assume that*

$$\Lambda - \Lambda \subset \langle \mathcal{G} \rangle_\mathbb{Z}, \qquad (4.2)$$

*where $\Lambda = \cup_{i\leq m}\Lambda_i$ and $\mathcal{G} == \bigcup_{k=0}^{\infty} Q^{-k}(\Lambda_i - \Lambda_i)$ for some $i \leq m$. Then overlap coincidence of $\mathcal{T}$ implies strong coincidence of $\mathbf{\Lambda}$. Furthermore overlap coincidence of $\mathcal{T}$ implies simultaneous coincidence of $\mathbf{\Lambda}$.*

Proof. For any $i, j \leq m$, take $v_i \in \Lambda_i$ and $v_j \in \Lambda_j$. Let $\alpha = v_j - v_i$. By the assumption,

$$\alpha \in \Lambda_j - \Lambda_i \subset \langle \mathcal{G} \rangle_\mathbb{Z}\,.$$

From Theorem 2.6 and Remark 3.5, $\mathcal{G} = \langle \mathcal{G} \rangle_\mathbb{Z}$. Therefore there is $M \in \mathbb{N}$ such that $Q^M\alpha \in \Lambda_1 - \Lambda_1$. Overlap coincidence implies that

$$\mathrm{dens}\ \mathcal{T} \cap (\mathcal{T} - Q^n\alpha) \longrightarrow 1 \qquad (4.3)$$

as $n \to \infty$ [19]. Since $\mathcal{T}$ has uniform patch frequencies [14], the density is not dependent on the choice of the van Hove sequence. So we can take a van Hove sequence

$\{\mathrm{supp}(\Omega^n(T_i - c_i + v_i))\}_{n\geq 1}$. Then we observe that $\mathcal{T}$ admits strong coincidence, otherwise the density is bounded away from one.

Furthermore, we claim that $\mathbf{\Lambda}$ admits simultaneous coincidence. Fix $k \leq m$ and $v_k \in \Lambda_k$. For any $i \leq m$, choose $v_i \in \Lambda_i$. Let $\alpha_i = v_k - v_i$. By the admissibility of $\mathbf{\Lambda}$, $\cap_{i\leq m}\mathrm{supp}(T_i - c_i + v_k)$ has non-empty interior. Let

$$P_n = \bigcap_{i\leq m} \mathrm{supp}(\Omega^n(T_i - c_i + v_i + \alpha_i)) \quad \text{for } n \in \mathbb{N}.$$

Then $\{P_n\}_{n\geq 1}$ is a van Hove sequence and $\Omega^n(T_k - c_k + v_k)$ is a patch in $\mathcal{T}$. From the overlap coincidence of $\mathcal{T}$, for each $i \leq m$

$$\frac{\mathrm{vol}(\Omega^n(T_k - c_k + v_k) \cap \Omega^n(T_i - c_i + v_i + \alpha_i) \cap P_n)}{\mathrm{vol}(P_n)} \overset{n\to\infty}{\longrightarrow} 1\,.$$

Thus

$$\begin{aligned} 1 - & \frac{\mathrm{vol}\left(\left(\bigcap_{i\leq m, i\neq k} \Omega^n(T_k - c_k + v_k) \cap \Omega^n(T_i - c_i + v_i + \alpha_i)\right) \cap P_n\right)}{\mathrm{vol}(P_n)} \\ \leq & \sum_{i\leq m, i\neq k} \left(1 - \frac{\mathrm{vol}\left(\Omega^n(T_k - c_k + v_k) \cap \Omega^n(T_i - c_i + v_i + \alpha_i) \cap P_n\right)}{\mathrm{vol}(P_n)}\right) \longrightarrow 0\ . \end{aligned}$$

It implies that there exists $n \in \mathbb{N}$ such that

$$\bigcap_{i\leq m, i\neq k} \Omega^n(T_k - c_k + v_k) \cap \Omega^n(T_i - c_i + v_i + \alpha_i) \cap P_n) \neq \emptyset\,.$$

Thus

$$\bigcap_{i\leq m, i\neq k} \Omega^n(T_k - c_k) \cap \Omega^n(T_i - c_i) \neq \emptyset$$

and the claim follows. □

**Corollary 4.2.** *Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ for which $\Xi(\mathcal{T})$ is a Meyer set. Let $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i\leq m}$ be an associated admissible substitution Delone multi-color set with a trivial height group. Then overlap coincidence of $\mathcal{T}$ implies simultaneous coincidence as well as strong coincidence of $\mathbf{\Lambda}$.*

Proof. The trivial height group assumption implies $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle_{\mathbb{Z}}$. □

We provide a lemma.

**Lemma 4.3.** *Let $A$ be a $m \times m$ integer matrix whose characteristic polynomial is irreducible. Then the entries of an eigenvector of $A$ are linearly independent over $\mathbb{Q}$.*

PROOF. Let $\alpha$ be an eigenvalue of $A$ and $(v_1, v_2, \ldots, v_m) \in \mathbb{Q}(\alpha)^m$ be a corresponding eigenvector. Denote by $\alpha_i$ $(i = 1, \ldots, m)$ the conjugates of $\alpha$. Applying Galois conjugate map $\tau_i$ which sends $\alpha$ to $\alpha_i$, we obtain $d$ different eigenvectors $(\tau_i(v_1), \ldots, \tau_i(v_m))$. Since $\alpha_i$ $(i = 1, \ldots, m)$ are all distinct, the corresponding eigenvectors are linearly independent over $\mathbb{C}$. So $B := (\tau_i(v_j))$ is invertible. However if $v_1, v_2, \ldots, v_m$ are linearly dependent over $\mathbb{Q}$, then the column vectors of $B$ are linearly dependent over $\mathbb{C}$ which is a contradiction. □

Now we consider an 1-dim substitution tiling and recall the following lemma from [18, Lemma 6.34] and [6, Thm. 12.1]).

**Lemma 4.4.** *[18, 6] Let $\sigma$ be a primitive irreducible substitution and consider its suspension tiling of $\mathbb{R}$ with an expansion factor $\beta$, the Perron Frobenius root of the substitution matrix $M_\sigma$. Let $\mathbf{\Lambda}$ be an associated substitution Delone multi-color set whose points are taken from the left end of tiles in the suspension tiling. Then we have*

$$\langle \Lambda_i - \Lambda_i \mid i \le m \rangle_{\mathbb{Z}} = \langle (\cup_{i \le m} \Lambda_i) - (\cup_{i \le m} \Lambda_i) \rangle_{\mathbb{Z}}. \tag{4.4}$$

Corollary 4.2 and Lemma 4.4 give a combinatorial result on Pisot substitution sequences: the overlap coincidence implies strong and simultaneous coincidence for 1-dim irreducible Pisot substitutions. This result agrees with the earlier results of [6, 4].

**Corollary 4.5.** *Let $\sigma$ be the irreducible Pisot substitution over letters $\{1, \ldots, m\}$ whose natural suspension tiling satisfies overlap coincidence. Then there are $L \in \mathbb{N}$ and $M \in \mathbb{N}$ such that prefixes of length $M$ of $\sigma^L(1), \sigma^L(2), \ldots, \sigma^L(m)$ has the same number of each letter $j$ for $1 \le j \le m$, and all ends with an identical letter.*

PROOF. As in Lemma 4.4, we choose the left end points as the set of control points. Then it fulfills the conditions of the admissibility and the trivial height group of Theorem 4.1. Theorem 4.1 shows that the suspension tiling satisfies simultaneous coincidence by taking $n$-th iterates of all prototiles. Since the suspension lengths of prototiles form a left eigenvector of $M_\sigma$, they are linearly independent over $\mathbb{Q}$ by Lemma 4.3. Counting the number of tile types in $n$-th iterate of each tile up to the tile where simultaneous coincidence occurs, we can claim the result. □

**Example 4.6.** We give an example of a substitution tiling for which an associated admissible substitution Delone multi-color set has a trivial height group, but it admits neither overlap coincidence nor strong coincidence (simultaneous coincidence). Consider the substitution $\sigma$ over four letters $\{a, b, A, B\}$ defined by

$$a \to aB, b \to a, A \to Ab, B \to A.$$

This is $\mathbb{Z}/2\mathbb{Z}$-extension of Fibonacci substitution. The suspension tiling dynamics does not satisfy overlap coincidence as it is computed in [3]. The suspension lengths of capital letter and non capital letter are the same. Introduce a letter to the letter involution $\tau$ which interchanges capital letter to non capital letter and vice versa. Then $\tau$ and $\sigma$ commutes from $\tau(\sigma^n(a)) = \sigma^n(A)$ for all $n \in \mathbb{N}$, we see that $\sigma$ does not satisfy prefix (nor suffix) strong coincidence. However taking the left end points as reference points, the height group is trivial which is confirmed by collecting all return vectors.

**Example 4.7.** Consider the substitution $\sigma$ over six letters $\{a, b, c, A, B, C\}$ defined by

$$a \to aB, b \to aC, c \to a, A \to Ab, B \to Ac, C \to A.$$

This is $\mathbb{Z}/2\mathbb{Z}$-extension of the Rauzy substitution. The suspension tiling dynamics of $\sigma$ satisfies overlap coincidence which is checked by the algorithm in [3]. By the same reasoning as Example 4.6, this substitution does not satisfy prefix nor suffix strong coincidence. Taking the left end points and collecting all return vectors, we can compute that the height group is equal to $\mathbb{Z}/2\mathbb{Z}$.

On the other hand, taking the tile map $\gamma$ which sends

$$a \to B, b \to C, c \to a, A \to A, B \to A, c \to A,$$

the set of control points is admissible and satisfies simultaneous coincidence. One can also check that the height group associate to this control points is trivial. So this gives an example of Corollary 4.2 for a reducible substitution.

**Remark 4.8.** Nakaishi [16] claimed that unimodular Pisot substitutions with prefix strong coincidence generate pure discrete spectrum of the dynamical system. It would mean that strong coincidence implies overlap coincidence in the associated substitution tilings in $\mathbb{R}$. So together with Corollary 4.5 the equivalence between overlap coincidence and strong coincidence could be established. We hope directly to observe the result of [16], the implication from strong coincidence to overlap coincidence in the framework of this paper.

**Acknowledgment**

The authors would like to thank the referees for the valuable comments and references. This research was supported by Basic Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011150) and the Japanese Society for the Promotion of Science (JSPS), Grant in aid 21540010. The second author is grateful for the support of KIAS for this research.

## References


[1] S.Akiyama, Strong coincidence and overlap coincidence, submitted.

[2] S. Akiyama, M. Barge, V. Berthé, J.-Y. Lee and A. Siegel, On the Pisot substitution conjecture, Preprint.

[3] S. Akiyama and J.-Y. Lee, Algorithm for determining pure pointedness of self-affine tilings, *Adv. Math.* **226** (2011), 2855–2883.

[4] V. Baker, M. Barge and J. Kwapisz, Geometric realization and coincidence for reducible non-unimodular Pisot tiling spaces with an application to shifts, *J. Instit. Fourier.* **56(7)** (2006), 2213–2248

[5] M. Barge and B. Diamond, *Coincidence for substitutions of Pisot type*, Bull. Soc. Math. France **130** (2002), no. 4, 619–626.

[6] M. Barge and J. Kwapisz, Geometric theory of unimodular Pisot substitutions, *Amer. J. Math.* **128(5)** (2006), 1219–1282.

[7] M. Barge, S. Stimac and R. F. Williams, Pure discrete spectrum in substitution tiling spaces, *Discrete and Continuous Dynamical Systems -A*, **2** (2013), 579–597.

[8] P. Arnoux and S. Ito, Pisot substitutions and Rauzy fractals, *Bull. Belg. Math. Soc. Simon Stevin* **8** (2001), 181–207.

[9] V. Canterini and A. Siegel, *Geometric representation of substitutions of Pisot type*, Trans. Amer. Math. Soc. **353 (12)** (2001), 5121–5144.

[10] M. Dekking, The spectrum of dynamical systems arising from substitutions of constant length, *Z. Wahrsch. Verw. Gebiete* **41** (1978), 221–239.

[11] M. Hollander and B. Solomyak, Two-symbol Pisot substitutions have pure discrete spectrum, *Ergodic Theory Dynam, Systems* **23**(2) (2003), 533–540.

[12] J. C. Lagarias, Meyer's concept of quasicrystal and quasiregular sets, *Commm. Math. Phys.* **179** (1996), 365–376.

[13] J.-Y. Lee, Substitution Delone sets with pure point spectrum are inter-model sets, *Journal of Geometry and Physics* **57** (2007), 2263-2285.

[14] J.-Y. Lee, R. V. Moody, and B. Solomyak, Consequences of pure point diffraction spectra for multiset substitution systems, *Discrete Comp. Geom.* **29** (2003), 525–560.

[15] J.-Y. Lee and B. Solomyak, Pisot family self-affine tilings, discrete spectrum, and the Meyer property. *Discrete and Continuous Dynamical Systems - A.* **32** (2012), 935 - 959.

[16] K. Nakaishi, Pisot conjecture and Rauzy fractals, Preprint.

[17] A. Siegel, Répresentation des systèmes dynamique substitutifs non unimodulaires, *Ergodic Theory Dynam. Systems* **23**(4) (2003), 1247–1273.

[18] B. Sing, Pisot substitutions and beyond, *Ph. D. Thesis* (2006).

[19] B. Solomyak, Dynamics of self-similar tilings, *Ergodic Th. Dynam. Sys.* **17** (1997), 695–738. Corrections to ' Dynamics of self-similar tilings', *ibid.* **19** (1999), 1685.

[20] V. F. Sirvent and B. Solomyak, Pure discrete spectrum for one-dimensional substitution systems of Pisot type. *Canad. Math. Bull.* **45**(4) (2002) 697–710. Dedicated to Robert V. Moody.