A tt^*-bundle associated with a harmonic map from a Riemann surface into a sphere

Sanae Kurosu[a], Katsuhiro Moriya[b]

[a] *Department of Mathematics, Faculty of Science Division I, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, JAPAN*
[b] *Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba-shi, Ibaraki 305-8571, JAPAN*

Abstract

A tt^*-bundle is constructed by a harmonic map from a Riemann surface into an n-dimensional sphere. This tt^*-bundle is a high-dimensional analogue of a quaternionic line bundle with a Willmore connection. For the construction, a flat connection is decomposed into four parts by a fiberwise complex structure.

Keywords: tt*-bundle, harmonic map, the Clifford algebra
2010 MSC: 53C43, 81R12

1. Introduction

A tt^*-*bundle* is a real vector bundle equipped with a family of flat connections, parametrized by a circle. The present paper delivers a tt^*-bundle derived from a harmonic map from a Riemann surface to an n-dimensional sphere.

The notion of tt^*-bundles is introduced by Schäfer [10] as a simple solution to a generalized version of the equation of *topological-antitopological fusion*, introduced by Cecotti and Vafa [2], in terms of real differential geometry. A topological-antitopological fusion of a topological field theory model is a special geometry structure on a Frobenius manifold. As a geometric interpretation of a special geometry structure on a quasi-Frobenius manifold, Dubrovin [6] showed that a solution to the equation is locally a *pluriharmonic map* from an n-dimensional quasi-Frobenius manifold to the symmetric space $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$.

Schäfer [10] showed that an admissible pluriharmonic map from a simply connected complex manifold M to a symmetric space $\mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q)$, and that to $\mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(p,q)$ with $p+q=r$, gives rise from a *metric* tt^*-bundle. A harmonic map from a Riemann surface to $\mathrm{SU}(1,1)/S(\mathrm{U}(1)\times\mathrm{U}(1)) \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is obtained by the generalized Weierstrass representation formula by Dorfmeister, Pedit, and Wu [5]. The Gauss map of a spacelike surface of constant mean curvature in the Minkowski space $\mathbb{R}^{2,1}$ is a harmonic map from a Riemann surface to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The Sym-Bobenko formula (Bobenko [1], Dorfmeister and Haak [4]) connects a surface and its Gauss map. Applying these formulae, Dorfmeister, Guest, and Rossman [3] gave the description of the quantum cohomology of $\mathbb{C}P^1$. The quantum cohomology of $\mathbb{C}P^1$ provides a solution to the third Painlevé equation.

A surface of constant mean curvature in $\mathbb{R}^3$ is an interesting research subject in the theory of surfaces. Its Gauss map is a harmonic map from a Riemann surface to the two-dimensional sphere S^2. It is impossible to write S^2 as a symmetric space $\mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q)$ or $\mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(p,q)$. This led the authors to find a tt^*-bundle for a harmonic map into S^2. The theory of a quaternionic line bundle with a Willmore connection by Ferus, Leschke, Pedit, and Pinkall [8] provides a way to construct a tt^*-bundle for a harmonic map from a Riemann surface into S^2. This method is extended and a tt^*-bundle associated with a harmonic map from a Riemann surface into S^n $(n \geq 2)$ is obtained (Theorem 4.1).

2. tt^*-bundles

We recall a tt^*-bundle (Schäfer [10]).

Let M be a complex manifold with complex structure J^M. For a one-form ω on M, we define a one-form $*\omega$ on M by $*\omega := \omega \circ J^M$. Let E be a trivial real vector bundle of rank n over M, ∇ a connection on E, and S a one-form with values in the real endomorphisms of E. A one-form S is considered as a one-form with values in n-by-n real matrices. Define a family of connections $\{\nabla^\theta\}_{\theta\in\mathbb{R}}$ on E by

$$\nabla^\theta := \nabla + (\cos\theta)S + (\sin\theta) * S.$$

The curvature of ∇^θ is

$$\begin{gathered}
d^{\nabla^\theta} \circ \nabla^\theta \\
= d^\nabla \circ \nabla + (\cos\theta) d^\nabla S + (\sin\theta) d^\nabla * S \\
+ ((\cos\theta)S + (\sin\theta) * S) \wedge ((\cos\theta)S + (\sin\theta) * S) \\
= d^\nabla \circ \nabla + (\cos\theta) d^\nabla S + (\sin\theta) d^\nabla * S \\
+ (\cos\theta)^2 S \wedge S + \cos\theta \sin\theta (S \wedge *S + *S \wedge S) + (\sin\theta)^2 * S \wedge *S \\
= d^\nabla \circ \nabla + (\cos\theta) d^\nabla S + (\sin\theta) d^\nabla * S \\
+ \frac{1+\cos 2\theta}{2} S \wedge S + \frac{\sin 2\theta}{2}(S \wedge *S + *S \wedge S) + \frac{1-\cos 2\theta}{2} * S \wedge *S \\
= d^\nabla \circ \nabla + \frac{1}{2} S \wedge S + \frac{1}{2} * S \wedge *S \\
+ (\cos\theta) d^\nabla S + (\sin\theta) d^\nabla * S \\
+ \frac{\cos 2\theta}{2}(S \wedge S - *S \wedge *S) + \frac{\sin 2\theta}{2}(S \wedge *S + *S \wedge S).
\end{gathered}$$

A vector bundle E with ∇ and S is called a tt^*-bundle if ∇^θ is flat for all $\theta \in \mathbb{R}$. By the preceding calculation, a vector bundle E with ∇ and S is a tt^*-bundle, if and only if

$$\begin{gathered}
d^\nabla \circ \nabla + S \wedge S = 0, \quad d^\nabla S = 0, \quad d^\nabla * S = 0, \\
S \wedge S = *S \wedge *S, \quad S \wedge * S = - * S \wedge S.
\end{gathered}$$

Indeed,

$$\begin{gathered}
(S \wedge S - *S \wedge *S)(X, Y) \\
= S(X)S(Y) - S(Y)S(X) - S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X) \\
= -S(X)S(J^M J^M Y) + S(J^M J^M Y)S(X) \\
- S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X) \\
= -S(X)S(J^M J^M Y) + S(J^M Y)S(J^M X) \\
+ S(J^M J^M Y)S(X) - S(J^M X)S(J^M Y) \\
= -(S \wedge *S + *S \wedge S)(X, J^M Y)
\end{gathered}$$

for any tangent vectors X, Y of M. Hence, $S \wedge S = *S \wedge *S$ is equivalent to $S \wedge *S = - * S \wedge S$. Then, a vector bundle E with ∇ and S is a tt^*-bundle, if and only if

$$d^\nabla \circ \nabla + S \wedge S = 0, \quad d^\nabla S = 0, \quad d^\nabla * S = 0, \quad S \wedge S = *S \wedge *S$$

(see Schäfer [10], Proposition 1).

Assume that E with ∇ and S forms a tt^*-bundle. Define F as the complexification of E, that is, $F := \mathbb{C} \otimes E$. Denote the complex-linear extensions of ∇ and S by the same notations respectively. Define a family of connections $\{\nabla^\mu\}_{\mu \in \mathbb{C}\setminus\{0\}}$ of F by

$$\nabla^\mu = \nabla + \frac{1}{\mu} C + \mu \bar{C}, \quad C = \frac{1}{2}(S - i * S). \tag{1}$$

Then C is a $(1,0)$-form on M with values in complex linear endmorphisms of F. The tt^*-bundle E with ∇ and S is the real part of F with ∇^μ if and only if $|\mu| = 1$.

Proposition 2.1. *For each $\mu \in \mathbb{C} \setminus \{0\}$, the connection ∇^μ is flat.*

Proof. As E with ∇ and S is a tt^*-bundle, it follows that

$$\begin{aligned} d^\nabla C = 0, \quad d^\nabla \bar{C} = 0, \\ C \wedge C = \frac{1}{4}(S \wedge S - iS \wedge *S - i * S \wedge S - *S \wedge *S) = 0, \\ C \wedge \bar{C} = \frac{1}{4}(S \wedge S + iS \wedge *S - i * S \wedge S + *S \wedge *S) = \frac{1}{2}(S \wedge S + iS \wedge *S). \end{aligned}$$

Then

$$\begin{aligned} d^{\nabla^\mu} \circ \nabla^\mu &= d^\nabla \circ \nabla + \left(\frac{1}{\mu} C + \mu \bar{C}\right) \wedge \left(\frac{1}{\mu} C + \mu \bar{C}\right) \\ &= d^\nabla \circ \nabla + C \wedge \bar{C} + \bar{C} \wedge C \\ &= d^\nabla \circ \nabla + S \wedge S = 0. \end{aligned}$$

Hence ∇^μ is flat. □

Adding the assumption in Proposition 2.1, we assume that there exists a hermitian pseudo-metric h on F, and a metric connection ∇ with respect to h, such that

$$h(C(X)a, b) = h(a, \bar{C}(\bar{X})b),$$

where a, $b \in \Gamma(F)$, and X is a vector field of type $(1,0)$ on M. Then $(F, \nabla, C, \bar{C}, h)$ becomes a harmonic bundle defined in Schäfer [11].

3. Decomposition of a connection

We obtain a condition for a map from a Riemann surface into a sphere, to become a harmonic map, by decomposing a flat connection into four parts.

Let $C\ell_n$ be the Clifford algebra associated with $\mathbb{R}^n$ and the quadratic form $x_1^2+x_2^2+\cdots+x_n^2$ (see Lawson and Michelsohn [9]). The Clifford algebra $C\ell_n$ is the algebra generated by an orthonormal basis $e_1, \dots, e_n$ subject to the relation

$$e_ie_j + e_je_i = -2\delta_{ij}.$$

Then $C\ell_n$ is identified with $\mathbb{R}^{2^n}$. The set

$$\{a \in \mathbb{R}^n \subset C\ell_n \,|\, a^2 = -1\}$$

is an $(n-1)$-dimensional unit sphere $S^{n-1} \subset \mathbb{R}^n \subset C\ell_n \cong \mathbb{R}^{2^n}$.

Let M be a Riemann surface with complex structure J^M and V be the trivial associate bundle of a principal $C\ell_n$-bundle, with right $C\ell_n$ action, over M. We denote the set of smooth sections of V by $\Gamma(V)$ and the fiber of V at p by V_p. Let $\Omega^m(V)$ be the set of V-valued m-forms on M for every non-negative integer m. Then $\Omega^0(V) = \Gamma(V)$. Let W be another trivial associate bundle of a principal $C\ell_n$-bundle, with right $C\ell_n$ action, over M. We denote by $\mathrm{Hom}(V, W)$ the $C\ell_n$-homomorphism bundle from V to W. Let N be a smooth section of the Clifford endomorphism bundle $\mathrm{End}(V)$ of V such that $-N_p \circ N_p$ is the identity map Id_p on V_p for every $p \in M$. The section N is a complex structure at each fiber of V. We have a splitting $\mathrm{End}(V) = \mathrm{End}(V)_+ \oplus \mathrm{End}(V)_-$, where

$$\mathrm{End}(V)_+ = \{\xi \in \mathrm{End}(V) : N\xi = \xi N\},$$
$$\mathrm{End}(V)_- = \{\xi \in \mathrm{End}(V) : N\xi = -\xi N\}.$$

This splitting induces a decomposition of $\xi \in \mathrm{End}(V)$ into $\xi = \xi_+ + \xi_-$, where $\xi_+ = (\xi - N\xi N)/2 \in \mathrm{End}(V)_+$ and $\xi_- = (\xi + N\xi N)/2 \in \mathrm{End}(V)_-$.

Let $T^*M \otimes_{\mathbb{R}} V$ be the tensor bundle of the cotangent bundle T^*M of M and V over real numbers. We set $*\omega = \omega \circ J^{TM}$ for every $\omega \in \Omega^1(V)$. We have a splitting $T^*M \otimes_{\mathbb{R}} V = KV \oplus \bar{K}V$, where

$$KV = \{\eta \in T^*M \otimes_{\mathbb{R}} V : *\eta = N\eta\}, \ \ \bar{K}V = \{\eta \in T^*M \otimes_{\mathbb{R}} V : *\eta = -N\eta\}.$$

This splitting induces the *type decomposition* of $\eta \in T^*M \otimes_{\mathbb{R}} V$ into $\eta = \eta' + \eta''$, where $\eta' = (\eta - N * \eta)/2 \in KV$ and $\eta'' = (\eta + N * \eta)/2 \in \bar{K}V$.

Let C be the right trivial Clifford bundle over M with fiber $C\ell_n$. We identify a smooth map $\phi\colon M \to C\ell_n$ with a smooth section $p \mapsto (p, \phi(p))$ of C. The bundle $\mathrm{End}(C)$ is identified with C, by the identification of $\xi_p \in \mathrm{End}(C)_p$ with $P_p \in C_p$ such that $\xi_p(1) = P_p$ for every $p \in M$. We assume that N takes values in $\mathbb{R}^n \subset C\ell_n$. Then N is considered as a map from M to $S^{n-1} \subset \mathbb{R}^n$. Then $T^*M \otimes_{\mathbb{R}} C$ decomposes as

$$T^*M \otimes_{\mathbb{R}} C = (KC)_+ \oplus (KC)_- \oplus (\bar{K}C)_+ \oplus (\bar{K}C)_-.$$

According to this decomposition, a connection $\nabla\colon \Gamma(C) \to \Omega^1(C)$ of the Clifford bundle C decomposes as

$$\begin{gathered}
\nabla = \partial^\nabla + A^\nabla + \bar{\partial}^\nabla + Q^\nabla, \\
\nabla'\colon \Gamma(C) \to \Gamma(KC), \ \ \nabla'\phi = (\nabla\phi)', \\
\nabla''\colon \Gamma(C) \to \Gamma(\bar{K}C), \ \ \nabla''\phi = (\nabla\phi)'', \\
\partial^\nabla\colon \Gamma(C) \to \Gamma((KC)_+), \ \ \partial^\nabla\phi = (\nabla'\phi)_+, \\
A^\nabla \in \Gamma(\mathrm{Hom}(C, (KC)_-)), \ \ A^\nabla\phi = (\nabla'\phi)_-, \\
\bar{\partial}^\nabla\colon \Gamma(C) \to \Gamma((\bar{K}C)_+), \ \ \bar{\partial}^\nabla\phi = (\nabla''\phi)_+, \\
Q^\nabla \in \Gamma(\mathrm{Hom}(C, (\bar{K}C)_-)), \ \ Q^\nabla\phi = (\nabla''\phi)_-,
\end{gathered}$$

where ϕ is any smooth section of C. We see that A^∇ and Q^∇ are tensorial, that is, $A^\nabla \in \Gamma(\mathrm{Hom}(C, (KC)_-))$ and $Q^\nabla \in \Gamma(\mathrm{Hom}(C, (\bar{K}C)_-))$. The sections A^∇ and Q^∇ are called the *Hopf fields* of ∇' and ∇'' respectively.

We denote by d the trivial connection on C.

Lemma 3.1. *A map $N\colon M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$ is a harmonic map, if and only if $d * A^d = 0$.*

Proof. The Hopf field A^d satisfies the equation

$$\begin{aligned}
A^d\phi =& \frac{1}{2}\left[(d' + Jd'J)\right]\phi \\
=& \frac{1}{4}\left[d - J*d + J(d - J*d)J\right]\phi \\
=& \frac{1}{4}\{(d\phi) - N*(d\phi) \\
& + [N(dN)\phi - d\phi] + [*(dN)\phi + N*d\phi]\} \\
=& \frac{1}{4}\left[N(dN) + *(dN)\right]\phi
\end{aligned}$$

for every $\phi \in \Gamma(C)$. Hence

$$d * A^d = \frac{1}{4}(dN \wedge *dN + Nd * dN).$$

Hence $d * A^d = 0$ if and only if

$$dN \wedge *dN + Nd * dN = 0.$$

For an isothermal coordinate (x, y) such that $x + yi$ is a holomorphic coordinate, a map $N \colon M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$ is a harmonic map if and only if

$$\Delta N = -(N_{xx} + N_{yy})dx \wedge dy = |dN|^2 N$$

(see Eells and Lemaire [7]). We have

$$\begin{aligned} d * dN &= d * (N_x\, dx + N_y\, dy) = d(-N_x\, dy + N_y\, dx) \\ &= -(N_{xx} + N_{yy})dx \wedge dy = \Delta N, \\ dN \wedge * dN &= (N_x\, dx + N_y\, dy) \wedge (-N_x\, dy + N_y\, dx) = (-N_x^2 - N_y^2)dx \wedge dy \\ &= (|N_x|^2 + |N_y|^2)dx \wedge dy = |dN|^2, \end{aligned}$$

where the Clifford multiplication is used. Hence, N is a harmonic map if and only if $d * A^d = 0$. □

4. Harmonic maps into a sphere

We construct a tt^*-bundle for a harmonic map from a Riemann surface to an n-dimensional sphere.

Let M be a Riemann surface with complex structure J^M. For a one-form ω on M, define a one-form $*\,\omega$ on M by $*\,\omega := \omega \circ J^M$. For one-forms ω and η on M with values in $C\ell_n$, we have the relation

$$*\,\omega \wedge *\,\eta = \omega \wedge \eta.$$

Indeed, for a basis E_1, E_2 of a tangent space of M with $J^M E_1 = E_2$, we have

$$\begin{aligned} &(\omega \wedge \eta)(qE_1 + rE_2, sE_1 + tE_2) \\ &= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)), \\ &(*\,\omega \wedge *\,\eta)(qE_1 + rE_2, sE_1 + tE_2) = (\omega \wedge \eta)(qE_2 - rE_1, sE_2 - tE_1) \\ &= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)), \end{aligned}$$

where q, r, s, $t \in \mathbb{R}$.

Let $F := M \times \mathbb{R}^{2^n} \cong M \times C\ell_n$. For a map $N\colon M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$, define a one-form S on M with values in $C\ell_n$ by

$$S := \frac{1}{4}(*\,dN + N\,dN).$$

Lemma 4.1. *N is a harmonic map if and only if the one-form S satisfies $d * S = 0$.*

Proof. Since we have

$$4\,d * S = d(-dN + N * dN) = dN \wedge *dN + N\,d * dN = 4d * A^d,$$

this lemma follows from Lemma 3.1. □

Theorem 4.1. *A vector bundle F with $\nabla := d - S$ and S is a tt^*-bundle.*

Proof. We see that

$$\begin{aligned}
4\,dS = d*\,dN + dN \wedge dN &= dN \wedge dN + N\,dN \wedge *dN,\\
16S \wedge S &= (*\,dN + N\,dN) \wedge (*\,dN + N\,dN)\\
&= *\,dN \wedge *\,dN + *dN \wedge N\,dN + N\,dN \wedge *\,dN + N\,dN \wedge N\,dN\\
&= dN \wedge dN + N\,dN \wedge *\,dN + N\,dN \wedge *\,dN + dN \wedge dN\\
&= 2(dN \wedge dN + N\,dN \wedge *\,dN).
\end{aligned}$$

Hence $dS = 2S \wedge S$ holds.

Lemma 4.1 and a direct calculation yield

$$\begin{aligned}
\nabla^\theta &= d + (\cos\theta - 1)S + (\sin\theta) * S,\\
&d^{\nabla^\theta} \circ \nabla^\theta\\
&= (\cos\theta - 1)dS + ((\cos\theta - 1)S + (\sin\theta) * S) \wedge ((\cos\theta - 1)S + (\sin\theta) * S)\\
&= (\cos\theta - 1)dS + (\cos\theta - 1)^2 S \wedge S + (\cos\theta - 1)(\sin\theta) S \wedge *S\\
&\quad + (\sin\theta)(\cos\theta - 1) * S \wedge S + (\sin\theta)^2 * S \wedge *S\\
&= (\cos\theta - 1)dS - 2(\cos\theta - 1)S \wedge S = 0.
\end{aligned}$$

Hence F with ∇ and S is a tt^*-bundle. □

For a harmonic map from a Riemann surface to S^2, we have two tt^*-bundles. One is the tt^*-bundle in Theorem 4.1. The other is that in the theory of quaternionic holomorphic line bundles (see [8]). These do not coincide directly as the fiber of the former is $C\ell_3$ and that of the latter is $C\ell_2$.

Acknowledgements.

The authors are very grateful to Claus Hertling for the comments to a preliminary version of this paper.

References

[1] A. I. Bobenko, All constant mean curvature tori in $\mathbf{R}^3$, S^3, H^3 in terms of theta-functions, Math. Ann. 290 (1991) 209–245.

[2] S. Cecotti, C. Vafa, Topological–anti-topological fusion, Nuclear Phys. B 367 (1991) 359–461.

[3] J. Dorfmeister, M. Guest, W. Rossman, The tt^* structure of the quantum cohomology of $\mathbb{C}\mathrm{P}^1$ from the viewpoint of differential geometry, Asian J. Math. 14 (2010) 417–437.

[4] J. Dorfmeister, G. Haak, Meromorphic potentials and smooth surfaces of constant mean curvature, Math. Z. 224 (1997) 603–640.

[5] J. Dorfmeister, F. Pedit, H. Wu, Weierstrass type representation of harmonic maps into symmetric spaces, Comm. Anal. Geom. 6 (1998) 633–668.

[6] B. Dubrovin, Geometry and integrability of topological-antitopological fusion, Comm. Math. Phys. 152 (1993) 539–564.

[7] J. Eells, L. Lemaire, Another report on harmonic maps, Bull. London Math. Soc. 20 (1988) 385–524.

[8] D. Ferus, K. Leschke, F. Pedit, U. Pinkall, Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori, Invent. Math. 146 (2001) 507–593.

[9] H. B. Lawson, M-L. Michelsohn, Spin geometry, Princeton Mathematical Series, 38 Princeton University Press, Princeton, NJ, 1989.

[10] L. Schäfer, tt^*-geometry and pluriharmonic maps, Ann. Global Anal. Geom. 28 (2005) 285–300.

[11] L. Schäfer, Harmonic bundles, topological-antitopological fusion and the related pluriharmonic maps, J. Geom. Phys. 56 (2006) 830–842.