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WEIGHTED LOSS FUNCTIONS FOR SET ESTIMATION AND TESTING HYPOTHESES

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From the decision-theoretic viewpoint, using a weighted loss we compare the risks of testing procedures in the location and scale parameter cases. We also get numerically the minimax solution of Bayes testing procedures *w.r.t.* a parameter of the prior distribution, under the weighted loss.

Key words and phrases: Bayes testing procedure, loss function, Neyman-Pearson test, p-value, risk.

1. Introduction

In the problem of testing hypotheses, we usually derive the conclusion whether the hypothesis is accepted or not at the given level, but do not give the degree of evidence to support the conclusion. On the other hand, from the viewpoint of the decision theory, the probabilities of type I and II errors are represented as the risk with respect to the 0-1 loss (see Lehmann (1986)). But, unfortunately the concept like p-value, which is popular in the area of statistical application, can not be described as a form of the risk. So, instead of the 0-1 loss, the quadratic loss and weighted loss are considered as candidates when solving the problem of testing hypotheses (see, e.g., Hwang *et al.* (1992) and Robert and Casella (1994)). For a one-sided problem of testing the normal mean, the comparison among the Neyman-Pearson (N-P) test, Bayes testing procedures and p-value was made (see, Hwang *et al.* (1992)).

Suppose that an n-dimensional random vector $\boldsymbol{X}$ is distributed according to a density function $f(\boldsymbol{x}, \theta)$, where $\theta \in \Theta \subset \mathbf{R}^1$. Let Θ_0 and Θ_1 be disjoint subsets of Θ. Then we regard the problem of testing the hypothesis H: $\theta \in \Theta_0$ against the alternative K: $\theta \in \Theta_1$ as that of estimating the indicator $\chi_{\Theta_0}(\theta)$ of Θ_0.

First we define the weighted loss function by

$$\ell_\lambda(\theta, d) := c(\theta)|\chi_{\Theta_0}(\theta) - d|^\lambda \quad (d \in \mathbf{R}^1; \lambda > 0), \tag{1.1}$$

where c is a nonnegative function on Θ, and the decision procedure by a measurable function φ satisfying $0 \le \varphi(\boldsymbol{x}) \le 1$, $\boldsymbol{x} \in \mathbf{R}^n$ (see also Robert and Casella (1994)). We also define the risk of φ as

$$R_\lambda(\theta, \varphi) := E_\theta[\ell_\lambda(\theta, \varphi)] = \begin{cases} c(\theta)E_\theta\left[|1 - \varphi(\boldsymbol{X})|^\lambda\right] & \text{for } \theta \in \Theta_0, \\ c(\theta)E_\theta\left[\{\varphi(\boldsymbol{X})\}^\lambda\right] & \text{for } \theta \in \Theta_1. \end{cases}$$

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In the one-sided problem of testing hypothesis on normal mean, Hwang *et al.* (1992) compared the risks based on the quadratic loss among the N-P test, p-value, and Bayes testing procedures, drawing figures of the behavior of those risks.

In this paper, we also consider the one-sided problem of testing the hypothesis H: $\theta \leq \theta_0$ against the alternative K: $\theta > \theta_0$ when θ is a location parameter. Since, in a neighborhood of the boundary $\theta = \theta_0$, both of the hypotheses would be nearly acceptable, it is desirable for the loss to take small values. For a moderately far area from $\theta = \theta_0$, it is also desirable for the loss to make a sensitive response. So, it is natural to choose as $c(\theta)$ the function which takes a minimum value at $\theta = \theta_0$ and is increasing in $|\theta - \theta_0|$. Here, letting $c(\theta) = |\theta - \theta_0|^\gamma$ $(\gamma \geq 0)$ and $\lambda = 2$ in (1.1), we have the loss function

$$L^{(\gamma)}(\theta, d) := |\theta - \theta_0|^\gamma \{\chi_{\Theta_0}(\theta) - d\}^2 \quad (\gamma \geq 0), \tag{1.2}$$

where $\Theta_0 = (-\infty, \theta_0]$. Indeed, for the neighborhood, the loss (1.2) with $\gamma > 0$ is smaller under both of H and K than the quadratic loss $L^{(0)}$. For the moderately far area from $\theta = \theta_0$, the loss (1.2) with $\gamma > 0$ also gives a more sensitive response than $L^{(0)}$. Thus the loss (1.2) seems to be more natural than $L^{(0)}$. Then we have as the risk of a decision procedure φ

$$R^{(\gamma)}(\theta, \varphi) := E_\theta[L^{(\gamma)}(\theta, \varphi)] = |\theta - \theta_0|^\gamma E_\theta\left[\{\chi_{\Theta_0}(\theta) - \varphi(\boldsymbol{X})\}^2\right]. \tag{1.3}$$

In the case when θ (> 0) is a scale parameter, we consider the problem of testing the hypothesis H: $\theta \leq \theta_0$ against the alternative K: $\theta > \theta_0$, where $\theta_0 > 0$. For a simple calculation, letting $c(\theta) = (\theta/\theta_0)^\gamma$ $(\gamma \geq 0)$ and $\lambda = 2$ in (1.1), we have the loss function

$$\tilde{L}^{(\gamma)}(\theta, d) := (\theta/\theta_0)^\gamma \{\chi_{\Theta_0}(\theta) - d\}^2 \quad (\gamma \geq 0), \tag{1.4}$$

thus the risk of a decision procedure φ is given by

$$\tilde{R}^{(\gamma)}(\theta, \varphi) := E_\theta[\tilde{L}^{(\gamma)}(\theta, \varphi)] = (\theta/\theta_0)^\gamma E_\theta\left[\{\chi_{\Theta_0}(\theta) - \varphi(\boldsymbol{X})\}^2\right].$$

In this case, one may consider as other weights instead of the above $c(\theta)$

$$c_1^{(\gamma)}(\theta) := (\max\{\theta/\theta_0, \theta_0/\theta\})^\gamma \qquad (\gamma \geq 0),$$
$$c_2^{(\gamma)}(\theta) := (\max\{\theta/\theta_0, \theta_0/\theta\} - 1)^\gamma \qquad (\gamma \geq 0).$$

In the one-sided problems of testing hypotheses, we numerically compare the risks of the N-P test, p-value, and Bayes testing procedures, and obtain the minimax solution of Bayes ones *w.r.t.* a parameter of the prior distribution.

2. Risk with respect to the loss $L^{(\gamma)}$: Location parameter case

In this section we consider the location parameter case. Suppose that a real random variable X is distributed according to a density function $f(x - \theta)$ (*w.r.t.*

the Lebesgue measure), where $\theta \in \Theta \subset \mathbf{R}^1$. Let π be a prior density of θ (*w.r.t.* the Lebesgue measure). Then we consider the problem of testing the hypothesis H: $\theta \leq \theta_0$ against the alternative K: $\theta > \theta_0$. Letting $\Theta_0 = (-\infty, \theta_0]$, from (1.2) we obtain as the risk of a decision procedure φ

$$R^{(\gamma)}(\theta, \varphi) = E_\theta[L^{(\gamma)}(\theta, \varphi)] = |\theta - \theta_0|^\gamma E_\theta[\{\chi_{\Theta_0}(\theta) - \varphi(\boldsymbol{X})\}^2]$$

and as its Bayes risk *w.r.t.* π

$$r_\pi(\varphi) = E_\pi[R^{(\gamma)}(\theta, \varphi)] = \int_\Theta R^{(\gamma)}(\theta, \varphi)\pi(\theta)d\theta.$$

Without loss of generality we assume that $\theta_0 = 0$. Then the Bayes risk is given by

$$\begin{aligned} r_\pi^{(\gamma)}(\varphi) &= \int_{-\infty}^{\infty} |\theta|^\gamma \pi(\theta) \left\{ \int_{-\infty}^{\infty} \{\chi_{\Theta_0}(\theta) - \varphi(x)\}^2 f(x-\theta)dx \right\} d\theta \\ &= \int_{-\infty}^{\infty} \Bigg[\{1 - \varphi(x)\}^2 \int_{-\infty}^{0} |\theta|^\gamma \pi(\theta) f(x-\theta)d\theta \\ &\qquad + \varphi^2(x) \int_0^\infty \theta^\gamma \pi(\theta) f(x-\theta)d\theta \Bigg] dx. \end{aligned} \tag{2.1}$$

Put

$$A := \int_{-\infty}^{0} |\theta|^\gamma \pi(\theta) f(x-\theta)d\theta, \quad B := \int_0^\infty \theta^\gamma \pi(\theta) f(x-\theta)d\theta.$$

The decision procedure φ minimizing the Bayes risk (2.1) is given by

$$\varphi_\pi^*(x) = \frac{A}{A+B} = \int_{-\infty}^{0} |\theta|^\gamma f(x-\theta)\pi(\theta)d\theta \Big/ \int_{-\infty}^{\infty} |\theta|^\gamma f(x-\theta)\pi(\theta)d\theta, \tag{2.2}$$

which is called the Bayes decision procedure. Since the posterior density of θ given $X = x$ is

$$p(\theta \mid x) = f(x-\theta)\pi(\theta) \Big/ \int_{-\infty}^{\infty} f(x-\theta)\pi(\theta)d\theta,$$

it follows from (2.2) that

$$\varphi_\pi^*(x) = \int_{-\infty}^{0} |\theta|^\gamma p(\theta \mid x)d\theta \Big/ \int_{-\infty}^{\infty} |\theta|^\gamma p(\theta \mid x)d\theta \tag{2.3}$$

(see also Robert and Casella (1994)).

Here, one of the competitors of the Bayes decision procedures is given as follows. Let $C(\alpha)$ be a rejection region of a test of level α. For given $\boldsymbol{X} = \boldsymbol{x}$,

$$p(\boldsymbol{x}) := \inf_{\boldsymbol{x} \in C(\alpha)} \alpha$$

is called the p-value.

Example 2.1 (Normal distribution). Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (*i.i.d.*) random variables with the normal distribution $N(\theta, 1)$. Let π_σ be a prior density of the normal distribution $N(0, \sigma^2)$. Since $\bar{X} = (1/n)\sum_{i=1}^n X_i$ is normally distributed as $N(\theta, 1/n)$, it follows that the posterior distribution of θ given $\bar{X} = \bar{x}$ is the normal one

$$N\left(\frac{n\sigma^2\bar{x}}{n\sigma^2+1}, \frac{\sigma^2}{n\sigma^2+1}\right).$$

(i) Case $\gamma = 0$. In this case, the risk $R^{(0)}$ is the expected quadratic loss, and is treated by Hwang *et al.* (1992). Then, for $\alpha = 0.01, 0.05, 0.10$, the risk of the N-P test takes very small values on Θ_0 and extremely large ones on Θ_1. This seems to arise from the facts that the N-P test takes only two values 0 and 1 and the hypothesis H is protected by restricting the probability of type I error to being less or equal to the level α. From the test-theoretic viewpoint, the UMP or UMP unbiased (UMPU) tests derived from the fundamental lemma of N-P are appropriate, but, in the set estimation problem treated here, the risk of the N-P test is affected by its support for the hypothesis H. On the other hand, the risks of the p-value and the Bayes procedure are symmetric about the boundary point $\theta = 0$ between Θ_0 and Θ_1 and relatively small all over the domain Θ (see Figure 2.1). Thus, the p-value and the Bayes procedure are fair with both of the hypothesis H and the alternative K and seem to be preferable to the N-P test. Note that the risk of the N-P test is symmetric around $\theta = 0$ for $\alpha = 0.50$.

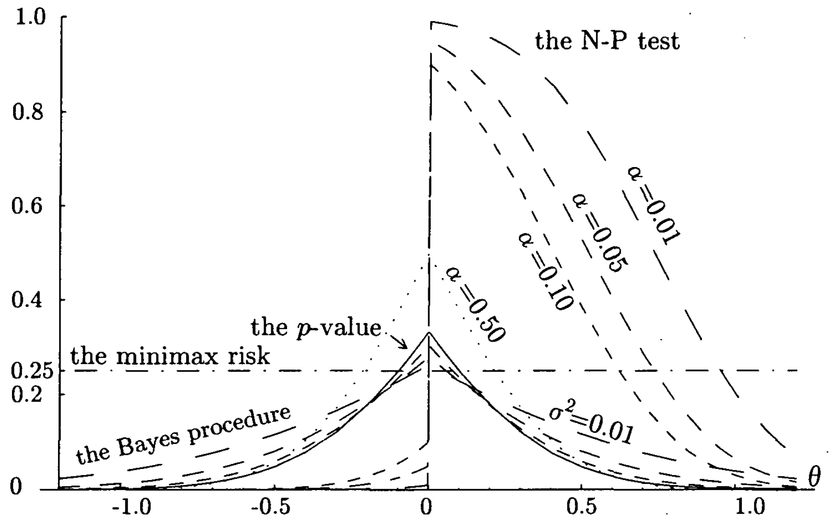

Figure 2.1. The risks of the Bayes procedures, the N-P tests and the p-value under the quadratic loss $L^{(0)}$ when $n = 10$ (Hwang *et al.* (1992)). The solid line is the risk of the p-value. The risks of the N-P tests also are given for $\alpha = 0.01$ (the longest dashes), $0.05, 0.10$ (the shortest dashes), 0.50 (dots). The risks of the Bayes procedures also are given for the prior variances $\sigma^2 = 0.01$ (the longest dashes), $0.03, 0.10$ (the shortest dashes). The constant risk 0.25 is the risk of the minimax procedure $\varphi_0 = 1/2$ that is the limit of the Bayes procedures when $\sigma^2 \to 0$.

(ii) Case $\gamma = 2$. From (2.3) we have the Bayes procedure

$$\varphi^*_{\pi_\sigma}(\bar{x}) = 1 - \int_0^\infty \theta^2 p(\theta \mid \bar{x}) d\theta \Big/ \int_{-\infty}^\infty \theta^2 p(\theta \mid \bar{x}) d\theta, \tag{2.4}$$

where $p(\theta \mid \bar{x})$ is a posterior density of θ given $\bar{X} = \bar{x}$, i.e., a normal density with a posterior mean $\mu_{\bar{x}} := E(\theta \mid \bar{x}) = n\sigma^2\bar{x}/(n\sigma^2+1)$ and variance $v_{\bar{x}}^2 := V(\theta \mid \bar{x}) = \sigma^2/(n\sigma^2+1)$. Then we have

$$\int_{-\infty}^\infty \theta^2 p(\theta \mid \bar{x}) d\theta = \mu_{\bar{x}}^2 + v_{\bar{x}}^2 = \frac{\sigma^2\left\{n^2\sigma^2\bar{x}^2 + (n\sigma^2+1)\right\}}{(n\sigma^2+1)^2}. \tag{2.5}$$

Letting ϕ be a standard normal density, we obtain $z\phi(z) = -\phi'(z)$ and $z^2\phi(z) = \phi''(z) + \phi(z)$. Hence

$$\begin{aligned}\int_0^\infty \theta^2 p(\theta \mid \bar{x}) d\theta &= \int_0^\infty \theta^2 \frac{1}{v_{\bar{x}}}\phi\left(\frac{\theta - \mu_{\bar{x}}}{v_{\bar{x}}}\right) d\theta = \int_{-\mu_{\bar{x}}/v_{\bar{x}}}^\infty (\mu_{\bar{x}} + v_{\bar{x}} z)^2 \phi(z) dz \\ &= \int_{-\mu_{\bar{x}}/v_{\bar{x}}}^\infty \left\{\mu_{\bar{x}}^2\phi(z) - 2\mu_{\bar{x}}v_{\bar{x}}\phi'(z) + v_{\bar{x}}^2(\phi''(z) + \phi(z))\right\} dz \\ &= (v_{\bar{x}}^2 + \mu_{\bar{x}}^2)\Phi\left(\frac{\mu_{\bar{x}}}{v_{\bar{x}}}\right) + \mu_{\bar{x}}v_{\bar{x}}\phi\left(\frac{\mu_{\bar{x}}}{v_{\bar{x}}}\right),\end{aligned} \tag{2.6}$$

where Φ is the cumulative distribution function of the standard normal distribution $N(0,1)$. Since $\mu_{\bar{x}}v_{\bar{x}} = n\sigma^3\bar{x}/(n\sigma^2+1)^{3/2}$, it follows from (2.4) to (2.6) that

$$\varphi^*_{\pi_\sigma}(\bar{x}) = 1 - \Phi\left(\frac{n\sigma\bar{x}}{\sqrt{n\sigma^2+1}}\right) - \frac{n\sigma\bar{x}\sqrt{n\sigma^2+1}}{n^2\sigma^2\bar{x}^2 + (n\sigma^2+1)}\phi\left(\frac{n\sigma\bar{x}}{\sqrt{n\sigma^2+1}}\right).$$

The indicator of the acceptance region of the uniformly most powerful (UMP) test is given by

$$\varphi_{\mathrm{NP}}(\bar{x}) = \chi_{(-\infty,u_\alpha]}\left(\sqrt{n}\bar{x}\right) = \begin{cases} 1 & \text{for } \bar{x} \le u_\alpha/\sqrt{n} \\ 0 & \text{for } \bar{x} > u_\alpha/\sqrt{n} \end{cases},$$

from the fundamental lemma of N-P, where α is the significance level of the UMP test and u_α is the upper 100α percentage point of the standard normal distribution $N(0,1)$. Here, note that the N-P test φ_{NP} is not the indicator of the rejection region $\{\boldsymbol{x} \mid \sqrt{n}\bar{x} \le u_\alpha\}^c = \{\boldsymbol{x} \mid \sqrt{n}\bar{x} > u_\alpha\}$ which is usually called the test function, but that of the acceptance region $\{\boldsymbol{x} \mid \sqrt{n}\bar{x} \le u_\alpha\}$. The p-value for the value $\sqrt{n}\bar{x}$ of the UMP test statistic $\sqrt{n}\bar{X}$ is given by

$$p(\bar{x}) = P_0\left\{\sqrt{n}\bar{X} > \sqrt{n}\bar{x}\right\} = \Phi\left(-\sqrt{n}\bar{x}\right).$$

On the other hand, in this paper, we are concerned with the decision-theoretic estimation problem of the indicator $\chi_{\Theta_0}(\theta)$ of the set Θ_0 which specifies the hypothesis H, and thus we require the estimators (decision procedures) to take the expected values close to 1 under the hypothesis H and the expected values close to 0 under the alternative K. By reversing 0 and 1 in the test function, the expected value of the N-P test φ_{NP} under the hypothesis H will be more than or equal to $1-\alpha$ and thus close to 1. On the other hand the risk of $\varphi^*_{\pi_\sigma}(\bar{x})$ is given as follows. Let Z be a normally distributed random variable with $N(0,1)$. For all $\theta \in \mathbf{R}^1$,

$$(2.7) \quad R^{(2)}(\theta, \varphi^*_{\pi_\sigma}) = \theta^2 E_Z\left[\left\{\Phi\left(\frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)}{\sqrt{n\sigma^2+1}}\right) + \frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)\sqrt{n\sigma^2+1}}{\{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)\}^2+(n\sigma^2+1)}\phi\left(\frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)}{\sqrt{n\sigma^2+1}}\right)\right\}^2\right].$$

It is also shown that the risks of the N-P test $\varphi_{\mathrm{NP}}(\bar{x})$ and the p-value $p(\bar{x})$ are given by

$$R^{(2)}(\theta, \varphi_{\mathrm{NP}}) = \theta^2\Phi\left((u_\alpha - \sqrt{n}\theta)\operatorname{sgn}\theta\right),$$
$$R^{(2)}(\theta, p) = \theta^2 E_Z\left[\Phi\left(Z-\sqrt{n}|\theta|\right)^2\right],$$

for all $\theta \in \mathbf{R}^1$, where $\operatorname{sgn}\theta$ is the sign of θ. Note that $R^{(2)}(\theta, \varphi^*_{\pi_\sigma})$ and $R^{(2)}(\theta, p)$ are symmetric about $\theta = 0$. The behavior of the risks is given in Figure 2.2.

Comparing the cases (i) and (ii), we see that the risks are not smooth at $\theta = 0$ in (i), but they are always smooth in (ii) (see Figures 2.1 and 2.2). In particular, the risk of the N-P test is not continuous at $\theta = 0$ in the case (i). Since, in a neighborhood of $\theta = 0$, both of the hypotheses would be nearly acceptable, it is desirable for the risk to take a small value, and that for a moderately far area

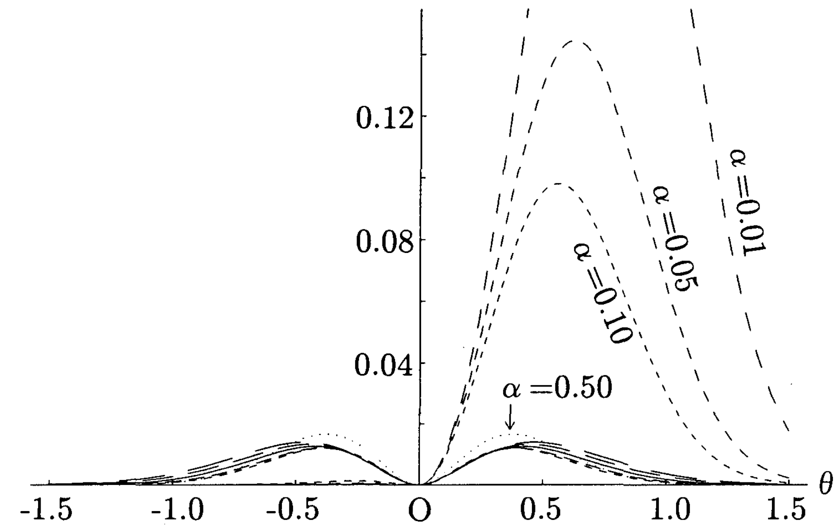

Figure 2.2. The risks of the Bayes procedures, the N-P tests and the p-value under the weighted loss $L^{(2)}$ when $n = 10$. The risks of the N-P tests also are given for $\alpha = 0.01$ (the longest dashes), 0.05, 0.10 (the shortest dashes), 0.50 (dots). The risks of the Bayes procedures also are given for the prior variances $\sigma^2 = 0.14^2$ (the longest dashes), $0.16^2, 0.30^2, 0.60^2$ (the shortest dashes).

from $\theta = 0$ the risk has a sensitive response (see Figure 2.2). Indeed, in the sense, as is seen in Figures 2.1 and 2.2, the weighted loss $L^{(2)}$ seems to be more suitable than the quadratic loss $L^{(0)}$. In Figure 2.2 as well as in Figure 2.1, the risks of $\varphi^*_{\pi_\sigma}$ and the p-value are small as a whole, while the risk of φ_{NP} is smaller than the others for $\theta \leq 0$, but is extremely larger than those for $\theta > 0$. The difference between the risks $R^{(2)}$ and $R^{(0)}$ of the Bayes procedures *w.r.t.* the weighted loss $L^{(2)}$ and the quadratic one $L^{(0)}$, respectively, is the behavior at the point $\theta = 0$ where $R^{(2)}$ is smooth and takes zero value, while $R^{(0)}$ is not smooth and positive, and $R^{(2)}$ seems to be less than $R^{(0)}$ as a whole (see Figures 2.1 and 2.2).

Next, since, for $\sigma = 0.3, 0.6$,

$$\sup_\theta R^{(2)}(\theta, \varphi^*_{\pi_\sigma}) < \sup_\theta R^{(2)}(\theta, p),$$

the Bayes procedures are better than the others (see also Figure 2.3). So, we consider the minimax solution of σ, that is, the prior variance σ_0^2 such that

$$\inf_\sigma \sup_\theta R^{(2)}(\theta, \varphi^*_{\pi_\sigma}) = \sup_\theta R^{(2)}(\theta, \varphi^*_{\pi_{\sigma_0}}).$$

Since the risk $R^{(2)}(\theta, \varphi^*_{\pi_\sigma})$ of the Bayes procedure is symmetric about $\theta = 0$, it is enough to restrict the value of parameter θ to the interval $[0, \infty)$. But, it is difficult to obtain analytically the minimax solution from (2.7). So, we get a numerical solution as follows. For some given value of σ, we get a numerical solution $\theta = \theta(\sigma)$ of the equation

$$\frac{\partial}{\partial\theta} R^{(2)}(\theta, \varphi^*_{\pi_\sigma}) = 0.$$

Hence, first we take the values of σ at suitable intervals, and plot the maximum value of $R^{(2)}(\theta(\sigma), \varphi^*_{\pi_\sigma})$ over them on graph (see Figure 2.4). Second we find the value σ_1 of σ minimizing the maximum value of $R^{(2)}(\theta(\sigma), \varphi^*_{\pi_\sigma})$. Third, for the values around $\sigma = \sigma_1$ at shorter intervals than the first ones, we repeat the above procedure. Finally we obtain the approximate value of the minimax solution (see

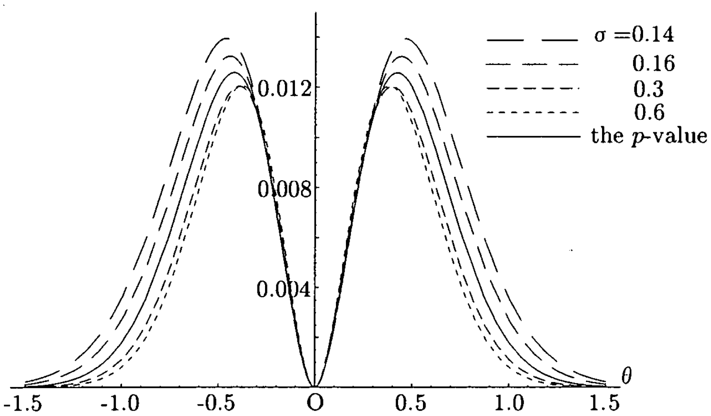

Figure 2.3. The risks of the Bayes procedures and the p-value when $n = 10$.

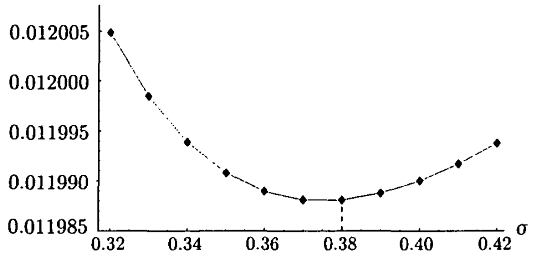

Figure 2.4. The maximum risk $\sup_\theta R^{(2)}(\theta, \varphi^*_{\pi_\sigma})$ of the Bayes procedure for given values of σ ($\sigma = 0.32$ (0.01) 0.42, $n = 10$).

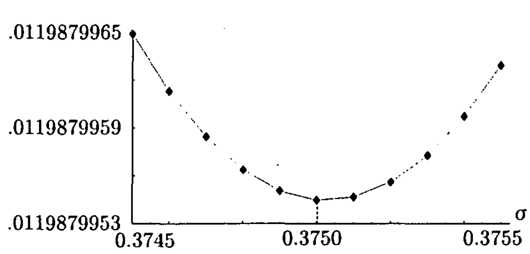

Figure 2.5. The minimax solution $\sigma_0 \approx 0.375$ and the minimax risk $R^{(2)}(0.388, \varphi^*_{\pi_{\sigma_0}}) \approx 0.01199$ ($\sigma = 0.3745$ (0.0001) $0.3755, n = 10$).

Table 2.1. The minimax solution and comparison between the minimax risk of the Bayes procedure and the maximum risk of the p-value.

n	σ_0	The minimax risk of the Bayes		The maxi-risk of the p-value
1	1.186	$R^{(2)}(1.228, \varphi^*_{\pi_{\sigma_0}}) \approx 0.11988$	$<$	$R^{(2)}(1.333, p) \approx 0.12569$
3	0.685	$R^{(2)}(0.709, \varphi^*_{\pi_{\sigma_0}}) \approx 0.03996$	$<$	$R^{(2)}(0.770, p) \approx 0.04190$
5	0.530	$R^{(2)}(0.549, \varphi^*_{\pi_{\sigma_0}}) \approx 0.02398$	$<$	$R^{(2)}(0.596, p) \approx 0.02514$
7	0.448	$R^{(2)}(0.464, \varphi^*_{\pi_{\sigma_0}}) \approx 0.01713$	$<$	$R^{(2)}(0.504, p) \approx 0.01796$
10	0.375	$R^{(2)}(0.388, \varphi^*_{\pi_{\sigma_0}}) \approx 0.01199$	$<$	$R^{(2)}(0.422, p) \approx 0.01257$
15	0.306	$R^{(2)}(0.317, \varphi^*_{\pi_{\sigma_0}}) \approx 0.00799$	$<$	$R^{(2)}(0.344, p) \approx 0.00838$
20	0.265	$R^{(2)}(0.275, \varphi^*_{\pi_{\sigma_0}}) \approx 0.00599$	$<$	$R^{(2)}(0.298, p) \approx 0.00628$

Figure 2.5), and can compare the values between the maximum risk of the Bayes procedure and the maximum risk of the p-value (see Table 2.1).

(iii) Case $\gamma = 1$. In a similar way to the case (ii), we have the Bayes procedure

$$\varphi^*_{\pi_\sigma}(\bar{x}) = 1 - \frac{n\sigma\bar{x}\Phi\left(\frac{n\sigma\bar{x}}{\sqrt{n\sigma^2+1}}\right) + \sqrt{n\sigma^2+1}\phi\left(\frac{n\sigma\bar{x}}{\sqrt{n\sigma^2+1}}\right)}{n\sigma\bar{x}\left\{2\Phi\left(\frac{n\sigma\bar{x}}{\sqrt{n\sigma^2+1}}\right) - 1\right\} + 2\sqrt{n\sigma^2+1}\phi\left(\frac{n\sigma\bar{x}}{\sqrt{n\sigma^2+1}}\right)}.$$

Then we obtain the risks of the Bayes procedure $\varphi^*_{\pi_\sigma}(\bar{x})$, the N-P test $\varphi_{\mathrm{NP}}(\bar{x})$ and the p-value $p(\bar{x})$ as follows. For $\theta \in \mathbf{R}^1$,

$$\begin{aligned} R^{(1)}(\theta, \varphi^*_{\pi_\sigma}) &= |\theta| \\ &\times E_Z\Bigg[\left\{\sqrt{n}\sigma(Z - \sqrt{n}|\theta|)\Phi\left(\frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)}{\sqrt{n\sigma^2+1}}\right) + \sqrt{n\sigma^2+1}\phi\left(\frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)}{\sqrt{n\sigma^2+1}}\right)\right\}^2 \Big/ \\ &\qquad \left\{\sqrt{n}\sigma(Z - \sqrt{n}|\theta|)\left\{2\Phi\left(\frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)}{\sqrt{n\sigma^2+1}}\right) - 1\right\}\right. \\ &\qquad\qquad \left. + 2\sqrt{n\sigma^2+1}\phi\left(\frac{\sqrt{n}\sigma(Z-\sqrt{n}|\theta|)}{\sqrt{n\sigma^2+1}}\right)\right\}^2\Bigg], \end{aligned}$$

$$R^{(1)}(\theta, \varphi_{\mathrm{NP}}) = |\theta|\Phi((u_\alpha - \sqrt{n}\theta)\mathrm{sgn}\theta), \quad R^{(1)}(\theta, p) = |\theta|E_Z[\{\Phi(Z - \sqrt{n}|\theta|)\}^2].$$

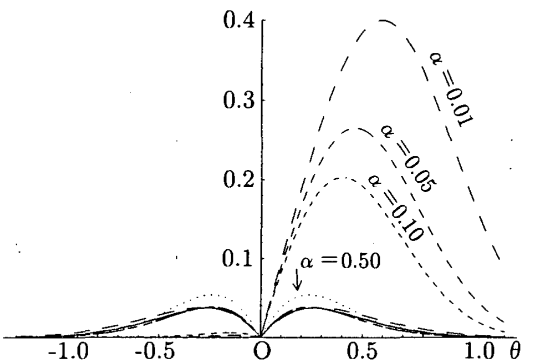

Figure 2.6. The risks of the Bayes procedures, the N-P tests and the p-value when $n = 10$. The risks of the N-P tests also are given for $\alpha = 0.01$ (the longest dashes), $0.05, 0.10$ (the shortest dashes), 0.50 (dots). The risks of the Bayes procedures also are given for the prior variances $\sigma^2 = 0.18^2$ (the longest dashes), $0.26^2, 1.50^2, 3.00^2$ (the shortest dashes).

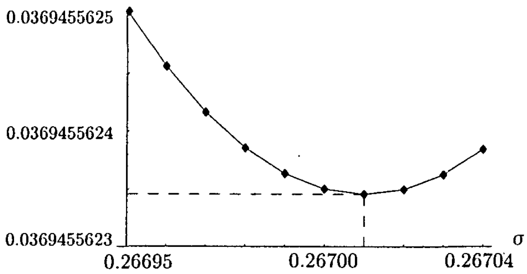

Figure 2.7. The minimax solution $\sigma_0 \approx 0.267$ and the minimax risk $R^{(1)}(0.259, \varphi^*_{\pi_{0.267}}) \approx 0.03695$ ($\sigma = 0.26695$ (0.00001) $0.26704, n = 10$).

Table 2.2. The minimax solution and comparison between the minimax risk of the Bayes procedure and the maximum risk of the p-value.

n	σ_0	The minimax risk of the Bayes		The maxi-risk of the p-value
1	0.844	$R^{(1)}(0.818, \varphi^*_{\pi_{\sigma_0}}) \approx 0.116832$	$<$	$R^{(1)}(0.813, p) \approx 0.116880$
5	0.378	$R^{(1)}(0.366, \varphi^*_{\pi_{\sigma_0}}) \approx 0.052249$	$<$	$R^{(1)}(0.363, p) \approx 0.052271$
10	0.267	$R^{(1)}(0.259, \varphi^*_{\pi_{\sigma_0}}) \approx 0.036946$	$<$	$R^{(1)}(0.257, p) \approx 0.036961$
15	0.218	$R^{(1)}(0.211, \varphi^*_{\pi_{\sigma_0}}) \approx 0.030166$	$<$	$R^{(1)}(0.210, p) \approx 0.030178$
20	0.189	$R^{(1)}(0.183, \varphi^*_{\pi_{\sigma_0}}) \approx 0.026124$	$<$	$R^{(1)}(0.182, p) \approx 0.026135$
25	0.169	$R^{(1)}(0.164, \varphi^*_{\pi_{\sigma_0}}) \approx 0.023366$	$<$	$R^{(1)}(0.163, p) \approx 0.023376$

Note that $R^{(1)}(\theta, \varphi^*_{\pi_\sigma})$ and $R^{(1)}(\theta, p)$ are symmetric about $\theta = 0$. The risks $R^{(1)}(\theta, \varphi^*_{\pi_\sigma})$, $R^{(1)}(\theta, \varphi_{\mathrm{NP}})$ and $R^{(1)}(\theta, p)$ are continuous in θ, but they are not differentiable at $\theta = 0$ (see Figure 2.6). The tendency of the risks are similar to that in the case (ii). In a similar way to the case (ii), we also get the approximate value of the minimax solution of the Bayes procedure (see Figure 2.7), and can compare the values between the minimax risk of the Bayes procedure and the maximum risk of the p-value (see Table 2.2).

Example 2.2 (Exponential distribution). Suppose that $X_1, \ldots, X_n$ are *i.i.d.* random variables according to the exponential distribution with the density $p(x, \theta) = e^{-(x-\theta)}$ $(x > \theta; \theta \in \mathbf{R}^1)$. Let π_τ be a prior density of the uniform distribution $U(-\tau, \tau)$. Since the joint density of $\boldsymbol{X} := (X_1, \ldots, X_n)$ is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}, \theta) = e^{-n(\bar{x}-\theta)} \chi_{(\theta,\infty)}(x_{(1)}) \propto e^{n\theta} \chi_{(-\infty, x_{(1)})}(\theta)$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $x_{(1)} := \min_{1 \le i \le n} x_i$, it follows that the posterior density of θ, given $X_{(1)} = x_{(1)}$, is

$$p(\theta \mid x_{(1)}) = \frac{e^{n\theta}\chi_{(-\infty, x_{(1)})}(\theta)\frac{1}{2\tau}\chi_{(-\tau,\tau)}(\theta)}{\int e^{n\theta}\chi_{(-\infty, x_{(1)})}(\theta)\frac{1}{2\tau}\chi_{(-\tau,\tau)}(\theta)d\theta} = \frac{e^{n\theta}\chi_{(-\tau,\tau\wedge x_{(1)})}(\theta)}{\int_{-\tau}^{\tau\wedge x_{(1)}} e^{n\theta}d\theta},$$

where $\tau \wedge x_{(1)} = \min\{\tau, x_{(1)}\}$. If γ is a non-negative integer, it follows from (2.2) that the Bayes procedure is given by

$$\varphi_\pi^*\left(x_{(1)}\right) = \frac{\int_{-\tau}^{\min\{0,\tau,x_{(1)}\}} |\theta|^\gamma e^{n\theta}d\theta}{\int_{-\tau}^{\tau\wedge x_{(1)}} |\theta|^\gamma e^{n\theta}d\theta} = \begin{cases} 1 & \text{for } x_{(1)} \le 0, \\ \dfrac{\Gamma(\gamma+1;n\tau)}{\Gamma(\gamma+1;n\tau)+\Gamma^*(\gamma+1;nx_{(1)})} & \text{for } 0 < x_{(1)} \le \tau, \\ \dfrac{\Gamma(\gamma+1;n\tau)}{\Gamma(\gamma+1;n\tau)+\Gamma^*(\gamma+1;n\tau)} & \text{for } x_{(1)} > \tau, \end{cases} \tag{2.8}$$

where

$$\Gamma(x;a) := \int_0^a t^{x-1}e^{-t}dt = (x-1)! - e^{-a}\sum_{i=0}^{x-1}\frac{(x-1)!}{(x-i-1)!}a^{x-i-1},$$

$$\Gamma^*(x;a) := \int_0^a t^{x-1}e^{t}dt = (-1)^x(x-1)! + e^{a}\sum_{i=0}^{x-1}(-1)^i\frac{(x-1)!}{(x-i-1)!}a^{x-i-1}$$

$$(x = 1, 2, \dots; a > 0).$$

(i) Case $\gamma = 0$. From (2.8) we have the Bayes procedure

$$\varphi_\pi^*(x_{(1)}) = \begin{cases} 1 & \text{for } x_{(1)} \le 0, \\ \dfrac{b-1}{be^{nx_{(1)}}-1} & \text{for } 0 < x_{(1)} \le \tau, \\ \dfrac{1}{b+1} & \text{for } x_{(1)} > \tau, \end{cases} \tag{2.9}$$

where $b := b(n,\tau) = e^{n\tau}$. The N-P test is the indicator of the acceptance region of the uniformly powerful test of level α given by

$$\varphi_{\mathrm{NP}}(x_{(1)}) = \begin{cases} 1 & \text{for } x_{(1)} \le \frac{1}{n}\log\frac{1}{\alpha}, \\ 0 & \text{for } x_{(1)} > \frac{1}{n}\log\frac{1}{\alpha}, \end{cases}$$

and also the p-value is

$$p(x_{(1)}) = P_0\{X_{(1)} > x_{(1)}\} = \int_{0\vee x_{(1)}}^\infty ne^{-nt}dt = \begin{cases} 1 & \text{for } x_{(1)} \le 0, \\ e^{-nx_{(1)}} & \text{for } x_{(1)} > 0, \end{cases}$$

where $0 \vee x_{(1)} = \max\{0, x_{(1)}\}$. The risks of the Bayes procedure, the N-P test and the p-value are given by

$$(2.10)\quad R^{(0)}(\theta, \varphi_\pi^*) = \begin{cases} b^2 e^{n\theta}\left\{2(b-1)\log\left(\dfrac{b}{b+1}\right) + \dfrac{2b^2+b-2}{(b+1)^2}\right\} & \text{for } \theta \le 0, \\ -e^{n\theta}\left\{2b(b-1)^2\log\left(\dfrac{b^2-1}{b^2-be^{-n\theta}}\right) + \dfrac{b(2b^2-3)}{(b+1)^2}\right\} & \\ \quad +(b-1)^2\dfrac{2be^{n\theta}-1}{be^{n\theta}-1} & \text{for } 0 < \theta \le \tau, \\ \dfrac{1}{(b+1)^2} & \text{for } \theta > \tau, \end{cases}$$

$$(2.11)\quad R^{(0)}(\theta, \varphi_{\mathrm{NP}}) = \begin{cases} \alpha e^{n\theta} & \text{for } \theta \le 0, \\ 1 - \alpha e^{n\theta} & \text{for } 0 < \theta \le \frac{1}{n}\log\frac{1}{\alpha}, \\ 0 & \text{for } \theta > \frac{1}{n}\log\frac{1}{\alpha}, \end{cases}$$

$$(2.12)\quad R^{(0)}(\theta, p) = \begin{cases} \frac{1}{3}e^{n\theta} & \text{for } \theta \le 0, \\ \frac{1}{3}e^{-2n\theta} & \text{for } \theta > 0. \end{cases}$$

From (2.10) to (2.12), the risks of the Bayes procedure $\varphi_\pi^*(x_{(1)})$, the N-P test $\varphi_{\mathrm{NP}}(x_{(1)})$ and the p-value $p(x_{(1)})$ are illustrated as Figures 2.8 and 2.9. As is seen in them, the tendency of their risks is similar to the case $\gamma = 0$ in testing hypothesis on the normal mean (see Figure 2.1).

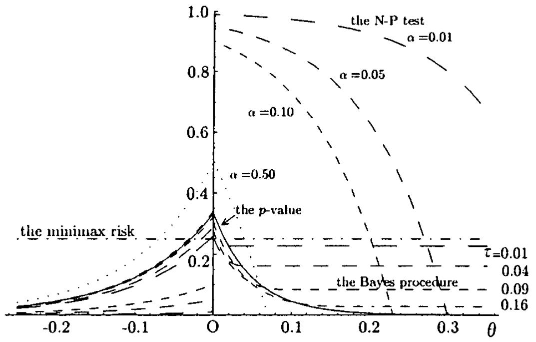

Figure 2.8. The risks of the Bayes procedures, the N-P tests and the p-value when $n = 10$.

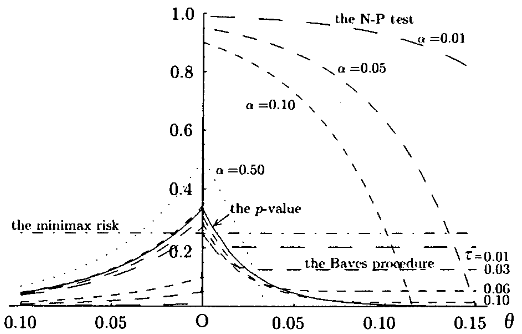

Figure 2.9. The risks of the Bayes procedures, the N-P tests and the p-value when $n = 20$.

(ii) Case $\gamma = 1$. From (2.8) we have the Bayes procedure

$$\varphi_\pi^*(x_{(1)}) = \begin{cases} 1 & \text{for } x_{(1)} \le 0, \\ \dfrac{1 - e^{-n\tau}(n\tau+1)}{2 + e^{nx_{(1)}}(nx_{(1)}-1) - e^{-n\tau}(n\tau+1)} & \text{for } 0 < x_{(1)} \le \tau, \\ \dfrac{1 - e^{-n\tau}(n\tau+1)}{2 + e^{n\tau}(n\tau-1) - e^{-n\tau}(n\tau+1)} & \text{for } x_{(1)} > \tau, \end{cases}$$

which yields its risk

(2.13)

$$R^{(1)}(\theta,\varphi_\pi^*)=\begin{cases} -\theta\left[\dfrac{1+e^{n\tau}(n\tau-1)}{2+e^{n\tau}(n\tau-1)-e^{-n\tau}(n\tau+1)}\right]^2 e^{-n(\tau-\theta)} \\ -\theta\int_0^{n\tau}\left[\dfrac{1+e^{v}(v-1)}{2+e^{v}(v-1)-e^{-n\tau}(n\tau+1)}\right]^2 e^{-v+n\theta}dv \quad \text{for } \theta\le 0, \\ \theta\left[\dfrac{1-e^{-n\tau}(n\tau+1)}{2+e^{n\tau}(n\tau-1)-e^{-n\tau}(n\tau+1)}\right]^2 e^{-n(\tau-\theta)} \\ +\theta\int_{n\theta}^{n\tau}\left[\dfrac{1-e^{-n\tau}(n\tau+1)}{2+e^{v}(v-1)-e^{-n\tau}(n\tau+1)}\right]^2 e^{-v+n\theta}dv \\ \qquad\qquad\qquad\qquad \text{for } 0<\theta\le\tau, \\ \theta\left[\dfrac{1-e^{-n\tau}(n\tau+1)}{2+e^{n\tau}(n\tau-1)-e^{-n\tau}(n\tau+1)}\right]^2 \quad \text{for } \theta>\tau. \end{cases}$$

Both of the risks of the N-P test and the p-value are equal to (2.11) and (2.12) multiplied by $|\theta|$. As is seen in Figures 2.10 and 2.11, the behavior of the risks of the Bayes procedures, the N-P tests and the p-value are similar to the normal case (iii) in Example 2.1 (see Figure 2.6).

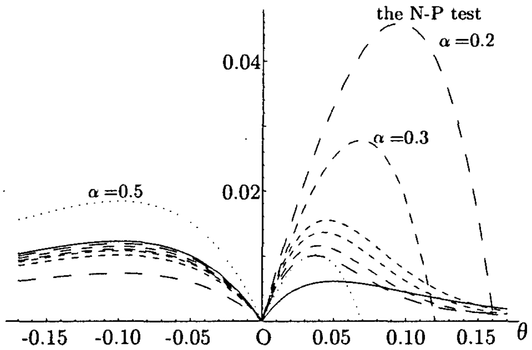

Figure 2.10. The risks of the Bayes procedures, the N-P tests and the p-value when $n = 10$.

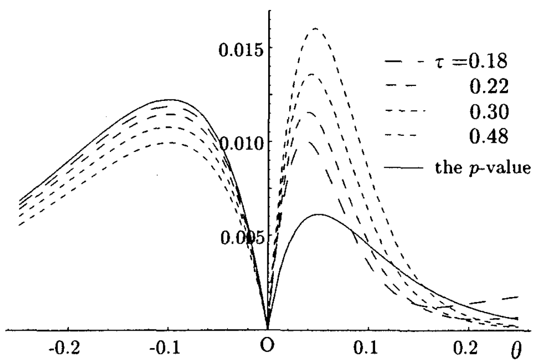

Figure 2.11. The risks of the Bayes procedures and the p-value when $n = 10$.

(iii) Case $\gamma = 2$. From (2.8) we have the Bayes procedure

(2.14)

$$\varphi_\pi^*(x_{(1)})=\begin{cases} 1 & \text{for } x_{(1)}\le 0, \\ \dfrac{2-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}{e^{nx_{(1)}}\{(nx_{(1)})^2-2nx_{(1)}+2\}-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}} & \text{for } 0<x_{(1)}\le\tau, \\ \dfrac{2-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}{e^{n\tau}\{(n\tau)^2-2n\tau+2\}-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}} & \text{for } x_{(1)}>\tau, \end{cases}$$

which yields its risk

(2.15) $$R^{(2)}(\theta,\varphi_\pi^*)=$$

$$\begin{cases} \theta^2\left[\dfrac{e^{n\tau}\{(n\tau)^2-2n\tau+2\}-2}{e^{n\tau}\{(n\tau)^2-2n\tau+2\}-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}\right]^2 e^{-n(\tau-\theta)} \\ \quad +\theta^2\int_0^{n\tau}\left[\dfrac{e^v(v^2-2v+2)-2}{e^v(v^2-2v+2)-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}\right]^2 e^{-v+n\theta}dv \\ \hfill \text{for } \theta\le 0, \\ \theta^2\left[\dfrac{2-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}{e^{n\tau}\{(n\tau)^2-2n\tau+2\}-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}\right]^2 e^{-n(\tau-\theta)} \\ \quad +\theta^2\int_{n\theta}^{n\tau}\left[\dfrac{2-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}{e^v(v^2-2v+2)-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}\right]^2 e^{-v+n\theta}dv \\ \hfill \text{for } 0<\theta\le\tau, \\ \theta^2\left[\dfrac{2-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}{e^{n\tau}\{(n\tau)^2-2n\tau+2\}-e^{-n\tau}\{(n\tau)^2+2n\tau+2\}}\right]^2 \qquad \text{for } \theta>\tau. \end{cases}$$

Both of the risks of the N-P test and the p-value are equal to (2.11) and (2.12) multiplied by θ^2. As is illustrated in Figures 2.12 and 2.13, the behavior of the risks of the Bayes procedures, the N-P tests and the p-value are similar to the normal case (ii) in Example 2.1 (see Figure 2.2).

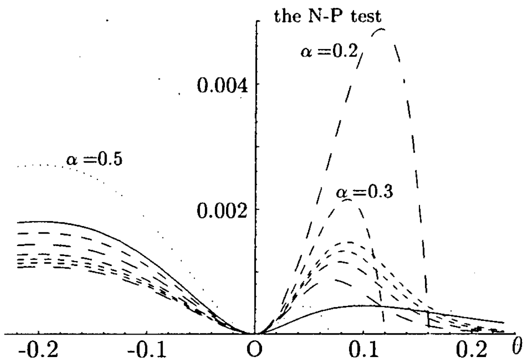

Figure 2.12. The risks of the Bayes procedures, the N-P tests and the p-value when $n = 10$.

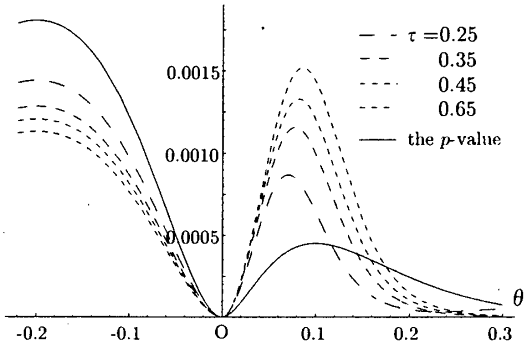

Figure 2.13. The risks of the Bayes procedures and the p-value when $n = 10$.

3. Risk with respect to the loss $\tilde{L}^{(\gamma)}$: Scale parameter case

In this section we consider the scale parameter case. Suppose that a real random variable X is distributed according to a density function $(1/\theta)f(t/\theta)$ (*w.r.t.* the Lebesgue measure), where $\theta > 0$. Let π be a prior density of θ (*w.r.t.* the Lebesgue measure). Then we consider the problem of testing the hypothesis H: $\theta \le \theta_0$ against the alternative K: $\theta > \theta_0$. Letting $\Theta_0 = (0, \theta_0]$, from (1.4) we obtain as the risk of a decision procedure φ

$$\tilde{R}^{(\gamma)}(\theta,\varphi) = E_\theta[\tilde{L}^{(\gamma)}(\theta,\varphi)] = \left(\frac{\theta}{\theta_0}\right)^\gamma E_\theta[\{\chi_{\Theta_0}(\theta)-\varphi(X)\}^2]$$

and as its Bayes risk *w.r.t.* π

$$\tilde{r}_\pi^{(\gamma)}(\varphi) = E_\pi[\tilde{R}^{(\gamma)}(\theta,\varphi)] = \int_\Theta \tilde{R}^{(\gamma)}(\theta,\varphi)\pi(\theta)d\theta.$$

Without loss of generality we assume that $\theta_0 = 1$. In a similar way to the location parameter case, we have the Bayes procedure

$$\varphi_\pi^*(x) = \int_0^1 \theta^\gamma \frac{1}{\theta} f\left(\frac{x}{\theta}\right)\pi(\theta)d\theta \Big/ \int_0^\infty \theta^\gamma \frac{1}{\theta} f\left(\frac{x}{\theta}\right)\pi(\theta)d\theta. \tag{3.1}$$

Since the posterior density of θ, given $X = x$, is also given by

$$p(\theta \mid x) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right)\pi(\theta) \Big/ \int_0^\infty \frac{1}{\theta} f\left(\frac{x}{\theta}\right)\pi(\theta)d\theta,$$

it follows from (3.1) that

$$\varphi_\pi^*(x) = \int_0^1 \theta^\gamma p(\theta \mid x)d\theta \Big/ \int_0^\infty \theta^\gamma p(\theta \mid x)d\theta. \tag{3.2}$$

Example 3.1 (Normal standard deviation). Suppose that $X_1, \dots, X_n$ are *i.i.d.* random variables with the normal distribution $N(0, \sigma^2)$. Let $\tau := 1/(2\sigma^2)$. Suppose that the prior density $\tilde{\pi}_a$ is the gamma density

$$\tilde{\pi}_a(\tau) = b^a \tau^{a-1} e^{-b\tau}/\Gamma(a) \quad (\tau > 0;\ b = 2a - 2 > 0)$$

whose distribution is denoted by $G(a, b)$. Then the prior density of σ is

$$\pi_a(\sigma) = \frac{b^a}{2^{a-1}\Gamma(a)} \sigma^{-2a-1} e^{-b/(2\sigma^2)} \quad (\sigma > 0).$$

Put $Y := \sum_{i=1}^n X_i^2$. Since Y/σ^2 is distributed according to the chi-square distribution with n degrees of freedom, the posterior density of σ, given $Y = y$, is given by

$$p(\sigma \mid y) = \frac{\sigma^{-(n+2a+1)} e^{-(b+y)/(2\sigma^2)}}{\int_0^\infty \sigma^{-(n+2a+1)} e^{-(b+y)/(2\sigma^2)} d\sigma}.$$

Then it follows from (3.2) that the Bayes procedure is given by

$$\begin{aligned} \varphi_{\pi_a}^*(y) &= \frac{\int_0^1 \sigma^\gamma p(\sigma \mid y)d\sigma}{\int_0^\infty \sigma^\gamma p(\sigma \mid y)d\sigma} = 1 - \frac{\int_1^\infty \sigma^\gamma p(\sigma \mid y)d\sigma}{\int_0^\infty \sigma^\gamma p(\sigma \mid y)d\sigma} \\ &= 1 - \frac{\int_1^\infty \sigma^{-(n+2a-\gamma+1)} e^{-(b+y)/(2\sigma^2)} d\sigma}{\int_0^\infty \sigma^{-(n+2a-\gamma+1)} e^{-(b+y)/(2\sigma^2)} d\sigma} \\ &= 1 - \frac{\int_0^{b+y} t^{(n+2a-\gamma)/2-1} e^{-t/2} dt}{\int_0^\infty t^{(n+2a-\gamma)/2-1} e^{-t/2} dt} \\ &= 1 - F(b+y; (n+2a-\gamma)/2, 1/2) \\ &= Q(b+y; (n+2a-\gamma)/2, 1/2), \end{aligned} \tag{3.3}$$

where $F(\cdot\,; (n+2a-\gamma)/2, 1/2)$ and $Q(\cdot\,; (n+2a-\gamma)/2, 1/2)$ are the cumulative distribution function (c.d.f.) and the upper probability of the gamma distribution

$G((n+2a-\gamma)/2, 1/2)$, respectively. If we take a and γ such that $n+2a-\gamma$ is a natural number, we rewrite from (3.3)

$$\varphi^*_{\pi_a}(y) = 1 - F_{n+2a-\gamma}(b+y) = Q_{n+2a-\gamma}(b+y),$$

since $G((n+2a-\gamma)/2, 1/2)$ is equal to the chi-square distribution $\chi^2_{n+2a-\gamma}$ with $n+2a-\gamma$ degrees of freedom, where $F_{n+2a-\gamma}$ and $Q_{n+2a-\gamma}$ are its c.d.f. and its upper probability, respectively.

Now we consider the problem of testing the hypothesis H: $\sigma \le \sigma_0$ against the alternative K: $\sigma > \sigma_0$. Then the N-P test and the p-value are given by

$$\varphi_{\mathrm{NP}}(y) = \chi_{(0,\chi^2_\alpha(n)]}(y), \tag{3.4}$$

$$p(y) = P_1\{Y \ge y\} = 1 - F_n(y) = Q_n(y), \tag{3.5}$$

respectively, where $\chi^2_\alpha(n)$ is the upper 100α percentage point of the chi-square distribution χ^2_n, and F_n and Q_n are its c.d.f. and its upper probability, respectively.

(i) Case $\gamma = 0$. It follows from (3.3) to (3.5) that the risks of the Bayes procedure $\varphi^*_{\pi_a}(y)$, the N-P test $\varphi_{\mathrm{NP}}(y)$ and the p-value $p(y)$ are

$$\tilde{R}^{(0)}(\sigma, \varphi^*_{\pi_a}) = \begin{cases} E_{Z_n}[\{F(b+\sigma^2 Z_n; (n+2a)/2, 1/2)\}^2] & \text{for } \sigma \le 1, \\ E_{Z_n}[\{Q(b+\sigma^2 Z_n; (n+2a)/2, 1/2)\}^2] & \text{for } \sigma > 1, \end{cases}$$

$$\tilde{R}^{(0)}(\sigma, \varphi_{\mathrm{NP}}) = \begin{cases} E_\sigma[\{\chi_{(\chi^2_\alpha(n),\infty)}(Y)\}^2] = Q_n(\chi^2_\alpha(n)/\sigma^2) & \text{for } \sigma \le 1, \\ E_\sigma[\{\chi_{(0,\chi^2_\alpha(n)]}(Y)\}^2] = F_n(\chi^2_\alpha(n)/\sigma^2) & \text{for } \sigma > 1, \end{cases} \tag{3.6}$$

$$\tilde{R}^{(0)}(\sigma, p) = \begin{cases} E[\{F_n(\sigma^2 Z_n)\}^2] & \text{for } \sigma \le 1, \\ E[\{Q_n(\sigma^2 Z_n)\}^2] & \text{for } \sigma > 1, \end{cases} \tag{3.7}$$

where Z_n is a random variable according to the chi-square distribution χ^2_n. The behavior of the risks of $\varphi^*_{\pi_a}(y)$, $\varphi_{\mathrm{NP}}(y)$ and $p(y)$ are given in Figures 3.1 and 3.2. As is seen in them, the tendency of their risks is similar to the case $\gamma = 0$ in testing hypothesis about normal mean (see Figure 2.1), and the behavior of the risk of the Bayes procedure seems to be stable.

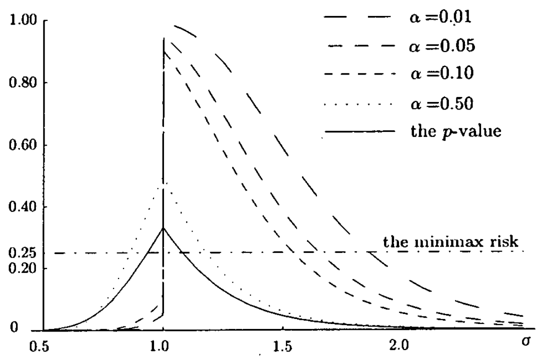

Figure 3.1. The risks of the N-P tests and the p-value when $n = 10$.

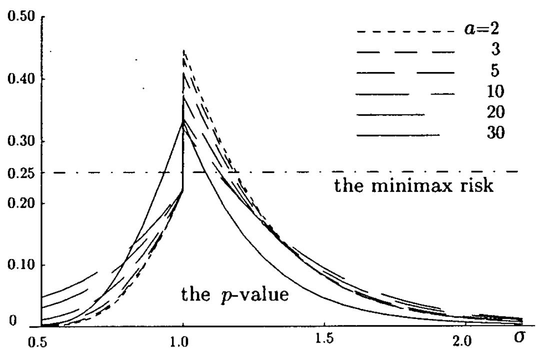

Figure 3.2. The risks of the Bayes procedures and the p-value when $n = 10$.

(ii) Case $\gamma = 2$. In a similar way to the case (i), it follows from (3.3) that

$$\begin{aligned}\varphi^*_{\pi_a}(y) &= 1 - F(b+y; (n+2a-2)/2, 1/2)\\ &= Q(b+y; (n+2a-2)/2, 1/2),\end{aligned}$$

and its risk is

$$\tilde{R}^{(2)}(\sigma, \varphi^*_{\pi_a}) = \begin{cases} \sigma^2 E_{Z_n}[\{F(b+\sigma^2 Z_n; (n+2a-2)/2, 1/2)\}^2] & \text{for } \sigma \le 1,\\ \sigma^2 E_{Z_n}[\{Q(b+\sigma^2 Z_n; (n+2a-2)/2, 1/2)\}^2] & \text{for } \sigma > 1.\end{cases}$$

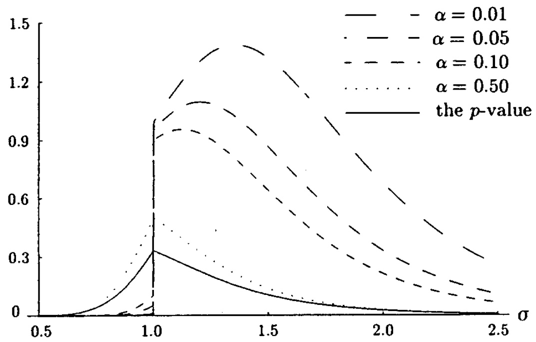

Figure 3.3. The risks of the N-P tests and the p-value when $n = 10$.

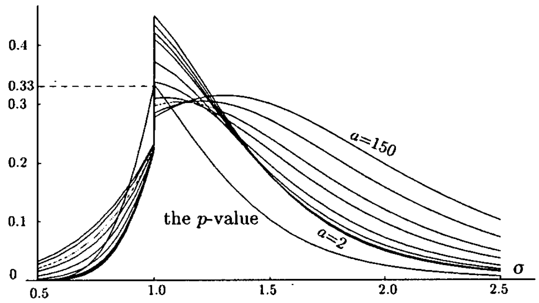

Figure 3.4. The risks of the Bayes procedure and the p-value when $n = 10$ ($a = 2, 3, 4, 5, 10, 20, 40, 60, 100, 150$).

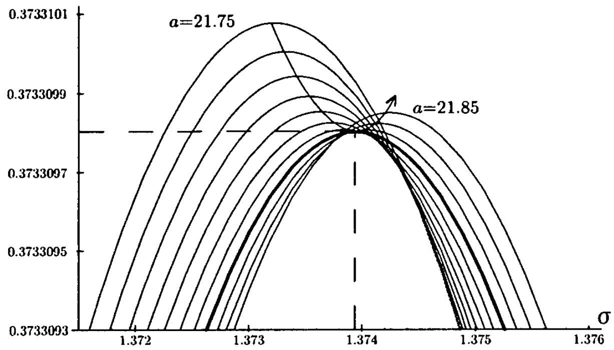

Figure 3.5. The minimax risk of the Bayes procedure $\tilde{R}^{(2)}(1.37, \varphi^*_{\pi_{21.8}}) \approx 0.37331$ ($n = 5, a = 21.50$ (0.01) 21.85).

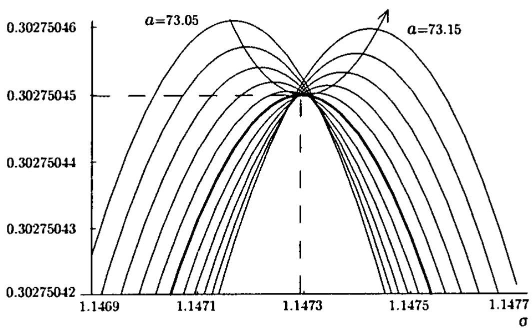

Figure 3.6. The minimax risk of the Bayes procedure $\tilde{R}^{(2)}(1.15, \varphi^*_{\pi_{73.1}}) \approx 0.30275$ ($n = 10$, $a = 73.05$ (0.01) 73.15).

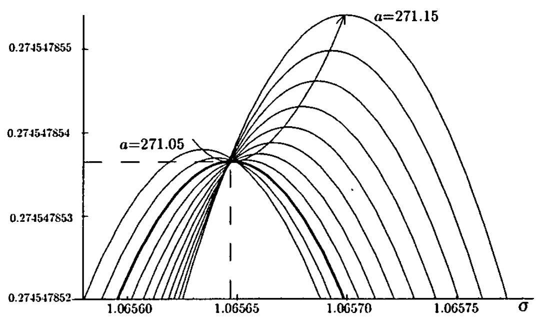

Figure 3.7. The minimax risk of the Bayes procedure $\tilde{R}^{(2)}(1.07, \varphi^*_{\pi_{271.1}}) \approx 0.27455$ ($n = 20$, $a = 271.05$ (0.01) 271.15).

The risks of the N-P test and the p-value are equal to (3.6) and (3.7) multiplied by σ^2, respectively. The behavior of the risks of $\varphi^*_{\pi_a}(y)$, $\varphi_{\mathrm{NP}}(y)$ and $p(y)$ are given in Figures 3.3 and 3.4, which are seen to be a little bit different from Figures 3.1 and 3.2.

In a similar way to the location parameter case, we can obtain numerically the minimax solution. If $n=5$, then

$$\begin{aligned}\inf_a \sup_\sigma \tilde{R}^{(2)}(\sigma, \varphi^*_{\pi_a}) &= \tilde{R}^{(2)}(1.37, \varphi^*_{\pi_{21.8}}) \approx 0.37331 \\ &> 0.33333 = \sup_\sigma \tilde{R}^{(2)}(\sigma, p) = \tilde{R}^{(2)}(1, p),\end{aligned}$$

which means that the maximum risk of the minimax Bayes procedure is bigger than that of the p-value (see Figure 3.5). But, for large n, it is not so (see Figure 3.6 for $n=10$ and Figure 3.7 for $n=20$).

4. Remarks

In the previous sections we discuss the risk of testing procedures under the weighted loss from the viewpoint of set estimation. In the cases of the normal distribution $N(\theta, 1)$ and the exponential distribution with a location parameter θ, we consider the one-sided problem of testing the hypothesis $H: \theta \leq \theta_0$ against the alternative $K: \theta > \theta_0$. Without loss of generality we assume that $\theta_0 = 0$. As is stated in Example 2.1, the weighted loss $L^{(2)}$ seems to be more suitable than the quadratic loss $L^{(0)}$. Indeed, for example, the risks of the Bayes procedures are smooth and take zero value at $\theta = 0$, which means that, in a neighborhood of $\theta = 0$, both of the hypotheses would be nearly acceptable, and the risks are comparatively sensitive for a moderately far area from $\theta = 0$.

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