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On the Number of Edges in a Minimum C_6-Saturated Graph

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Abstract A graph is said to be C_k-saturated if it contains no cycles of length k but does contain such a cycle after the addition of any edge in the complement of the graph. Determining the minimum size of C_k-saturated graphs is one of the interesting problems on extremal graphs. The exact minimum sizes are known for $k = 3, 4$ and 5, but only general bounds are shown for $k \geq 6$. This paper deals with bounds of the minimum size when $k = 6$. It is shown that the minimum size of a C_6-saturated graph on n vertices is no greater than $\lfloor \frac{3n-3}{2} \rfloor$ and no less than $\lceil \frac{7n}{6} \rceil - 2$. This lower bound, which is first proposed exclusively for the C_6-saturated graphs, significantly improves the best previously known lower bound.

Keywords C_6-saturated graph, extremal graph, forbidden graph, cycle-free

Mathematics Subject Classification (2000) 05C35 · 05C38

1 Introduction

For a graph family $\mathcal{F}$, a given graph G is said to be $\mathcal{F}$*-saturated* if G contains no copy of member $F \in \mathcal{F}$, but for any edge e in the complement of G, the graph adding edge e to G contains a copy of $F \in \mathcal{F}$. The minimum and maximum numbers of edges in an $\mathcal{F}$-saturated graph with n vertices are denoted by $\mathrm{sat}(n, \mathcal{F})$ and $\mathrm{ex}(n, \mathcal{F})$, respectively. For convenience, if $\mathcal{F} = \{F\}$, then we denote F-saturated, $\mathrm{sat}(n, F)$ and $\mathrm{ex}(n, F)$. In 1941, P. Turán determined $\mathrm{ex}(n, K_p)$, where K_p stands for a complete graph of p vertices. In 1964, Erdös, Hajnal and Moon [11] introduced the parameter $\mathrm{sat}(n, F)$ and determined the value of $\mathrm{sat}(n, K_p)$. There are a few graphs as F for which $\mathrm{sat}(n, F)$ is known exactly. For example, stars [15], paths [9,10,12,15], and bipartite graphs[4] are studied for F.

Cycle-saturated graphs have been investigated by various researchers. Let C_k denote a cycle on k vertices. Exact values of $\mathrm{sat}(n, C_k)$ are known for a few k. Since $C_3 \cong K_3$, $sat(n, C_3) = n-1$ holds by the result of Erdös, Hajnal and Moon [11]. Ollmann [17] continued to consider the case of C_4, and proved that $\mathrm{sat}(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$ for $n \geq 5$. Later, Tuza[18] provided a shortened proof of this result. Recently, Chen solved the case of C_5 [3,5]. She established that $\mathrm{sat}(n, C_5) = \lceil \frac{10}{7}(n-1) \rceil$. For the case of C_n, Bondy [2] first showed that $\mathrm{sat}(n, C_n) \geq \lceil \frac{3n}{2} \rceil$. This result was later improved to $\mathrm{sat}(n, C_n) = \lfloor \frac{3n+1}{2} \rfloor$ for $n \geq 20$ by some researchers [6–8,16]. No other exact values seem to be known. Faudree, Faudree and Schmitt [14] said that finding exact value of $\mathrm{sat}(n, C_k)$ for some k is quite difficult. For a general k, some bounds of $\mathrm{sat}(n, C_k)$ are discussed. Barefoot et al. [1] proved that $n(1 + \frac{1}{2k+8}) \leq \mathrm{sat}(n, C_k) \leq c_k n + \mathrm{O}(n)$ for $k \geq 6$, where c_k is $2 - \frac{1}{2^{k/2}-2}$ if k is even, and $2 - \frac{1}{3 \cdot 2^{(k-3)/2}-2}$ if k is odd. They also showed tighter bounds for some specified k. Füredi and

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Kim [13] improved their lower bound to $(1+\frac{1}{k+2})n-1 < \mathrm{sat}(n,C_k) < (1+\frac{1}{k-4})n+\binom{k-4}{2}$ for $k \geq 7$ and $n \geq 2k-5$. However, the gap still exists between the lower bound and the upper bound. Thus, it remains an interesting problem to determine non-trivial bounds of $\mathrm{sat}(n,C_k)$ for some k.

In this paper, we discuss new bounds for $\mathrm{sat}(n,C_6)$. The best previously known upper bound of $\mathrm{sat}(n,C_6)$ was $\frac{3n}{2}$ for $n \geq 11$ due to Barefoot et al. [1]. They constructed C_6-saturated graphs for any $n(\geq 11)$ by using an idea of constructing C_k-builders. This idea is also applied in our discussion and leads to a new upper bound

$$\mathrm{sat}(n,C_6) \leq \lfloor \frac{3n-3}{2} \rfloor$$

when $n \geq 9$. At first sight, our result has not much modification. However, this new upper bound provides a tight bound for small n. A lower bound for $\mathrm{sat}(n,C_6)$ can be yielded by employing the result for $\mathrm{sat}(n,C_k)$ of Barefoot et al. [1]. Unfortunately, to our knowledge, no other lower bound of C_6 is known. In order to improve the lower bound, we introduce a new idea: instead of dealing with C_6-saturated graphs directly, we first analyze the number of edges in a C_6-saturated graph whose minimum degree is no less than two, and then propose a lower bound for $\mathrm{sat}(n,C_6)$. By using this new idea, we obtain a new lower bound

$$\mathrm{sat}(n,C_6) \geq \lceil \frac{7n}{6} \rceil - 2$$

when $n \geq 6$ that improves the lower bound $\frac{21}{20}n$ given in Barefoot et al.[1]. We believe that our lower bound is first proposed for the case of C_6. Our result also implies that the bounds given by Füredi and Kim [13] hold for $k=6$.

2 Preliminaries

Given a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set. The cardinalities of $V(G)$ and $E(G)$ are called the *order* and *size* of G, respectively. Especially, we denote $e(G) = |E(G)|$. For a subset A of $V(G)$, $G[A]$ represents the subgraph of G induced by A. We denote a path through vertices $u_1, u_2, ..., u_k$ by $P(u_1, u_2, ..., u_k)$, and a cycle through $u_1, u_2, ..., u_k, u_1$ by $C_k(u_1, u_2, ..., u_k, u_1)$. Ignoring the set of vertices, a cycle on k vertices will be denoted by C_k.

The distance between two vertices u and v is denoted by $\mathrm{dist}(u,v)$. If A and B are two subsets of $V(G)$ we denote by $e(A,B)$ the number of edges of $E(G)$ with one end vertex in A and the other in B, that is,

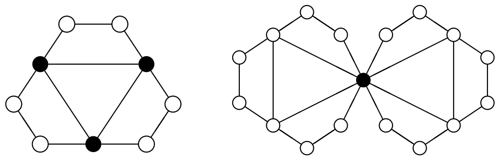
$$e(A,B) = |\{(u,v) \in E(G) \mid u \in A, v \in B\}|.$$

Given a vertex $v \in V(G)$, we denote by $N(v)$ the set of neighbors of v, that is, $N(v) = \{u \in V(G) \mid (v,u) \in E(G)\}$, and by $d(v)$ the cardinality of $N(v)$, called the *degree* of v. Let $\delta(G) = \min\{d(v) \mid v \in V(G)\}$ be the minimum degree.

3 An upper bound

This section slightly modifies the upper bound for $\mathrm{sat}(n,C_6)$ from [1] by constructing C_6-saturated graphs. Note that a graph G is C_6-saturated if and only if there is no path of length 5 joining any adjacent two vertices, while there is a path of length 5 joining any nonadjacent two vertices.

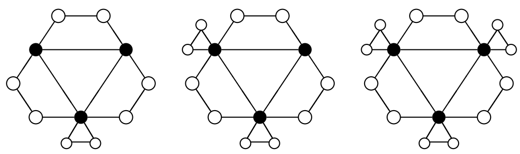
As mentioned in Barefoot et al. [1], a C_k-saturated graph G is said to be a *C_k-builder* if G has a vertex, called a *joint vertex*, such that if the joint vertices of two copies of G are identified then the resulting graph is still C_k-saturated. Note that a graph obtained from s $(s \geq 1)$ copies of a C_k-builder by identifying joint vertices is also C_k-saturated. For example, the graph B_1 in Fig. 1 (a), shown originally in [1], is a C_6-builder, where one of the bold vertices is a joint vertex. This fact can be verified by the existence of paths of length 2 and 3 from the joint vertex to every vertex nonadjacent to the joint vertex, and by the existence of paths with length 1, 3, and 4 from the joint vertex to every vertex adjacent to the joint vertex. The graph B_2 in Fig. 1 (b) is obtained from two copies of B_1 by identifying one of the bold vertices as joint vertices. Moreover, a graph attaching one triangle (i.e., K_3) to



(a) B_1 (b) B_2

Fig. 1 An example of C_6-builder; (a) a C_6-builder shown originally in [1]. (b) a graph B_2 identified joint vertices of two copies of B_1.

B_1 at each of some of the bold vertices is also C_6-saturated, because there exists a path of length 3 from a bold vertex to every other vertex. Such graphs are denoted by $B_\triangle, B_{\triangle^2}$ and $B_{\triangle^3}$ as shown in Fig. 2.

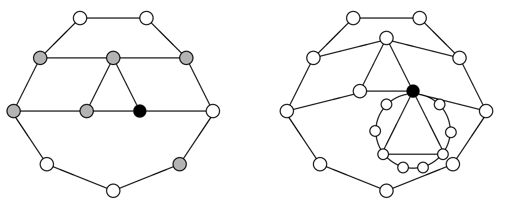


(a) $B_\triangle$ (b) $B_{\triangle^2}$ (c) $B_{\triangle^3}$

Fig. 2 Graphs attaching one triangle to B_1 at each of some of the bold vertices.

We now introduce another notation. Given a C_k-builder G_1, a C_k-saturated graph G is called a *C_k-base graph w.r.t. G_1* if G has a vertex, called a *base vertex*, such that if the base vertex and the joint vertex of G_1 are identified then the resulting graph is still C_k-saturated. It is obvious that a C_k-builder is a base graph w.r.t. itself, with a base vertex identical to its joint vertex. We can see that $B_\triangle$, $B_{\triangle^2}$ and $B_{\triangle^3}$ are also C_6-base graph w.r.t. B_1.

Proposition 1 *Let B_1 and $\underline{B}_0$ are graphs in Fig. 1 (a) and Fig. 3 (a), respectively. Then, $\underline{B}_0$ is a C_6-base graph w.r.t. B_1.*



(a) $\underline{B}_0$ (b) Attaching B_1 to $\underline{B}_0$

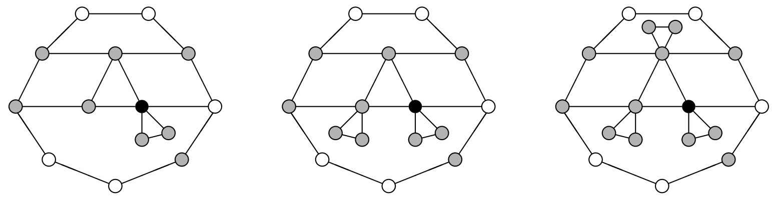
Fig. 3 C_6-base graph w.r.t B_1.

Proof It can be verified that $\underline{B}_0$ is a C_6-saturated graph.

Let the bold vertex in Fig. 3 (a) be a base vertex. We can verify that $\underline{B}_0$ has a path of length 2 from this base vertex to each gray vertex. In addition, $\underline{B}_0$ also has both paths of length 3 and 4 from the base vertex to each white vertex. On the other hand, when we define one of the bold vertices as a joint vertex of B_1, we can verify that B_1 has a path of length 3 between the joint vertex and any other vertex, and also has a path of length 1 or 2 between them. Hence, the graph identifying the joint vertex of B_1 with the base vertex of $\underline{B}_0$, as shown in Fig. 3 (b), has a path of length 5 between nonadjacent vertices, which implies that it is C_6-saturated. $\square$

In a similar discussion, we can obtain three other C_6-base graphs w.r.t B_1.

Corollary 1 *Let $\underline{B}_\triangle, \underline{B}_{\triangle^2}$ and $\underline{B}_{\triangle^3}$ be a graph represented in Fig. 4 (a), (b) and (c), respectively. Then, these graphs are C_6-base graphs w.r.t. B_1 in Fig. 1 (a), where the base vertices are represented by bold ones.*



(a) $\underline{B}_\triangle$ (b) $\underline{B}_{\triangle^2}$ (c) $\underline{B}_{\triangle^3}$

Fig. 4 Graphs attaching one triangle (triangles) to $\underline{B}_0$. There exists a path of length 3 from a base vertex to each gray vertex, while there exist both paths of length 3 and 4 from a base vertex to each white vertex.

Theorem 1 $\mathrm{sat}(n, C_6) \le \lfloor \frac{3n-3}{2} \rfloor$, *when $n \ge 9$.*

Proof When n is odd, Barefoot et al. [1] has shown this upper bound as follows. When $n = 8s + r$ for $r \in \{1, 3, 5, 7\}$, a graph obtained by attaching $s - 1$ copies of B_1 to the C_6-base graph B_1, $B_\triangle$, $B_{\triangle^2}$ or $B_{\triangle^3}$ has order $8s + r$ and size $12s + \frac{3r-3}{2}$. Hence, we have $\mathrm{sat}(n, C_6) \le 12s + \frac{3r-3}{2} = 12 \times \frac{n-r}{8} + \frac{3r-3}{2} = \frac{3n-3}{2}$.

Now let n be even. When $n = 10$, the graphs in the Fig. 5 are C_6-saturated with sizes 13. When $n \ge 12$ and $n = 8s+r$ for some $r \in \{2, 4, 6, 8\}$, a C_6-saturated graph with order n and size $12s + \lfloor \frac{3r-3}{2} \rfloor$ can be constructed using C_6-base graphs w.r.t B_1. If $r = 4$, a graph obtained by attaching $s-1$ copies of B_1 to the C_6-base graph $\underline{B}_0$ has order $n = 8(s-1)+12 = 8s+4$ and size $12(s-1) + 16 = 12s + 4$. If $r = 6$, a graph attaching $s-1$ copies of B_1 to $\underline{B}_\triangle$ has order $n = 8(s-1) + 14 = 8s + 6$ and size $12(s-1) + 19 = 12s + 7$. If $r = 8$, a graph obtained by attaching $s-1$ copies of B_1 to $\underline{B}_{\triangle^2}$ has order $n = 8(s-1)+16 = 8s+8$ and size $12(s-1)+22 = 12s+10$. Finally, if $r = 2$, a graph obtained by attaching $s-2$ copies of B_1 to $\underline{B}_{\triangle^3}$ has order $n = 8(s-2)+18 = 8s+2$ and size $12(s-2)+25 = 12s+1$. We then estimate, for even n, $\mathrm{sat}(n, C_6) \le 12s + \lfloor \frac{3r-3}{2} \rfloor = 12 \times \frac{n-r}{8} + \lfloor \frac{3r-3}{2} \rfloor = \frac{3n}{2} - \frac{3r}{2} + \lfloor \frac{3r-3}{2} \rfloor = \lfloor \frac{3n-3}{2} \rfloor$. $\square$

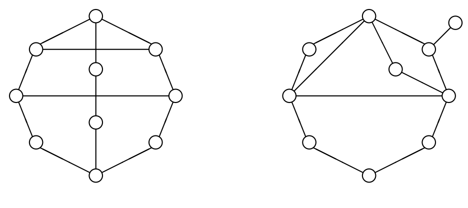


Fig. 5 Minimum C_6-saturated graphs with order 10.

Remark 1 By enumerating all the graphs whose sizes are less than $\lfloor \frac{3n-3}{2} \rfloor$ with small n, our computer search verifies $\mathrm{sat}(n, C_6) = \lfloor \frac{3n-3}{2} \rfloor$ for $n = 9, 10$ and 11. Moreover, minimum C_6-saturated graphs with order 9 are only B_1 shown in Fig. 1 (a) and graphs in Fig 6. When $n = 10$, the minimum C_6-saturated graphs are the only graphs in Fig. 5.

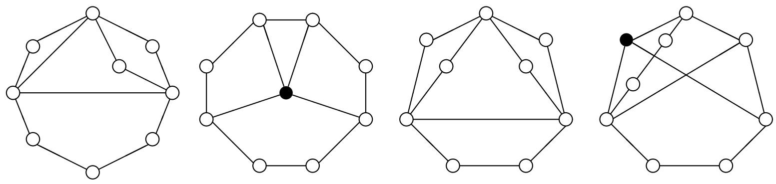


Fig. 6 Minimum C_6-saturated graphs with order 9. Bold vertices imply that they act as joint vertices for C_6-builders.

4 A lower bound

This section gives a lower bound of $\mathrm{sat}(n, C_6)$.

Theorem 2 $\mathrm{sat}(n, C_6) \geq \lceil \frac{7n}{6} \rceil - 2$ *holds.*

To obtain this lower bound, we first estimate the number of edges for a C_6-saturated graph whose minimum degree is greater than 2.

Theorem 3 *Assume that G is a C_6-saturated graph with order n. If $\delta(G) \geq 2$, then we have $e(G) \geq \lceil \frac{5n-7}{4} \rceil$.*

4.1 Proof of Theorem 3

It is obvious when $\delta(G) \geq 3$, since $e(G) = \sum_{v \in V(G)} d(v)/2 \geq 3n/2$. Thus, we shall focus on the case of $\delta(G) = 2$. To prove Theorem 3, for simplicity, we deal with only C_6-saturated graphs without any end block consisting of a cycle, that is, without cycles containing exactly one cut vertex. If an end block consists of a cycle with length more than 4, it does not form a cycle of length 6 when adding some chord in this cycle. If an end block consists of a cycle C with length 3, our lower bound is not affected, because the subgraph H of G deleting C is also C_6-saturated and

$$e(G) = e(H) + 3 \geq \frac{5(n-2)-7}{4} + 3 \geq \frac{5n-7}{4}$$

holds. Thus, without loss of generality, this subsection assumes the following.

Assumption 1 *A graph G is a C_6-saturated graph with order n and with no end block consisting of a cycle.*

We provide some additional notations in order to prove Theorem 3. Suppose that a vertex $r \in V(G)$ has the minimum degree, i.e., $d(r) = \delta(G)$. We call such a vertex r a *root*, and let

$$L^i \stackrel{\text{def}}{=} \{u \in V(G) \mid \mathrm{dist}(r, u) = i\}.$$

Note that $L^0 = \{r\}$ and $L^i = \emptyset$ for $i > 5$, because G is C_6-saturated, that is to say, a pair of nonadjacent vertices is connected by a path of length five. Thus, $V(G)$ can be partitioned into L^0, L^1, L^2, L^3, L^4 and L^5. We further partition each L^i $(i = 1, 2, 3, 4, 5)$ into

$$L^i_S \stackrel{\text{def}}{=} \{u \in L^i \mid e(\{u\}, L^{i-1} \cup L^i) = 1\}, \text{ and } L^i_{\overline{S}} \stackrel{\text{def}}{=} L^i \setminus L^i_S.$$

Our lower bound of $e(G)$ is estimated by summing degrees of vertices. We denote the sum of degrees of vertices in $X (\subseteq V(G))$ by

$$D(X) \stackrel{\text{def}}{=} \sum_{x \in X} d(v).$$

Edges incident to $v \in L^i$ are divided into three classes; edges between L^{i-1} and v, edges between L^i and v, and edges between v and L^{i+1}. According to these classes, we divide $d(v)$ into three parameters:

$$\begin{aligned} d^l(v) &\overset{\text{def}}{=} |\{(u,v) \in E(G) \mid u \in L^{i-1}\}|, \\ d^m(v) &\overset{\text{def}}{=} |\{(u,v) \in E(G) \mid u \in L^{i}\}|, \text{ and} \\ d^r(v) &\overset{\text{def}}{=} |\{(u,v) \in E(G) \mid u \in L^{i+1}\}|. \end{aligned}$$

We also section $D(X)$ for $X(\subseteq V(G))$ into

$$D^l(X) \overset{\text{def}}{=} \sum_{v \in X} d^l(v), \quad D^m(X) \overset{\text{def}}{=} \sum_{v \in X} d^m(v), \text{ and } D^r(X) \overset{\text{def}}{=} \sum_{v \in X} d^r(v).$$

Note that if $X \subseteq L^i$, then $D^l(X) = e(L^{i-1}, X)$, $D^m(X) = 2e(X,X) + e(X, L^i \setminus X)$, and $D^r(X) = e(X, L^{i+1})$ holds. Especially, for $i = 0,1,2,3,4$

$$D^r(L^i) = D^l(L^{i+1}) = e(L^i, L^{i+1}) \geq |L^{i+1}| \tag{1}$$

holds since $d^l(v) \geq 1$ for each $v \in V(G) \setminus \{r\}$. From the definition of L^i_S, we have $d^l(v) + d^m(v) = 1$ for any vertex $v \in L^i_S$ $(i = 1,2,3,4,5)$. Thus, we obtain

$$D^l(L^i_S) = D^l(L^i_S) + D^m(L^i_S) = |L^i_S|. \tag{2}$$

The followings are basic properties for degrees.

Proposition 2 *$D^l(L^i) + D^m(L^i) \geq |L^i| + |L^i_{\overline{S}}|$ holds for $i = 1,2,3,4,5$.*

Proof We have $D^l(L^i_{\overline{S}}) + D^m(L^i_{\overline{S}}) \geq 2|L^i_{\overline{S}}|$ from the definition of $L^i_{\overline{S}}$. Thus, together with (2), we obtain

$$\begin{aligned} D^l(L^i) + D^m(L^i) &= D^l(L^i_S) + D^m(L^i_S) + D^l(L^i_{\overline{S}}) + D^m(L^i_{\overline{S}}) \\ &\geq |L^i_S| + 2|L^i_{\overline{S}}| \\ &= |L^i| + |L^i_{\overline{S}}|. \end{aligned}$$

□

Proposition 3 *For $i \in \{1,2,3,4\}$, $D^r(L^i) \geq \max\{|L^i_S|, |L^{i+1}|\}$ holds if $d(v) \geq 2$ for each $v \in L^i_S$.*

Proof For $v \in L^i_S$, it follows form $d^l(v) + d^m(v) = 1$ and $d(v) \geq 2$ that $d^r(v) \geq 1$. Thus, we have $D^r(L^i) \geq |L^i_S|$. Together with (1), we obtain the desired relation. □

For $\hat{L} \subseteq L^i$ and $X \subseteq L^{i-1}$, denote the set of vertices $\hat{L} \cap (\bigcup_{v \in X} N(v))$ by $\hat{L}(X)$. Especially, if $X = \{u\}$, we denote $\hat{L}(u)$ instead of $\hat{L}(\{u\})$.

When we estimate the lower bound of $e(G)$ with $\delta(G) = 2$, consider two cases of whether we can choose a root r satisfying $D^m(L^1) = 0$ or not. At first, we deal with the case of $D^m(L^1) \neq 0$ for any root r with $d(r) = 2$. Namely, every vertex of degree two is in a triangle (i.e. K_3). To establish this case, the technique proposed by [18] is adopted.

Proposition 4 *For a C_6-saturated graph G with $\delta(G) = 2$, let $A \subseteq \{v \in V(G) \mid d(v) = 2\}$. Assume that every vertex in A is in a triangle and has no neighbor with degree 2. At least one of the following is satisfied:*
(i) We can choose a root $r \in A$ such that $L^1 = \{x_1, x_2\}$ and $L^2 = \{t_1, t_2\}$ satisfy $L^2(x_1) = \{t_1\}$ and $L^2(x_2) = \{t_2\}$, where $t_1 \neq t_2$;
(ii) $D(X) \geq 3|X|$ holds for $X = V(G) \setminus \{v \in V(G) \mid d(v) = 2, v \notin A\}$.

Proof Let G' be the subgraph induced by the vertices in $V(G) \setminus A$. Call an edge of G' red if it forms a triangle with some vertex in A, and call the other edges of G' blue. An edge of G incident to a vertex in A is called white. Let $d_{\mathrm{red}}(v)$, $d_{\mathrm{blue}}(v)$, and $d_{\mathrm{white}}(v)$ be the number of red edges, of blue edges, and of white edges, respectively, incident to v.

Partition $V(G')$ into $X_1, \ldots, X_l$, where each X_i induces a connected component in the graph consisting of only red edges. Since every vertex in A is contained in a triangle, such a vertex corresponds to one red edge. We define

$$Y_i \stackrel{\mathrm{def}}{=} \{v \in A \mid \exists u_1, u_2 \in X_i, (v, u_1), (v, u_2) \in E(G)\}$$

and

$$V_i \stackrel{\mathrm{def}}{=} X_i \cup Y_i.$$

Note that $V_1, \ldots, V_l$ is a partition of $V(G)$.

If $V_i \cap A = \emptyset$, then V_i is a singleton of a vertex v that does not incidence with any red edge. Such a vertex v has degree more than 3 or belongs to the set $\{u \in V(G) \mid d(u) = 2, u \notin A\}$. Therefore, if $D(V_i) \geq 3|V_i|$ for any V_i with $A \cap V_i \neq \emptyset$, then the condition (ii) is satisfied.

We now consider the case that there is a $V_{\hat{i}}$ whose average degree is less than 3, that is, $D(V_{\hat{i}}) = \sum_{v \in V_{\hat{i}}} (d_{\mathrm{red}}(v) + d_{\mathrm{blue}}(v) + d_{\mathrm{white}}(v)) < 3|V_{\hat{i}}|$. We assume there are k red edges in $G[X_{\hat{i}}]$. Thus, $\sum_{v \in X_i} d_{\mathrm{red}}(v) = 2k$. Since $4|Y_{\hat{i}}| = \sum_{v \in V_{\hat{i}}} d_{\mathrm{white}}(v)$, we have $2k + \sum_{v \in X_{\hat{i}}} d_{\mathrm{blue}}(v) + 4|Y_{\hat{i}}| < 3|V_{\hat{i}}| = 3(|Y_{\hat{i}}| + |X_{\hat{i}}|)$. Because $X_{\hat{i}}$ forms connected components by only red edges, $|X_{\hat{i}}| \leq k + 1$ holds. Thus, we have

$$2k + \sum_{v \in X_{\hat{i}}} d_{\mathrm{blue}}(v) + |Y_{\hat{i}}| < 3|X_{\hat{i}}| \leq 3k + 3 \tag{3}$$

Moreover, it follows from the definition of Y_i that $|Y_{\hat{i}}| \geq k$, which implies that $3k + \sum_{v \in X_{\hat{i}}} d_{\mathrm{blue}}(v) < 3|X_{\hat{i}}| \leq 3k + 3$. Hence, $\sum_{v \in X_{\hat{i}}} d_{\mathrm{blue}}(v) \leq 2$ and $|X_{\hat{i}}| = k + 1$. Thus, red edges in $G[X_{\hat{i}}]$ forms a spanning tree.

Next, we prove $|Y_{\hat{i}}| = k$. Suppose that $|Y_{\hat{i}}| \geq k + 1$. Then, there exists a red edge (u_1, u_2) corresponding to two vertices $y_1, y_2 \in Y_{\hat{i}}$. By joining y_1 and y_2, there exists a cycle $C_6(y_1, v_1, v_2, v_3, v_4, y_2, y_1)$. Without loss of generality, we assume that $v_1 = u_1$ and $v_4 = u_2$. By substituting $|Y_{\hat{i}}| \geq k + 1$ to (3), we obtain $\sum_{v \in X_{\hat{i}}} d_{\mathrm{blue}}(v) \leq 1$, which implies that the cycle $C_6(y_1, v_1, v_2, v_3, v_4, y_2, y_1)$ does not contain any blue edge. If $v_2, v_3 \in X_{\hat{i}}$, $(v_1, v_2), (v_2, v_3), (v_3, v_4)$ are red edges. If $v_2 \in Y_{\hat{i}}$, there exists a red edge (v_1, v_3). Similarly, if $v_3 \in Y_{\hat{i}}$, there exists a red edge (v_2, v_4). In any case, we conclude there exists a cycle through u_1 and u_2 consisting of only red edges, which contradicts the fact that red edges in $G[X_{\hat{i}}]$ forms a spanning tree.

Hence, we obtain $|Y_{\hat{i}}| = k$, which implies each red edge in $G[X_{\hat{i}}]$ exactly corresponds to only one vertex in $Y_{\hat{i}}$. Because each leaf s of the spanning tree consisting of red edges in $G[X_{\hat{i}}]$ is a neighbor of a vertex in A, $d(s) \geq 3$ holds from the assumption. Then, such a leaf s needs to be incident with blue edges, since $d_{\mathrm{red}}(s) = 1$, $d_{\mathrm{white}}(s) = 1$. It follows from $\sum_{v \in X_{\hat{i}}} d_{\mathrm{blue}}(v) \leq 2$ that there are exactly two leaves on the spanning tree. Hence, the spanning tree is actually a path and any inner vertex v in this path satisfies $d(v) = d_{\mathrm{red}}(v) + d_{\mathrm{white}}(v) = 2 + 2$.

Furthermore, we prove $k = 1$. Suppose $k \geq 2$. Then, there exist three vertices $u_1, u_2, u_3 \in X_{\hat{i}}$ such that there exist red edges (u_1, u_2) and (u_2, u_3). Thus, there also exist white edges $(y_1, u_1), (y_1, u_2)$ and $(y_2, u_2), (y_2, u_3)$, for some $y_1, y_2 \in Y_{\hat{i}}$. Joining y_1 and y_2, we have to have a cycle $C_6(y_1, v_1, v_2, v_3, v_4, y_2, y_1)$. Clearly, v_1 has to be either u_1 or u_2. When $v_1 = u_1$, we have to have $v_4 = u_2$ or $v_4 = u_3$. If $v_4 = u_2$, then $v_3 = u_3$, since u_2 is not a leaf. However, this implies a contradiction since a cycle $C_6(u_1, y_1, u_2, y_2, u_3, v_2, u_1)$ is created without adding any extra edge. On the other hand, if $v_4 = u_3$, then clearly, $v_3 \neq u_2$ and $v_2 \neq u_2$. However, this also leads a contradiction since a cycle $C_6(u_1, y_1, u_2, u_3, v_3, v_2, u_1)$ is created without adding any extra edge. Thus, $v_1 = u_2$. In this case, we have to have $v_2 = u_1$ and $v_4 = u_3$.

However, this contradicts since a cycle $C_6(u_1, v_3, u_3, y_2, u_2, y_1, u_1)$ is created without adding any extra edge. Hence, we can conclude that $k = 1$.

Conclusively, the desirable root r is the unique vertex in $Y_{\hat{i}}$. Initiated at this root, $L^1 = \{x_1, x_2\}$. Then, there are two blue edges (x_1, t_1) and (x_2, t_2), that is, $L^2(x_1) = t_1$ and $L^2(x_2) = t_2$. If $t_1 \ = t_2$, joining r and t_1, we fail to obtain a C_6 cycle. Thus r satisfies the desired condition (i). □

Proposition 5 *For a C_6-saturated graph G, assume that $\delta(G) = 2$, $e(G) < \frac{3n}{2}$ and every vertex with degree 2 is in a triangle. For any $u \in L^3_S$, $L^4(u) \nsubseteq L^4_S$ holds by choosing an appropriate root.*

Proof Since $e(G) < \frac{3n}{2}$, Proposition 4 implies that we can choose a root r satisfying the condition (i) by setting $A = \{v \in V(G) \mid d(v) = 2\}$ which does not violate the assumption of Proposition 4 since G satisfies Assumption 1. Without loss of generality, we assume $u \in L^3(L^2(x_2))$. Joining x_1 and u, we have a cycle $C_6(x_1, v_1, v_2, v_3, v_4, u, x_1)$. Since $u \in L^3_S$, v_4 belongs to L^2 or L^4. Suppose that $v_4 \in L^2$. Then, $v_4 = t_2$. Since x_2 can be neither v_1 nor v_2, we have $v_1 = t_1 \in L^2$. In this case, both of v_2 and v_3 do not become r or x_2. However, this contradicts since a $C_6(x_1, t_1, v_2, v_3, t_2, x_2, x_1)$ can be created without adding any extra edge. Hence, we have to have $v_4 \in L^4$. Moreover, we have $v_3 \in L^3 \cup L^4$ since $\mathrm{dist}(x_1, v_3) \leq 3$. Hence, we obtain $v_4 \in L^4_{\bar{S}}$. □

The above property implies that we have $e(L^3_S, L^4_{\bar{S}}) \geq |L^3_S|$. Thus, we have $D^l(L^4_{\bar{S}}) \geq |L^3_S|$.

Lemma 1 $e(G) \geq \frac{5n-5}{4}$ *holds when $\delta(G) = 2$ and every vertex with degree 2 is in a triangle.*

Proof We estimate $\sum_{i=0}^{5} D(L^i)$ by calculating degrees of vertices in each hierarchy when $e(G) < \frac{3n}{2}$.

Let a root r satisfy the condition (i) of Proposition 4. It is clear that

$$D(L^0) + D(L^1) + D^l(L^2) = 10.$$

From (1) and Proposition 2, we obtain

$$D^r(L^2) + D^l(L^3) + D^m(L^3) \geq 2|L^3| + |L^3_{\bar{S}}|.$$

Combining Proposition 2 and $D^r(L^3) = D^l(L^4) = D^l(L^4_S) + D^l(L^4_{\bar{S}}) \geq |L^4_S| + |L^3_S|$ which can be deduced from Proposition 5, we obtain

$$\begin{aligned} D^r(L^3) + D^l(L^4) + D^m(L^4) &\geq (|L^4_S| + |L^3_S|) + (|L^4| + |L^4_{\bar{S}}|) \\ &= 2|L^4| + |L^3_S|. \end{aligned}$$

On the other hand, (1) and Proposition 2 bring

$$D^r(L^3) + D^l(L^4) + D^m(L^4) \geq 2|L^4| + |L^4_{\bar{S}}|.$$

Thus, we have

$$D^r(L^3) + D^l(L^4) + D^m(L^4) \geq 2|L^4| + \max\{|L^3_S|, |L^4_{\bar{S}}|\}.$$

Using Proposition 3 we have $D^r(L^4) \geq \max\{|L^4_S|, |L^5|\}$, and it follows $D(L^5) = 2|L^5|$ since $d(v) \geq 2$ for all v.

By summing up these inequalities, we obtain

$$\begin{aligned} 2e(G) &= \sum_{i=0}^{5} D(L^i) \\ &= (D(L^0) + D(L^1) + D^l(L^2)) + (D^r(L^2) + D^l(L^3) + D^m(L^3)) \\ &\quad + (D^r(L^3) + D^l(L^4) + D^m(L^4)) + D^r(L^4) + D(L^5) \\ &\geq 10 + (2|L^3| + |L^3_{\bar{S}}|) + (2|L^4| + \max\{|L^3_S|, |L^4_{\bar{S}}|\}) + \max\{|L^4_S|, |L^5|\} + 2|L^5| \\ &\geq 10 + \frac{5(|L^3| + |L^4| + |L^5|)}{2} = 10 + \frac{5(n-5)}{2}. \end{aligned}$$

□

Next, we treat the case in which there exists a root r such that $D^m(L^1) = 0$. That is to say, there exists a root r that is not in a triangle. In this case, we need to estimate the summation of degrees more precisely. We further partition L^i_S ($i = 2, 3$) into

$$L^i_{SC} \stackrel{\text{def}}{=} \{u \in L^i_S \mid L^{i+1}(u) \subseteq L^{i+1}_S\}$$

and

$$L^i_{S\overline{C}} \stackrel{\text{def}}{=} \{u \in L^i_S \mid L^{i+1}(u) \not\subseteq L^{i+1}_S\}.$$

Then, we have

$$e(L^i_{S\overline{C}}, L^{i+1}_{\overline{S}}) \geq |L^i_{S\overline{C}}|. \tag{4}$$

Recall that $L^3_{S\overline{C}}(L^2_{\overline{S}}) = L^3_{S\overline{C}} \cap (\bigcup_{v \in L^2_{\overline{S}}} N(v))$. Thus, we have

$$e(L^3_{S\overline{C}}(L^2_{\overline{S}}), L^4_{\overline{S}}) \geq |L^3_{S\overline{C}}(L^2_{\overline{S}})|. \tag{5}$$

Lemma 2 $D^r(L^2) + D^l(L^3) + D^m(L^3) \geq 2|L^3| + \max\{|L^2_{S\overline{C}}|, |L^3_{\overline{S}}|\}$ *holds.*

Proof Together with (2) and (4), we estimate

$$\begin{aligned} D^r(L^2) = D^l(L^3) &= D^l(L^3_S) + D^l(L^3_{\overline{S}}) \\ &= |L^3_S| + e(L^2, L^3_{\overline{S}}) \\ &\geq |L^3_S| + e(L^2_{S\overline{C}}, L^3_{\overline{S}}) \\ &\geq |L^3_S| + |L^2_{S\overline{C}}|. \end{aligned}$$

Combining Proposition 2, we obtain

$$\begin{aligned} D^r(L^2) + D^l(L^3) + D^m(L^3) &\geq |L^3_S| + |L^2_{S\overline{C}}| + |L^3| + |L^3_{\overline{S}}| \\ &= 2|L^3| + |L^2_{S\overline{C}}|. \end{aligned} \tag{6}$$

On the other hand, from (1) and Proposition 2, we obtain

$$D^r(L^2) + D^l(L^3) + D^m(L^3) \geq 2|L^3| + |L^3_{\overline{S}}|. \tag{7}$$

We can rewrite the two lower bounds (6) and (7) into a compact form as the desirable inequality. □

To consider a finer structure, we partition L^3_{SC} into

$$L^3_{SCD} \stackrel{\text{def}}{=} \{u \in L^3_{SC} \mid \forall w \in L^5(L^4(u)), N(w) \subseteq L^4(L^3_{SC}) \vee d(w) = 2\}$$

and

$$L^3_{SC\overline{D}} \stackrel{\text{def}}{=} \{u \in L^3_{SC} \mid \exists w \in L^5(L^4(u)), N(w) \not\subseteq L^4(L^3_{SC}) \wedge d(w) > 2\}.$$

Lemma 3 *When $d(v) \geq 2$ for any $v \in L^5$, $D(L^5) \geq 2|L^5| + \frac{1}{2}|L^3_{SC\overline{D}}|$ holds.*

Proof Define

$$L^5_{\overline{D}} \stackrel{\text{def}}{=} \{u \in L^5 | N(u) \not\subseteq L^4(L^3_{SC}), d(u) \geq 3\}.$$

By using the definition of $L^3_{SC\overline{D}}$, for a given $u \in L^3_{SC\overline{D}}$, we can find a $w \in L^5(L^4(u))$ for which $N(w) \not\subseteq L^4(L^3_{SC})$ and $d(w) \geq 3$. Namely, $w \in L^5_{\overline{D}}$. Hence, we can find at least one $w \in L^5_{\overline{D}}$ for any given $u \in L^3_{SC\overline{D}}$, which implies that

$$e(L^4(L^3_{SC\overline{D}}), L^5_{\overline{D}}) \geq |L^4(L^3_{SC\overline{D}})| \geq |L^3_{SC\overline{D}}|. \tag{8}$$

On the other hand, if $w \in L^5_{\overline{D}}$ then we have

$$e(L^4 \setminus L^4(L^3_{SC\overline{D}}), \{w\}) + d^m(w) \geq e(L^4 \setminus L^4(L^3_{SC}), \{w\}) + d^m(w) \geq 1,$$

because $N(w) \nsubseteq L^4(L^3_{SC})$. As a result,

$$e(L^4 \setminus L^4(L^3_{SC\overline{D}}), L^5_{\overline{D}}) + D^m(L^5_{\overline{D}}) \geq |L^5_{\overline{D}}| \tag{9}$$

holds. Combining (8) and (9), we obtain

$$\begin{aligned}
D(L^5_{\overline{D}}) &= \frac{1}{2}D(L^5_{\overline{D}}) + \frac{1}{2}(D^l(L^5_{\overline{D}}) + D^m(L^5_{\overline{D}})) \\
&\geq \frac{1}{2}D(L^5_{\overline{D}}) + \frac{1}{2}(e(L^4(L^3_{SC\overline{D}}), L^5_{\overline{D}}) + e(L^4 \setminus L^4(L^3_{SC\overline{D}}), L^5_{\overline{D}}) + D^m(L^5_{\overline{D}})) \\
&\geq \frac{3}{2}|L^5_{\overline{D}}| + \frac{1}{2}(|L^3_{SC\overline{D}}| + |L^5_{\overline{D}}|) \\
&= 2|L^5_{\overline{D}}| + \frac{1}{2}|L^3_{SC\overline{D}}|.
\end{aligned}$$

Since $d(v) \geq 2$ for $v \in L^5$, we conclude

$$\begin{aligned}
D(L^5) &= D(L^5 \setminus L^5_{\overline{D}}) + D(L^5_{\overline{D}}) \\
&\geq 2|L^5 \setminus L^5_{\overline{D}}| + 2|L^5_{\overline{D}}| + \frac{1}{2}|L^3_{SC\overline{D}}| \\
&= 2|L^5| + \frac{1}{2}|L^3_{SC\overline{D}}|.
\end{aligned}$$

□

We next estimate a bound for $D^r(L^3) + D^l(L^4) + D^m(L^4)$.

Proposition 6 *Let $L^i_P \stackrel{\text{def}}{=} \{u \in L^i \mid d^m(u) = 0\}$. If $D^m(L^1) = 0$, then $L^4(u) \nsubseteq L^4_S$ holds for any $u \in L^3_S(L^2_P)$ with $d(u) \geq 2$.*

Proof Suppose it is false, that is to say, for some $u \in L^3_S(L^2_P)$, $L^4(u) \subseteq L^4_S$ holds. Since $u \in L^3_S$ and $d(u) \geq 2$, there exists $w \in L^4(u)$. If we add (r, w) to $E(G)$, a cycle, say $C_6(r, v_1, v_2, v_3, v_4, w, r)$ should exist.

Since $w \in L^4(u) \subseteq L^4_S$, $v_4 \in L^3 \cup L^5$ holds. If $v_4 \in L^5$, then $\text{dist}(r, v_4) > 4$ and it violates the construction of the cycle C_6. Thus, we have $v_4 \in L^3$, which, together with $w \in L^4_S$, implies $v_4 = u$. By using the similar technique, we prove $v_3 \in L^2$. In addition, $u \in L^3_S(L^2_P)$ and $(v_3, u) \in E(G)$ together imply $v_3 \in L^2_P$. We can also say $v_2 \in L^1$ in a similar way. However, if $v_2 \in L^1$, then we need to have $v_1 \in L^1$. Thus, we have $(v_1, v_2) \in E(G)$, which violates the condition $D^m(L^1) = 0$. □

Proposition 6 implies that there exists a $w \in L^4_{\overline{S}}$ with $(u, w) \in E(G)$ for any $u \in L^3_S(L^2_P)$. Since $L^2_S \subseteq L^2_P$, we have $e(L^3_S(L^2_S), L^4_{\overline{S}}) \geq |L^3_S(L^2_S)|$.

Lemma 4 *If $D^m(L^1) = 0$ and $d(v) \geq 2$ for any $v \in L^3$, then $D^r(L^3) + D^l(L^4) + D^m(L^4) \geq 2|L^4| + |L^3_S(L^2_S)| + |L^3_{S\overline{C}}(L^2_{\overline{S}})|$ holds.*

Proof Proposition 6 implies

$$e(L^3_S(L^2_S), L^4_{\overline{S}}) \geq |L^3_S(L^2_S)|.$$

Together with (2) and (5), we estimate

$$\begin{aligned}
D^r(L^3) = D^l(L^4) &= D^l(L^4_S) + D^l(L^4_{\overline{S}}) \\
&\geq D^l(L^4_S) + e(L^3_S, L^4_{\overline{S}}) \\
&\geq |L^4_S| + e(L^3_S(L^2_S), L^4_{\overline{S}}) + e(L^3_{S\overline{C}}(L^2_{\overline{S}}), L^4_{\overline{S}}) \\
&\geq |L^4_S| + |L^3_S(L^2_S)| + |L^3_{S\overline{C}}(L^2_{\overline{S}})|,
\end{aligned}$$

where the penultimate inequality can be deduced from the fact that L_S^3 is partitioned into $L_S^3(L_S^2)$ and $L_S^3(L_{\overline{S}}^2)$. Combining Proposition 2, we obtain

$$\begin{aligned} D^r(L^3)+D^l(L^4)+D^m(L^4) &\geq (|L_S^4|+|L_S^3(L_S^2)|+|L_{S\overline{C}}^3(L_{\overline{S}}^2)|)+(|L^4|+|L_{\overline{S}}^4|) \\ &= 2|L^4|+|L_S^3(L_S^2)|+|L_{S\overline{C}}^3(L_{\overline{S}}^2)|. \end{aligned}$$

□

In order to give another lower bound for $D^r(L^3)+D^l(L^4)+D^m(L^4)$, we introduce notions

$$L_{\overline{S}P}^4 \stackrel{\text{def}}{=} \{u \in L_{\overline{S}}^4 | e(\{u\}, L^3) \geq 2\},$$

and

$$L_{\overline{S}P3}^4 \stackrel{\text{def}}{=} \{u \in L_{\overline{S}P}^4 | e(\{u\}, L^3 \cup L^4) \geq 3\}.$$

Proposition 7 *If $|L^1|=2$, then $L^4(L^3(u)) \cap L_{\overline{S}P}^4 \neq \emptyset$ holds for any $u \in L_{SC}^2$.*

Proof Since $u \in L_S^2$, there exists an $\hat{x} \in L^1$ with $(\hat{x},u) \notin E(G)$. By joining $\hat{x}$ and u, a cycle $C_6(\hat{x},v_1,v_2,v_3,v_4,u,\hat{x})$ should exist. When $v_4 \in L^1$, we have $v_1 \in L^2$ since $v_1 \neq r$. However, this case creates $C_6(\hat{x},v_1,v_2,v_3,v_4,r,\hat{x})$ without adding any extra edge. Hence, we have to have $v_4 \in L^3$. Since $u \in L_{SC}^2$, we have $v_4 \in L_S^3$, which implies $v_3 \in L^4$ and then $v_2 \in L^3$. Hence, we end this proof with $v_3 \in L_{\overline{S}P}^4$. □

Proposition 8 *If $|L^1|=2$, then $|\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}| \leq 2$ holds for any $u \in L^4$. Moreover, when $D^m(L^1)=0$ and $|\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}|=2$ holds, we have $u \in L_{\overline{S}P3}^4$.*

Proof Assume that three distinct elements x,y,z belong to $\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}$ for some $u \in L^4$. Since $|L^1|=2$, we can immediately claim that at least two of x,y,z are adjacent to either x_1 or x_2. Without loss of generality, assume $(x_2,y),(x_2,z) \in E(G)$, then there exist some $y_1,z_1 \in L_S^3$ for which $(y,y_1),(z,z_1) \in E(G)$ and $(y_1,u),(z_1,u) \in E(G)$ since $u \in L^4(L^3(y))$ and $u \in L^4(L^3(z))$. Now the contradiction can be seen from the fact that $C_6(x_2,y,y_1,u,z_1,z,x_2)$ exists without adding any extra edge. Hence, $|\{v \in L_{SC}^2 | u \in L^4(v)\}| \leq 2$.

We next consider the case of $|\{v \in L_{SC}^2 | u \in L^4(v)\}|=2$. Let $x,y \in L_{SC}^2$ such that $u \in L^4(L^3(x)) \cap L^4(L^3(y))$. Obviously, there exist some $x_1,y_1 \in L_S^3$ for which $(x,x_1),(y,y_1) \in E(G)$ and $(x_1,u),(y_1,u) \in E(G)$. Then, $u \in L_{\overline{S}P}^4$. Assume that $u \notin L_{\overline{S}P3}^4$. Namely, $(L^3 \cup L^4) \cap N(u)=\{x_1,y_1\}$. By joining r and u, we have to have a cycle $C_6(r,v_1,v_2,v_3,v_4,u,r)$. If $v_4 \in L^5$, then $\text{dist}(v_4,r)>4$, which contradicts. Thus, we have $v_4=x_1$ or $v_4=y_1$. Namely, $v_4 \in L_S^3(L_S^2)$. We can also notice that $v_1 \in L^1$ and $v_2 \in L^2$ since $D^m(L^1)=0$. Therefore, $v_3 \in L^2 \cup L^3$, which however, implies $v_4 \in L_{\overline{S}}^3$ or $v_4 \in L^3(L_{\overline{S}}^2)$. This contradicts to $v_4 \in L_S^3(L_S^2)$. Hence, $u \in L_{\overline{S}P3}^4$ □

Proposition 9 *If $|L^1|=2$, then $|L_{\overline{S}P}^4|+|L_{\overline{S}P3}^4| \geq |L_{SC}^2|$ holds.*

Proof Since $|\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}| \leq 2$ for any $u \in L_{\overline{S}P}^4$, we divide $L_{\overline{S}P}^4$ into subsets U_0,U_1 and U_2 as

$$\begin{aligned} U_0 &= \{u \in L_{\overline{S}P}^4 \mid |\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}|=0\} \\ U_1 &= \{u \in L_{\overline{S}P}^4 \mid |\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}|=1\} \\ U_2 &= \{u \in L_{\overline{S}P}^4 \mid |\{v \in L_{SC}^2 | u \in L^4(L^3(v))\}|=2\}. \end{aligned}$$

From Proposition 7, it can be seen that for any $v \in L_{SC}^2$, v corresponds to some $u \in L_{\overline{S}P}^4$, where $u \in U_1$ or $u \in U_2$. Additionally, by the definitions of U_1 and U_2 we immediately obtain

$$|U_1|+2|U_2| \geq |L_{SC}^2|.$$

On the other hand, from Proposition 8 and the definition of $L^4_{\bar{S}P3}$, it follows $|L^4_{\bar{S}P3}| \geq |U_2|$. Hence, we can complete the proof by noting that

$$|L^4_{\bar{S}P}| + |L^4_{\bar{S}P3}| \geq (|U_1| + |U_2|) + |U_2| \geq |L^2_{SC}|.$$

□

Lemma 5 *If $|L^1| = 2$, then $D^r(L^3) + D^l(L^4) + D^m(L^4) \geq 2|L^4| + |L^4_{\bar{S}}| + |L^2_{SC}|$ holds.*

Proof we have

$$\begin{aligned} &D^r(L^3) + (D^l(L^4) + D^m(L^4)) \\ &\geq (|L^4| + |L^4_{\bar{S}P}|) + (|L^4| + |L^4_{\bar{S}}| + |L^4_{\bar{S}P3}|) \\ &= 2|L^4| + |L^4_{\bar{S}P}| + |L^4_{\bar{S}}| + |L^4_{\bar{S}P3}| \\ &\geq 2|L^4| + |L^4_{\bar{S}}| + |L^2_{SC}|, \end{aligned}$$

where the last inequality holds due to Proposition 9. □

Finally, we give a lower bound for $D^l(L^2) + D^m(L^2)$.

Proposition 10 *If $\delta(G) = 2$ and there exists a root r that is not in a triangle, then we can select a root r' with $d(r') = 2$ such that $D^m(L^1) = 0$ and $|L^3_{SCD}(u)| \leq 1$ for any $u \in L^2_{\bar{S}}$.*

Proof We need only consider the root r that does not satisfy the condition for r'. That is to say, we assume that for a root r there exists $u \in L^2_{\bar{S}}$ such that two distinct vertices y, z belong to $L^3_{SCD}(u)$. Since $y, z \in L^3_{SC}$ it immediately follows that there exist some $y_1, z_1 \in L^4_S$ such that $(y, y_1), (z, z_1) \in E(G)$.

Joining y and z we would have a cycle, say $C_6(y, v_1, v_2, v_3, v_4, z, y)$. We first prove $v_1 \in L^4$. Suppose if $v_1 \in L^2$, then we have to have $v_1 = u$ and $v_4 \in L^4$. Furthermore, $v_3 \in L^5$ holds since $v_4 \in L^4_{\bar{S}}$. Therefore we have $\mathrm{dist}(v_1, v_3) > 2$, which contradicts to the construction of C_6. Hence, we can say $v_1 \in L^4$ and $v_2 \in L^5$. In a similar way, we can say $v_4 \in L^4$ and $v_3 \in L^5$. Without loss of generality, we can set $v_1 = y_1$ and $v_4 = z_1$. Since $y, z \in L^3_{SCD}(u)$ and $(v_2, v_3) \in E(G)$, which implies $N(v_2) \not\subseteq L^4(L^3_{SC})$ and $N(v_3) \not\subseteq L^4(L^3_{SC})$, we obtain $d(v_2) = d(v_3) = 2$.

We now take v_2 as a root. It is obviously that $D^m(L^1) = 0$. If there exists some h such that $(y_1, h), (z_1, h) \in E(G)$, then we obtain a $C_6(y_1, h, z_1, z, u, y, y_1)$ without adding any extra edge, as is shown in Fig. 7. Hence, we have $d^m(z_1) = 0$. Furthermore, we are going to prove that for root v_2, $|L^3_{SCD}(u)| \leq 1$ for any $u \in L^2_{\bar{S}}$. Suppose it is false, then there exists $u' \in L^2_{\bar{S}}$ such that two distinct vertices y', z' belong to $L^3_{SCD}(u')$. As the similar discussion, we have $y'_1, z'_1 \in L^4_S$ such that $(y', y'_1)(z', z'_1) \in E(G)$ and $y'_2, z'_2 \in L^5$ such that $(y'_2, z'_2) \in E(G)$ and $d(y'_2) = d(z'_2) = 2$. We next consider a cycle $C_6(y'_2, u_1, u_2, u_3, u_4, v_3, y'_2)$ created by joining v_3 and y'_2. Notice that $u_1 = y'_1$ or $u_1 = z'_2$, but we can prove neither holds as follows. If $u_1 = y'_1$, then we have to have $u_2 = y'$ and $u_3 = u'$. Since $d(v_3) = 2$, v'_4 should be z_1, which leads a contradiction to $d^m(z_1) = 0$. If $u_1 = z'_2$, then it is obviously that $u_2 = z'_1$, $u_3 = z'$ and $u_4 = u'$. Since z_1 is the unique vertex adjacent to v_3 in L^2, u' has to be z_1. Namely, $u' \in L^3_{SC}(v_3)$, since $z_1 \in L^3_S(v_3)$. On the other hand, Proposition 6 implies that $L^3_{SC}(v_3) = \emptyset$ since $v_3 \in L^2_P$, which also contradicts. Therefore v_2 is a desired root r'.

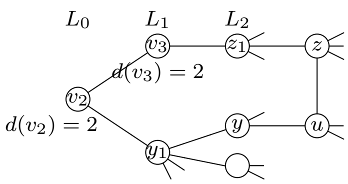


Fig. 7 When we set v_2 a root.

□

The above property implies that $|L^2_{\overline{S}}| \geq |L^3_{SCD}|$ holds by choosing an appropriate root. By combining this result and Proposition 2, we have

$$D^l(L^2) + D^m(L^2) \geq |L^2| + |L^2_{\overline{S}}| \geq |L^2| + \frac{1}{2}(|L^2_{\overline{S}}| + |L^3_{SCD}|),$$

which conducts the following property.

Proposition 11 *If $\delta(G) = 2$, $D^l(L^2)+D^m(L^2) \geq |L^2|+\frac{1}{2}(|L^2_{\overline{S}}|+|L^3_{SCD}|)$ holds by choosing an appropriate root.*

Lemma 6 $e(G) \geq \frac{5n-7}{4}$ *holds when $\delta(G) = 2$ and a root r with $D^m(L^1) = 0$ exists.*

Proof Suppose that we have selected a root r satisfying the condition of Proposition 10. Since r is the only vertex in L^0 and there are two vertices in L^1, we have

$$D(L^0) + D(L^1) = D(L^0) + D^l(L^1) + D^r(L^1) \geq 4 + |L^2|.$$

Combining Lemmas 11, 2, 4, 5, 3 and 3, we conclude

$$\begin{aligned}
2e(G) &= \sum_{i=0}^{5} D(L^i), \\
&= (D(L^0) + D(L^1)) + (D^l(L^2) + D^m(L^2)) + (D^r(L^2) + D^l(L^3) + D^m(L^3)) \\
&\quad +(D^r(L^3) + D^l(L^4) + D^m(L^4)) + D^r(L^4) + D(L^5) \\
&\geq (4 + |L^2|) + (|L^2| + \frac{1}{2}(|L^2_{\overline{S}}| + |L^3_{SCD}|)) + (2|L^3| + \max\{|L^2_{S\overline{C}}|, |L^3_{\overline{S}}|\}) \\
&\quad +(2|L^4| + \max\{|L^3_S(L^2_S)| + |L^3_{S\overline{C}}(L^2_{\overline{S}})|, |L^4_{\overline{S}}| + |L^3_{S\overline{C}}|\}) + \max\{|L^4_S|, |L^5|\} + (2|L_5| + \frac{1}{2}|L^3_{SC\overline{D}}|) \\
&\geq 4 + \frac{5(|L^2| + |L^3| + |L^4| + |L^5|)}{2} = 4 + \frac{5(n-3)}{2}
\end{aligned}$$

□

Consequently, we obtain the result of Theorem 3 by combining the results of Lemma 1 and Lemma 6.

4.2 Proof of Theorem 2

In order to prove Theorem 2, we only consider the case of $\delta(G) = 1$ for a minimum C_6-saturated graph. We denote by S a set of vertices of degree 1 and $N(S) \stackrel{\text{def}}{=} \bigcup_{v\in S} N(v)$. Let $\tilde{G}$ be the subgraph induced by $V(G)\setminus S$, that is, $\tilde{G} = G[V(G)\setminus S]$, and $\tilde{n} = |V(\tilde{G})|(= n-|S|)$. Note that, since G is C_6-saturated, (i) $\tilde{G}$ is also C_6-saturated; (ii) $\delta(\tilde{G}) \geq 2$; (iii) $|S| = |N(S)|$ holds. The property (iii) follows from $N(u) \cap N(v) = \emptyset$ for any $u, v \in S$. The properties (i)(ii) imply that $e(\tilde{G})$ can be bounded below by $\frac{5\tilde{n}-7}{4}$ from Theorem 3. Thus, we have

$$e(G) = e(\tilde{G}) + |S| \geq \frac{5\tilde{n}-7}{4} + |S| = \frac{5n - |S| - 7}{4}. \tag{10}$$

We now estimate another bound of $e(G)$.

Lemma 7 $e(G) \geq n + \frac{|S|-5}{2}$ *holds.*

Proof In order to avoid confusion, we denote the degree of v in $\tilde{G}$ by $\tilde{d}(v)$. Let $A = \{v \in N(S) \mid \tilde{d}(v) = 2\}$.

If $A = \emptyset$, then $\tilde{d}(v) \geq 3$ for any $v \in N(S)$. Thus, we have

$$e(\tilde{G}) \geq \frac{3|N(S)| + 2(\tilde{n} - |N(S)|)}{2} = \tilde{n} + \frac{|S|}{2}.$$

If $A \neq \emptyset$, we choose a root $r \in A$ and define $\tilde{L}^i = \{u \in V(\tilde{G}) \mid \mathrm{dist}(r, u) = i\}$. Since G is C_6-saturated, $\tilde{L}^i = \emptyset$ for $i \geq 5$ and $\tilde{L}^4 \cap N(S) = \emptyset$. At first we consider the case of $D^m(\tilde{L}^1) \neq 0$ for any root $r \in A$, that is to say, every vertex in A is in a triangle in $\tilde{G}$. Note that any neighbor u of $r \in A$ satisfies $\tilde{d}(u) \geq 3$, because by joining u and $v \in S$ with $(v, r) \in E(G)$ we have to have C_6. Thus, A satisfies the assumption of Proposition 4. If the condition (i) of Proposition 4 holds, we have

$$
\begin{aligned}
e(\tilde{G}) &= (D(\tilde{L}^0) + D(\tilde{L}^1) + D^l(\tilde{L}^2) + D^r(\tilde{L}^2) + D(\tilde{L}^3) + D(\tilde{L}^4))/2 \\
&\geq (10 + |\tilde{L}^3| + 2|\tilde{L}^3| + 2|\tilde{L}^4|)/2 \\
&= (2\tilde{n} + |\tilde{L}^3|)/2 \geq (2\tilde{n} + |N(S)| - 5)/2 = \tilde{n} + (|S| - 5)/2.
\end{aligned}
$$

If the condition (ii) of Proposition 4 holds, we have

$$
e(\tilde{G}) \geq \frac{3|N(S)| + 2(\tilde{n} - |N(S)|)}{2} = \tilde{n} + \frac{|S|}{2},
$$

since $N(S) \subseteq V(\tilde{G}) \setminus \{v \in V(\tilde{G}) \mid \tilde{d}(v) = 2, v \neq A\}$.

Finally, we treat the case in which there exists a root $r \in A$ with $D^m(\tilde{L}^1) = 0$. For any vertex $v \in \tilde{L}^2 \cap N(S)$, there exists a path of length 4 from r to v, which, together with $D^m(\tilde{L}^1) = 0$, implies that $\tilde{d}^m(v) \geq 1$. Thus, $D^m(\tilde{L}^2) \geq |\tilde{L}^2 \cap N(S)|$ holds. Hence, we have

$$
\begin{aligned}
e(\tilde{G}) &= (D(\tilde{L}^0) + D(\tilde{L}^1) + D^l(\tilde{L}^2) + D^m(\tilde{L}^2) + D^r(\tilde{L}^2) + D(\tilde{L}^3) + D(\tilde{L}^4))/2 \\
&\geq (4 + 2|\tilde{L}^2| + |\tilde{L}^2 \cap N(S)| + |\tilde{L}^3| + 2|\tilde{L}^3| + 2|\tilde{L}^4|)/2 \\
&= (2\tilde{n} - 2 + |\tilde{L}^2 \cap N(S)| + |\tilde{L}^3|)/2 \geq (2\tilde{n} - 2 + |N(S)| - 3)/2 = \tilde{n} + (|S| - 5)/2.
\end{aligned}
$$

Consequently, we obtain $e(G) = e(\tilde{G}) + |S| \geq n + \frac{|S|-5}{2}$. □

By combining this result and (10),

$$
e(G) \geq \max\{\frac{5n - |S| - 7}{4}, n + \frac{|S| - 5}{2}\} \geq \frac{7n}{6} - 2.
$$

5 Conclusion

In this paper, we discuss the bounds for $\mathrm{sat}(n, C_6)$. The obtained bounds, up to our knowledge, are the best known ones so far. Besides, we foresee two promising avenues of research in the near future.

First, although the refined upper bound seems only a minor improvement of the previous one, by computer search it proves that this upper bound coincides with $\mathrm{sat}(n, C_6)$ when $n = 9$, 10 and 11. We might conjecture $\mathrm{sat}(n, C_6) = \lfloor \frac{3n-3}{2} \rfloor$, when $n \geq 9$. In order to reach the equality, the lower bound needs improving. This requires that when we analyze the lower bound of the minimum number of edges in a C_6-saturated graph with $\delta(G) = 2$ each hierarchy $L^i (i = 1, 2, ..., 5)$ should be partitioned into more finer structures and that once such a lower bound is obtained a better calculation method should be also developed to estimate the lower bound of $\mathrm{sat}(n, C_6)$.

Second, as we have seen, the lower bound of $\mathrm{sat}(n, C_6)$ can be obtained from the lower bound of the number of edges in a minimum C_6-saturated graph with $\delta(G) = 2$, which is meanwhile the key idea of this paper. However, the application of this idea may not be limited to the case of C_6-saturated graphs. Generally, when improving the lower bound of $\mathrm{sat}(n, C_k)$ for some k, we can first estimate the lower bound of the number of edges in a minimum C_k-saturated graph with $\delta(G) = 2$ for some k. And then, by using the obtained lower bound we can continue to improve the lower bound of $\mathrm{sat}(n, C_k)$ for some k.

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