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An Empirical Research and Development of Teaching Materials for Mathematics to Foster Rich Creativity

- An Attempt to Make Materials from "Sangaku (Mathematics Tablet)"of Kon'nou Shrine -

Hideyo MAKISHITA

Junior and Senior High School at Komaba,
University of Tsukuba, Faculty of Mathematics

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Abstract: *Wasan (traditional Japanese mathematics) as well as their important spin-offs the Sangaku (mathematics tablets) discussed in this article are a part of Japan's unique cultural heritage. They were written on Ema (wooden votive tablets) and presented as offerings at shrines and temples in order to thank the gods.*

In the present article, I first will introduce a brief history of Wasan, with strong emphasis on the development of Sangaku, which is followed by two examples of Sangaku problems from Kon'ou Shrine and their modern mathematical solutions.

It is my sincere hope that the elegance of Wasan's way of looking at things and way of thinking, the very essence of mathematics in Edo culture, will be brought to life once again in modern mathematics education.

Key Words: *Sangaku* (mathematics tablet), Kon'nou Shrine, Formula of *Keishigen*, Formula of *Soukogen*, Formula of *Shousa*

1 Introduction

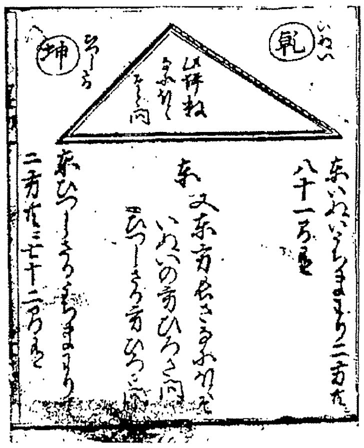
Western mathematics was introduced in the Japanese school system at the beginning of the Meiji Period. In order to distinguish between the mathematics of the Edo Period and Western mathematics, the mathematics of the Edo Period is referred to as *Wasan* (Japanese mathematics). With the introduction of mathematics as Western mathematics in Japanese schools, the number of people learning *Wasan* steadily declined. At present, with *Wasan* researchers at the forefront, research on *Wasan* is being conducted actively and, through writings and events, *Wasan* culture is being spread to the world. In Heisei 20 (2008), events were held on a national scale in commemoration of the 300th anniversary of the death of Takakazu Seki, a man who is referred to as a "sage of mathematics."At the same time, *Wasan* is also increasingly being reevaluated. Furthermore, one increasingly sees reports of practical lesson trials or other research which attempts to make use of *Wasan* in mathematics education.

The custom of making "*Sangaku* offerings," which is discussed in this article, is a part of Japan's unique mathematics culture which began around the middle of the Edo Period. It is thought that in Edo, the political, economic and cultural center of Japan, an enormous number of *Sangaku* were dedicated as offerings. It is written in "*Sangaku Shinteiki*" (Yoshitaka Muraki, Enpou 8 (1681)) that *Sangaku* were being dedicated as offerings throughout Edo at the time. It has also been noted elsewhere that *Sangaku* were dedicated as offerings at Meguro Fudo (Ryusenji), near our school. From the publication of "*Shinpeki Sanpou*" (Sadatsugu Fujita, Kansei 1 (1789)), a collection of *Sangaku*, and others, we can surmise that a great number of people from all walks of life held an active interest in *Sangaku*.

However, most of the *Sangaku* dedicated in the *ema* halls of shrines and temples have been damaged by the weathering of time or destroyed by fires and wars; to the point that only about 1,000 *Sangaku* survive intact to this day in all of Japan. Especially in Tokyo, where such a large number of *Sangaku* had been presented as offerings, for reasons such as those listed above, the number of surviving *Sangaku* is extremely small. In Iwate Prefecture, Yamagata Prefecture and other parts of the Tohoku Region and in Nagano Prefecture, there are a large number of surviving *Sangaku* and, in turn, *Wasan* research in these areas is very active.

2 *Wasan* and *Idai Keishou*(The Passing on of Difficult Problems)

When introducing *Wasan,* it is first necessary to introduce the *Wasan* treatise, "*Jinkouki.*"Written and published by Mitsuyoshi Yoshida in Kan'ei 4 (1627), "*Jinkouki,*" is a mathematical treatise which is well-known to many people. Mitsuyoshi Yoshida, a member of the Kyoto Suminokura Clan of merchants, had studied the Chinese mathematical treatise, "*Sanpou Tousou,*" under the tutelage of father and son, Ryoui Suminokura and Soan Suminokura. Soan Suminokura is the man best known for publishing the great reproduction of classical writings, "*Sagabon.*" Yoshida, using "*Sanpou Tousou*" as a model, created mathematical problems which were intimately related to the realities of day-to-day life in Japan at the time, publishing them as "*Jinkouki.*"From its illustrations to its overall style, "*Jinkouki*" is an exceptional book which draws from the literary legacy of the great work, "*Sagabon.*" "*Jinkouki*" was also a lifestyle manual explaining, in both minute and considerate detail, methods for using the calculation tool called the "soroban" (abacus), which was necessary for people in their daily lives at that time. As the abacus was coming into common use across Japan, a slew of *Wasan* treatises imitating "*Jinkouki*" also began to appear. Also, because these imitations of "*Jinkouki*" continued to be published without relent, Yoshida revised "*Jinkouki*" several times. In "Revised *Jinkouki,*" published in Kan'ei 18 (1641), Yoshida included 12 mathematics problems, without answers, at the back of the book, writing, "There are those out there who are teaching mathematics at the level of "*Jinkouki.*" People studying mathematics probably have no way of knowing whether their teacher is competent or not, so I will teach you a method for judging the ability of your teacher. I shall write here twelve problems without answers. You may judge the ability of your teacher by whether or not he can solve these problems."These problems are called "*idai* (problems left behind)" or "*konomi* (favorites)."



An Actual Problem (*Idai* from "Revised Jinkouki")

There is a right triangle whose perpendicular side and vertical side have a sum value of 81 and whose perpendicular side and horizontal side have a sum value of 72. Find the area of this triangle and the lengths of its three sides.

The solutions to Yoshida's *idai* were revealed 12 years later, in Shou'ou 2 (1653), in "*Sanryouroku,*" which was written by a young mathematician named Tomosumi Enami. In imitation of Yoshida, Enami also wrote his own set of 8 *idai* problems in the back of his book. Enami's publication stimulated other mathematicians to publish their works with solutions to the *idai* of "*Jinkouki*" along their own sets of *idai.* Examples include "*Enpou Shikanki,*" written by

Shunjuu Hatsusaka in 1657, "*Sanpou Ketsugishou*," written by Yoshinori Isomura in 1659 and "*Sanpou Kongenki*," written by Masaoki Satoh in 1669.

So began the mathematical question and answer "relay," in which one would publish a *Wasan* treatise with solutions to previous *idai* while also including new *idai* of one's own creation. This relay process is called "*idai keishou*" (the passing on of difficult problems). It goes without saying that these *idai* became progressively more difficult, while at the same time spurring the development of new kinds of arithmetical operations.

For example, Takakazu Seki (ca. Kan'ei 19 (1642) – Houei 5 (1708)) devised the "*Boushohou* Method,[1]" a method for algebraic calculation on paper, in order to solve an *idai* from Kazuyuki Sawaguchi's "*Kokin Sanpouki*" (Kanbun 11 (1671)). The result of this was later introduced in Seki's "*Hatsubi Sanpou*." It appears that many *Wasan* mathematicians at the time did not fully understand *boushohou* (the Formula of *Tenzan*), leading Seki's disciple, Katahiro Takabe, to explain, based on concrete problems, the Tenzan Formula in greater detail within the pages of "*Hatsubi Sanpou Endangenkai*." It can rightly be said that "*idai keishou*" contributed greatly to the development of *Wasan*.

3 The Offering of *Sangaku*

The term *Sangaku* refers to *ema* (votive tablets) on which mathematical problems were written and which were dedicated at shrines and temples. It is said that the custom of offering *Sangaku* began in the Kanbun Era (around 1660), in the middle of the Edo Period. Common people of the Edo Period dedicated *Sangaku* as expressions of gratitude to the gods for having been able to solve mathematical problems and as prayers to be able to apply themselves ever more to their studies. With shrines and temples, as centers of social discourse for common people at the time, also being places for showing one's achievements, there also appeared Sangaku which were dedicated with only *idai*, without answers. About 1,000 *Sangaku* exist to this day.[2] The custom of offering *Sangaku* is considered to be a completely unique aspect of Japanese culture for which nothing similar exists anywhere in the world.

As can also be understood from the background which produced *Sangaku*, the mathematical culture of the Edo Period was extremely advanced, had diverse contents ranging from mathematical games to fully-fledged mathematics and also offers us a wealth of content and topics which are applicable to the modern mathematics taught in schools. In the Meiji Period, Japan made the conversion to Western mathematics, so few people now know of *Wasan*. Amidst such a backdrop, however, *Wasan*, with its multitude of topics intimately related to daily life, is being reevaluated, as is the case with the application of *Wasan* to the mathematics taught in school. At the Japan Society for Mathematical Education research symposiums, too, an increasing number of reports of *Wasan* being put into actual practice in the classroom are being submitted. In addition, the non-profit organization "*Wasan*" is engaged in activities to bring this long-forgotten Edo Period cultural tradition back to life in modern mathematics education. One example of their activities, based upon the *Wasan* custom of offering *Sangaku,* is the holding of a "'Let's Make *Sangaku!*' Competition," with the best works being dedicated, as fine examples of *Sangaku*, at Kanda Shrine in the Ochanomizu District of Tokyo. I would like, by all means, to apply *Sangaku* and *Wasan*, both invaluable cultural assets left to us by our ancestors, to the mathematics classroom.

3.1. *Sanpou Shoujo* (Arithmetic Girl)

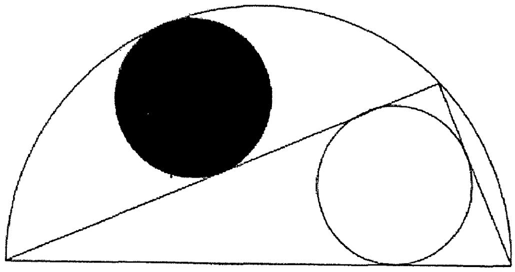
1 *Boushohou* would later develop into what is called the *Tenzan* Formula. In terms of Western mathematics, it is comparable to literal expressions.

2 If one includes other *Sangaku* which are mentioned in existing records or other literature, the number becomes significantly larger.

When first showing *Wasan* and *Sangaku* to students, I recommend introducing them to the work of historical fiction, "*Sanpou Shoujo*" (Endoh Hiroko, Chikuma Shobo Ltd., 2006). Having students read this book is very effective in helping students understand the cultural elements of *Wasan* in the Edo Period. The title, "*Sanpou Shoujo*, originally belonged to a treatise on *Wasan.* Written and published by Touzo Chiba in An'ei 4 (1775), it is a book about mathematics. Recently, the book "Reading the *Wasan* Treatise, '*Sanpou Shoujo*'" was published by Chikuma Shobo (Hiroshi Kotera, Chikuma Shobo, Ltd., 2009). It contains detailed introductions of the *Wasan* problems appearing in the book.

Now, in the historical novel "Sanpou Shoujo," there is a passage in which a young samurai, Sannosuke Mizuno, is triumphantly carrying a *Sangaku* through a crowd of sightseers on his way to a shrine, where he intends to dedicate it. Seeing the *Sangaku,* local girl Aki, having been taught the same problem by her father and, being quite fond of arithmetic, cannot help but whisper, "Your answer is wrong." Hearing this, Mizuno's samurai companions quickly surround Aki and press her to explain what about the answer is wrong. Aki, however, simply states, in an unwavering tone of voice, the correct answer. Mizuno then realizes his error, gives up on dedicating the *Sangaku* and willingly returns home. That is how the passage goes.

The problem in the passage is as laid out below.

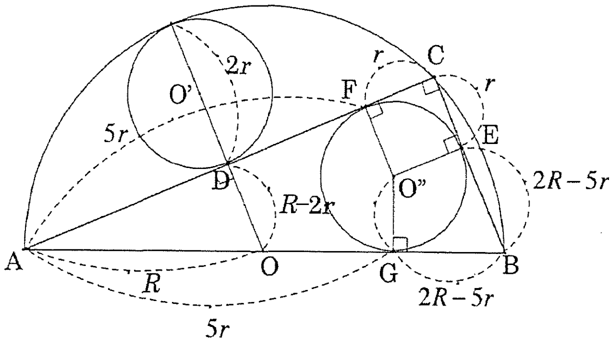


The Problem from "*Sanpou Shoujo*"

The problem on the *Sangaku* reads, "As shown in the picture, there is a triangle inscribed on a semicircle. If the radius of the circle inscribed on the triangle (white circle) matches the radius of the small circle in contact with the semicircle and one side of the triangle, what is the relationship between the diameter of the semicircle and that of the small circle?" This is a seemingly-simple, but somewhat difficult-to-solve problem. It is very easy to read, and I think it will be a good source of material for introducing *Sangaku* to students.

3.2. Solution to the Problem from "*Sanpou Shoujo*"

First, let us the assign the letters ABC to the sides of the triangle, O to the semicircle, O' to the small circle on the inside of the arc and O" to the inscribed circle, as shown in the illustration below. Here, let us refer to the radii of the small circle and semicircle as R and r, respectively.



The problem from "*Sanpou Shoujo*"

€

If we assign the letter D to the point of contact between O' and CA; then:

$$\mathrm{OD} = R - 2r$$

By triangle mid-segment theorem, we get:

$$\mathrm{BC} = 2(R - 2r)$$

If we then assign the letters E, F and G to the respective points of contact between O" and BC, CA and AB; then:

$$\mathrm{CE} = \mathrm{CF} = r$$

So: $\quad BE=BG=2(R-2r)-r=2R-5r$

$$AG=AF=2R-(2R-5r)=5r$$

Thus, $AC=6r$

Therefore, in terms of triangle ABC:

$$AB=2R, \quad BC=2(R-r), \quad CA=6r$$

So, by the Pythagorean Theorem, we get:

$$\{2(R-2r)\}^2+(6r)^2=(2R)^2$$

$$\Leftrightarrow \quad 4(R-2r)^2+36r^2=4R^2$$

$$\Leftrightarrow \quad -16Rr+16r^2+36r^2=0$$

$r>0$ which gives us:

$$4R=13r$$

From the above, we find that the relationship between the radii of the semicircle and the small circle is:

$$4\times 2R=13\times 2r$$

However, Sannosuke Mizuno's *Sangaku* had the mistaken answer,"$2R=3\times 2r$," written on it.

4 *Sangaku* of Kon'ou Shrine

Kon'ou Shrine was established at its present location by Kawasaki Tosa-no-kami Motoie as a local samurai shrine in Kanji 6 (1092), during the reign of the 73rd Horikawa Emperor.

Kon'ou Shrine

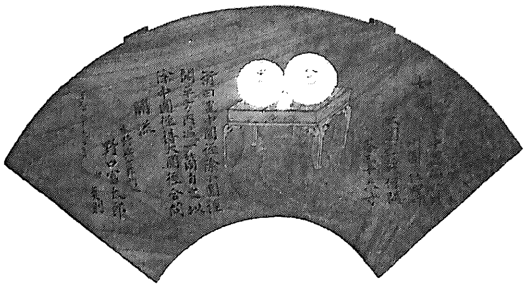
The Kawasaki Clan had been granted the family name of Shibuya for distinguished service rendered and the shrine itself had originally been called Shibuya Shrine. Later, the shrine came to be called Kon'ou Shrine in honor of Kono'oumaru Shibuya, who had served Yoshitomo Minamoto and was renowned for his distinguished service during the Hogen Rebellion. Construction on the present-day shrine building was begun in Keichou 17 (1612), during the reign of Hidetada Tokugawa, by Iemitsu Tokugawa's guardian, Tadatoshi Aoyama, and wet nurse, Lady Kasuga. Preserving the style of early-Edo Period architecture, the present-day shrine gate was added in the mid-Edo Period.

In Bunji 5 (1189), after the decline and fall of Yasuhira Fujiwara of Oshu, Yoritomo Minamoto dedicated a sword at the shrine in remembrance of Kon'oumaru's allegiance to his father. At the same time, Minamoto also transplanted the "Usuwasure-no-Sakura" (Usuwasure Cherry) tree from his estate in Kamegaya, Kamakura onto the shrine grounds, bestowing upon it the name "Kon'ou Sakura" (Kon'ou Cherry). This tree has been designated by Shibuya Ward as a natural monument. Kon'ou Shrine is located in one of the more serene areas of Shibuya and is also filled with many historical artifacts.

Here, I shall introduce three of the existing *Sangaku* of Kon'ou Shrine, along with translations of the texts of their problems into modern Japanese and examples of methods for solving them for use in junior and senior high school classes. I shall also, as much as possible, touch upon how the problems were solved using Wasan.

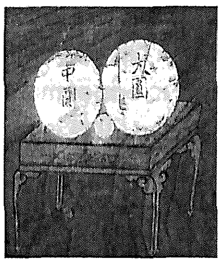
4.1. Tomitarou Noguchi: The *Sangaku* of Minamoto-no-Sadanori (*Sangaku* 1)

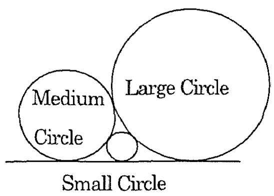
The following *Sangaku* was dedicated with the name Minamoto-no-Sadanori in Genji 1 (1864). Its fan-like shape is different from the rectangular shape most commonly seen in *Sangaku*, making it a distinctive and rare example. The problem presents 3 circles—one large, one medium and one small—and seeks to find the diameter of the large circle when those of the small and medium circle are given.



***Sangaku* 1, Genji 1 (1864)**

如図
中圓径九寸
小圓径四寸
大圓径幾何問
答三十六寸
術曰置中圓径除小圓径
開平方内減一箇自之以
除中圓径得大圓径合問
關流 水野興七郎門人
野口富太郎
源 貞則
元治元 甲子年十一月吉日





4.1.1. *Sangaku* Problem Text

As shown in the illustration, if the diameter of the medium circle[3] is 9 *sun* and the diameter of the small circle is 4 *sun*, what is the diameter of the large circle?

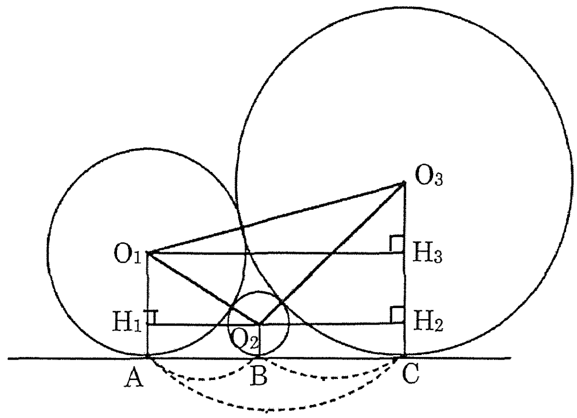
Answer: 36 *sun*.

Explanation (formula): first divide the segment of the medium circle by that of the small circle and take the square root of that number. Then, subtract 1 from that number and square the result. By dividing that number by the segment of the medium circle, we can find the segment of the large circle. The resulting number is equal to the diameter of the large circle.

Moreover, the *Sangaku* expresses the contact relationship between the large, medium and small spheres. That is, all three spheres are centered on a level plane.

4.1.2. A Modern Mathematical Solution to the Problem of *Sangaku* 1

Let a be the diameter of medium circle O_1, b be the diameter of small circle O_2 and x be the value of the diameter of large circle O_3.



Let A, B and C be the points of contact between the shared plane and the medium circle, small circle and large circle, respectively. Then:

$$AB+BC=AC$$

So, by the Pythagorean Theorem:

$$AB=O_2H_1=\sqrt{(a+b)^2-(a-b)^2}$$

$$BC=O_2H_2=\sqrt{(x+b)^2-(x-b)^2}$$

$$AC=O_1H_3=\sqrt{(x+a)^2-(x-a)^2}$$

Thus: $\sqrt{(a+b)^2-(a-b)^2}+\sqrt{(x+b)^2-(x-b)^2}$

$$=\sqrt{(x+a)^2-(x-a)^2}$$

$$2\sqrt{ab}+2\sqrt{bx}=2\sqrt{ax}$$

Thus, we get: $(\sqrt{a}-\sqrt{b})\sqrt{x}=\sqrt{ab}$

3 The text reads "circle," but the illustration depicts spheres. In *Wasan*, "segment" refers to diameter.

$$\sqrt{x} = \frac{\sqrt{ab}}{\sqrt{a} - \sqrt{b}}$$

$$x = \left(\frac{\sqrt{ab}}{\sqrt{a} - \sqrt{b}} \right)^2 \qquad (4.1)$$

In short, $a = 9$ (*sun*), $b = 4$ (*sun*) gives us:

$$x = \left(\frac{\sqrt{9 \times 4}}{\sqrt{9} - \sqrt{4}} \right)^2 = \left(\frac{6}{3-2} \right)^2 = 36 \text{ (sun)}$$

Moreover, the equation used (4.1) is:

$$x = \left(\frac{\sqrt{ab}}{\sqrt{a} - \sqrt{b}} \right)^2 = \frac{a}{\left(\sqrt{\frac{a}{b}} - 1 \right)^2}$$

This equation is exactly the same as what is described in the explanation (formula) written on the *Sangaku*.

4.1.3. A *Wasan* Solution to *Sangaku* 1

Wasan has formulas (*jojutsu*), which are used to solve problems. The formula used to solve this problem can be found as follows:

Solution Using *Jojutsu*

Answer: Let r_1, r_2, r_3 be the respective radii of the large circle, medium circle and small circle. From this, we use the formula:

$$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r_3}}$$

Thus we get,

$$\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{r_3}} - \frac{1}{\sqrt{r_2}}$$

$$\frac{1}{\sqrt{r_1}} = \frac{\sqrt{r_2} - \sqrt{r_3}}{\sqrt{r_2}\sqrt{r_3}}$$

$$\sqrt{r_1} = \frac{\sqrt{r_2}\sqrt{r_3}}{\sqrt{r_2} - \sqrt{r_3}} = \frac{\sqrt{r_2}}{\sqrt{\frac{r_2}{r_3}} - 1}$$

$$r_1 = \frac{r_2}{\left(\sqrt{\frac{r_2}{r_3}} - 1 \right)^2}$$

Explanation

From the above, by assigning r_1, r_2, r_3 to the respective radii of the large, medium and small circles, we get:

$$2\sqrt{r_2 r_3} + 2\sqrt{r_3 r_1} = 2\sqrt{r_1 r_2}$$

If we divide both sides of the equation by $2\sqrt{r_1 r_2 r_3}$,

we get: $$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r_3}}$$

4.1.4. How to Look at *Sangaku*

In general, *Sangaku* have an illustration (picture) drawn on them and the illustration itself is the problem.

(1) Problem text (2) Illustration (3) Answer (4) Explanation (*jojutsu* formula) (5) Date of dedication, name

The explanation/formula (4) lays out the method used to find the answer. These explanations often make use of *jutsu* (formulas) which are unique to *Wasan*, such as the Formula of *Ruien*, and contain parts of those formulas or their results.

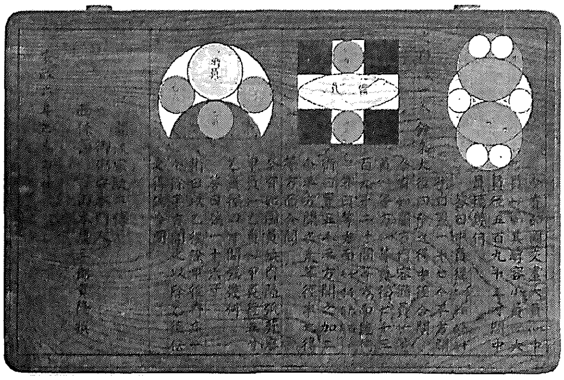
A collection of these formulas, "*Sanpou Jojutsu* (Formulas of Arithmetic" (Gazen Yamamoto, Tenpou 12 (1841)), was also published.

4.2. The *Sangaku* of Takataka Youzaburou Yamamoto (*Sangaku 2*)

The following *Sangaku* was dedicated in Ansei 6 (1859) by Takataka Youzaburou Yamamoto of the Saijou Domain[4] of Iyo Province in Shikoku. It

4 The Saijou Domain's official residence in Edo was located near present-day Aoyama Gakuin.It is thus thought that Yamamoto happened to dedicate it at Kon'ou Shrine, which was just across Tamakawa Road (National Route 246). The Saijou Domain was a

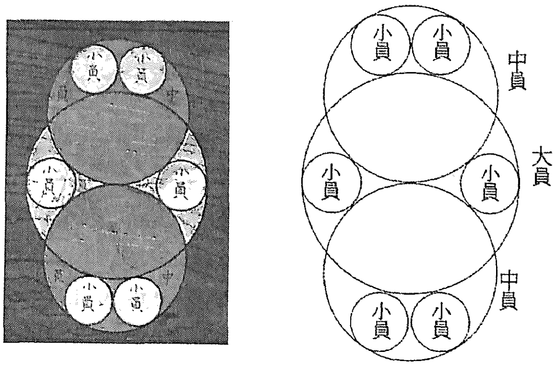
is thought that it was dedicated at Kon'ou Shrine because of the proximity of the shrine to the Saijou Domain's official residence in Edo. Also, as can also be seen from the photograph, this *Sangaku* has three problems.



***Sangaku 2,* Ansei 6 (1859)**

4.2.1. *Sangaku* Problem Text (Problem 1)

(第一問)
今有如図交畫大員一个
中員二个而其罅容小員六个
大員径五百九十三寸
問中員径幾何
答曰中員径四百六十三寸有奇
術曰置一十七个平方開
之内減一个餘乗大径四除之
得中径合問



小員;Small Circle,
中員;Medium Circle, 大員;Large Circle

subordinate domain of the Kishu Domain.

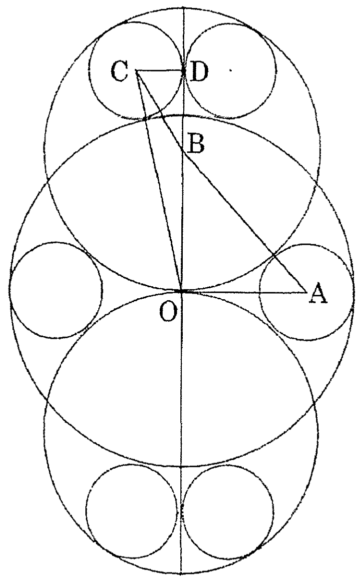
As shown in the illustration, there are three overlapping circles: a large circle and two medium circles. Six small circles are inscribed within the intervals where the circles do not overlap. If the diameter of the large circle is 593 *sun,* what is the diameter of the medium circles?

Answer: the diameter of the medium circles is 463 *sun* and a decimal number (remainder).

Explanation (formula): first, extract the square root of 17 and subtract 1. Multiplying that value by the diameter of the large circle and then dividing by 4 gives the diameter of the medium circle.
In other words,
Numerical formula:

$$\frac{593(\sqrt{17}-1)}{4} = 463.000409$$

(1) Solution to *Sangaku* 2 (Problem 1)



As shown in the illustration, let O(R) be the large circle, A(r) be one small circle, C(r) be another small circle and B(x) be the medium circle. Also, with D as the point of contact between the two small circles: BD=y
As for △OAB:

$$(R-r)^2 + x^2 = (x+r)^2 \qquad ①$$

As for △OCD:

$$r^2 + (x+y)^2 = (R+r)^2 \qquad ②$$

As for △BCD:

$$y^2 + r^2 = (x-r)^2 \qquad ③$$

In order to eliminate y from ③, we make

$$y = \sqrt{x^2 - 2rx}$$

and substitute in ②.

Then, in order to eliminate $2r$ from ①, we make

$$2r = \frac{R^2}{x+R}$$

and substitute in ②. As the result of these calculations, we get the equation:

$$x^3 + 2Rx^2 - R^2x - R^3 = 0$$

Although it is possible to solve this problem algebraically using a cubic function[5], because the value of R is so large, I opted to use a calculator. The result was:

$$x = 237.774, \ -164.5, \ -666.2$$

From the above: $x = 237.774$

The diameter of the medium circle, then, is approximately 475.5 (*sun*).

However, this value does not match what is written on the *Sangaku*.

Note: Using *Wasan*, the answer was sought by calculating with a cubic function, similar to the Horner Algorithm. In the Edo Period, when methods for calculation were limited to the *soroban* (abacus) or *sangi* (calculation sticks), people worked to reduce the number of necessary calculations.

In general, with the cubic equation:

$x^3 + ax^2 + bx + c = 0$ you have:

$$x^3 + ax^2 + bx + c = x\{x(x+a)+b\}+c$$

Looking at the number of calculations required, the left side,

$$x \times x \times x + a \times x \times x + b \times x + c = 0$$

has 5 of × and 3 of +, thus requiring a total of 8 calculations.

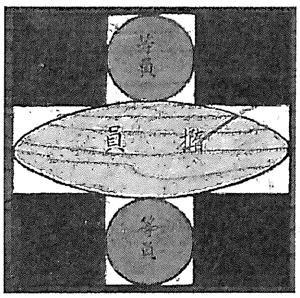
The right side, $x\{x(x+a)+b\}+c = 0$

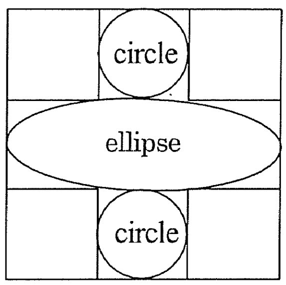
has 2 of × and 3 of +, thus requiring a total of 5 calculations.

It is thought that this problem, as well, was calculated in a similarly-devised manner and that, through such a process of calculation, a margin of error also emerged.

4.2.2. *Sangaku* Problem Text (Problem 2)

（第二問）
今有如図方員内容楕員一個
員二個等方四個得七千三
百九十二寸問等方面幾何
答曰等力面七千六百〇七寸有奇
術曰置五個平方開之加二個
平方開之乗等径半之得
平方面合問





As shown in the illustration, there are four squares of equal size inside of a larger square. Also, there are two circles of equal size between the small squares. There is an oval running along the apices of the four squares and in contact with the two circles.

If the diameter of the circles is 7392 *sun*, what is the length of the sides of the four squares?

Answer: 7607 *sun* and a remainder.

5 In general, polynomials with more than five dimensions cannot always be solved algebraically.

Explanation (formula): First take the square root of 5 and add 2. Then, take the square root of the resulting number and multiple it by half of the diameter of the circles.

Expressing the text of the explanation (formula) as an equation gives us the following:

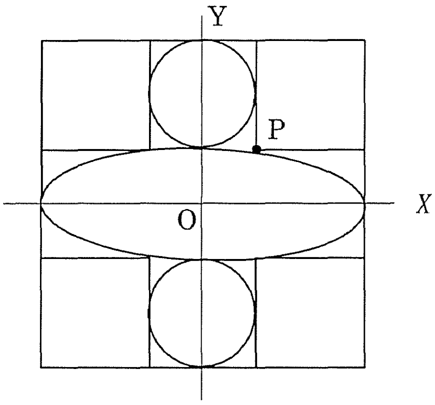
$$\text{Equation:} \sqrt{\sqrt{5}+2} \times \frac{7392}{2} = 7607.000117$$

(1) A Modern Solution to *Sangaku* 2 (Problem 2)

Let r be the radius of the circles and 2a be the length of the sides of the squares. Also, as shown in the illustration, let us establish an X-axis and a Y-axis running between the midpoints of the sides of the outer square. Let x be the side of one of the squares whose value we are seeking.

From the relationship between the lengths, we get:

$$2x + 2r = 2a$$



As shown in the illustration, when the small square in quadrant 1 is in contact with the oval, the coordinates of its apex, P, are P(r, $a-x$). So, the equation for the oval is:

$$\frac{X^2}{a^2} + \frac{Y^2}{(a-2r)^2} = 1$$

By eliminating a from the equation for the oval, using, $a = x + r$ and creating an equation of x and r,we get:

$$\frac{r^2}{(x+r)^2} + \frac{r^2}{(x-r)^2} = 1$$

$$x^4 - 4r^2x^2 - r^4 = 0$$

$$x^2 = \left(\sqrt{5}+2\right)r^2$$

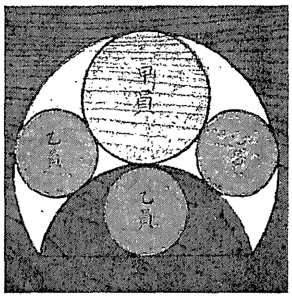
Thus, we get the answer:

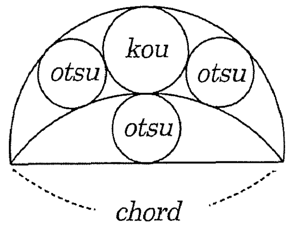
$$x = \sqrt{\sqrt{5}+2}r = \sqrt{\sqrt{5}+2}\,\frac{2r}{2}$$

This answer matches the answer given in the explanation (formula).

4.2.3. *Sangaku* Problem Text (Problem 3)

(第三問)
今有如図員鋏内隔弧背容
甲員一個乙員三個甲員径五寸
乙員径四寸問弦幾何
答曰弦一十六寸
術曰以乙径除甲径内減一個
餘平方開之以除乙径倍之得弦合問
關流宗統六傳
御粥安本門人
西條藩 山本庸三郎貴隆撰
安政六年己未四月





As shown in the illustration, there is an arc and 4 circles—one labeled *kou* and three labeled *otsu*, within a larger circle. If the diameter of *kou* is 5 *sun* and the diameter of *otsu* is 4 *sun*, then what is the length of the chord?

Answer: The length of the chord is 16 *sun*.

Explanation (formula):

$$chord = \frac{2otsu}{\sqrt{\frac{otus}{kou} - 1}}$$

* This is the length of the chord when the diameter of the circle *kou* is expressed as *kou* and that of circle *otsu* is expressed as *otsu*.

(1) Solution to *Sangaku* 2 (Problem 2)

Let A be the center point of circle *kou* and B be the center point of the circle *otsu* (on the right).

Also, as shown in the illustration, let C be the center point of the third circle, D be the center point of the outer circle, H be a segment of a perpendicular line drawn from B to the diameter of the outer circle, G be the point of contact between the central *otsu* circle and E and F be the start and end points of the chord whose length we are seeking.

The diameters of the *kou* circle, the *otsu* circle, the third circle and the outer circle are expressed as $2R$, $2r$, $2a$ and $2b$, respectively. Also, the length of the chord is expressed as x and the length of AH is expressed as y.

By the similarity of triangle:
By the Formula of *Keishigen*:

$$x^2 = 4\cdot 2r(2a-2r) \qquad ①$$

Because the power of point of G is equal for both the outer circle and the third circle:

$$(2R+2r)(2b-2R-2r) = 2r(2a-2r) \qquad ②$$

In short,

$$(2R+2r)\cdot 2b - \left\{(2R)^2 + 2\cdot 2R\cdot 2r + 2r\cdot 2a\right\} = 0 \qquad ③$$

By applying the *Wasan*'s Formula of like cosine law to triangle △ABD, we get:
By applying the Formula of *Soukogen* to triangle△ABD, we get:

$$(2b-2r)^2 = (2R+2r)^2 + (2b-2R)^2 - 4(2b-2R)y$$

$$(2R)^2 + 2R\cdot 2r - 2b\cdot 2R + 2b\cdot 2r - 2(2b-2R)y = 0 \qquad ④$$

By applying the *Wasan*'s Formula of like cosine law to triangle △ABC, we get:
By applying the Formula of *Soukogen* to triangle △ABC, we get:

$$(2a+2r)^2 = (2R+2r)^2 + (2a+2R)^2 - 4(2a+2R)y$$

$$(2R)^2 + 2R\cdot 2r + 2a\cdot 2R - 2a\cdot 2r - 2(2a+2R)y = 0 \qquad ⑤$$

If we eliminate y from ④ and ⑤, we get:

If we eliminate y by applying the *ijou* (ratio)s from ④ and ⑤, we get:

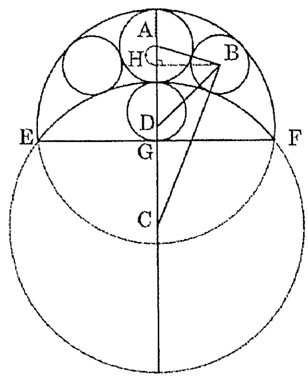
$$\left\{(2R)^2 + 2R\cdot 2r - 2b\cdot 2R + 2b\cdot 2r\right\}(2a+2R)$$

$$= \left\{(2R)^2 + 2R\cdot 2r + 2a\cdot 2R - 2a\cdot 2r\right\}(2b-2r)$$

Putting these into order gives us:

$$\left\{(2R)^2 + 2a\cdot 2R - 2a\cdot 2r\right\}\cdot 2b$$

$$-\left\{(2R)^3 + 2r\cdot(2R)^2 + 2a\cdot(2R)^2\right\} = 0 \qquad ⑥$$



If we eliminate the outer parts of the problem by applying the *ijou* (Ratio) from ③ and ⑥, we get: $(2R+2r)\left\{(2R)^3 + 2r\cdot(2R)^2 + 2a\cdot(2R)^2\right\}$

$$= \left\{(2R)^2 + 2\cdot 2R\cdot 2r + 2r\cdot 2a\right\}\left\{(2R)^2 + 2a\cdot 2R - 2a\cdot 2r\right\}$$

$$(2R-2r)\cdot(2a)^2 + \left\{(2R)^2 - 2\cdot 2R\cdot 2r\right\}\cdot 2a - (2R)^2\cdot 2r = 0$$

Factoring gives us:

$$\left\{2(2R-2r)a - 2R\cdot 2r\right\}(2a+2R) = 0 \qquad ⑦$$

Thus,

$$2a = \frac{2R\cdot 2r}{2R-2r}$$

Substituting this into ① gives us:

$$x^2 = 4\cdot 2r\left(\frac{2R\cdot 2r}{2R-2r} - 2r\right) = \frac{4(2r)^3}{2R-2r}$$

Thus,

$$x=\frac{2\cdot 2r}{\sqrt{\frac{2R}{2r}-1}}$$

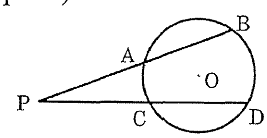
Input the value: $2R=5$ *sun*, $2r=4$ *sun*, so:

$$x=\frac{2\times 4}{\sqrt{\frac{5}{4}-1}}=16 \quad (sun)$$

Fit it the answer.

Note 1: *Houbeki* (Power of a Point)

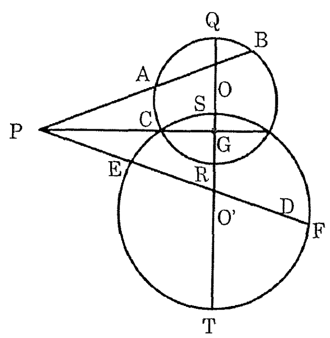
(1) When there is a fixed point, P, and a fixed circle, O, a straight line is drawn from P intersecting points A and B on circle O. PA×PB is then called ***houbeki*** (power of a point).



The *houbeki* is fixed in length. In other words, as shown in the above illustration:

$$PA\times PB=PC\times PD$$

(2) Furthermore, as shown in the illustration, another circle, O', is drawn with shared points C and D.



$$PA\times PB=PC\times PD=PE\times PF \qquad (*)$$

Also, as shown in the illustration, a straight line, QT, is drawn through the centers of circle O and circle O' and the points of intersection are labeled R and S.

Also, by labeling the point of intersection between PD and QT as G, we find from (*) that the *houbeki* for point G is equal, giving us:

$$QG\times GR=SG\times GT$$

This thus leads us to equation ②.

In short,

$$(2R+2r)(2b-2R-2r)=2r(2a-2r) \qquad ②$$

Note 2: The Formula of *Keishigen*

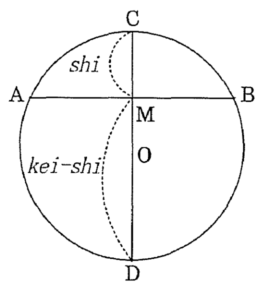
As shown in the illustration, there are two points, A and B, forming a chord AB, on the circumference of a circle O.

A bisector is drawn perpendicular to chord AB and its points of intersection with the circle are labeled C and D. Its point of intersection with the chord is labeled M. Segment CM is called *shi*.

The formula expressing the relationship between the diameter of Circle O, the chord and the *shi*,

$$chord^2=4shi(kei-shi)$$

This is called the **Formula of *Keishigen***.



Proof: From $\triangle BCM \backsim \triangle DBM$,

$$CM : BM=BM : DM$$

Thus, $BM^2=CM\times DM$

$$\frac{chord^2}{4}=shi(kei-shi)$$

In short,

$$chord^2=4shi(kei-shi)$$

Note 3: The Formula of *Soukogen*

This formula is the same as the Law of Cosines that we study in Math I.

For example, applying the Law of Cosines to triangle
$\triangle$ABD gives us:

$$(2b-2r)^2=(2R+2r)^2+(2b-2R)^2$$
$$-2(2R+2r)(2b-2R)\cos\angle \mathrm{BAD} \quad (**)$$

In other words,

$$\cos\angle \mathrm{BAD}=\frac{\mathrm{AH}}{\mathrm{BA}}=\frac{y}{\frac{1}{2}(2R+2r)}=\frac{2y}{2R+2r}$$

Thus, the above equation (**) gives us:

$$(2b-2r)^2$$
$$=(2R+2r)^2+(2b-2R)^2-4(2b-2r)y$$

In such a way, *Wasan* created an equation without using cosines and applied it as a formula.

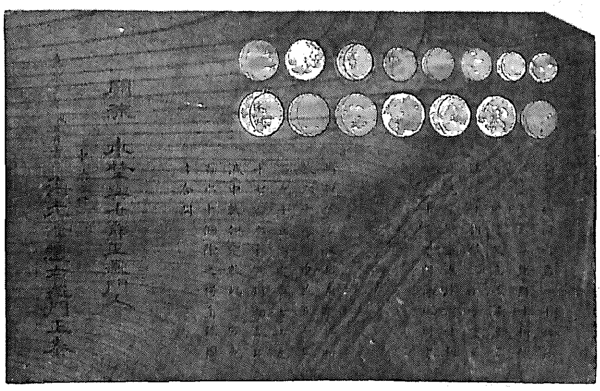
This is called the **Formula of *Soukogen*.**

Note 4: *Ijou* (Ratio)

The term *ijou* (ratio) refers to multiplying diagonally. Knowing the formula itself doesn't do much for you, but when performing variations in order to eliminate variables, as in formulas ④ and ⑤, it provides an invaluable perspective. *Wasan* mathematicians had a wide array of such invaluable perspectives at their disposal.

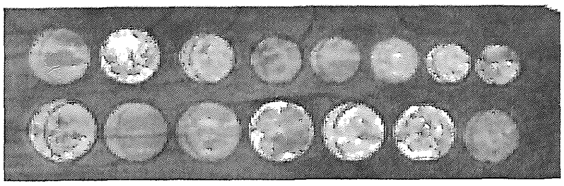
4.3. The *Sangaku* of Masayasu Ebisawashouemon (*Sangaku* 3)

The following *Sangaku* was dedicated in Kaei 3 (1850) by Masayasu Ebisawashouemon, a pupil of Shoudo Mizunokousitirou, who was a follower of Seki. This *Sangaku* features a sequential problem which was created by assigning the sequential indicators $\{a_n\}$ to the names of the Twenty-Eight Mansions used in astronomy and astrology, in order from the east. The Twenty-Eight Mansions divide the sky ecliptic into the four cardinal directions, with seven mansions in each of the four quadrants: 角 (Horn, Spica), 亢 (Neck, Virgo), 氐 (Root, Libra), 房 (Room, Libra), 心 (Heart, Anatares), 尾 (Tale, Scorpius), and 箕 (Winnowing Basket, Sagittarius) in the east; 斗 (Dipper, Sagittarius), 牛 (Ox, Capricornus), 女 (Girl, Aquarius), 虚 (Emptiness, Aquarius), 危 (Danger, Pegasus), 室 (Encampment, Pegasus), and 壁 (Wall, Algenib Pegasus) in the north; 奎 (Legs, Andromeda), 婁 (Bond, Aries), 胃 (Stomach, Aries), 昴 (Hairy Head, Pleiades), 畢 (Net, Taurus), 觜 (Turtle Beak, Orion), and 参 (Three Stars, Orion) in the west; and 井 (Well, Gemini), 鬼 (Ogre, Cancer), 柳 (Will, Hydra), 星 (Star, Alphard), 張 (Extended Net, Crater), 翼 (Wings, Corvus), and 軫 (Chariot, Corvus) in the south. Because convenient sequential indicators like $\{a_n\}$ did not exist at the time, such a method of sequencing was used.



***Sangaku 3*, Kaei 3 (1850)**

今有如圖宿名一十五球只云角亢二球
周寸相併一十六寸又云心尾箕三球周
寸相併三十寸重云虚危室壁奎五球周
寸相併六十三寸問角球周寸幾何
答七寸七分六厘三毛三糸一忽一微有奇
術曰依方程招差術得初數六十九個
中數五千三百九十五箇
定数七万九千七百六十個
列初數以減中數加定數以一万九百六十個
除之得角球周寸合問
關流　水埜興七郎正衛門人
中渋谷村
嘉永三年　戌　五月吉日
海老澤摠右衛門正泰



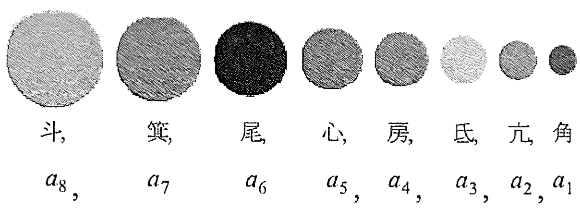
4.3.1. The Text of the Problem

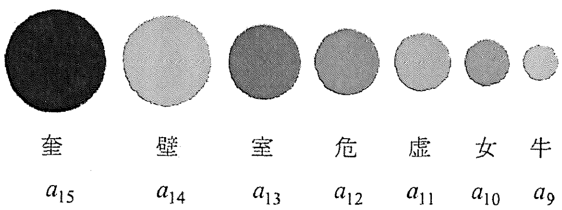
As shown in the illustration, there are 15 spheres with the names of 15 of the Twenty-Eight Mansions. The sum of the circumferences of the two spheres, a_1 and a_2, is 16 *sun;* the sum of the circumferences of the three spheres, a_5, a_6 and a_7, is 30 *sun*. In addition, the sum of the circumferences of the five spheres, a_{11}, a_{12}, a_{13}, a_{14} and a_{15} is 63 *sun*. Find the circumference of sphere a_1.

Answer: 7.76321 *sun* and a minute remainder.

4.3.2. Solution to the *Sangaku* Problem

This problem seeks to find the circumference of the first sphere (角) in a row of 15 spheres—as shown in the following illustration—when the sums of the circumferences of sets of two, three and five of the spheres are given sequentially.





By assigning sequential indicators, a_n, to the problem in the text, we get the following conditions:

$$\begin{cases} a_1 + a_2 = 16 \\ a_5 + a_6 + a_7 = 30 \\ a_{11} + a_{12} + a_{13} + a_{14} + a_{15} = 63 \end{cases}$$

By examining the make-up of the sequence of numbers that meet these conditions, we can deduce the pattern:

$$a_n = an^2 + bn + c \quad (n=1,\ 2,\ 3\mathrm{L}\ )$$

$$\begin{cases} 5a + 3b + 2c = 16 \\ 110a + 18b + 3c = 30 \\ 855a + 65b + 5c = 63 \end{cases}$$

Solving this, we get: $a = -\dfrac{69}{10960}$, $b = \dfrac{1079}{2192}$, $c = \dfrac{997}{137}$

Thus, the circumference of the sphere (角) we are seeking, a_1, is:

$$a_1 = -\frac{69}{10960} + \frac{1079}{2192} + \frac{997}{137} = 7.763321\mathrm{L}$$

This matches the answer on the *Sangaku*.

Note: Looking at the explanation (formula), we see, "First number: 69; middle number: 5395; last number: 79760," with the line:

" 列初數以減中數加定數."

So,(first number) – (middle number)＋(last number).

This is followed by the line:

"以一万九百六十個除之."

So,

$$\frac{\text{middle number-first number+last number}}{10960}$$

In other words:

$$\frac{5395 - 69 + 79760}{10960} = \frac{-69}{10960} + \frac{5395}{10960} + \frac{79760}{10960}$$
$$= -\frac{69}{10960} + \frac{1079}{2192} + \frac{997}{137}$$
$$= 7.763321\mathrm{L}$$

Lastly we have the line:

“得角球周寸合問.”

Which means: this is how the length of the circumference of the sphere, 角, was found.

Note: The Formula of *Shousa*

The Formula of *Shousa* is a formula for determining the coefficient of an integral function.

5 In Closing

The contents of the *Sangaku* of Kon’ou Shrine are well-suited to the mathematical content taught in secondary education. The *Sangaku* are well-preserved and the characters are legible, making it easy for students to understand what they are doing. With the kind understanding of the chief priest of the shrine, students at my school have, over several semesters, been able to observe the *Sangaku* directly as part of their field work for thematic studies and such.

I myself, having thought for quite some time that the content of *Wasan* may become a breakthrough ushering in a new era of mathematics education, have conducted demonstrative research into *Wasan,* including the usage of *Wasan* geometry teaching materials, the making of *Sangaku* and so on. I have also published my results through various organs, including this school’s Research for Super Science High School, the Japan Society for Mathematical Education, Nationwide *Wasan* Research Symposiums, and in this school’s article collections. This year, I was fortunate enough to receive a grant-in-aid for scientific research, with which I, in order to turn the section of “The Development of Mathematics Teaching Materials Which Foster Greater Creativity, with Empirical Research” concerning the *Sangaku* of Kon’ou Shrine into teaching material, I edited the content into an article. In the process, I presented my research in progress at the Fifth Nationwide *Wasan* Research Symposium (Nagasaki), where I received a lot of valuable knowledge and opinions.

Actually observing genuine *Sangaku* alongside my students while doing field work, I feel an impression of how wonderful *Wasan* is, and, with that impression, a strong sense of interest in the fact that such mathematical research was carried out during the Edo Period. Furthermore, I felt that my students’ experience of making *Sangaku*, that is, having my students make *Sangaku*, added a new layer of depth to the content of my normal classes.

In addition, at last year’s Komaba Akademeia, we were planning to hold public lectures for adults entitled, “A look at Edo Period mathematics in Shibuya—unraveling the distinctive Japanese custom of *Sangaku* using junior high and high school mathematics,” based upon the content of this article. What will modern adults think about *Wasan* and *Sangaku* culture? I would like to give a full report about that on another occasion.

At last, *Tenchi Meisatsu* was editied by Mr. Toh Ubukata, and candidated for the *Naoki Award.* This is a novel which features Wasan, especially the Sangaku of Kon’nou Shrine. In editing *Tenchi Meisatsu*, I knew my book *Sangaku Dojou* was referenced by him. So I am very glad of the expanding of Edo culture *Wasan*, and that many people, especially many students will get to be familiar with mathematics.

Acknowledgements

The author would like to thank his colleagues for their valuable comments and input. In addition, I have received appropriate advice from a lot of persons.

In addition, on the event of my article’s publication in this school’s article collection, I was fortunate enough to receive tremendously valuable instruction on solving *Sangaku* problems from noted *Wasan* researcher Hiroshi KOTERA (faculty of mathematics, Todaiji Gakuen). Mr. KOTERA was also extremely helpful when our students conducted the field-work on the *Sangaku* of Nara Prefecture for their Kansai Regional Research activity. I would like to express my humblest gratitude to Mr. KOTERA.

Also, for the reprinting of "Jinkouki" in modern Japanese, I received instruction from Takashi FUKUDA, of this school's a faculty member of the Japanese department. Also, for this article's English language title, Takao HACHIMIYA, of this school's a faculty member of the English department, was kind enough to help me make revisions on numerous occasions. I would like to take this opportunity to express my humblest gratitude to them both.

In 2010, I attended to the international conferences, which were the 5th East Asia Regional Conference on Mathematics Education (EARCOM) in Tokyo Japan, and The 15th Asian Technology Conference in Mathematics (ATCM) in Kuala Lumpur Malaysia, to make presentations with a part of the paper. Mr. Gessl who is a native speaker in my school's a faculty member of the English department was kind enough to help me correct the paper. I would like to express my humblest gratitude to him.

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Related

1. Kon'ou Shrine is located at 3-5-12, Shibuya, Shibuya Ward, Tokyo Prefecture.
2. The characters used for the texts of the *Sangaku* were written using a font available on a computer. Also, the line breaks in the text were normalized.

This research paper is an excerpt from "The Development of Mathematics Teaching Materials Which Foster Greater Creativity, with Empirical Research" by Hideyo MAKISHITA, for which he received a grant-in-aid for scientific research from the Japan Society for the Promotion of Science (Research No. 21913011).