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A Measure of Inference in Classical and Intuitionistic Logics

by

Mamoru KANEKO and Nobu-Yuki SUZUKI

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UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

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Mamoru Kaneko† and Nobu-Yuki Suzuki‡

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Abstract

This paper presents a measure of inference in classical and intuitionistic logics in the Gentzen-style sequent calculi. The measure for a proof of a sequent is the width of the proof tree, that is, the number of leaves of the proof tree. Then the measure for a sequent is the minimum value of the widths of possible proofs of the sequent; if it is unprovable, the assigned value is $+\infty$. It counts the indispensable cases for possible proofs of a sequent. By this measure, we can separate between sequents easy to be proved and ones difficult; we can go further than provability and/or unprovability. It is motivated by some economics/game theory problem (bounded rationality). However, it would be not straightforward to obtain the exact value of this measure for a given sequent. In this paper, we will develop a method of calculating the value of the measure. We will apply our measure to various classes of problems, for example, to evaluate the difficulty of proving contradictory sequents. We also exemplify our measure with a problem of game theoretical decision making.

1. Introduction

This paper presents a measure of inference in classical and intuitionistic logics in the Gentzen-style sequent calculus (Gentzen [7]). The definition of the measure takes two steps: For each proof (tree) P, we measure the width, i.e., the number of leaves, of P. Then the measure of inference assigns, to a given sequent $\sigma = \Gamma \to \Theta$, the minimum value of such widths of possible proofs of σ, if σ is provable, and if it is unprovable,

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†Institute of Policy and Planning Sciences, University of Tsukuba, Ibaraki 305-8573, Japan (kaneko@shako.sk.tsukuba.ac.jp)

‡Department of Mathematics, Faculty of Science, Shizuoka University, Ohya, Shizuoka 422-8529, Japan (smnsuzu@ipc.shizuoka.ac.jp)

the assigned value is $+\infty$. It counts the indispensable cases for possible proofs of sequent σ. By this measure, we represent the degree of difficulty in proving a given sequent. Although our problem is highly proof-theoretic, we are motivated by some general problems in game theory/economics. In Section 1, we explain, first, our motivation; and second, the contribution of this paper. Lastly, we present one game theoretic example, to which our theory will be applied in Section 5.

1.1. General Motivation

Our problem of measuring inferences for a given sequent is related to various fields in mathematical logic but is arising in game theory/economics, as mentioned above. It is an analysis of proofs further than provability. Provability of a sequent $\sigma = \Gamma \to \Theta$ is defined by existence of a proof of σ. The literature of logic has largely focussed on this definition, except for computational complexity to be mentioned presently. This confines us to the scope of provability and/or unprovability, ignoring difficulty to reach a proof. When the required size of a possible proof is very large, we may not reach a proof (an example is discussed in Woodin [29]). We study this difficulty in this paper.

If the objective of mathematics is to find a new theorem together with its proof, it could be irrelevant to evaluate the required size of a proof. However, since mathematics itself is constructed by human activities, we may not reach very complicated proofs, while such theorems exist independent of human activities. Such theorems are not separated in a clear-cut manner from easy to difficult ones; they form a spectrum. Therefore, a study of such as spectrum could be informative to mathematics itself.

One target of game theory/economics is to study human behavior and decision-making in a game/social situation. It is more directly related to human activities than mathematics. The importance of "*bounded rationality*" has been emphasized in the economics literature since Simon [24]. Simon himself criticized the assumption of "*super-rationality*" for economic agents' decision making, but touched only a particular perceptual form of "bounded rationality". Since then, only scattered approaches have been given. The problem of *logical omniscience/omnipotence* (cf., Weingartner [28]) may be regarded as the counterpart of "super-rationality" in philosophical logic. We intend to touch a central part of "bounded rationality".

Our approach is related to the computational complexity in computer sciences. This evaluates the (limiting) performance of an algorithm (program); typically, it is formulated as a question of how the required time and memory size increase as the length of input data increases - - see Goldreich [8]. It is natural from the viewpoint of computer sciences to evaluate and compare required algorithms (programs). The principal question in the approach of "proof complexity" ("the lengths of proofs") is along the same line; the literature has focussed on the size of a required algorithm - - see Pudlák [23]. In these approaches, algorithms are compared by their limiting behaviors, while we focus on measuring inferences required for each fixed single (perhaps, small) instance of a sequent but not the performance of an algorithm. After the development of our theory, we may connect our approach with that of proof complexity, but this paper does not address this question.

Our approach is well understood from the viewpoint of "bounded rationality" in game theory/economics. For this, we should mention two related literatures: *epistemic logics of shallow (interpersonal) depths*; and *inductive game theory*.

In the game theory literature, it has often been treated as a non-mathematical assumption that the structure (rules, payoff functions) of a game is common knowledge among the participating players. This was regarded as necessary. Kaneko-Nagashima [11], [12] formulated an infinitary epistemic (predicate) logic to discuss the problem of common knowledge explicitly. For the literature, see Meyer-van der Hoek [21], Fagin *et al.* [6], and Kaneko, *et al.* [18]. The literature may be regarded as about "super rationality" and "omniscience/omnipotence" in a social context.

A departure from this literature was taken by Kaneko-Suzuki [13], [14], [15]: They developed a theory where interpersonal nesting structures can be restricted; for example, only depth of interpersonal nesting is 2, i.e., a player thinks about the beliefs of the other player, but does not go further than this depth. This approach treats limitations on *interpersonal* beliefs and inferences, while we target to study limitations on *intrapersonal* inferences within one player. Since classical and intuitionistic logic logics form the central parts of their epistemic logics, the approach here can be extended and connected to their epistemic logics. We give one game theoretical example in Section 1.3.

Consider a sequent $\sigma = \Gamma \rightarrow \Theta$ from the viewpoint of epistemic logic, where a player has σ in his mind. Logic can discuss inferences from basic beliefs Γ to their logical consequences Θ, but has no capability of discussing the source of such basic beliefs Γ. Kaneko-Kline [10] has constructed inductive game theory (IGT), to look for a source for such basic beliefs in players' experiences with trials/errors, in repeating a social (game) situation. This experiential source together with other cognitive postulates suggests that it is difficult to obtain precise basic beliefs through experiences with trials/errors. The subject of the present paper is closely related to this line of research: Once the measure of inference is appropriately defined, we can use it to describe difficulty in inference to obtain basic beliefs from a player's experiences.

1.2. Contributions of the Present Paper

We define the measure of inference, denoted by $\eta_{\mathrm{L}*}$, in classical and intuitionistic logics $\mathrm{L} = \mathrm{CL}$ or IL in Gentzen's [7] sequent calculus. We have four types of those measures, depending upon $\mathrm{L} = \mathrm{CL}$ or IL, and with or without cuts, $* = \mathrm{w}$ or f, i.e., $\eta_{\mathrm{CLw}}, \eta_{\mathrm{CLf}}, \eta_{\mathrm{ILw}}$ and η_{ILf}.

We are interested in giving a method to calculate the exact value $\eta_{\mathrm{L}*}(\sigma)$ for an arbitrary given sequent $\sigma = \Gamma \to \Theta$. Finding a proof P of σ is not enough for the calculation of $\eta_{\mathrm{L}*}(\sigma)$, since it gives only its upper bound for $\eta_{\mathrm{L}*}(\sigma)$. We give the lower bound method (LB-method) to calculate $\eta_{\mathrm{L}*}(\sigma)$. Specifically, we define a function β_{L} over the set of sequents so that its value $\beta_{\mathrm{L}}(\sigma)$ for each sequent σ is calculated by looking at the syntactical structures of σ. We will prove (Theorem 4.4) that this function β_{L} gives a lower bound $\beta_{\mathrm{L}}(\sigma)$ of $\eta_{\mathrm{L}*}(\sigma)$ for any sequent σ. This β_{L} gives often the exact value of $\eta_{\mathrm{L}*}(\sigma)$. Then, we will apply this LB-method certain classes of problems.

Our concern is to measure the number of inference steps represented by $\eta_{\mathrm{L}*}(\sigma)$ as the indispensable contents included in σ. The LB-method is to estimate this value from σ itself. These problems are related to the question of what the class of possible proofs of σ is, but we do not aim to construct a proof itself. In this sense, our theory is not about proof-search (cf., Pym-Ritter [22]).

One is a problem of game theoretical decision making. This is related to the original motivation, which we describe in Section 1.3. This example will be used to motivate the formulation of our logic; we take conjunctions and disjunctions of finite sets of formulae, rather than to two formulae. Since a lot of assumptions (or basic beliefs) are used in game theoretic practices, this language is more convenient than that with binary conjunctions and disjunctions. We take the width of a proof for the definition of our measure $\eta_{\mathrm{L}*}$ for the same reason. The will be discussed in Section 5.

In Section 6, a contradictory statement is evaluated by our measure $\eta_{\mathrm{L}*}$. Evaluations may differ significantly in the two cases with cuts and without cuts. In general, the value of $\eta_{\mathrm{L}*}$ is smaller when cuts are allowed. Also, $\eta_{\mathrm{L}*}(\Gamma \to \ )$ and $\eta_{\mathrm{L}*}(\Gamma \to \neg A \wedge A)$ may differ, though those sequents are equivalent with respect to provability. The difference is somewhat parallel to the difference between the measures η_{Lw} and η_{Lf}. We give also an evaluation, by $\eta_{\mathrm{L}*}$, of a specific contradictory statement arising in economics.

In Section 7, we will discuss a modification of $\eta_{\mathrm{L}*}$ when we allow only binary conjunctions and disjunctions. In Section 9, we will give other problems and various remarks.

1.3. A Game with Large and Small Stores

Consider the situation where a large store, 1 (a supermarket), and a small store, 2 (a minimart) are competing. Now, we consider decision-making by store 2. The small store has the subjective understanding of the situation: Store 1 is large enough to ignore store 2 for 1's decision-making, but store 2's profits are influenced by 1's choice. His understanding is described by Tables 1.1 and 1.2. That is, store 2 understands that the situation is described by (g_1, g_2). Store 1 has only three alternative actions, and his payoff is determined by his own choice. On the other hand, 1 has 10 alternative actions, and the resulting payoffs are determined by the choices of both stores 1 and 2.

In this game, store 2 has a "dominant action", $\mathbf{s}_{10}$, which gives the highest payoff whatever 1 chooses. To achieve this understanding, he compares the payoff from $\mathbf{s}_{10}$

with that from $\mathbf{s}_1, ..., \mathbf{s}_9$ in all the three cases of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$; hence it needs at least $9 \times 3 = 27$ comparisons.

Table 1.1; g_1

$\mathbf{a}_1$	6000
$\mathbf{a}_2$	2000
$\mathbf{a}_3$	1000

Table 1.2; g_2

$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\cdots$	$\mathbf{s}_9$	$\mathbf{s}_{10}$
1	2	3	$\cdots$	9	10
5	6	7	$\cdots$	13	14
5	7	9	$\cdots$	21	23

Store 2 has an alternative decision procedure (criterion): First, he predicts the choice by store 1, and, using his prediction he chooses an action. For this procedure, he needs 2 comparisons to predict that 1's choice would be $\mathbf{a}_1$, and then he needs again at least 9 comparisons to verify that $\mathbf{s}_{10}$ is the best given the prediction $\mathbf{a}_1$. In this case, the minimum number of required comparisons is 11. Hence, there is a "trade-off": The concentration on his own payoff matrix does not require interpersonal inferences, but interpersonal considerations may simplify his decision-making with respect to the number of payoff comparisons. We do not explicitly consider interpersonal epistemic aspects involved here, but the results given in the present paper can be extended in epistemic logics of shallow depths in Kaneko-Suzuki [13], [14], [15].

The above argument can be described in terms of the measure $\eta_{\mathrm{L}*}$ of inference. In Section 5, an example where store 2 has no dominant strategy is also considered.

2. Classical and Intuitionistic Logics

Here we present classical and intuitionistic logics CL and IL. We adopt the following list of primitive symbols:

countably infinite number of propositional variables: $\mathbf{p}_0, \mathbf{p}_1, ...$;

logical connective symbols: $\neg$ (not), $\supset$ (implies), $\wedge$ (and), $\vee$ (or);

parentheses: (,); *comma*: , ; and *braces* { , }.

In some examples, we use different propositional variables and use lower case letters p, q, etc. to denote those variables. We define *formulae* inductively: (o): any propositional variable p is a formula; (i): if C, D are formulae, so are $(C \supset D)$ and $(\neg C)$; (ii): if Φ is a finite set of formulae with its cardinality $|\Phi| \geq 2$, then $(\wedge\Phi)$ and $(\vee\Phi)$ are formulae. We denote the set of all formulae by $\mathcal{P}$.

The conjunctive and disjunctive symbols $\wedge$ and $\vee$ are applied to a finite nonempty set Φ of formulae with $|\Phi| \geq 2$. Since Φ is given as $\{A_1, ..., A_l\}$ $(l \geq 2)$ with set-theoretical identification, we need commas and braces as primitive symbols. This deviates from the standard formulation of formulae, on which we will comment in Section 7. We may write $A \wedge B, A \vee B$ and $A \vee B \vee C$ for $\wedge\{A, B\}, \vee\{A, B\}$ and $\vee\{A, B, C\}$, etc., when these are easier. We often abbreviate the parentheses $(,)$ when it causes no confusion.

Let Γ, Θ be finite (possibly empty) sets of formulae in $\mathcal{P}$. Using auxiliary symbol $\to$, we introduce a new expression $\Gamma \to \Theta$, which we call a *sequent*. We abbreviate (set-theoretical) braces, for example, $\{A\} \cup \Gamma \to \Theta \cup \{B\}$ is written as $A, \Gamma \to \Theta, B$, and also, $\Gamma \cup \Delta \to \Theta \cup \Lambda$ is abbreviated as $\Gamma, \Delta \to \Theta, \Lambda$. We note that in expression $A, \Gamma \to \Theta, B$, we allow Γ, Θ to contain A, B, respectively, and that in $\Gamma, \Delta \to \Theta, \Lambda$, they may have nonempty intersections. Nevertheless, this sequent is identified by the triple of the set $\Gamma \cup \Delta$, $\to$, and the set $\Theta \cup \Lambda$. We will make a stipulation on those expressions and "side formulae".

The logical inferences are governed by one axiom schema and various inference rules.

Axiom Schema (Initial Sequents): $A \to A$, where A is any formula.

Structural Rules: The following inference rules are called the *thinning* and *cut*:

$$\frac{\Gamma \to \Theta}{\Delta, \Gamma \to \Theta, \Lambda}\ (th)$$

$$\frac{\Gamma \to \Theta, A \qquad A, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\ (cut)$$

In (th), the sets Δ and Λ may be empty. Formulae in Δ, Λ are called *thinning formulae*. The formula A in (cut) is called the *cut-formula*.

Operational Rules:

$$\frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ (\neg \to) \qquad \frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A}\ (\to \neg)$$

$$\frac{\Gamma \to \Theta, A \qquad B, \Delta \to \Lambda}{A \supset B, \Gamma, \Delta \to \Theta, \Lambda}\ (\supset\to) \qquad \frac{A, \Gamma \to B, \Theta}{\Gamma \to A \supset B, \Theta}\ (\to\supset)$$

$$\frac{A, \Gamma \to \Theta}{\wedge\Phi, \Gamma \to \Theta}\ (\wedge \to) \text{ where } A \in \Phi \qquad \frac{\{\Gamma \to \Theta, A : A \in \Phi\}}{\Gamma \to \Theta, \wedge\Phi}\ (\to \wedge)$$

$$\frac{\{A, \Gamma \to \Theta : A \in \Phi\}}{\vee\Phi, \Gamma \to \Theta}\ (\vee \to) \qquad \frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, \vee\Phi}\ (\to \vee) \text{ where } A \in \Phi.$$

The uppersequents of $(\supset\to)$, $(\to \wedge)$ and $(\vee \to)$ form sets of sequents. In the operational rules, we say that the formula(s) to be changes in the uppersequent(s) the *side formula*(s), and that the formula formed in the lower sequent is the *principal formula*.

For examples, in $(\to \wedge)$, all A in Φ are the side formulae, and $\wedge\Phi$ is the principal formula. It would be convenient to denotes, by $I[A]$, the combination of an inference rule I with its principal formula A.

Although we allow that sets appearing in sequents may have intersections, we make the following:

Stipulation S: For each inference rule, the cut or side formula(e) does not belong to the neighboring sets in the upper sequents of the rule, for example, $A \notin \Theta$ and $A \notin \Delta$, and $A \notin \Gamma$ in $(\wedge \to)$ and $A \notin \Theta$ in $(\to \wedge)$.

Due to (th), this stipulation affects neither provability nor our measure; only the depth of a proof may be affected slightly. However, this stipulation enables us to determine an immediate "descendant" uniquely, which will be stated and used in Section 8.

A *proof* P in CL is defined as a triple $(X, <; \psi)$ with the following properties:

(i): $(X, <)$ is a finite tree, and its immediate predecessor relation is denoted by $<_\iota$;[1]

(ii): ψ is a function which associates a sequent $\psi(x) = \Delta \to \Lambda$ to each node $x \in X$;

(a): for any leaf (maximal node) x in $(X, <)$, $\psi(x)$ is an instance of the axiom;

(b): for any non-leaf $x \in X$,

$$\frac{\{\psi(x') : x <_\iota x'\}}{\psi(x)}\ I \tag{2.1}$$

is an instance of one inference rule.

The same inference may be used several times in a proof P. To avoid ambiguity caused by such multiple uses of the same inference, we identify the address of an application of an inference I of (2.1) by the lower node x in P, which we call an *application* of inference I ($I[A]$ with the specification of its principal formula).

Let $\sigma = \Gamma \to \Theta$ be a sequent. We say that $P = (X, <; \psi)$ is a *proof of* σ in CL iff P is a proof in CL with $\psi(x_0) = \sigma$ for the root x_0 of $(X, <)$. We say that σ is *provable in* CL iff there is a proof of σ in CL, denoted by $\vdash_{\mathrm{CL}} \sigma$.

The above axiom schema and inference rules form classical logic CL. Intuitionistic logic IL is obtained from CL by giving the restriction that the succedent of each sequent has cardinality at most 1. See Gentzen [7] and Kleene [19]. A proof and provability for IL are defined in the parallel manner with this restriction. Since a proof in IL is a proof in CL, $\vdash_{\mathrm{IL}} \sigma$ implies $\vdash_{\mathrm{CL}} \sigma$.

A proof P in L = CL or IL is said to be *cut-free* iff P has no applications of (cut). The following theorem by Gentzen [7] (see also Kleene [19] and Takeuti [25]) plays an important role in this paper.

Theorem 2.1 (Cut-Elimination for CL and IL). Let L be CL or IL. If $\vdash_{\mathrm{L}} \sigma$, then there is a cut-free proof P of σ in L.

[1] We call t' a *predecessor* of t iff $t < t'$; and an *immediate* predecessor of t iff $t <_\iota t'$.

We will define our measure of inference focussing on the width of a proof. The width of a proof matters only with the four rules $(cut), (\supset\rightarrow), (\rightarrow\wedge), (\vee\rightarrow)$, since each of the other inference rules has only one uppersequent. Given a sequent $\sigma = \Gamma \rightarrow \Theta$, we estimate the size of a possible proof of σ by looking at σ. Here, (cut) becomes an obstacle in estimating it from a given sequent σ. Nevertheless, we define and study our measure of inference in the cases both with and without (cut)'s, since we would like to study also the role of (cut). For this purpose, we need to refine Theorem 2.1, which will be stated in Section 8.3.

3. Measure of Inference $\eta_{\mathrm{L}*}$

Here, we define the measure of inference $\eta_{\mathrm{L}*}$ for each $\sigma = \Gamma \rightarrow \Theta$, and give various examples to illustrate it. In order to obtain the exact values of $\eta_{\mathrm{L}*}$, we need some method of calculating the value of $\eta_{\mathrm{L}*}(\sigma)$, which will be discussed in Section 4.

For each proof $P = (X, <; \psi)$ in L = CL or IL, we define

$$\eta(P) = \text{ the number of the leaves of the tree } (X, <). \tag{3.1}$$

Thus, η measures the width of P, and ignores its depth[2,3]. Later, we will explain why we care about the width of a proof rather than the total number of applications of inferences including the depth of a proof. This is also related to the choice of our language allowing applications of $\wedge$ and $\vee$ to finite sets of formulae, which will be discussed in Section 7.

The width of a proof increases with the applications of $(cut), (\supset\rightarrow)$, $(\rightarrow\wedge)$, and $(\vee\rightarrow)$, but not with other inferences. To evaluate $\eta(P)$, we should pay attention to these inferences; one evaluation method will be discussed in Section 4. We note that since η depends upon a single proof, we do not put subscript L to η.

Our ultimate goal is to study the measure for a sequent, rather than a proof.

Definition 3.1. We define the *measure* η_{Lf} *of inference for a sequent* $\sigma = \Gamma \rightarrow \Theta$ in logic L = CL or IL as follows:

$$\eta_{\mathrm{Lf}}(\sigma) = \begin{cases} \min\{\eta(P) : P \text{ is a cut-free proof of } \sigma \text{ in L}\} & \text{if } \vdash_{\mathrm{L}} \sigma \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

By eliminating "cut-free" in (3.2), we have the other measure $\eta_{\mathrm{Lw}}(\sigma)$. The expression $\eta_{\mathrm{L}*}(\sigma)$ denotes either $\eta_{\mathrm{Lf}}(\sigma)$ or $\eta_{\mathrm{Lw}}(\sigma)$.

[2] Urquhart [27] and Arai [1]counted the number of all sequents in a proof. Then, they studied the computational complexities of various specific sets of problem instances.

[3] We do count all the initial sequents which may be identical as sequents. For some problems, it could be more natural to count identical initial sequents only once. But this is less basic than our counting form, and also is more complex. In this paper, we adopt the way of counting all initial sequents.

The measure $\eta_{\mathrm{L}*}$ counts the indispensable contents of the sequent σ to be proved. In other words, by tracing upwards in a proof P, we would meet an initial sequent, and if this initial sequent occurs in any proof of σ, we regard it as indispensable for σ. Technically, we take the minimum number of leaves of proofs of σ, since there may be multiple proofs of a given σ with different number of initial sequents; these will be presently exemplified by a few examples. When σ is unprovable in logic L, we defined $\eta_{\mathrm{L}*}(\sigma) = +\infty$. An unprovable case may be regarded as a limit of large proofs.

Since logic L is CL or IL, we have four types of measures;

$$\eta_{\mathrm{CLw}}(\sigma),\ \eta_{\mathrm{CLf}}(\sigma),\ \eta_{\mathrm{ILw}}(\sigma) \text{ and } \eta_{\mathrm{ILf}}(\sigma). \tag{3.3}$$

We will concentrate on these measures; we will briefly mention the corresponding measures for epistemic logics of shallow depths in Section 9.

Since a proof in IL is a proof in CL, we have the inequalities in (1) of Lemma 3.1. Since a cut-free proof in L is a proof in L, we have the second assertion.

Lemma 3.1.(1): $\eta_{\mathrm{CLw}}(\sigma) \leq \eta_{\mathrm{ILw}}(\sigma)$ and $\eta_{\mathrm{CLf}}(\sigma) \leq \eta_{\mathrm{ILf}}(\sigma)$;

(2): $\eta_{\mathrm{Lw}}(\sigma) \leq \eta_{\mathrm{Lf}}(\sigma)$ for L = CL, IL.

Examples for strict inequalities in (1) will be given in Example 4.2. Theorem 6.2 will give one sequent σ so that $\eta_{\mathrm{Lw}}(\sigma) < \eta_{\mathrm{Lf}}(\sigma)$ for L = CL, IL.

Now, we consider of how $\eta_{\mathrm{L}*}$ works, using simple examples of sequents.

Example 3.1. Consider the sequent $\sigma = p \supset q \to p \supset q$ (p, q propositional variables): Since σ itself is a proof, and $\eta_{\mathrm{L}*}$ takes positive integers as its values by (3.2), we have $\eta_{\mathrm{L}*}(\sigma) = 1$. But we have the following proof:

$$Q_1 : \quad \cfrac{\cfrac{p \to p \quad q \to q}{p, p \supset q \to q}\,(\supset\to)}{p \supset q \to p \supset q}\,(\supset\to)$$

Here, $\eta(Q_1) = 2$. We do not need to look into the contents of two occurrences of $p \supset q$ in σ; we treat them as "chunks", and each occurrence has a "companion."

The next example shows that there may be different proofs of a sequent σ with the same width, which means that the indispensable contents are not uniquely determined. Also, this example shows that it is not straightforward to find the exact value of $\eta_{\mathrm{L}*}(\sigma)$.

Example 3.2. Consider $\sigma = p, p \supset q, r, r \supset q \to q$. We have two different proofs (among others) of σ:

$$Q_2 : \quad \cfrac{\cfrac{p \to p \qquad q \to q}{p, p \supset q \to q}\,(\supset\to)}{p, p \supset q, r, r \supset q \to q}\,(th) \qquad Q_3 : \quad \cfrac{\cfrac{r \to r \qquad q \to q}{r, r \supset q \to q}\,(\supset\to)}{p, p \supset q, r, r \supset q \to q}\,(th)$$

We may regard either $p, p \supset q$ or $r, r \supset q$ as superfluous to obtain σ, but we cannot discard both simultaneously. We expect $\eta_{\mathrm{L}*}(\sigma) = 2$ from $\eta(Q_2) = \eta(Q_3) = 2$. We can prove this fact as follows: It suffices to show that $\eta(P) \geq 2$ for any proof P of σ. Indeed, let P be a cut-free proof of σ. Then P has at least one application of ($\supset\rightarrow$). Hence, $\eta(P) \geq 2$. When P has at least one cut, we have already $\eta(P) \geq 2$, since (*cut*) has two branches. Hence, $\eta_{\mathrm{L}*}(\sigma) = 2$.

The above proof of $\eta_{\mathrm{L}*}(\sigma) = 2$ is specific to this example. It would be quite inconvenient to find this kind of argument in each example. Also, typically, the value $\eta_{\mathrm{L}*}(\sigma)$ is larger than 2, and it would not be easy to construct such arguments for those cases. Therefore, we would like to have some method of calculating the exact value of $\eta_{\mathrm{L}*}(\sigma)$ for σ. We will develop one such method in Section 4.

It is easy to see that $\eta_{\mathrm{L}*}$ satisfies the following inequalities along the inference rules. But these inequalities do not help us obtain the exact value of $\eta_{\mathrm{L}*}(\sigma)$ for σ.

Lemma 3.2.(0): $\eta_{\mathrm{L}*}(\Delta, \Gamma \rightarrow \Theta, \Lambda) \leq \eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta)$;

(1*l*): $\eta_{\mathrm{L}*}(\neg A, \Gamma \rightarrow \Theta) \leq \eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, A)$;

(1*r*): $\eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, \neg A) \leq \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta)$.

(2*l*): $\eta_{\mathrm{L}*}(\wedge\Phi, \Gamma \rightarrow \Theta) \leq \min\limits_{A \in \Phi} \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta)$;

(2*r*): $\eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, \wedge\Phi) \leq \sum\limits_{A \in \Phi} \eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, A)$;

(3*l*): $\eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, \vee\Phi) \leq \min\limits_{A \in \Phi} \eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, A)$;

(3*r*): $\eta_{\mathrm{L}*}(\vee\Phi, \Gamma \rightarrow \Theta) \leq \sum\limits_{A \in \Phi} \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta)$;

(4*l*): $\eta_{\mathrm{L}*}(A \supset B, \Gamma, \Delta \rightarrow \Theta, \Lambda) \leq \eta_{\mathrm{L}*}(\Gamma \rightarrow A, \Theta) + \eta_{\mathrm{L}*}(B, \Delta \rightarrow \Lambda)$;

(4*r*): $\eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta, A \supset B) \leq \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta, B)$.

Proof. We prove only (2*l*). If $\nvdash_{\mathrm{L}} A, \Gamma \rightarrow \Theta$ for all $A \in \Phi$, then $\min_{A \in \Phi} \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta) = +\infty$ and thus we have the assertion. Suppose $\vdash_{\mathrm{L}} A, \Gamma \rightarrow \Theta$ for some $A \in \Phi$. Let P be a proof of $A', \Gamma \rightarrow \Theta$ with $A' \in \Phi$ and $\eta(P) = \eta_{\mathrm{L}*}(A', \Gamma \rightarrow \Theta) = \min_{A \in \Phi} \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta)$. We add one more inference to P as follows:

$$\frac{P}{\wedge\Phi, \Gamma \rightarrow \Theta} \; (\wedge \rightarrow)$$

We denote this proof by P'. Then, $\eta(P') = \eta(P) = \eta_{\mathrm{L}*}(A', \Gamma \rightarrow \Theta) = \min_{A \in \Phi} \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta)$. By the definition (3.2), we have $\eta_{\mathrm{L}*}(\wedge\Phi, \Gamma \rightarrow \Theta) \leq \min_{A \in \Phi} \eta_{\mathrm{L}*}(A, \Gamma \rightarrow \Theta)$. ■

If the above inequalities hold with equalities, the assertions of the above lemma help us calculate $\eta_{\mathrm{L}*}(\Gamma \rightarrow \Theta)$. However, only sometimes, these hold with equalities. Now, we give two more examples to show these difficulties.

Example 3.3. Consider $\sigma = \wedge\Phi \to \wedge\Phi$ and $\sigma' = \Phi \to \wedge\Phi$ with $\Phi = \{p_1, ..., p_{10}\}$. The two sequents are deductively equivalent, but differ with respect to $\eta_{\mathrm{L}*}$. First, we observe $\eta_{\mathrm{L}*}(\wedge\Phi \to \wedge\Phi) = 1$ but $\sum_{p\in\Phi} \eta_{\mathrm{L}*}(\wedge\Phi \to p) = 10$ since $\eta_{\mathrm{L}*}(\wedge\Phi \to p) = 1$ for all $p \in \Phi$. Hence, $(2r)$ of Lemma 3.2 holds for σ with strict inequality. On the other hand, the sequent $\Phi \to \wedge\Phi$ is proved as follows:

$$\dfrac{\left\{ \dfrac{p \to p}{\Phi \to p}\ (th) \right\}_{p\in\Phi}}{\Phi \to \wedge\Phi}\ (\to \wedge)$$

Hence $\eta_{\mathrm{L}*}(\Phi \to \wedge\Phi) \leq 10$ by the definition (3.2) of $\eta_{\mathrm{L}*}$. In fact, $\eta_{\mathrm{L}*}(\Phi \to \wedge\Phi) = 10$ will be proved in Section 4. Thus, $(2r)$ holds for σ' with equality, i.e., $\eta_{\mathrm{L}*}(\Phi \to \wedge\Phi) = \sum_{p\in\Phi} \eta_{\mathrm{L}*}(\Phi \to p)$.

Example 3.4. Consider the sequent $\sigma = p_0, p_0 \supset \wedge\Phi \to \wedge\Phi$, where $\Phi = \{p_1, ..., p_{10}\}$. This sequent has two proofs:

$$\dfrac{\left\{ \dfrac{p_0 \to p_0 \qquad \dfrac{p_t \to p_t}{\wedge\Phi \to p_t}\ (\wedge \to)}{p_0, p_0 \supset \wedge\Phi \to p_t}\ (\supset\to) \right\}_{p_t\in\Phi}}{p_0, p_0 \supset \wedge\Phi \to \wedge\Phi}\ (\to \wedge) \qquad \dfrac{p_0 \to p_0 \qquad \wedge\Phi \to \wedge\Phi}{p_0, p_0 \supset \wedge\Phi \to \wedge\Phi}\ (\supset\to).$$

Thus, the first proof has value $\eta(P) = 20$, and the second has only 2. In fact, we will prove $\eta_{\mathrm{Lw}}(\sigma) = \eta_{\mathrm{Lf}}(\sigma) = 2$ in Section 4.

4. The Lower Bound Method

As already mentioned in several examples in Section 3, the behavior of $\eta_{\mathrm{L}*}$ is complex, and it is not easy to calculate the exact value $\eta_{\mathrm{L}*}(\sigma)$ for a given sequent $\sigma = \Gamma \to \Theta$. A mechanical method of calculation is expected only for some class of sequents[4]. Here, we will provide one method of finding the exact value $\eta_{\mathrm{L}*}(\sigma)$.

4.1. The Lower Bound Method

The following lemma is a small and straightforward observation, but explains our motivation to introduce a lower bound function.

Lemma 4.1 (LB-Method). Let β be a function assigning a natural number to every sequent σ. For any sequent σ, if (1) $\beta(\sigma) \leq \eta_{\mathrm{L}*}(\sigma)$ and (2) $\beta(\sigma) = \eta(P)$ for some proof P of σ, then $\beta(\sigma) = \eta_{\mathrm{L}*}(\sigma)$.

[4] This was discussed in Kaneko-Suzuki [16] for epistemic logics of shallow depths with the intuitionistic base logic.

For a given σ, if we have (1) and (2), then we get the exact value of $\eta_{\mathrm{L}*}(\sigma)$. A poor example of β is given as $\beta(\sigma) = 1$ if $\vdash_{\mathrm{L}} \sigma$; and $\beta(\sigma) = +\infty$ otherwise. However, this β does not help us identify $\eta_{\mathrm{L}*}(\sigma)$: If we have $\eta(P) > 1$ for any proof P of σ, then this β says nothing; and if we find a proof P of a sequent σ with $\eta(P) = 1$, it automatically implies $\eta_{\mathrm{L}*}(\sigma) = 1$. We look for a more accurate β. A requirement for such a function β is to be calculated only using the information included in σ. Indeed, we will give such a function, denoted by β_{L}, in Section 4.3 and will prove (1) for any σ.

Our basic idea to define the function β_{L} is to count the occurrences of subformulae $\wedge\Phi$, $\vee\Psi$ and $A \supset C$ in σ which are necessarily the principal formulae of some applications of $(\rightarrow \wedge)$, $(\vee \rightarrow)$ and $(\supset\rightarrow)$ in any proof of σ. It is a strength of sequent calculi CL and IL that we can estimate, to a certain extent, the applications of $(\rightarrow \wedge)$, $(\vee \rightarrow)$ and $(\supset\rightarrow)$ by looking only at σ. For this estimation, first, let us connect the width of a tree with the number of branches in the tree.

Lemma 4.2. Let $(X, <)$ be a finite tree with the nonterminal nodes $1, ..., k$ with m_t branches at nonterminal node t. Let $n_1, ..., n_\ell$ be the nonterminal nodes for which $m_{n_t} > 1$ for all $t = 1, ..., \ell$. Then the number of terminal nodes of $(X, <)$ is given as

$$\sum_{t=1}^{\ell} m_{n_t} - (\ell - 1). \tag{4.1}$$

Proof. We prove the assertion by induction on ℓ. Let $\ell = 1$. Then a tree has only one nonterminal node with has multiple branches. Then the number of terminal nodes is $m_{n_1} = m_{n_1} - (\ell - 1)$.

Now, we assume the induction hypothesis that (4.1) holds for any tree with ℓ nonterminal nodes with multiple branches. Now, let $(X, <)$ be a tree with $\ell + 1$ nonterminal nodes with multiple branches. We choose one nonterminal node, called $n_{\ell+1}$, with multiple branches but not $(n_{\ell+1} < n_t)$ for all $t = 1, ..., \ell$. By eliminate all the branches at $n_{\ell+1}$ and their predecessor, we have have a tree, $(X', <')$ with ℓ nonterminal nodes with multiple branches. By the induction hypothesis, the number of terminal nodes of $(X', <')$ is given as $\sum_{t=1}^{\ell} m_{n_t} - (\ell - 1)$. Now, we return to the original tree $(X, <)$. The node $n_{\ell+1}$, which is a terminal node in $(X', <')$, has $m_{n_{\ell+1}}$ branches and $m_{n_{\ell+1}}$ terminal nodes as its predecessors. That is, one terminal node is eliminated, but $m_{n_{\ell+1}}$ terminal nodes are newly added. Thus, the number of terminal nodes of $(X, <)$ is calculated as:

$$(\sum_{t=1}^{\ell} m_{n_t} - (\ell - 1)) + (m_{n_{\ell+1}} - 1) = \sum_{t=1}^{\ell+1} m_{n_t} - (\ell + 1 - 1).$$

Now, we have (4.1) for $\ell + 1$. ■

The following lemma is an immediate consequence of Lemma 4.2. This observation is suggestive for the definition β_{L}.

Lemma 4.3. Let P be a proof in L = CL or IL. We list all the applications of $(\to \wedge)$, $(\vee \to)$, $(\supset\to)$, (cut) in P as

(1): $(\to \wedge)[\wedge\Phi_1], ..., (\to \wedge)[\wedge\Phi_{\ell_1}]$; (2): $(\vee \to)[\vee\Psi_1], ..., (\vee \to)[\vee\Psi_{\ell_2}]$;

(3): $(\supset\to)[A_1 \supset B_1], ..., (\supset\to)[A_{\ell_3} \supset B_{\ell_3}]$; (4): $(cut_1), ..., (cut_{\ell_4})$.

Then, $\eta(P) = \sum_{t=1}^{\ell_1} |\Phi_t| + \sum_{t=1}^{\ell_2} |\Psi_t| + 2(\ell_3 + \ell_4) - (\ell_1 + \ell_2 + \ell_3 + \ell_4 - 1)$.

Note that some of the above applications have the same principal formulae.

4.2. Occurrences and Signs

To define the lower bound function β_{L} and to state the main theorem (Theorem 4.4), we use various proof-theoretic concepts. They have been known in the folklore of proof theory, but are usually discussed in an informal manner. Since our concern is directly related to the structure of a proof, we need to introduce them explicitly.

First, we introduce a linear order $\prec$ over the set of all formulae $\mathcal{P}$. We stipulate that each finite nonempty set Φ of formulae is ordered by $\prec$: When Φ has m elements, we may write $\Phi = \{A_1, ..., A_m\}$ following the order $\prec$, i.e., $A_1 \prec ... \prec A_m$.

We will use the concept of an *occurrence* ξ at three levels, which indicates the address of a formula B relative to (i) a formula A; (ii) a sequent σ; and (iii) a proof P. Only (i) requires a careful consideration, and $(ii), (iii)$ are straightforward. Since (i) and (ii) are need for Theorem 4.4, we give only (i) and (ii). (iii) will be needed for a proof of Theorem 4.4 and will be given in Section 8.

Consider one example:

$$A = \neg(\vee\{p_1, p_2, B\} \supset \wedge\{p_1, B \supset p_2\}), \tag{4.2}$$

where $B = \wedge\{p_1, p_2, p_3\}$. The subformulae in $\vee\{p_1, p_2, B\}$ and $\wedge\{p_1, B \supset p_2\}$ are ordered by $\prec$. This A has two occurrences of B, and we would like to separate each without ambiguity. The left occurrence of B is identified in the following manner:

$$\xi = [A \mid \epsilon \cdot \neg \cdot (\supset, -1) \cdot (\vee, 3) : B]. \tag{4.3}$$

The first A is the *reference formula,* and the last B is the *target formula*. The sequence in the middle is the address of B relative to A. The *null* symbol ϵ indicates the outermost viewpoint. Then, we go through the immediate subformulae to the left B in (4.2). Since $\supset, \wedge, \vee$ have multiple immediate subformulae, $(\supset, -1)$ is used to indicate the premise of $\supset$, and $(\supset, 1)$ indicates the conclusion. The expression $(\vee, 3)$ indicates to choose the third disjunct of $\vee\{p_1, p_2, B\}$. These expressions are concatenated by $\cdot$. The right occurrence of B in (4.2) is given as $\xi' = [A \mid \epsilon \cdot \neg \cdot (\supset, 1) \cdot (\wedge, 2) \cdot (\supset, -1) : B]$. The trivial example is $[A \mid \epsilon : A]$, which is the occurrence of the target formula A relative to the reference formula A itself.

Formally, we define an occurrence relative to a reference formula A by induction from the outermost viewpoint of A:

(0): the occurrence of A relative to A itself is $\xi = [A \mid \epsilon : A]$.

Suppose that an occurrence of B relative to A is already defined as $\xi = [A \mid \alpha : B]$. Then, the occurrence of an immediate subformula C of B is defined as follows:

(1): if $B = \neg C$, the occurrence of this C is given as $[A \mid \alpha \cdot \neg : C]$;

(2): if $B = C \supset D$, the occurrences of these C and D are given as $[A \mid \alpha \cdot (\supset, -1) : C]$ and $[A \mid \alpha \cdot (\supset, 1) : D]$;

(3): if $B = \wedge\Phi = \wedge\{C_1, ..., C_k\}$ (or $B = \vee\Phi = \vee\{C_1, ..., C_k\}$), the occurrence of $C = C_l$ in Φ $(1 \leq l \leq k)$ is $[A \mid \alpha \cdot (\wedge, l) : C]$ ($[A \mid \alpha \cdot (\vee, l) : C]$, respectively).

We can verify that the left occurrence of B in (4.2) is calculated by those steps and given in (4.3). We denote the reference formula A and target formula B of $\xi = [A \mid \alpha : B]$ by $\rho[\xi]$ and $\tau[\xi]$.

We can also talk about an occurrence ξ of B in A in a sequent $\sigma = \{C_1, ..., C_j, ..., C_m\} \to \{D_1, ..., D_k\}$, where $A = C_j$. Here $C_1, ..., C_j, ..., C_m$ and $D_1, ..., D_k$ are ordered by $\prec$. The address of $A = C_j$ in σ is described by $[\sigma \mid (a, j)]$, stating that A occurs as the j-th formula C_j in the antecedent of σ. Let occurrence $\xi = [A \mid \alpha : B]$ be given. Then, we have the occurrence ξ in $C_j = A$ relative to sequent σ as:

$$[\sigma \mid (a, j)] \cdot \xi. \tag{4.4}$$

If the indicated place is D_j in the succedent of σ, then it is given as $[\sigma \mid (s, j)] \cdot \xi$. The occurrence $[\sigma \mid (a, j)] \cdot \xi$ or $[\sigma \mid (s, j)] \cdot \xi$ relative to a sequent σ is denoted by ζ. Its reference formula $\rho[\zeta]$ and target formula $\tau[\zeta]$ are defined by $\rho[\zeta] = \rho[\xi]$ and $\tau[\zeta] = \tau[\xi]$. Since ζ contains all the information of ξ, it is enough to refer to ζ when we talk about an occurrence relative to σ.

The sign (positive or negative) of an occurrence ζ in a sequent σ or a formula A is unambiguously defined as follows: We assign 1 or -1 to each component of the address of an occurrence ζ in σ as follows:

$$1 \text{ to } [\sigma \mid (s, j)], \varepsilon, (\wedge, j), (\vee, j), (\supset, 1) \text{ and } -1 \text{ to } [\sigma \mid (a, j)], (\neg), (\supset, -1),$$

where $j = 1, ...$ Then, regarding concatenation $\cdot$ as multiplication $\times$, we can calculate the sign of an occurrence in σ. For example, when ζ is given as (4.4) and ξ is given as (4.3), we have

$$\begin{aligned} sgn[\zeta] &= -sgn(\xi) = -sgn([A \mid \epsilon \cdot \neg \cdot (\supset, -1) \cdot (\vee, 3); B]) \\ &= (-1) \times 1 \times (-1) \times (-1) \times 1 = -1. \end{aligned}$$

A strength of the sequent calculus is the sign-preserving property that once a formula occurs in a proof P, all descendants have the same sign. We will state this property as Theorem 8.1 in Section 8.1 after we define the concept of descendants.

4.3. The Lower Bound Function β_{L}

Now, we isolate some occurrences in a sequent σ using the concepts introduced above. We say that an occurrence ζ' is a *companion* of another occurrence ζ in σ iff $\tau[\zeta] = \tau[\zeta']$ and ζ, ζ' have opposite signs. Then, we say that an occurrence ζ in σ is *legitimate* iff

(i): ζ has no companions;

(ii): $\tau[\zeta]$ of ζ is expressed as either $\wedge\Phi$, $\vee\Psi$ or $A \supset C$;

(iii): if $\tau[\zeta] = \wedge\Phi$, ζ is positive in σ, and if $\tau[\zeta] = \vee\Psi$ or $A \supset C$, it is negative in σ.

The sequent $\sigma = p_0, p_0 \supset \wedge\Phi \to \wedge\Phi$ of Example 3.4 has two occurrences of $\wedge\Phi$; one is negative and the other is positive; they are companions. Hence, neither is legitimate. This sequent has only one legitimate occurrence $p_0 \supset \wedge\Phi$.

Consider a set μ of legitimate occurrences $\zeta_1, ..., \zeta_m$ in σ. Let the target formulae $\tau[\zeta_1], ..., \tau[\zeta_m]$ be

$$\wedge\Phi_1, ..., \wedge\Phi_{\ell_1}, \vee\Psi_1, ..., \vee\Psi_{\ell_2}, A_1 \supset B_1, ..., A_{\ell_3} \supset B_{\ell_3}, \tag{4.5}$$

where some of them may be identical. Then we define

$$w(\mu) = \sum_{t=1}^{\ell_1} |\Phi_t| + \sum_{t=1}^{\ell_2} |\Psi_t| + 2\ell_3 - (\ell_1 + \ell_2 + \ell_3 - 1). \tag{4.6}$$

Here, μ may the empty set $\emptyset$; then $w(\emptyset) = -(-1) = 1$.

By $w(\mu)$ of (4.6), we estimate the number $\eta(P)$ of Lemma 4.3. But this depends upon the choice of μ. It may be too much to list all legitimate occurrences in σ. For example, the sequent $p, p \supset q, r, r \supset q \to q$ of Example 3.2 has two legitimate occurrences, i.e., $p \supset q$ and $r \supset q$; each single occurrence is enough to obtain this sequent. Hence, we should consider a certain subset of legitimate occurrences in σ. It is chosen so that the other occurrences could be replaced by new propositional variables while keeping its provability. For $p, p \supset q, r, r \supset q \to q$, we can replace the legitimate occurrence $p \supset q$ (or $r \supset q$) by a new propositional variable p' but still we have $\vdash_{\mathrm{L}} p, p', r, r \supset q \to q$ (or $p, p \supset q, r, p' \to q$).

In general, we should take care of nesting occurrences. Therefore, we consider a set μ of legitimate occurrences $\zeta_1, ..., \zeta_m$ in σ satisfying

$$\text{if } \zeta_t \text{ is included in a legitimate occurrence } \zeta, \text{ then } \zeta \in \mu. \tag{4.7}$$

Thus, μ is upward closed. Let $\zeta'_1, ..., \zeta'_k$ be the other legitimate occurrences in σ each of which is maximal in the sense of nesting, i.e., each ζ'_t $(t = 1, ..., k)$ includes no occurrences of $\zeta'_1, ..., \zeta'_k$ but ζ'_t itself. However, we allow some of $\zeta'_1, ..., \zeta'_k$ to occur in $\zeta_1, ..., \zeta_m$. Let $q_1, ..., q_k$ be new propositional variables not occurring in σ. Then, we define

$\sigma^* = \Gamma^* \to \Theta^*$ to be the sequent obtained from σ by replacing $\zeta'_1, ..., \zeta'_k$ by $q_1, ..., q_k$. We say that a set of legitimate occurrences $\zeta_1, ..., \zeta_m$ satisfying (4.7) is *genuine* iff

$$\vdash_{\mathrm{L}} \ \Gamma^* \to \Theta^*. \tag{4.8}$$

Note that this definition depends upon the choice of CL or IL. We denote the set of all genuine sets by $\mathbf{M}_{\mathrm{L}}(\sigma)$.

For $\sigma = p, p \supset q, r, r \supset q \to q$ of Example 3.2, either $p \supset q$ or $r \supset q$ can be replaced by a new propositional variable p'. Hence, $\mathbf{M}_{\mathrm{L}}(\sigma) = \{\{\zeta, \zeta'\}, \{\zeta\}, \{\zeta'\}\}$, where $\tau[\zeta] = p \supset q$ and $\tau[\zeta'] = r \supset q$.

Now, we take the minimal value of $w(\mu)$ over $\mathbf{M}_{\mathrm{L}}(\sigma)$, i.e.,

$$\beta_{\mathrm{L}}(\sigma) = \min_{\mu \in \mathbf{M}_{\mathrm{L}}(\sigma)} w(\mu). \tag{4.9}$$

Now, we state the main theorem of this section: We have (1) of Lemma 4.1 for any sequent σ. We postpone its proof to Section 8, since a proof needs a few more proof theoretical concepts. The theorem holds for L = CL,IL.

Theorem 4.4 (Lower Bound Function β_{L}). $\beta_{\mathrm{L}}(\sigma) \leq \eta_{\mathrm{Lw}}(\sigma)$ for any sequent $\sigma = \Gamma \to \Theta$.

By Lemma 3.1.(2), we have $\beta_{\mathrm{L}}(\sigma) \leq \eta_{\mathrm{Lw}}(\sigma) \leq \eta_{\mathrm{Lf}}(\sigma)$; an example for a strict inequality will be given in Section 6. The dependence of $\beta_{\mathrm{L}}(\sigma)$ and $\eta_{\mathrm{Lw}}(\sigma)$ upon L = CL or IL will be shown in Example 4.2.

The lower bound function β_{L} often gives a good estimate of $\eta_{\mathrm{L}*}$. For $\sigma = p, p \supset q, r, r \supset q \to q$ of Example 3.2, $\mathbf{M}_{\mathrm{L}}(\sigma) = \{\{\zeta, \zeta'\}, \{\zeta\}, \{\zeta'\}\}$. Hence, $\beta_{\mathrm{L}}(\sigma) = \min_{\mu \in \mathbf{M}_{\mathrm{L}}(\sigma)} w(\mu) = 2$. Using, Theorem 4.4, we have $\beta_{\mathrm{L}}(\sigma) = 2 \leq \eta_{\mathrm{Lw}}(\sigma) \leq \eta_{\mathrm{Lf}}(\sigma)$. Since we already gave a cut-free proof P of σ with $\eta(P) = 2$, we have $\eta_{\mathrm{Lw}}(\sigma) = \eta_{\mathrm{Lf}}(\sigma) = 2$ by Lemma 4.1.

In Example 3.3, $\sigma' = \Phi \to \wedge\Phi$ has a unique legitimate occurrence $\tau[\zeta] = \wedge\Phi$. Hence, $\beta_{\mathrm{L}}(\sigma') = 10$. Since we gave a proof P of σ' with $\eta(P) = 10$, we have $\eta_{\mathrm{L}*}(\sigma') = 10$.

In the sequent $\sigma = p_0, p_0 \supset \wedge\Phi \to \wedge\Phi$ of Example 3.4, $\wedge\Phi$ has a companion. Hence, σ has a unique legitimate occurrence $\tau[\zeta] = p_0 \supset \wedge\Phi$, which constitutes also a unique genuine set. Hence, $\beta_{\mathrm{L}}(\sigma) = 2$, and $\eta_{\mathrm{L}*}(\sigma) = 2$.

In those examples, the LB-method provided the exact value of $\eta_{\mathrm{L}*}(\sigma)$ independently of CL or IL. An example showing the difference between $\eta_{\mathrm{CL}*}$ and $\eta_{\mathrm{IL}*}$ is $\to p \vee (\neg p)$ (p is a propositional variable): $\eta_{\mathrm{CLw}}(\sigma) = \eta_{\mathrm{CLf}}(\sigma) = 1$, but $\eta_{\mathrm{ILw}}(\sigma) = \eta_{\mathrm{ILf}}(\sigma) = +\infty$. We do not need Theorem 4.4 for this calculation. The next example is a less trivial one. Theorem 4.4 provides the exact values $\eta_{\mathrm{Lw}}(\sigma), \eta_{\mathrm{Lf}}(\sigma)$.

Example 4.1 (Dependence upon L = CL or IL). Consider $\sigma = p_0 \vee (\neg p_0), p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1$; let $\zeta_1, \zeta_2, \zeta_3$ be the three legitimate occurrences in the antecedent of σ. Let q be a new propositional variable. Then, $\vdash_{\mathrm{CL}} q, p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1$, but not in IL. Thus, $\{\zeta_2, \zeta_3\}$ is a genuine set in CL, but not in IL; and $\{\zeta_1, \zeta_2, \zeta_3\}$ is a unique genuine set in IL. Hence, $\beta_{\mathrm{CL}}(\sigma) = 2\times 2-(2-1) = 3$ and $\beta_{\mathrm{IL}}(\sigma) = 3\times 2-(3-1) = 4$. We find proofs P_1 and P_2 of σ, respectively, in CL and IL so that $\eta(P_1) = 3$ and $\eta(P_2) = 4$. Indeed, P_1 and P_2 are given as follows:

$$P_1 : \dfrac{\dfrac{\dfrac{\dfrac{p_0 \to p_0}{\to p_0, \neg p_0}(\to\neg) \quad p_1 \to p_1}{p_0 \supset p_1 \to p_1, \neg p_0}(\supset\to) \quad \dfrac{p_1 \to p_1}{p_1, p_0 \supset p_1 \to p_1}(th)}{p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1}(th)}{p_0 \vee \neg p_0, p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1}(th)$$

$$P_2 : \dfrac{\dfrac{\dfrac{p_0 \to p_0 \quad p_1 \to p_1}{p_0, p_0 \supset p_1 \to p_1}(\supset\to)}{p_0, p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1}(th) \quad \dfrac{\dfrac{\neg p_0 \to \neg p_0 \quad p_1 \to p_1}{\neg p_0, \neg p_0 \supset p_1 \to p_1}(\supset\to)}{\neg p_0, p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1}(\mathrm{th})}{p_0 \vee \neg p_0, p_0 \supset p_1, \neg p_0 \supset p_1 \to p_1}(\to\supset)$$

5. An Application to the Game with Small and Large Stores

Here, we study decision making by small store 2 in the game in Section 1.3 in terms of measure $\eta_{\mathrm{L}*}$. Recall that (g_1, g_2) given by Tables 1.1 and 1.2 express store 2's understanding of the situation. We consider two *decision-making criteria* for store 2 :

(i): the *dominant-strategy* criterion; and (ii): the *prediction-decision* criterion.

With (i), store 2 concentrates on Table 1.2 for his decision-making, while with (ii), he first uses Table 1.1 to predict what 1 would choose, and then he chooses an action based on Table 1.2. Criterion (ii) requires store 2 to think about (predict) 1's choice; it needs interpersonal inferences. In the game theory literature, it is regarded as more complex than (i). On the other hand, the number of steps required for (ii) is significantly smaller than that for (i): We meet a "trade-off" between complexity of inferences involving interpersonal thinking and that of the number of pure intrapersonal inferences.

In this paper, we do not consider the epistemic structures required for (i) and (ii), which problem was discussed in Kaneko-Suzuki [13]. We will discuss those problems also with the measure of inference for epistemic logics of shallow depths in [16].

Let us formulate (i) and (ii) in our propositional language. We introduce propositional variables as follows: $p(\mathbf{a}_l, \mathbf{a}_{l'})$, $l, l' = 1, 2, 3$; and $q(\mathbf{s}_r, \mathbf{s}_{r'} \mid \mathbf{a}_l)$, $l = 1, 2, 3$ and $r, r' = 1, ..., 10$. They are possible (strict) preferences by stores 1 and 2 : $p(\mathbf{a}_l, \mathbf{a}_{l'})$ means that 1 prefers $\mathbf{a}_l$ to $\mathbf{a}_{l'}$; and $q(\mathbf{s}_r, \mathbf{s}_{r'} \mid \mathbf{a}_l)$ means that conditional upon the choice $\mathbf{a}_l$ by 1, store 2 prefers strictly $\mathbf{s}_r$ to $\mathbf{s}_{r'}$.

Game (g_1, g_2) : Tables 1.1 and 1.2 are expressed as the sets of preferences:

$$\hat{g}_1 = \{p(\mathbf{a}_l, \mathbf{a}_{l'}), \neg p(\mathbf{a}_{l'}, \mathbf{a}_l) : l, l' = 1, 2, 3 \text{ with } l < l'\}; \tag{5.1}$$

$$\hat{g}_2 = \{q(\mathbf{s}_r, \mathbf{s}_{r'} \mid \mathbf{a}_l), \neg q(\mathbf{s}_{r'}, \mathbf{s}_r \mid \mathbf{a}_l) : r > r' \text{ and } l = 1, 2, 3\}. \tag{5.2}$$

The first states that store 1 prefers action $\mathbf{a}_l$ to $\mathbf{a}_{l'}$ if $l < l'$, and the second states that when 1 chooses $\mathbf{a}_l$, store 2 would prefer $\mathbf{s}_r$ to $\mathbf{s}_{r'}$ with $r > r'$. Thus, Tables 1.1 and 1.2 are expressed in terms of preferences induced by the numerical payoffs.

Now, let us go to two decision-criteria.

Dominant-Strategy (DS) Criterion: It is formulated as $\wedge\{q(\mathbf{s}_r, \mathbf{s}_{r'} \mid \mathbf{a}_l) : r' \neq r$ and $l = 1, 2, 3\}$, which is denoted by $\mathrm{Dom}_2(\mathbf{s}_r)$. It suggests to choose an action $\mathbf{s}_r$ so that he prefers it to the other actions whatever 1 chooses. Looking at Table 1.2, we find that $\mathbf{s}_{10}$ is a unique dominant strategy. This claim is formulated as the sequent and $\vdash_{\mathrm{L}} \hat{g}_2 \to \mathrm{Dom}_2(\mathbf{s}_{10})$. A proof of $\hat{g}_2 \to \mathrm{Dom}_2(\mathbf{s}_{10})$ is given as

$$P: \quad \cfrac{\left\{ \cfrac{q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_l) \to q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_l)}{\hat{g}_2 \to q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_l)}\ (th) \right\}_{r' \neq 10 \text{ and } l=1,2,3}}{\hat{g}_2 \to \wedge\{q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_l) : r' \neq 10 \text{ and } l = 1, 2, 3\}}\ (\to \wedge) \tag{5.3}$$

Here, $\eta(P) = 3 \times 9 = 27$. Also, we have $\beta_{\mathrm{L}}(\hat{g}_2 \to \mathrm{Dom}_2(\mathbf{s}_{10})) = 27$, since $\mathrm{Dom}_2(\mathbf{s}_{10})$ is a unique legitimate occurrence and constitutes a unique genuine set. Hence, we have $\eta_{\mathrm{L}*}(\hat{g}_2 \to \mathrm{Dom}_2(\mathbf{s}_{10})) = 27$ by the LB-method (Lemma 4.1 and Theorem 4.4). In fact, we can add $\hat{g}_1$ to this statement, i.e., $\eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2 \to \mathrm{Dom}_2(\mathbf{s}_{10})) = 27$.

The DS-criterion may be incapable in recommending any action in some situations. To see this, we consider the following variant.

Game (g_1, g_2') : We replace Table 1.2 by Table 1.2$'$; if store 1 takes action $\mathbf{a}_3$, store 2's payoff ordering is completely opposite to that for the other choices by 1. Then, there is no dominant strategy for store 2.

Table 1.2$'$; g_2'

	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\cdots$	$\mathbf{s}_9$	$\mathbf{s}_{10}$
$\mathbf{a}_1$	1	2	3	$\cdots$	9	10
$\mathbf{a}_2$	11	12	13	$\cdots$	20	21
$\mathbf{a}_3$	-1	-2	-3	$\cdots$	-9	-10

The set of preferences $\hat{g}_2'$ is defined based on this table in a parallel manner as (5.2). Since $\nvdash_{\mathrm{L}} \hat{g}_2' \to \mathrm{Dom}_2(\mathbf{s}_{10})$, we have $\eta_{\mathrm{L}*}(\hat{g}_2' \to \mathrm{Dom}_2(\mathbf{s}_{10})) = +\infty$. However, it holds that $\vdash_{\mathrm{L}} \hat{g}_2' \to \neg\vee_r \mathrm{Dom}_2(\mathbf{s}_r)$, which store 2 may notice. In this sequent, only $\vee_r \mathrm{Dom}_2(\mathbf{s}_r)$ is a legitimate occurrence. It holds that $\beta_{\mathrm{L}}(\hat{g}_2' \to \neg\vee_r \mathrm{Dom}_2(\mathbf{s}_r)) = 10$. We have a proof of this sequent:

$$\cfrac{\cfrac{\left\{ \cfrac{\circ \quad \circ \quad \circ}{\hat{g}_2', \mathrm{Dom}_2(\mathbf{s}_r) \to} \right\}_{r=1,\ldots,10}}{\hat{g}_2', \vee_r \mathrm{Dom}_2(\mathbf{s}_r) \to}\ (\vee \to)}{\hat{g}_2' \to \neg \vee_r \mathrm{Dom}_2(\mathbf{s}_r)}\ (\to \neg)$$

Thus, $\eta_{\mathrm{L}*}(\hat{g}_2' \to \neg\vee_r \mathrm{Dom}_2(\mathbf{s}_r)) = 10$ by the LB-method. Again, we can add $\hat{g}_1$ to this statement, i.e., $\eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2' \to \neg\vee_r \mathrm{Dom}_2(\mathbf{s}_r)) = 10$.

The above is interpreted as follows: Store 2 finds the DS-criterion does not allow any action by checking his payoff matrix. He needs 10 comparisons to have this knowledge. In game (g_1, g_2), the DS-criterion does not require interpersonal thinking about 1's decision-making, but in game (g_1, g_2'), this criterion has no power for decision-making. Therefore, store 2 needs something else: One procedure is to predict what 1 would choose. The prediction-decision criterion is such a procedure. From the viewpoint of our measure η_{L^*}, however, we can even compare the required inferences for the DS-criterion and prediction-decision criteria in (g_1, g_2).

Prediciton-Decision (PD) Criterion: The prediction part is formulated as $\wedge\{p(\mathbf{a}_l, \mathbf{a}_{l'}) : l' \neq l\}$, denoted by $\mathrm{Bt}_1(\mathbf{a}_l)$, where $l = 1, 2, 3$.That is, store 2 thinks that 1 would choose the best action. The prediction-decision criterion is formulated as $\wedge_l[\mathrm{Bt}_1(\mathbf{a}_l) \supset \wedge_{r' \neq r} q(\mathbf{s}_r, \mathbf{s}_{r'} \mid \mathbf{a}_l)]$, denoted by $\mathrm{PD}_2(\mathbf{s}_r)$. This means that for each prediction (unique in our example), $\mathbf{s}_r$ is best preferred to other alternative actions. This PD-criterion was an instance of the general form given in Kaneko-Suzuki [13][5]. Since we adopt classical and intuitionistic logics, we do not explicitly include epistemic structures.

In game (g_1, g_2), store 2 with this criterion looks at 1's preferences $\hat{g}_1$ as well as his own preferences $\hat{g}_2$. He predicts that 1 would prefer $\mathbf{a}_1$ to $\mathbf{a}_2, \mathbf{a}_3$ (so he would choose $\mathbf{a}_1$), and he finds that $\mathbf{s}_{10}$ is better than the other alternative actions. The same holds for the game (g_1, g_2'). These are described as

$$\vdash_{\mathrm{L}} \hat{g}_1, \hat{g}_2 \to \mathrm{PD}_2(\mathbf{s}_{10}) \text{ and } \vdash_{\mathrm{L}} \hat{g}_1, \hat{g}_2' \to \mathrm{PD}_2(\mathbf{s}_{10}). \tag{5.4}$$

Either sequent has two legitimate occurrences: $\wedge_l[\mathrm{Bt}_1(\mathbf{a}_l) \supset \wedge_{r' \neq 10} q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_l)]$ and $\wedge_{r' \neq 10} q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_1)$. Neither $\mathrm{Bt}_1(\mathbf{a}_2)$ nor $\mathrm{Bt}_1(\mathbf{a}_3)$ holds for g_1, which implies that $\wedge_{r' \neq 10} q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_2)$ and $\wedge_{r' \neq 10} q(\mathbf{s}_{10}, \mathbf{s}_{r'} \mid \mathbf{a}_3)$ can be replaced by new propositional variables without destroying provability of (5.4). Hence, these constitute a unique genuine set. Hence,

$$\beta_{\mathrm{L}}(\hat{g}_1, \hat{g}_2 \to \mathrm{PD}_2(\mathbf{s}_{10})) = \beta_{\mathrm{L}}(\hat{g}_1, \hat{g}_2' \to \mathrm{PD}_2(\mathbf{s}_{10})) = (3 + 9) - (2 - 1) = 11. \tag{5.5}$$

Actually, we can find a proof P of each sequent with $\eta(P) = 11$. Hence, we have $\eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2 \to \mathrm{PD}_2(\mathbf{s}_{10})) = \eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2' \to \mathrm{PD}_2(\mathbf{s}_{10})) = 11$.

The above argument for (5.5) looks slightly different from that given in Section 1.1, though the resulting values are the same. In fact, the argument in Section 1.1 can be expressed so that $\mathbf{a}_2$ and $\mathbf{a}_3$ are not chosen, rather than that $\mathbf{a}_1$ is chosen. Then, it is exactly the same as the calculation for (5.5).

[5]A general formulation requires the existence of a prediction in addition to the above implication part. It is not important in this example, we ignore the existence requirement of a prediction for simplicity.

Let us summarize the above arguments:

Table 5.1

	game: (g_1, g_2)	game: (g_1, g_2')
DS-criterion	$\eta_{\mathrm{L}*}(\hat{g}_2 \to \mathrm{Dom}_2(\mathbf{s}_{10})) = 27$	$\eta_{\mathrm{L}*}(\hat{g}_2' \to \neg \vee_r \mathrm{Dom}_2(\mathbf{s}_r)) = 10$
PD-criterion	$\eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2 \to \mathrm{PD}_2(\mathbf{s}_{10})) = 11$	$\eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2 \to \mathrm{PD}_2(\mathbf{s}_{10})) = 11$

In sum, the DS-criterion works for (g_1, g_2), but not for (g_1, g_2'). This requires no interpersonal thinking, and is often regarded as more "natural" than requiring interpersonal one. In (g_1, g_2), the PD-criterion suggests the same action, but requires a much smaller number of comparisons. This difference could be larger by increasing the number of alternative actions by store 1. Hence, interpersonal thinking is less reliable but simplifies decision making. In game (g_1, g_2'), the DS-criterion does not work. For store 2 to notice this fact, he should check $\hat{g}_2' \to \neg \vee_r \mathrm{Dom}_2(\mathbf{s}_r)$, which needs at least 10 steps. Once he notices this fact, he may go to the PD-criterion, which requires 11 steps to find a decision. We can combine these two logical arguments:

$$\eta_{\mathrm{L}*}(\hat{g}_1, \hat{g}_2' \to (\neg \vee_r \mathrm{Dom}_2(\mathbf{s}_r)) \wedge \mathrm{PD}_2(\mathbf{s}_{10}))) = 10 + 11 = 21.$$

That is, store 2 first verifies that the DS-criterion recommends no action, and then he uses the PD-criterion and finds one recommendation.

6. Evaluations of Some Contradictory Statements

Contradiction-freeness is one important criterion for an axiomatic system. For set-theory, Woodin [29] discussed the possibility that an axiomatic system may contain a contradiction from the objective point of view, but we may not find the system is contradictory when a proof to reach a contradiction is too large. This is relevant also for game theory/economics in that the beliefs owned by a player may be contradictory but he himself does not notice it. When measure $\eta_{\mathrm{L}*}$ takes a large value to derive a contradiction, it would be difficult for a game player (us) to notice it. Thus, it is important to analyze the behavior of $\eta_{\mathrm{L}*}$ for contradictory statements. Here, we give some general considerations, and then a specific problem arising in economics.

6.1. General Considerations

In logic L = CL or IL, a contradictory statement is formulated as either $\vdash_{\mathrm{L}} \Gamma \to$ or $\vdash_{\mathrm{L}} \Gamma \to \neg A \wedge A$ for some (any) A. With respect to provability, they are equivalent, but they differ in general with respect to our measure of inference.

Theorem 6.1: Suppose $\vdash_{\mathrm{L}} \Gamma \to \ $. Then, for any formula A,

$$\eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A) \leq \eta_{\mathrm{Lw}}(\Gamma \to \ ) \leq \eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A) + 1. \tag{6.1}$$

Proof. First, we have the inequality: $\eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A) \leq \eta_{\mathrm{Lw}}(\Gamma \to \ )$, since $\Gamma \to \neg A \wedge A$ is derived from $\Gamma \to \ $ by (th). Next, we show that the difference between $\eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A)$ and $\eta_{\mathrm{Lw}}(\Gamma \to \ )$ is at most 1. Indeed, since $\eta_{\mathrm{Lw}}(\neg A \wedge A \to \ ) = 1$ and

$$\dfrac{\dfrac{\cdots}{\Gamma \to \neg A \wedge A} \qquad \dfrac{\cdots}{\neg A \wedge A \to}}{\Gamma \to} \ (cut),$$

we have $\eta_{\mathrm{Lw}}(\Gamma \to \ ) \leq \eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A) + \eta_{\mathrm{Lw}}(\neg A \wedge A \to \ ) = \eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A) + 1$. ■

We conjecture that $\eta_{\mathrm{Lw}}(\Gamma \to \neg A \wedge A) = \eta_{\mathrm{Lw}}(\Gamma \to \ )$, but so far, we have succeeded in neither proving this equality nor finding a counter example. The inequality corresponding to (6.1) does not hold for the measure η_{Lf}, which is the next subject.

Let $A_t = \wedge\{p_t, q\}$ for $t = 1, ..., m$, and $B_t = \vee\{p_t, q\}$ for $t = 1, ..., n$, where $p_1, ...,$ and q are all propositional variables. Then, consider the following sequents:

$$\begin{aligned} \sigma_0 &= \quad \vee\{A_1, ..., A_m\} \quad \to \quad \wedge\{B_1, ..., B_n\}; \\ \sigma_1 &= \vee\{A_1, ..., A_m\}, \neg \wedge \{B_1, ..., B_n\} \to \quad ; \\ \sigma_2 &= \vee\{A_1, ..., A_m\}, \neg \wedge \{B_1, ..., B_n\} \to \neg q \wedge q. \end{aligned} \tag{6.2}$$

Those sequents are provable and deductively equivalent in L = CL, IL. The sequents σ_1 and σ_2 are the targets of the present discussion. However, σ_0 is closely related to these sequents, and is used to directly show the difference between η_{Lw} and η_{Lf}. We have the following theorem, which will be proved below.

Theorem 6.2: (0): $\beta_{\mathrm{L}}(\sigma_0) = \beta_{\mathrm{L}}(\sigma_1) = \beta_{\mathrm{L}}(\sigma_2) = m + n - 1$;

(1): $\eta_{\mathrm{Lw}}(\sigma_0) = \eta_{\mathrm{Lw}}(\sigma_1) = \eta_{\mathrm{Lw}}(\sigma_2) = m + n$;

(2): $\eta_{\mathrm{Lf}}(\sigma_0) = \eta_{\mathrm{Lf}}(\sigma_1) = m \times n$ and $\eta_{\mathrm{Lf}}(\sigma_2) = m + n$.

This theorem has various implications other than the exemplification of Theorem 6.1. When $m, n \geq 2$ and not $(m = n = 2)$, we have $\eta_{\mathrm{Lf}}(\sigma_0) = \eta_{\mathrm{Lf}}(\sigma_1) = m \times n > \eta_{\mathrm{Lf}}(\sigma_0) = \eta_{\mathrm{Lw}}(\sigma_1) = m + n$, which states that Lemma 3.1.(2) holds in strict inequality. Also, $m \times n$ is much larger than $m + n$ for large m, n.

The inequality $\eta_{\mathrm{Lw}}(\sigma_0) = m + n < \eta_{\mathrm{Lf}}(\sigma_0) = m \times n$ shows a difference caused by (cut)'s[6]. On the other hand, $\eta_{\mathrm{Lf}}(\sigma_1) = m \times n > \eta_{\mathrm{Lf}}(\sigma_2) = m + n$ assumes no (cut)'s

[6] This argument is reminiscent of Boolos [3]: In a first-order tableau system with equality and no (cut)'s, he presented one example where if (cut)'s are additionally available, its poof became much smaller than the original proof.

and is caused by additional $\neg q \wedge q$. This additional $\neg q \wedge q$ plays the same role as (cut) in proving these sequents.

Here, the LB-method may not give even an approximation of $\eta_{\mathrm{Lf}}(\sigma_0) = \eta_{\mathrm{Lf}}(\sigma_1)$. When the LB-method is not available, we would have a difficulty to evaluate the exact value of $\eta_{\mathrm{L}*}$. Indeed, to prove (1) and (2), we cannot directly use the LB-method. Nevertheless, we use it in various steps of those proofs.

Proof of Theorem 6.2: In each sequent, $\vee\{A_1, ..., A_m\}, \wedge\{B_1, ..., B_n\}$ are legitimate occurrences, and they form a unique genuine set. Hence, $\beta_{\mathrm{L}}(\sigma_0) = \beta_{\mathrm{L}}(\sigma_1) = \beta_{\mathrm{L}}(\sigma_2) = m + n - (2 - 1) = m + n - 1$.

Here, we prove only $\eta_{\mathrm{Lf}}(\sigma_1) = m \times n$ and $\eta_{\mathrm{Lw}}(\sigma_1) = m + n$. For $\eta_{\mathrm{Lf}}(\sigma_1) = m \times n$, it suffices to show that (i) there is a cut-free proof P_1 of σ_1 with $\eta(P_1) = m \times n$; and (ii) $\eta_{\mathrm{Lf}}(\sigma_1) \geq m \times n$.

For (i), we give a cut-free proof P_1 of σ_1 with $\eta(P_1) = m \times n$:

$$\cfrac{\cfrac{\left\{ \cfrac{\left\{ \cfrac{\cdots}{A_l \to B_k} \right\}_{l=1,...m}}{\vee\{A_1,...,A_m\} \to B_k}\ (\vee\to) \right\}_{k=1,...,n}}{\vee\{A_1,...,A_m\} \to \wedge\{B_1,...,B_k\}}\ (\to\wedge)}{\vee\{A_1,...,A_m\},\neg\wedge\{B_1,...,B_n\} \to}\ (\neg\to), \quad \text{where the top left is } \cfrac{\cfrac{q \to q}{q \to \vee\{p_k,q\}}}{A_l \to B_k}$$

Consider (ii). Let P be a cut-free proof of σ_1. Denote the occurrences of $\vee\{A_1, ..., A_m\}$ and $\wedge\{B_1, ..., B_n\}$ in σ in P by ζ_1 and ζ_2. In σ_1, $\vee\{A_1, ..., A_m\}$ and $\wedge\{B_1, ..., B_n\}$ are legitimate occurrences. The uppermost ancestors of ζ_1 and ζ_2 are the principal formulae of $(\vee \to)$ and $(\to \wedge)$, respectively. A lowest occurrence of such an application is expressed as

$$\frac{\{A_t, \Delta \to \Lambda : t = 1, ..., m\}}{\vee\{A_1, ..., A_m\}, \Delta \to \Lambda}\ (\vee \to) \text{ or } \frac{\{\Delta \to \Lambda, B_t : t = 1, ..., n\}}{\Delta \to \Lambda, \wedge\{B_1, ..., B_n\}}\ (\to \wedge). \tag{6.3}$$

Consider the left case. The right case can be treated in a parallel manner. Then, Δ contains $\neg \wedge \{B_1, ..., B_n\}$ or Λ contains $\wedge\{B_1, ..., B_n\}$. Consider an upper sequent $A_t, \Delta \to \Lambda$. Since P is cut-free, the sequent $A_t, \Delta \to \Lambda$ can be expressed as

$$A_t, \neg \wedge \{B_1, ..., B_n\} \to \wedge\{B_1, ..., B_n\}; \text{ or } A_t \to \wedge\{B_1, ..., B_n\}. \tag{6.4}$$

Consider the case of the first sequent. The second case is similar. Some uppermost ancestors of one occurrence of $\wedge\{B_1, ..., B_n\}$ in this sequent is the principal formula of $(\to \wedge)$. Hence, the subtree determined by $A_t, \Delta \to \Lambda$ has at least n leaves. Since this holds for each $A_t, \Delta \to \Lambda$, we have $\eta(P) \geq m \times n$. For the case of (6.4), we can prove the same inequality in a similar manner.

Now, let us prove $\eta_{\mathrm{Lw}}(\sigma_1) = m + n$. We find a proof P_1' of σ_1 with $\eta(P_1') = m + n$:

$$\cfrac{\cfrac{\cfrac{\left\{ \cfrac{q \to q}{A_k \to q}\ (\wedge\to) \right\}_{k=1,...,m}}{\vee\{A_1,...,A_m\} \to q}\ (\vee\to) \quad \cfrac{\left\{ \cfrac{q \to q}{q \to B_l}\ (\to\vee) \right\}_{l=1,...,n}}{q \to \wedge\{B_1,...,B_n\}}\ (\to\wedge)}{\vee\{A_1,...,A_m\} \to \wedge\{B_1,...,B_n\}}\ (cut)}{\vee\{A_1,...,A_m\},\neg\wedge\{B_1,...,B_n\} \to}\ (\neg\to)$$

We show $\eta_{\mathrm{Lw}}(\sigma_1) \geq m+n$. By (0), $\beta_{\mathrm{L}}(\sigma_1) = m+n-1$. Hence, $\eta_{\mathrm{Lw}}(\sigma_1) \geq m+n-1$. Let P be an arbitrary proof of σ_1. If it has no (*cut*), then $\beta(P) \geq m \times n$ by $\eta_{\mathrm{Lf}}(\sigma_1) = m \times n$. Let P contains at least one (*cut*). Then, P has already $(\to \vee)$ and $(\to \wedge)$ with principal formula $\vee\{A_1, ..., A_m\}$ and $\wedge\{B_1, ..., B_n\}$. Hence, $\beta(P) \geq m+n+2-(3-1) = m+n$. Hence, we have $\eta_{\mathrm{Lw}}(\sigma_1) \geq m+n$. ■

The sequent σ_2 has a cut-free proof P with $\eta(P) = m+n$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\left\{\dfrac{q \to q}{q \to B_l}\ (\to\vee)\right\}_{l=1,\ldots,n}}{q \to \wedge\{B_1,\ldots,B_n\} \to}\ (\to\wedge)}{q, \neg\wedge\{B_1,\ldots,B_n\} \to}\ (\neg\to)}{\neg\wedge\{B_1,\ldots,B_n\} \to \neg q}\ (\to\neg)}{\vee\{A_1,\ldots,A_m\}, \neg\wedge\{B_1,\ldots,B_n\} \to \neg q}\ (th) \quad \dfrac{\dfrac{\left\{\dfrac{q \to q}{A_k \to q}\ (\wedge\to)\right\}_{k=1,\ldots,m}}{\vee\{A_1,\ldots,A_m\} \to q}\ (\vee\to)}{\vee\{A_1,\ldots,A_m\}, \neg\wedge\{B_1,\ldots,B_n\} \to q}\ (th)}{\vee\{A_1,\ldots,A_m\}, \neg\wedge\{B_1,\ldots,B_n\} \to \neg q \wedge q}\ (\to \wedge)$$

This additional $\neg q \wedge q$ in the succedent plays a similar role to the cut-formula.

6.2. Inferences for a Cyclical Contradiction

In economics, cyclical preferences have been discussed a lot. They do not allow the decision maker (DM) to choose the best preferred action: Cyclical preferences are contradictory in this sense (see Tversky [26] and MacCrimmon-Larsson [20] for possible sources and difficulties arising). Logically speaking, cyclical preferences themselves are not contradictory; under some additional conditions such as transitivity and asymmetry, cyclical preferences yield a contradiction. As already stated, if it takes a too long time to find a contradiction in either sense, DM could not find it. Here, we discuss this problem from the viewpoint of our measure of inference.

Suppose that DM faces a problem to choose one from k alternatives $\mathbf{a}_1, ..., \mathbf{a}_k$. He has two types of beliefs:

(1) basic preferences comparing consecutive alternatives;

(2) additional properties for preferences.

To describe those, we prepare propositional variables p_{rs} $(1 \leq r, s \leq k, r \neq s)$; each p_{rs} is intended to mean that DM strictly prefers alternative $\mathbf{a}_r$ to $\mathbf{a}_k$.

For (1), DM makes direct comparisons only between $\mathbf{a}_r$ to $\mathbf{a}_{r+1}$ ($k+1$ is understood as 1). Then, we suppose that he prefers $\mathbf{a}_r$ to $\mathbf{a}_{r+1}$ for all $r = 1, ..., k$. These preferences are called the *basic preferences* over k alternatives: Then let

$$\Delta_k = \{p_{12}, p_{23}, ..., p_{(k-1)k}\} \cup \{p_{k1}\}. \tag{6.5}$$

He has a cycle of preferences over those alternatives.

The set Δ_k of basic preferences constitutes raw data for him. This lacks preference comparisons between remote alternatives, $\mathbf{a}_r$ and $\mathbf{a}_s$ $(|r-s| > 1)$. DM may compensate for such missing parts by some beliefs of general properties for preferences. Here, we assume transitivity and asymmetry for his preferences:

$$\Pi_k^T = \{p_{rs} \wedge p_{st} \supset p_{rt} : 1 \leq r, s, t \leq k\}; \tag{6.6}$$

$$\Pi_k^A = \{p_{rs} \supset (\neg p_{sr}) : 1 \leq r, s \leq k\}. \tag{6.7}$$

The sets Π_k^T and Π_k^A describe the agent's beliefs of basic properties. DM makes direct comparisons between two consecutive alternatives, which are described by the set Δ_k of direct data. Indirect comparisons between nonconsecutive alternatives such as p_{13} are made from the raw data and his beliefs of basic properties. For example, he derives p_{13} from p_{12}, p_{23} using $p_{12} \wedge p_{23} \supset p_{13}$.

The set Δ_k of basic preferences itself is not contradictory. The union of three sets $\Delta_k, \Pi_k^T, \Pi_k^A$, yields a contradiction. Indeed, using the raw data Δ_k and transitivity Π_k^T successively, we have p_{1k}, which together with Π_k^A implies $\neg p_{k1}$. On the other hand, Δ_k contains p_{k1}; thus we have a contradiction, $\vdash_{\mathrm{L}} \Delta_k, \Pi_k^T, \Pi_k^A \rightarrow$. The value of measure $\eta_{\mathrm{L}*}$ is given in the following theorem, which is proved in the end of this section.

Theorem 6.3 (Cyclical Contradiction): Let $k \geq 2$. Then we have:

(1): $\beta_{\mathrm{L}}(\Delta_k, \Pi_k^T, \Pi_k^A \rightarrow\ ) = 2k - 2$;

(2): $\eta_{\mathrm{Lw}}(\Delta_k, \Pi_k^T, \Pi_k^A \rightarrow\ ) = \eta_{\mathrm{Lf}}(\Delta_k, \Pi_k^T, \Pi_k^A \rightarrow\ ) = 2k - 2$.

We show these claims in the cases of $k = 2, 3$, and give a sketch for a general case.

Let $k = 2$. Then, $\Delta_k = \{p_{12}, p_{21}\}, \Pi_2^T = \emptyset$ and $\Pi_2^A = \{p_{12} \supset (\neg p_{21}), p_{21} \supset (\neg p_{12})\}$. The raw data $\Delta_k = \{p_{12}, p_{21}\}$ and asymmetry Π_2^A already lead to a contradiction. In this case, each formula in Π_2^A is legitimate in the sequent $\sigma_2 = \Delta_2, \Pi_2^A \rightarrow$, but only one is enough to have a contradiction. Thus $\beta_{\mathrm{L}}(\sigma_2) = 2$, and we have a proof P of $\Delta_2, \Pi_2^A \rightarrow$ with $\eta(P_2) = 2$. Hence, $\eta_{\mathrm{Lf}}(\sigma_2) = \eta_{\mathrm{Lw}}(\sigma_2) = 2$ by the LB-method.

Let $k = 3$. To obtain p_{13}, we use Transitivity once, and then $\neg p_{31}$ by Asymmetry once, which contradicts his raw preference p_{31}. Transitivity consists of $p_{12} \wedge p_{23} \supset p_{13}$, and this has two legitimate occurrences $p_{12} \wedge p_{23} \supset p_{13}$ and $p_{12} \wedge p_{23}$ in the sequent $\Delta_3, \Pi_3^T, \Pi_3^A \rightarrow$. Hence, $\beta_{\mathrm{L}}(\Delta_3, \Pi_3^T, \Pi_3^A \rightarrow\ ) = 2 \times 3 - 2 = 4$. Since we can find a proof P of this sequent with $\eta(P) = 4$, and thus, by the LB-method, $\eta_{\mathrm{L}*}(\Delta_3, \Pi_3^T, \Pi_3^A \rightarrow\ ) = 4$.

In the above argument, many formulae in Π_k^T, Π_k^A are not used. To prove the theorem, we choose some subsets $\mathring{\Pi}_k^T, \mathring{\Pi}_k^A$ of Π_k^T, Π_k^A so that $\Delta_k, \mathring{\Pi}_k^T, \mathring{\Pi}_k^A$ is still contradictory: Indeed, let

$$\mathring{\Pi}_k^T = \{p_{1t} \wedge p_{t(t+1)} \supset p_{1(t+1)} : t = 2, ..., k - 1\} \text{ and } \mathring{\Pi}_k^A = \{p_{1k} \supset \neg p_{k1}\}.$$

These are enough for the above derivation of a contradiction from $\Delta_k, \Pi_k^T, \Pi_k^A$, i.e., $\vdash_{\mathrm{L}} \Delta_k, \mathring{\Pi}_k^T, \mathring{\Pi}_k^A \rightarrow$. We should count the legitimate occurrences in $\mathring{\Pi}_k^T, \mathring{\Pi}_k^A$ noting that each $p_{1t} \wedge p_{t(t+1)} \supset p_{1(t+1)}$ contains two legitimate occurrences in the sequent. Then, we have (1). For (2), we can construct a proof P of $\Delta_k, \Pi_k^T, \Pi_k^A \to$ with $\eta(P) = 2k-2$ by induction on k.

7. Binary Conjunctions and Disjunctions

We take conjunction $\wedge$ and disjunction $\vee$ directly for finite sets of formulae. But it is more standard in the literature of logic to apply $\wedge$ and $\vee$ to ordered pairs of formulae. Our choice is to avoid too many repeated applications of them, since conjunctive and/or disjunctive formulae consisting of many components often appear in game theoretical practices. Here, we give a brief discussion on how the choice of the different language affects the measure $\eta_{\mathrm{L}*}$ of inference.

Consider the subset $\mathcal{P}_b$ of $\mathcal{P}$ in which $\wedge$ and $\vee$ are applied only to binary sets of formulae. We denote logic $\mathrm{L} = \mathrm{CL}, \mathrm{IL}$ with $\mathcal{P}_b$ by L_b[7]. The theory developed in this paper is available for L_b; η_{L_b*} and β_{L_b} are defined in the same way only with the replacement of L by L_b. It is important to emphasize that the LB-method holds for them without any changes. However, we should ask how different $\eta_{\mathrm{L}*}$ and η_{L_b*} are. In this section, we give only brief discussions on this question.

Let us start with the following simple lemma: We say that a sequent $\sigma = \Gamma \to \Theta$ *is for* L_b iff $\Gamma \cup \Theta \subseteq \mathcal{P}_b$.

Lemma 7.1. $\eta_{\mathrm{L}_b*}(\sigma) = \eta_{\mathrm{L}*}(\sigma)$ for any sequent for L_b.

The case of $\eta_{\mathrm{L}_b\mathrm{f}}(\sigma) = \eta_{\mathrm{Lf}}(\sigma)$ is straightforward due to the subformula property of a cut-free proof, but the case of $\eta_{\mathrm{L}_b\mathrm{w}}(\sigma) = \eta_{\mathrm{Lw}}(\sigma)$ is not, and will be proved later.

To make a comparison between η_{L_b*} and $\eta_{\mathrm{L}*}$ for a sequent σ containing non-binary $\wedge$ and $\vee$, we translate σ into the other one by expressing the non-binary $\wedge$ and $\vee$ by repeated applications of binary $\wedge$ and $\vee$. Let $\{A_1, ..., A_m\}$ be any set of formula in $\mathcal{P}$ with $m \geq 2$, where $A_1, ..., A_m$ are ordered by $\prec$. Then, we define $\bigwedge\{A_1, ..., A_m\}$ and $\bigvee\{A_1, ..., A_m\}$ by induction on m as follows:

$$\begin{aligned} \bigwedge\{A_1, A_2\} &= \wedge\{A_1, A_2\}; \text{ and } \bigwedge\{A_1, ..., A_m\} = \wedge\{\bigwedge\{A_1, ..., A_{m-1}\}, A_m\}; \quad (7.1) \\ \bigvee\{A_1, A_2\} &= \vee\{A_1, A_2\}; \text{ and } \bigvee\{A_1, ..., A_m\} = \vee\{\bigvee\{A_1, ..., A_{m-1}\}, A_m\}. \end{aligned}$$

Then, we define the *translator* φ from $\mathcal{P}$ to $\mathcal{P}_b$ inductively as follow:

(o): $\varphi(p) = p$ for all propositional variables;

(i): $\varphi(\neg A) = \neg\varphi(A)$ and $\varphi(A \supset B) = \varphi(A) \supset \varphi(B)$;

[7] If we adopt the language where $\wedge$ and $\vee$ are applied to the ordered pairs of formulae, we will have again other small differences. In this case, we can give the restriction on a proof that no formula in the antecedent or succedent of each sequent occurs more than twice (cf., Došen [5]) Our logical system L_b is similar to the standard formulation with this restriction.

(ii): $\varphi(\wedge\{A_1, ..., A_m\}) = \bigwedge\{\varphi(A_1), ..., \varphi(A_m)\}$;

(iii): $\varphi(\vee\{A_1, ..., A_m\}) = \bigvee\{\varphi(A_1), ..., \varphi(A_m)\}$.

This φ is surjective but not injective, for example, $\varphi(\wedge\{A_1, A_2, A_3\}) = \varphi(\wedge\{\wedge\{A_1, A_2\}, A_3\})$ $= \bigwedge\{A_1, A_2, A_3\}$. Now, we can make a comparison between $\eta_{\mathrm{L}*}(\sigma)$ and $\eta_{\mathrm{L}_b*}(\varphi(\sigma))$ for a sequent σ in L.

We define $(\to \bigwedge)$ and $(\bigvee \to)$ in terms of repeated applications of $(\to \wedge)$ and $(\vee \to)$ as follows: for $\Phi_k = \{A_1, ..., A_k\}$, $k = 2, ..., m$,

$$\dfrac{\Gamma \to \Theta, A_1 \quad \Gamma \to \Theta, A_2}{\Gamma \to \Theta, \wedge\Phi_2}\ (\to \wedge)$$

$$\begin{array}{c} \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \end{array}$$

$$\dfrac{\dfrac{\dfrac{\circ \quad \circ \quad \circ}{\Gamma \to \Theta, \bigwedge \Phi_{m-2}}\ (\to \wedge) \quad \Gamma \to \Theta, A_{m-1}}{\Gamma \to \Theta, \bigwedge \Phi_{m-1}}\ (\to \wedge) \quad \Gamma \to \Theta, A_m}{\Gamma \to \Theta, \bigwedge \Phi_m}\ (\to \wedge)$$

In the parallel manner, $(\bigvee \to)$ is defined. Both width and depth of $(\to \bigwedge)$ (or $(\bigvee \to)$) are m, while $(\to \wedge)[\wedge\Phi_m]$ has width m and depth 2.

From any proof P of σ in logic L, we can obtain a proof P_b of $\varphi(\sigma)$ in logic L_b by replacing applications of $(\to \wedge)$ and/or $(\vee \to)$ by the corresponding $(\to \bigwedge)$ and/or $(\bigvee \to)$. This is stated in the following lemma.

Lemma 7.2. Let P be a proof of a sequent σ in logic L = CL, IL. Then, there is a proof P_b of $\varphi(\sigma)$ in L_b such that $\eta(P) = \eta(P_b)$.

The depth of P_b is typically much deeper than that of P.

It follows from Lemma 7.2 that $\eta_{\mathrm{L}_b*}(\varphi(\sigma)) \leq \eta_{\mathrm{L}^*}(\sigma)$ for any σ for L. Thus, if σ is a sequent for L_b, we have $\eta_{\mathrm{L}_b*}(\sigma) \leq \eta_{\mathrm{L}^*}(\sigma)$. On the other hand, we have $\eta_{\mathrm{L}_b*}(\sigma) \geq \eta_{\mathrm{L}*}(\sigma)$, since a proof of σ in L_b is also a proof in L. In sum, we have Lemma 7.1.

These lemmas may be interpreted as meaning that with respect to width, the choice of the non-binary language $\mathcal{P}$ or binary $\mathcal{P}_b$ does not matter. However, we have an example for $\eta_{\mathrm{L}_b*}(\varphi(\sigma)) < \eta_{\mathrm{L}^*}(\sigma)$.

Example 7.1. Consider $\sigma = \wedge\{p_1, ..., p_m\} \to \wedge\{p_1, ..., p_{m-1}\}$ in L. We can show $\eta_{\mathrm{L}*}(\sigma) = m-1$. However, $\varphi(\sigma) = \bigwedge\{p_1, ..., p_m\} \to \bigwedge\{p_1, ..., p_{m-1}\}$, and $\bigwedge\{p_1, ..., p_m\} = \wedge\{\bigwedge\{p_1, ..., p_{m-1}\}, p_m\}$. Then $\eta_{\mathrm{L}*}(\varphi(\sigma)) = 1$, since $\varphi(\sigma)$ is proved as follows:

$$\dfrac{\bigwedge\{p_1, ..., p_{m-1}\} \to \bigwedge\{p_1, ..., p_{m-1}\}}{\wedge\{\bigwedge\{p_1, ..., p_{m-1}\}, p_m\} \to \bigwedge\{p_1, ..., p_{m-1}\}}\ (\wedge \to).$$

We can state a sufficient condition for the equivalence between $\eta_{\mathrm{L}*}(\sigma) = \eta_{\mathrm{L}_b*}(\varphi(\sigma))$ in terms of β_{L} and β_{L_b}. The following holds.

Lemma 7.3. If $\beta_{\mathrm{L}}(\sigma) = \beta_{\mathrm{L}_b}(\varphi(\sigma))$ and $\eta_{\mathrm{L}*}(\sigma) = \beta_{\mathrm{L}}(\sigma)$, then $\eta_{\mathrm{L}*}(\sigma) = \eta_{\mathrm{L}_b*}(\varphi(\sigma))$.

Proof. By Theorem 4.4 for L_b, we have $\beta_{\mathrm{L}_b}(\varphi(\sigma)) \leq \eta_{\mathrm{L}_b*}(\varphi(\sigma))$. Thus, $\eta_{\mathrm{L}*}(\sigma) = \beta_{\mathrm{L}}(\sigma) = \beta_{\mathrm{L}_b}(\varphi(\sigma)) \leq \eta_{\mathrm{L}_b*}(\varphi(\sigma))$. By Lemma 7.2, we have $\eta_{\mathrm{L}*}(\sigma) \geq \eta_{\mathrm{L}_b*}(\varphi(\sigma))$. Thus, $\eta_{\mathrm{L}*}(\sigma) = \eta_{\mathrm{L}_b*}(\varphi(\sigma))$. ■

This means that as far as $\beta_{\mathrm{L}}(\sigma) = \beta_{\mathrm{L}_b}(\varphi(\sigma))$ and the LB-method gives the exact value for $\eta_{\mathrm{L}*}(\sigma)$, the two measures $\eta_{\mathrm{L}*}$ and η_{L_b*} are equivalent. Hence, our question is now when $\beta_{\mathrm{L}}(\sigma) = \beta_{\mathrm{L}_b}(\varphi(\sigma))$ holds. We have not succeeded in finding good sufficient conditions for this equivalence. Nevertheless, they take the same values in all the examples in this paper except for Example 7.1.

8. Proof of Theorem 4.4

We prepare a few more proof theoretical concepts in order to prove Theorem 4.4.

8.1. Ancestors and Descendants in a Proof

Consider a proof tree $P = (X, <; \psi)$. Let $[\sigma \mid (a, j)] \cdot \xi$ (or $[\sigma \mid (s, j)] \cdot \xi$) be an occurrence relative to the sequent $\sigma = \psi(x)$ ($x \in X$). Then, its occurrence relative to $P = (X, <; \psi)$ is defined by adding the address of x in P;

$$[P \mid x] \cdot [\sigma \mid (a, j)] \cdot \xi \text{ (or } [P \mid x] \cdot [\sigma \mid (s, j)] \cdot \xi). \tag{8.1}$$

We denote the occurrence of (8.1) by the symbol θ. The reference and target formulae are also denoted by $\rho[\theta]$ $(= \rho[\xi])$ and $\tau[\theta]$ $(= \tau[\xi])$.

Since an occurrence in (8.1) contains all relevant information about the address of $[\sigma \mid (a, j)] \cdot \xi$ or $[\sigma \mid (s, j)] \cdot \xi$, we can regard it as an occurrence relative to $\sigma = \psi(x)$. Conversely, when an occurrence ζ relative to a sequent σ is given and also a proof P of σ is given, ζ is regarded as an occurrence relative to P by adding $[P \mid x_0] \cdot \zeta$, where x_0 is the root of P. Hence, we use also symbol ζ when the reference sequent is well specified.

The introduction of an occurrence in a proof $P = (X, <; \psi)$ enables us to define a *descendant* and an *ancestor.* In each case of the 12 inference rules in Section 2, we need to define *immediate* descendants and ancestors. However, since these are similar, we give the definition only in the case of

$$\frac{A, \Gamma \to \Theta}{\wedge\Phi, \Gamma \to \Theta} \ (\wedge \to)[\wedge\Phi], \text{ where } A \in \Phi.$$

Suppose that this occurs in a proof $P = (X, <, \psi)$, and that the upper and lower sequents are given as $\psi(x) = \sigma$ and $\psi(x') = \sigma'$ $(x, x' \in X)$. Let B be a subformula occurring in $\psi(x) = A, \Gamma \to \Theta$. Consider an occurrence $\theta = [P \mid x] \cdot [\sigma \mid (a, t)] \cdot \xi$ or $[P \mid x] \cdot [\sigma \mid (s, t)] \cdot \xi$ with $\tau[\theta] = B$ and $\xi = [\rho[\theta] \mid \alpha : B]$.

I1: Let $\rho[\theta]$ be in Γ. Then, an *immediate descendant* θ' of θ is $[P \mid x'] \cdot [\sigma' \mid (a, t')] \cdot \xi$, where formula $\rho[\theta]$ is the t'-th formula in $\{\wedge\Phi\} \cup \Gamma$ with respect to $\prec$.

I2: Let $\rho[\theta]$ be the side formula A of $(\wedge \rightarrow)$. Then an *immediate descendant* θ' of θ is $[P \mid x'] \cdot [\sigma' \mid (a, t')] \cdot \xi'$, where $\wedge\Phi$ is the t'-th formula in $\{\wedge\Phi\} \cup \Gamma$ and $\xi' = [A \mid (\wedge, k) \cdot \alpha : B]$, i.e., A is the k-th formula in Φ.

I3: If θ is in the succedent of $\sigma = \psi(x)$, i.e., it is $[P \mid x] \cdot [\sigma \mid (s, t)] \cdot \xi$, then the immediate descendant θ' of θ is $\theta' = [P \mid x'] \cdot [\sigma' \mid (s, t)] \cdot \xi$.

Under Stipulation S in Section 2, the following lemma holds.

Lemma 8.1.(**Unique Immediate Descendant**): Consider an occurrence θ in P in logic L, which is neither a cut-formula nor in the endsequent. It has a unique immediate descendant.

We say that an occurrence θ' is called a *descendant* of θ in a proof P if there is a chain $\{\theta_0, ..., \theta_k\}$ so that $\theta_0 = \theta$, $\theta_k = \theta'$, and θ_t is an immediate descendant of θ_{t-1} for $t = 1, ..., k$.

When θ' is a descendant of θ, we say that θ is an *ancestor* of θ'. An immediate ancestor may not be uniquely determined. In the above (2) for $(\wedge \rightarrow)[\wedge\Phi]$, if Γ contains already $\wedge\Phi$, then the occurrence of A in the lower sequent has two immediate ancestors in the upper sequent. Also, since $(\rightarrow \wedge)$, $(\vee \rightarrow)$ or $(\supset\rightarrow)$ has multiple uppersequents, θ in its lowersequent may have multiple ancestors. However, Lemma 8.1 is enough for the proof of Theorem 4.4.

The sign of an occurrence $\theta = [P \mid x] \cdot [\sigma \mid (a, j)] \cdot \xi$ or $[P \mid x] \cdot [\sigma \mid (s, j)] \cdot \xi$ in $P = (X, <; \psi)$ is simply the sign of $[\sigma \mid (a, j)] \cdot \xi$ or $[\sigma \mid (s, j)] \cdot \xi$ in σ. Then, we have:

Theorem 8.2.(**Sign-Preservation Property**): Let P be a proof in L = CL or IL. Consider an occurrence θ in P. Every descendant of θ has the same sign as that of θ.

8.2. Proof of Theorem 4.4 for η_{Lf}

First, we prove Theorem 4.4 for the cut-free case, i.e., $\beta_{\mathrm{L}}(\sigma) \leq \eta_{\mathrm{Lf}}(\sigma)$. If σ is not provable, then $\eta_{\mathrm{Lf}}(\sigma) = +\infty$ and nothing should be proved. Let σ be provable in L.

Lemma 8.3 (Cut-Free Case). Let P be any cut-free proof of a given sequent σ in L. Then, there is a genuine set μ of legitimate occurrences in σ such that $w(\mu) \leq \eta(P)$.

Proof. Let ζ be a legitimate occurrence in σ. As remarked above, this ζ can be regarded as an occurrence relative to P. We say that ζ is P-*essential* iff there is some uppermost ancestor $\alpha[\zeta]$ of ζ in P so that its target formula is the principal formula of $(\rightarrow \wedge)$, $(\vee \rightarrow)$ or $(\supset\rightarrow)$. Let μ be the set of all P-essential legitimate occurrences in σ.

This μ satisfies condition (4.7): Indeed, suppose that $\zeta_t \in \mu$ is a suboccurrence of another legitimate occurrence $\zeta \neq \zeta_t$ in $\sigma = \psi(x_o)$. Consider the uppermost ancestor

$\alpha[\zeta_t]$ of ζ_t so that it is the principal formula of $(\to \wedge), (\vee \to)$ or $(\supset\to)$. The target formula $\tau[\zeta_t]$ is a subformula in a conjunct or a disjunct of $\tau[\zeta]$ if $\tau[\zeta]$ is expressed as $\wedge\Phi$ or $\vee\Psi$; or in A or B of $\tau[\zeta]$ if it is $A \supset B$. Hence, there is an application of $(\to \wedge), (\vee \to)$ or $(\supset\to)$ between ζ_t and $\alpha[\zeta_t]$ such that its principal formula is $\tau[\zeta]$. Hence, ζ is P-essential, so $\zeta \in \mu$.

Now, let $\zeta'_1, ..., \zeta'_k$ be the other legitimate occurrences in σ so that each ζ'_t is not a suboccurrence of any other $\zeta'_{t'}$ $(t' \neq t)$. Let $q_1, ..., q_k$ be new propositional variables. Then, we replace all ancestors of ζ'_t by q_t $(t = 1, ..., k)$, respectively, in proof P. We show that the replacement does not destroy the proof structure of P : Indeed, since P is cut-free and each ζ'_t has no companions, no uppermost ancestor is introduced by an initial sequent. By the definition of μ, all the ancestors of each ζ'_t are introduced by (th), $(\wedge \to)$ or $(\to \vee)$. The endsequent obtained by this replacement is denoted by σ^*. Hence, $\vdash_{\mathrm{L}} \sigma^*$. Hence, μ is a genuine set.

Finally, for each $\zeta \in \mu$, we have an application of $(\to \wedge)$, $(\vee \to)$ or $(\supset\to)$ so that its principal formula is $\tau[\zeta]$. For any $\zeta, \zeta' \in \mu$, if $\zeta \neq \zeta'$, the uppermost ancestors are different by Lemma 8.1. Hence, one occurrence ζ in μ has at least one distinguished application with its principal formula $\tau[\zeta]$. Hence, by the definition (4.6) of w and Lemma 4.3, we have $w(\mu) \leq \eta(P)$. ■

Proof of Theorem 4.4 for η_{Lf}. Let P be a cut-free proof of σ. Let μ be the genuine set of occurrences given in Lemma 8.3. Then, $w(\mu) \leq \eta(P)$. By (4.9), we have $\beta_{\mathrm{L}}(\sigma) = \min_{\mu' \in \mathbf{M}_{\mathrm{L}}(\sigma)} w(\mu') \leq w(\mu) \leq \eta(P)$. Since P is a cut-free proof of σ, we have $\beta_{\mathrm{L}}(\sigma) \leq \eta_{\mathrm{Lf}}(\sigma) = \min\{\eta(P) : P \text{ is a cut-free proof of } \sigma\}$. ■

8.3. Proof of Theorem 4.4 for η_{Lw}

The proof of Lemma 8.3 does not work when a proof P has a (cut). However, we can prove the same assertion for any proof P with (cut)'s, constructing a further argument based on Lemma 8.3. For this purpose, we should refine the cut-elimination theorem (Theorem 2.1).

Consider a proof P of σ with/without cuts. Also consider applications $(\to \wedge)[\wedge\Phi]$, $(\vee \to)[\vee\Psi]$ and/or $(\supset\to)[A \supset B]$ in P. Suppose that the endsequent σ has the descendants ζ_1, ζ_2, ζ_3 of the principal formulae $\wedge\Phi$, $\vee\Psi$ and/or $A \supset B$ of those applications. By Lemma 8.1, each descendant is uniquely determined. Hence, we can write $(\to \wedge)[\wedge\Phi]\langle\zeta_1\rangle$, $(\vee \to)[\vee\Psi]\langle\zeta_2\rangle$ and/or $(\supset\to)[A \supset B]\langle\zeta_3\rangle$. For example, $(\to \wedge)[\wedge\Phi]\langle\zeta_1\rangle$ means that in P, once an application of $(\to \wedge)[\wedge\Phi]$ occurs and ζ_1 is its descendant in the endsequent σ of P. By this concept, we can distinguish between two applications with the same principal formulae, e.g., when $(\to \wedge)[\wedge\Phi]$ occurs in two different places in P, they are distinguished by their descendants in the end sequent. We have the possibility that P has (cut)'s and the endsequent does not have a descendant of the principal formula of such an application. But this possibility is irrelevant in the following argument.

The following theorem can be proved in almost the same way as Gentzen's [7] proof of the cut-elimination theorems for CL and IL[8].

Theorem 8.4 (Refinement of the Cut-Elimination Theorem): Let L be CL or IL, and σ any sequent in L. Let ζ_1 be a positive occurrence in σ with $\tau[\zeta_1] = \wedge\Phi$, and let ζ_2, ζ_3 be negative occurrences in σ with $\tau[\zeta_2] = \vee\Psi, \tau[\zeta_3] = A \supset B$.

Let P be a proof of σ in L. Then, there is a cut-free proof P' of σ such that if P' has applications of $(\rightarrow \wedge)[\wedge\Phi]\langle\zeta_1\rangle, (\vee \rightarrow)[\vee\Psi]\langle\zeta_2\rangle$ and/or $(\supset\rightarrow)[A \supset B]\langle\zeta_3\rangle$, then P has applications of the same inferences with the same principal formulae and their descendants ζ_1, ζ_2 and/or ζ_3 in the endsequent σ.

Using this theorem, we can modify Lemma 8.3 in the case for a proof with (cut)'s. Once we have the following modification, the above proof of Theorem 4.4 is the same.

Lemma 8.3$'$ (Case with Cuts). Let P be any proof of a sequent σ with (cut)'s in L. Then, there is a genuine set μ of legitimate occurrences in σ such that $w(\mu) \leq \eta(P)$.

Proof. Let P' be a cut-free proof given by Theorem 8.4. Then, let μ be the set given in the proof of Lemma 8.3 for P'. Although μ is defined and proved to be a genuine set depending upon P', the definition of a genuine set does not depend upon the choice of a proof. Hence, μ is a genuine set.

Let $\zeta \in \mu$. Then, we have an application of $(\rightarrow \wedge)[\wedge\Phi]\langle\zeta\rangle, (\vee \rightarrow)[\vee\Psi]\langle\zeta\rangle$ or $(\supset\rightarrow)[A \supset B]\langle\zeta\rangle$ in P'. Then, the original P has the corresponding application by Theorem 8.4. By Lemma 8.1, if ζ and ζ' in μ are different, then the corresponding applications in P are different. Thus, by the definition (4.6) of w and Lemma 4.3, we have $w(\mu) \leq \eta(P)$. ■

9. Conclusions and Some Remarks

We have developed a theory of the measure of inference for classical logic CL and intuitionistic logic IL. In either L = CL or IL, the measure η_{L*} gives, to a given sequent σ, the minimum number of the widths of possible proofs - - it counts the number of indispensable contents included in σ. To calculate the exact value $\eta_{L*}(\sigma)$, we developed the LB-method in Section 4, and using it, we calculated the values for various examples. By these considerations, we have had certain important consequences both from the viewpoints of logic as well as game theory/economics.

In Section 5, we studied the degrees of difficulties coming from two different decision criteria: the dominant-strategy (DS) criterion and prediction-decision (PD) criterion. We have shown a trade-off between the difficulties caused by these decision criteria. The

[8] A proof will be sent upon request.

DS does not require interpersonal thinking; on the other hand, the PD does but may simplify a lot his thinking to reach a decision.

Section 6 is about a contradictory sequent. In the Gentzen-style sequent formulation, two sequents $\Gamma \rightarrow$ and $\Gamma \rightarrow \neg q \wedge q$ are equivalent with respect to provability. However, it was argued in Section 6.1 they may be different with respect to our measure $\eta_{\mathrm{L}*}$: In particular, the two cases with/without (*cut*)'s are quite different, and the succedent $\neg q \wedge q$ may perform in a similar way as (*cut*). In Section 6.2, we considered also the contradictory statement arising from economics. We can make the example so that the value of $\eta_{\mathrm{L}*}$ for the sequent is arbitrary large. It means that when the value is very large, the premises (antecedent) are contradictory from the objective point of view, while this fact may be difficult to be perceived.

We have a lot of different aspects as well as a lot of applications to be considered. Here, we will discuss a few aspects and applications to be emphasized.

(1): Refinement of the LB-Method: This works well in the examples in this paper in that $\beta_{\mathrm{L}}(\sigma)$ gives the exact value $\eta_{\mathrm{L}*}(\sigma)$ or approximates it. Only two exceptions are the sequents σ_0 and σ_1 given in (6.2) in that $\beta_{\mathrm{L}}(\sigma)$ is very different from $\eta_{\mathrm{Lf}}(\sigma)$. This is caused by the fact that the lower bound function β_{L} counts legitimate occurrences as a summation form. However, in the examples σ_0 and σ_1, the ancestors of legitimate occurrences are nesting, which requires us to count them as a multiplication form. The difference caused by nesting can be seen more severely in the application of the LB-method to the Pigeonhole Principle (see Buss [4], Arai [1]). A refinement of the LB-method toward this direction is an important open question.

(2): Computational Complexity and Proof Search: By (1), we may recall the literature of computational complexity and proof search. Once the measure of inference is well developed, we can use it for an analysis of computational complexity for various classes of problem instances as well as for proof search. This is a side problem along the line of our original motivation. Nevertheless, it would be important to think about this application.

(3): Epistemic Logics of Shallow Depths: This is closely related to the present research in motivations, and was briefly discussed in the game with large and small stores in Section 5. The extension of the measure $\eta_{\mathrm{L}*}$ itself to those epistemic logics is straightforward, though we have different possible ways of counting epistemic depths. Also, the function β_{L} and Theorem 4.4 can be extended to those logics. Then, we can discuss the trade-off mentioned above in a more explicit manner.

(4):Mechanical Method of Calculations: Lemma 3.2 gave decomposition properties along the principal formula of an application of an inference. However, it gave only decompositions with inequalities, and does not help the calculation of $\eta_{\mathrm{L}*}$. Therefore, we have developed the LB-method. However, when we restrict our attention to some class of sequents, we can expect those decompositions with equalities, perhaps, for intuitionistic logic L = IL. In fact, this is partially done for intuitionistic-based epistemic logics. Then we can calculate $\eta_{\mathrm{L}*}$ in a mechanical way, and expect to a mechanical construction of a proof.

(5): Hilbert-Style Logic: Measure $\eta_{\mathrm{L}*}$ is system-specific, as it depends upon the choice of a language. If we adopt a Hilbert-style formulation of classical or intuitionistic logics, then the value changes a lot. There are diverse formulations of Hilbert-style systems. In the case of classical logic, one system in Kaneko-Nagashima [11] may be well comparable with the sequent formulation of the present paper. A proof in a sequent calculus can be converted into the Hilbert-style formulation and *vice versa*. For this conversion, (*cut*) does play an important role. Since the LB-method works even for η_{CLw}, we would be able to compare our measure for that for the Hilbert-style formulation.

(6): Connections to Inductive Game Theory: From our research viewpoint, it is more direct to apply the theory to the induction process in inductive game theory of Kaneko-Kline [10]. For example, the inductive process itself was not formulated in [10], but it can be formulated as an algorithm from accumulated experiences to an individual view. This algorithm can be formulated as a set of beliefs of a player, and he infers his view based on this set of beliefs from his accumulated experiences. If our measure gives a large number, he would have a difficulty in constructing his view.

The present authors have already developed some of those problems - - some papers [16] and [17] will be available. Yet, we have a lot of open problems about the theory presented here. We expect a lot of further contributions along the line of the research given in this paper.

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