

# Comparison geometry of ancient solutions to the Ricci flow

Takumi Yokota  
(Doctoral Program in Mathematics)

Submitted to the Graduate School of  
Pure and Applied Sciences  
in Partial Fulfillment of the Requirements  
for the Degree of Doctor of Philosophy in  
Mathematics

at the  
University of Tsukuba

# Abstract

In this thesis, we study the geometric properties of the Ricci flow. We mainly restrict our attention to a special class of them consisting of ancient solutions. An ancient solution is a solution to Hamilton's Ricci flow equation which exists for an infinite time in the past. It plays a key role in the singularity analysis of the Ricci flow. Our main tool is Perelman's reduced volume which is an integral quantity known to be monotone non-increasing in the backward-time along the Ricci flow.

We first generalize the monotonicity of the reduced volume to the super Ricci flow. The super Ricci flow is a generalization of the Ricci flow as well as of Riemannian manifolds of non-negative Ricci curvature. This provides a unified approach to the comparison geometry of these two objects.

One of our main results is a gap theorem for ancient solutions to the Ricci flow which states that any ancient solution with the asymptotic limit of its reduced volume being sufficiently close to that of the Gaussian soliton must be isometric to the Euclidean space for all time. This is a natural generalization of Anderson's result for Ricci-flat manifolds. As a corollary, we also obtain a gap theorem for gradient shrinking Ricci solitons. This result confirms the recent conjecture of Carrillo–Ni.

Subsequently, we consider the monotone quantity discovered by Ecker–Knopf–Ni–Topping after Perelman's significant works. We prove that the asymptotic limit of this quantity is equal to that of Perelman's reduced volume for any ancient solution to the Ricci flow with bounded curvature. This provides a relation between these two monotone quantities defined for Ricci flows.

Most of the results of this thesis appear in author's papers [Yo, Yo2, Yo3, Yo4].

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Overview . . . . .	3
1.2	Main results . . . . .	4
<b>2</b>	<b>Comparison geometry of super Ricci flows</b>	<b>8</b>
2.1	Super Ricci flow . . . . .	8
2.2	Definition of the reduced volume . . . . .	12
2.3	Monotonicity of the reduced volume I . . . . .	14
2.4	Monotonicity of the reduced volume II . . . . .	18
<b>3</b>	<b>Three lemmas</b>	<b>20</b>
3.1	Preliminary estimates . . . . .	20
3.2	Asymptotic reduced volume . . . . .	22
3.3	Finiteness of fundamental group . . . . .	25
3.4	Reduced volume under Cheeger–Gromov convergence . . . . .	29
<b>4</b>	<b>A gap theorem for ancient solutions</b>	<b>32</b>
4.1	Technical lemma . . . . .	32
4.2	Proof of Theorem 4.1 . . . . .	34
<b>5</b>	<b>A gap theorem for gradient shrinkers</b>	<b>35</b>
5.1	Shrinking Ricci solitons . . . . .	35
5.2	Expanding Ricci solitons . . . . .	39
5.3	Concluding remarks . . . . .	41
<b>6</b>	<b>Asymptotic volume ratio under the Ricci flow</b>	<b>44</b>
<b>7</b>	<b>Asymptotic reduced volume</b>	<b>47</b>
7.1	Definition . . . . .	48
7.2	Proof of Theorem 7.1 . . . . .	51
<b>8</b>	<b>Harnack implies bounded curvature in dimension 2</b>	<b>56</b>
8.1	Hamilton’s point picking lemma . . . . .	57
8.2	Proof of Proposition 8.3 . . . . .	59
<b>A</b>	<b>Appendix</b>	<b>61</b>

# Chapter 1

## Introduction

### 1.1 Overview

In his fundamental paper [Ha], Hamilton introduced the following evolution equation for Riemannian metrics on a fixed manifold:

$$\frac{\partial}{\partial t}g = -2 \operatorname{Ric}(g(t)), \quad (1.1)$$

where  $\operatorname{Ric}(g(t))$  denotes the Ricci tensor of  $g(t)$ . We shall refer to a solution of this equation as a *Ricci flow*. He proved the short-time existence and uniqueness of the Ricci flow for any given initial metric on a compact manifold. Shortly later a drastically simplified proof was provided by DeTurck [De], and Shi [Sh] extended the short-time existence to complete metrics of bounded curvature on non-compact manifolds. The uniqueness problem on non-compact manifolds are discussed in Chen–Zhu [ChZh].

Helpful references for the basics of the Ricci flow are Chow–Knopf [Vol1], Chow–Lu–Ni [CLN] and Topping [Topp2].

In [Ha] and the subsequent paper [Ha2], Hamilton studied the Ricci flow and showed that the positivity of the Ricci curvature in dimension 3 and that of the curvature operator in arbitrary dimensions are preserved by the Ricci flow. This was done by applying the maximum principle to the evolution equations which the curvature tensor of the Ricci flow enjoys. Then the main results of these papers are that any Ricci flow starting an initial metric satisfying one of such curvature assumptions is deformed into a metric of positive constant curvature. This implies that the universal covering of a manifold admitting a Riemannian metric satisfying such curvature conditions is diffeomorphic to the sphere. In this direction, recent striking progresses are the works of Böhm–Wilking [BöWi] and Brendle–Schoen [BrSc].

After that, Hamilton considered to use the Ricci flow to study the geometry of general 3-dimensional closed manifolds. This approach made a success in dimension 2; see Chen–Lu–Tian [CLT] and the references therein. He then proposed a program of proving Thurston’s geometrization conjecture by means of the Ricci flow. This is the so-called Hamilton program.

In general, a Ricci flow on a closed manifold develops a singularity in finite time. If  $(M^n, g(t)), t \in [0, T)$  is a maximal solution to the Ricci flow with  $T < \infty$ , as was shown by Hamilton [Ha], the norm of the curvature tensor  $\operatorname{Rm}$  becomes arbitrarily large as  $t$  approaches the singular time  $T$ . (Subsequently, Sesum [Se] showed that the Ricci tensor

also blows-up in finite-time singularities of the Ricci flow; it is not known whether or not the same is true for the scalar curvature.)

In analyzing singularities of solutions of geometric PDEs, a strategy which we usually adopt is to take a blow-up limit; take a sequence  $\{(p_k, t_k)\}$  of points in  $M^n \times [0, T)$  such that  $Q_k := |\text{Rm}|(p_k, t_k)$  tends to infinity and consider the rescaled flows  $\tilde{g}_k(t) := Q_k g(Q_k^{-1} \cdot + t_k)$ ,  $t \in (-Q_k t_k, 0]$ .

If we choose the points  $(p_k, t_k)$  properly and the local collapse does not occur, by applying Hamilton's compactness theorem (Theorem 3.20), we are able to find a subsequence of  $\{(M^n, \tilde{g}_k(t), p_k)\}$  converging to the limit Ricci flow  $(M_\infty^n, g_\infty(\tau), p_\infty)$ ,  $t \in (-\infty, 0]$  which is an ancient solution to the Ricci flow. An *ancient solution* is, in Hamilton's terminology, a Ricci flow which exists on the infinite interval  $(-\infty, 0]$ . Ancient solutions are important objects in the study of singularities of the Ricci flow. To rule out the local collapse was the crucial missing step in the Hamilton program.

It was Perelman who made the breakthrough and proved the geometrization conjecture by completing a weak form of the Hamilton program in his papers [Pe, Pe2]. He established the no local collapsing result by introducing the integral quantity which he calls the reduced volume. The reduced volume is monotone non-decreasing in time along the Ricci flow and plays an important role in his argument [Pe, Pe2]. Detailed accounts of Perelman's argument are given in Kleiner–Lott [KL], Cao–Zhu [CaZhu] and Morgan–Tian [MoTi].

In this thesis, we are concerned with the geometric properties of ancient solutions to the Ricci flow which do not necessarily arise as blow-up limits of singularities of the Ricci flow. As hinted above, the main tool for our investigation is Perelman's reduced volume. In the next section, we list the main theorems of the present theses.

We remark here that the classification of 2-dimensional complete non-compact ancient solution to the Ricci flow with bounded curvature was obtained by Daskalopoulos–Sesum [DaSe] and Chu [Chu]. Recently, the classification of 2-dimensional compact ancient solutions was done by Daskalopoulos–Hamilton–Sesum [DHS]. However, we are far from a complete understanding of the whole picture of all ancient solutions in dimensions 3 (e.g. [CLN, Problem 9.75]) and higher.

## 1.2 Main results

In this thesis, we adopt the convention that the reduced volume is identically 1 for the Gaussian soliton. The *Gaussian soliton* is the trivial Ricci flow  $(\mathbb{R}^n, g_E)$  on the Euclidean space regarded as a gradient shrinking Ricci soliton  $(\mathbb{R}^n, g_E, \frac{|\cdot|^2}{4})$ .

In dealing with the reduced volume of an ancient solution  $(M, g(t))$ ,  $t \in (-\infty, 0]$ , we find it convenient to introduce the reverse-time parameter  $\tau := -t \in [0, \infty)$ . Now we state our main theorems of this thesis.

**Theorem 1.1** (Gap theorem for ancient solutions). *For any  $n \geq 2$ , there exists a constant  $\varepsilon_n > 0$  which depends only on  $n$  and satisfies the following: let  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  be an  $n$ -dimensional complete ancient solution to the Ricci flow with Ricci curvature bounded below. Suppose that the asymptotic limit of the reduced volume  $\lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau)$  is greater than  $1 - \varepsilon_n$  for some point  $p \in M$ . Then  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  is the Gaussian soliton  $(\mathbb{R}^n, g_E)$ , i.e., it is isometric to  $(\mathbb{R}^n, g_E)$  for all  $\tau \in [0, \infty)$ .*

In the statement above, the limit  $\tilde{\mathcal{V}}(g) := \lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau)$  will be referred to as the *asymptotic reduced volume* of the flow  $(M, g(\tau))$ . We will see in Section 3.2 below that  $\tilde{\mathcal{V}}(g)$  is independent of the choice of the base point  $p \in M$ .

The asymptotic reduced volume  $\tilde{\mathcal{V}}(g)$  is a Ricci flow analogue of asymptotic volume ratio of Riemannian manifolds of non-negative Ricci curvature. The *asymptotic volume ratio*  $\nu(g)$  of a complete Riemannian manifold  $(M^n, g)$  with  $\text{Ric}(g) \geq 0$  is defined as  $\nu(g) := \lim_{r \rightarrow \infty} \text{Vol} B(p, r) / \omega_n r^n$ , which is well-defined due to the Bishop–Gromov inequality (Theorem 2.11). Here  $\text{Vol} B(p, r)$  is the volume of the metric ball  $B(p, r)$  of center  $p \in M$  and radius  $r > 0$ , and  $\omega_n$  stands for the volume of the unit ball in the Euclidean space  $(\mathbb{R}^n, g_E)$ .

By regarding a Ricci-flat metric, i.e., a Riemannian metric whose Ricci tensor vanishes, as an ancient solution as in Theorem 1.1, we recover the following result, which is the motivation of the present work.

**Theorem 1.2** (Anderson [An, Gap Lemma 3.1], also Petersen [Pet]). *For any  $n \geq 2$ , there exists a constant  $\varepsilon_n > 0$  which satisfies the following: let  $(M^n, g)$  be an  $n$ -dimensional complete Ricci-flat Riemannian manifold. Suppose that the asymptotic volume ratio  $\nu(g)$  of  $(M^n, g)$  is greater than  $1 - \varepsilon_n$ . Then  $(M^n, g)$  is isometric to  $(\mathbb{R}^n, g_E)$ .*

On the way to the proof of Theorem 1.1, we establish several lemmas. Here we state one of them as a theorem, which is of independent interest.

**Theorem 1.3** (Finiteness of fundamental groups). *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient solution to the Ricci flow on  $M$  with Ricci curvature bounded below. If its asymptotic reduced volume  $\tilde{\mathcal{V}}(g)$  is strictly positive, then the fundamental group  $\pi_1(M)$  of  $M$  is finite.*

More generally, Theorem 1.3 is shown for super Ricci flows in Section 3.3 under certain assumptions. The super Ricci flow, introduced by McCann–Topping [McTo], is a generalization of the Ricci flow as well as of Riemannian manifolds of non-negative Ricci curvature. A few applications of Theorem 1.3 are offered in that section.

Next, we apply Theorem 1.1 above to gradient shrinkers. We call a triple  $(M^n, g, f)$  a *gradient shrinking Ricci soliton*, or *gradient shrinker*, when

$$\text{Ric}(g) + \text{Hess } f - \frac{1}{2\lambda}g = 0 \quad (1.2)$$

holds for some positive constant  $\lambda > 0$ . Shrinking Ricci solitons are typical examples of ancient solutions to the Ricci flow. We refer the reader to Cao’s survey article [Ca] for recent advances in the geometry of gradient Ricci solitons.

We always normalize the potential function  $f \in C^\infty(M)$  by adding a constant so that

$$\lambda(2\Delta f - |\nabla f|^2 + R) + f - n = 0 \quad \text{on } M, \quad (1.3)$$

where  $R$  denotes the scalar curvature of  $(M^n, g)$ . Since taking the trace of equation (1.2) yields  $R + \Delta f = n/2$ , (1.3) is equivalent to

$$\lambda(|\nabla f|^2 + R) - f = 0 \quad \text{on } M. \quad (1.4)$$

The left-hand sides of (1.3) and (1.4) is known to be constant (e.g. [Vol2-I, Proposition 1.15]). Then we define the *Gaussian density*  $\Theta(M)$  of  $(M^n, g, f)$  by

$$\Theta(M) := \int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g. \quad (1.5)$$

**Corollary 1.4** (Gap theorem for gradient shrinkers). *Let  $(M^n, g, f)$  be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below. Then*

- (1) *the Gaussian density  $\Theta(M^n)$  does not exceed 1.*
- (2) *Suppose that  $\Theta(M^n) > 1 - \varepsilon_n$ , then  $(M^n, g, f)$  is, up to scaling, the Gaussian soliton  $(\mathbb{R}^n, g_E, \frac{|\cdot|^2}{4})$ , i.e.,  $(M^n, g)$  is isometric to  $(\mathbb{R}^n, g_E)$ . Here the constant  $\varepsilon_n$  comes from Theorem 1.1.*

The statements in Corollary 1.4 are intimately related to the results of Carrillo–Ni [CaNi]. In particular, Corollary 1.4.(2) confirms their speculation that *the Gaussian density is 1 only for the Gaussian soliton* [CaNi]. See Remark 5.14 below.

After that, we consider another monotone quantity defined for the Ricci flow. This quantity  $I_{(p,0)}(r)$  was discovered and shown to be non-increasing in  $r > 0$  by Ecker–Knopf–Ni–Topping [EKNT]. We prove the following theorem which gives a relation between  $I_{(p,0)}(r)$  and Perelman’s reduced volume  $\tilde{V}_{(p,0)}(\tau)$ .

**Theorem 1.5.** *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient solution to the Ricci flow with bounded curvature. Then for any  $p \in M$ , we have*

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau) = \lim_{r \rightarrow \infty} I_{(p,0)}(r). \quad (1.6)$$

Our starting point of this work was provided by the viewpoint that the Ricci flow is a generalization of Einstein metrics; an *Einstein* metric is a Riemannian metric whose Ricci tensor is constant. We should mention the results of Fang–Zhang–Zhang [FZZ] and Ishida [Is], which are also the results of this direction. They regard a *non-singular solution* ([Ha7])  $(M, g(t)), t \in [0, \infty)$  to the normalized Ricci flow:

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}(g(t)) + \frac{2}{n} r g(t), \quad (1.7)$$

where  $r = r(t)$  denotes the averaged scalar curvature

$$r := \frac{\int_M R(\cdot, t) d\mu_{g(t)}}{\int_M 1 d\mu_{g(t)}},$$

as a generalization of Einstein metrics. They derive conclusions about the topology of the underlying manifold from the existence of a non-singular solution.

The organization of this thesis is as follows.

In Chapter 2, we review definitions and Perelman’s results in [Pe]. We will do this for super Ricci flows. The main theorem of this chapter is the monotonicity of Perelman’s reduced volume along the super Ricci flow satisfying a few natural assumptions. Two different proofs are presented as in the literature in Sections 2.3 and 2.4. This is why the difference lies in the assumptions of Theorems 1.1 and 1.5.

In Chapter 3, some lemmas which are required in the proof of the main theorem will be established. Among them, we prove that the asymptotic reduced volume  $\tilde{\mathcal{V}}(g)$  is

independent of the choice of the base point. Theorem 1.3 is also shown for super Ricci flows in this chapter.

In Chapter 4, we give a proof of Theorem 1.1.

In Chapter 5, we prove Corollary 1.4 and consider gradient expanding Ricci solitons with non-negative Ricci curvature.

In Chapter 6, we investigate the asymptotic volume ratio of Ricci flows of non-negative Ricci curvature. It is shown that the asymptotic volume ratio  $\nu(g(t))$  of the Ricci flow  $(M^n, g(t))$  is constant in  $t$  provided its Ricci curvature is non-negative and bounded.

In Chapter 7, Theorem 1.5 will be proved. Actually, we prove a theorem which slightly generalizes Theorem 1.5. The generalization is done in two directions; we prove Theorem 1.5 with the Ricci flow being replaced with the super Ricci flow and the reduced volume being replaced with its slight generalization considered in Section 2.4.

Finally, in Chapter 8, we prove a proposition about the Harnack inequality of the Ricci flow on surfaces. More precisely, we show that any complete Ricci flow on a surface with non-negative curvature satisfying Harnack inequality must have bounded curvature.

Appendix A is devoted to a detailed proof of Perelman's point picking lemma. This lemma will be used several times in the argument.

## Acknowledgments

The author expresses his sincere gratitude to his adviser Takao Yamaguchi for his constant encouragements, comments and discussions. He also thank Professors Koichi Nagano and Satoshi Ishiwata for stimulating conversations. He also would like to thank Professor Lei Ni for his interest in his paper [Yo2] and insightful comments. The author was supported in part by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

## Chapter 2

# Comparison geometry of super Ricci flows

In the present chapter, we recall and generalize the definitions and results established by Perelman in Sections 6 and 7 of his seminal paper [Pe] where he introduced a comparison geometric approach to the Ricci flow, which is often called the *reduced geometry*. The main result of this chapter is the monotonicity of Perelman's reduced volume  $\tilde{V}_{(p,0)}(\tau)$  along the super Ricci flow (Theorems 2.12 and 2.17). We present two different proofs.

A main application of the monotonicity of the reduced volume is the following no local collapsing result for the Ricci flow.

**Theorem 2.1** (No local collapsing theorem [Pe]). *Let  $(M^n, g(t))$ ,  $t \in [0, T)$  be a complete Ricci flow with Ricci curvature bounded below,  $T < \infty$  and*

$$\text{Vol}_{g(0)} B_{g(0)}(p, r_0) > \exists v_1 > 0 \quad \text{for all } p \in M.$$

*Then for any  $\rho > 0$ , there exists a constant  $\kappa = \kappa(g(0), T, \rho) > 0$  such that  $(M^n, g(t))$  is (weakly)  $\kappa$ -noncollapsing at any  $(p_*, \tau_*) \in M^n \times (T/2, T)$  and on any scale  $r \in (0, \rho]$ .*

See Definition 3.16 below for the definition of being  $\kappa$ -noncollapsing.

The main references for this chapter are, apart from Perelman's original paper [Pe], Chow et al. [Vol2-I], Kleiner–Lott [KL], Morgan–Tian [MoTi] and Ye [Ye2]. Among them, Ye [Ye2] paid careful attention to argue under the assumption of Ricci curvature bounded below rather than bounded sectional curvature (see also [EKNT, Appendix]). The assumption of Theorem 1.1 on the curvature of the Ricci flow  $(M^n, g(\tau))$  is the same as that considered in [Ye2]. We mainly follow the notation of [Vol2-I].

## 2.1 Super Ricci flow

As mentioned above, we would like to develop Perelman's reduced geometry in more general situation, that is, the super Ricci flow. This will provide us with a convenient setting for comparison geometry of the Ricci flow. A smooth one-parameter family of Riemannian metrics  $(M, g(\tau))$ ,  $\tau \in [0, T)$  is called a *super Ricci flow* when it satisfies

$$\frac{\partial}{\partial \tau} g \leq 2 \text{Ric}(g(\tau)). \tag{2.1}$$

Super Ricci flow was introduced by McCann–Topping [McTo] in their attempt to generalize the contraction property of heat equation in the Wasserstein spaces, which characterizes the non-negativity of the Ricci curvature of Riemannian metrics (see von Renesse–Sturm [ReSt]), to time-depending metrics. They give the following characterization of the super Ricci flow; see the original paper [McTo] for definitions and the precise statement.

**Theorem 2.2** (McCann–Topping [McTo]). *Let  $(M^n, g(\tau))$  be a one-parameter family of Riemannian metrics on a closed manifold  $M^n$  and  $u_1(\tau) = u_1(\cdot, \tau)$  and  $u_2 = u_2(\cdot, \tau)$  be two smooth positive solutions to the conjugate heat equation:*

$$\frac{\partial}{\partial \tau} u = \Delta_{g(\tau)} u - H u \quad (2.2)$$

with  $\int_M u_i(\cdot, \tau) d\mu_{g(\tau)} = 1$  for  $i = 1, 2$ . Then the following are equivalent.

- (1)  $(M^n, g(\tau))$  is a super Ricci flow.
- (2) The Wasserstein distance  $W_2^{g(\tau)}(u_1(\tau), u_2(\tau))$  between  $u_1(\cdot, \tau)$  and  $u_2(\cdot, \tau)$  induced by  $g(\tau)$  is non-increasing in  $\tau$ .

The Wasserstein geometry has been intriguing Riemannian geometers due to its potential connection to the geometry of Ricci curvatures. Consult Villani’s book [Vi] for this fascinating area of research.

Basic and important examples of super Ricci flows are

**Example 2.3.** (1) A solution to the backward Ricci flow equation  $\frac{\partial}{\partial \tau} g = 2 \operatorname{Ric}(g(\tau))$  and

- (2)  $g(\tau) := (1 + 2C\tau)g_0, \tau \in \left[0, \frac{1}{|C| - C}\right)$  for some fixed Riemannian metric  $g_0$  with Ricci curvature bounded from below by  $C \in \mathbb{R}$ .

Therefore, it can be said that the study of super Ricci flows includes those of (backward) Ricci flows and manifolds with Ricci curvature bounded from below.

We will see that it is straightforward to generalize Perelman’s reduced geometry to the super Ricci flow if we impose the following assumptions.

**Assumption 2.4.** Putting  $2h := \frac{\partial}{\partial \tau} g$  and  $H := \operatorname{tr}_{g(\tau)} h$ ,  $h$  satisfies

- (1) contracted second Bianchi identity  $2 \operatorname{div} h(\cdot) = \langle \nabla H, \cdot \rangle$  and
- (2) heat-like equation  $-\operatorname{tr}_{g(\tau)} \frac{\partial}{\partial \tau} h \geq \Delta_{g(\tau)} H$ , or equivalently,

$$-\frac{\partial}{\partial \tau} H \geq \Delta_{g(\tau)} H + 2|h|^2. \quad (2.3)$$

Clearly, the ones in Example 2.3 above satisfy Assumption 2.4. It is known that the evolution equation for the scalar curvature  $R$  under the Ricci flow  $g(\tau)$  is given by

$$-\frac{\partial}{\partial \tau} R = \Delta_{g(\tau)} R + 2|\operatorname{Ric}|^2 \quad (2.4)$$

(e.g. Hamilton [Ha], also Ehrlich [Eh]).

Due to the entropy formulae which we meet now, it seems quite natural to consider super Ricci flows which satisfy Assumption 2.4.

Let  $(M^n, g(\tau)), \tau \in [0, T)$  be a super Ricci flow  $\frac{\partial}{\partial \tau} g =: 2h \leq 2 \text{Ric}(g(\tau))$  on a closed manifold  $M$ . Put  $H := \text{tr}_{g(\tau)} h$ . Following Perelman [Pe, Section 3], we define the  $\mathcal{W}$ -entropy for a triple  $(g(\tau), f, \tau)$  by

$$\mathcal{W}(g(\tau), f, \tau) := \int_M \left[ \tau(|\nabla f|^2 + H) + f - n \right] u d\mu_{g(\tau)} \quad (2.5)$$

where  $f$  is a smooth function on  $M^n$ ,  $\tau > 0$  and  $u := (4\pi\tau)^{-n/2} e^{-f}$ .

**Proposition 2.5** ( $\mathcal{W}$ -entropy for the super Ricci flow). *Let  $(M^n, g(\tau)), \tau \in (0, T)$  be a super Ricci flow satisfying Assumption 2.4 on a closed manifold  $M^n$ . We evolve a positive function  $u := (4\pi\tau)^{-n/2} e^{-f}$  by the conjugate heat equation*

$$\left( \frac{\partial}{\partial \tau} - \Delta_{g(\tau)} + H \right) u \iff \frac{\partial f}{\partial \tau} = \Delta_{g(\tau)} f - |\nabla f|^2 + H - \frac{n}{2\tau}.$$

Then the  $\mathcal{W}$ -entropy  $\mathcal{W}(g(\tau), f, \tau)$  is non-increasing in  $\tau$ .

*Proof.* Following Perelman [Pe, Section 9], we first let

$$v := [\tau(2\Delta f - |\nabla f|^2 + H) + f - n] u \quad (2.6)$$

(cf. (1.3)). Then, integrating by parts, we have

$$\mathcal{W}(g, f, \tau) = \int_M v d\mu_g,$$

which leads us to the computation of the evolution equation of  $v$ .

We obtain the following equation for general evolving Riemannian metrics by routine calculation:

$$\begin{aligned} \left( \frac{\partial}{\partial \tau} - \Delta + H \right) v = & -2\tau \left[ \left| h + \text{Hess } f - \frac{1}{2\tau} g \right|^2 + (2\text{div } h - dH)(\nabla f) \right. \\ & \left. - \frac{1}{2} \left( \frac{\partial H}{\partial \tau} + \Delta H + 2|h|^2 \right) + (\text{Ric} - h)(\nabla f, \nabla f) \right] u \end{aligned}$$

(cf. [Pe, (9.1)]). In the derivation of the above equation, we used the derivative formula for the time-dependent Laplacian (e.g. [CLN, Lemma 2.30]):

$$\frac{\partial}{\partial \tau} \Delta_{g(\tau)} f = -2\langle h, \text{Hess } f \rangle + (dH - 2\text{div } h)(\nabla f) + \Delta_{g(\tau)} \frac{\partial f}{\partial \tau},$$

and the Bochner–Weitzenböck formula (e.g. [Vol1, Appendix B]):

$$\frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess } f|^2 + \langle \nabla \Delta f, \nabla f \rangle + \text{Ric}(\nabla f, \nabla f).$$

Then, using the well-known formula  $\frac{\partial}{\partial \tau} d\mu_{g(\tau)} = H d\mu_{g(\tau)}$ , we obtain

$$\begin{aligned} \frac{d}{d\tau} \mathcal{W}(g(\tau), f, \tau) &= \int_M \left( \frac{\partial}{\partial \tau} - \Delta + H \right) v d\mu_{g(\tau)} \\ &\leq -2\tau \int_M \left| h + \text{Hess } f - \frac{1}{2\tau} g \right|^2 u d\mu_{g(\tau)} \\ &\leq 0. \end{aligned} \quad (2.7)$$

This ends the proof. □

From (2.7), we simultaneously recover the entropy formulae of Perelman ( $h = \text{Ric}$ ) [Pe] and Ni ( $h = 0$ ) [Ni].

An application of the monotonicity of  $\mathcal{W}$ -functional for the Ricci flow is the following theorem due to Perelman [Pe, Theorem 4.1] (cf. Theorem 2.1).

**Theorem 2.6** (No local collapsing theorem improved [Pe], also [KL, Vol2-I]). *Let  $(M^n, g(t)), t \in [0, T)$  be a Ricci flow on a closed manifold  $M^n$  with  $T < \infty$ . Then for any  $\rho > 0$ , there exists a constant  $\kappa = \kappa(g(0), T, \rho) > 0$  such that*

$$R(\cdot, t) \leq r^{-2} \text{ on } B_{g(t)}(p, r) \implies \text{Vol}_{g(t)} B_{g(t)}(p, r) \geq \kappa r^n$$

for any  $p \in M$ ,  $t \in [0, T)$  and  $r \in (0, \rho]$ .

Another application of the monotonicity of  $\mathcal{W}$ -entropy is Topping's diameter estimate [Topp] for the Ricci flow. The  $\mathcal{W}$ -entropy is intimately tied to the Logarithmic Sobolev inequality (e.g. Ye [Ye]); see also Theorem 5.15 below.

We also have a similar formula for the super Ricci flow analogue of  $\mathcal{F}$ -entropy introduced in [Pe, Section 1]. Introducing this  $\mathcal{F}$ -functional enabled Perelman to give a gradient flow interpretation of the Ricci flow (see [Pe, Section 1]), although it is known that the Ricci flow is not a gradient flow (e.g. Müller [Mü]). This gradient flow formulation allows us to rule out non-trivial breathers, i.e., periodic solutions to the Ricci flow equation [Pe, Sections 2 and 3].

**Proposition 2.7** ( $\mathcal{F}$ -entropy for the super Ricci flow). *Let  $(M^n, g(\tau)), \tau \in (0, T)$  be a super Ricci flow satisfying Assumption 2.4 on a closed manifold  $M^n$ . We evolve a positive function  $u = e^{-f}$  by*

$$\left( \frac{\partial}{\partial \tau} - \Delta + H \right) u = 0 \iff \frac{\partial}{\partial \tau} f = \Delta_{g(\tau)} f - |\nabla f|^2 + H. \quad (2.8)$$

Then the  $\mathcal{F}$ -entropy

$$\mathcal{F}(g(\tau), f) := \int (|\nabla f|^2 + H) e^{-f} d\mu_{g(\tau)}$$

is non-increasing in  $\tau$ .

*Proof.* Let  $v' := [2\Delta f - |\nabla f|^2 + R] e^{-f}$ . Then, by a calculation similar to the one we have done in the proof of the previous proposition, we obtain

$$\begin{aligned} \left( \frac{\partial}{\partial \tau} - \Delta + H \right) v' &= -2 \left[ |h + \text{Hess } f|^2 + (2\text{div } h - dH)(\nabla f) \right. \\ &\quad \left. - \frac{1}{2} \left( \frac{\partial H}{\partial \tau} + \Delta H + 2|h|^2 \right) + (\text{Ric} - h)(\nabla f, \nabla f) \right] e^{-f}. \end{aligned}$$

Hence,

$$\frac{d}{d\tau} \mathcal{F}(g(\tau), f) = \frac{d}{d\tau} \int_M v' d\mu_{g(\tau)} \leq -2 \int_M |h + \text{Hess } f|^2 e^{-f} d\mu_{g(\tau)} \leq 0.$$

□

*Remark 2.8.* Alternating proofs for the monotonicity of  $\mathcal{W}$ -entropy and  $\mathcal{F}$ -entropy as well as the reduced volume which is defined below are provided by Topping [Topp3] and Lott [Lo], respectively. Their proofs are based on the Wasserstein geometric consideration of the space-time  $M \times [0, T]$ .

Throughout this thesis, we denote by  $(M^n, g(\tau)), \tau \in [0, T)$  a complete super, or backward Ricci flow on an  $n$ -manifold  $M^n$  satisfying Assumption 2.4. It is also assumed that the time-derivative  $\frac{\partial}{\partial \tau} g$  is bounded from below in each compact time interval, that is, for any compact interval  $[\tau_1, \tau_2] \subset [0, T)$ , we can find  $K = K(\tau_1, \tau_2) \geq 0$  such that  $-Kg(\tau) \leq \frac{\partial}{\partial \tau} g \leq 2 \text{Ric}(g(\tau))$  and hence

$$e^{K(\tau_2 - \tau)} g(\tau_2) \geq g(\tau) \geq e^{-K(\tau - \tau_1)} g(\tau_1)$$

for all  $\tau \in [\tau_1, \tau_2]$ . Although Assumption 2.4 looks too restrictive, the author's intention is a unified treatment of backward Ricci flows and Riemannian manifolds with non-negative Ricci curvature.

## 2.2 Definition of the reduced volume

In this section, we recall the definitions from [Pe, Sections 6 and 7]. To begin with, fix a point  $p \in M$ ,  $[\tau_1, \tau_2] \subset [0, T)$  and  $\bar{\tau} \in (0, T)$ .

**Definition 2.9.** Let  $\gamma : [\tau_1, \tau_2] \rightarrow M$  be a curve. We define the  $\mathcal{L}$ -length of  $\gamma$  and the  $\mathcal{L}$ -distance between two points  $(p, \tau_1)$  and  $(q, \tau_2) \in M \times [\tau_1, \tau_2]$ , respectively, by

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left( \left| \frac{d\gamma}{d\tau} \right|_{g(\tau)}^2 + H(\gamma(\tau), \tau) \right) d\tau$$

and

$$L_{(p, \tau_1)}(q, \tau_2) := \inf \mathcal{L}(\gamma).$$

Here we take the infimum over all curves  $\gamma : [\tau_1, \tau_2] \rightarrow M$  with  $\gamma(\tau_1) = p$  and  $\gamma(\tau_2) = q$ .

The lower bound of  $\frac{\partial}{\partial \tau} g$  guarantees that the  $\mathcal{L}$ -distance between any two points is achieved by a minimal  $\mathcal{L}$ -geodesic (e.g. Morgan–Tian [MoTi]). This is the only place where we employ the assumption on  $\frac{\partial}{\partial \tau} g$ . A curve  $\gamma(\tau)$  is called an  $\mathcal{L}$ -geodesic when it satisfies the Euler–Lagrange equation:

$$2\nabla_X X + \frac{X}{\tau} - \nabla H + 4h(X, \cdot) = 0, \quad \text{where} \quad X := \frac{d\gamma}{d\tau}(\tau). \quad (2.9)$$

Then the *reduced distance* and the *reduced volume* based at  $(p, 0)$  are defined, respectively, by

$$\ell_{(p, 0)}(q, \bar{\tau}) := \frac{1}{2\sqrt{\bar{\tau}}} L_{(p, 0)}(q, \bar{\tau})$$

and

$$\tilde{V}_{(p, 0)}(\bar{\tau}) := \int_M (4\pi\bar{\tau})^{-n/2} \exp(-\ell_{(p, 0)}(q, \bar{\tau})) d\mu_{g(\bar{\tau})}(q)$$

where  $d\mu_{g(\bar{\tau})}$  denotes the volume element induced by  $g(\bar{\tau})$ .

We can rewrite the reduced volume as

$$\tilde{V}_{(p,0)}(\bar{\tau}) = \int_{T_p M} (4\pi\bar{\tau})^{-n/2} \exp(-\ell_{(p,0)}(\mathcal{L}\exp_{\bar{\tau}}(V), \bar{\tau})) \mathcal{L}J_V(\bar{\tau}) dx_{g(0)}(V) \quad (2.10)$$

by pulling back the integrand by the  $\mathcal{L}$ -exponential map  $\mathcal{L}\exp_{\bar{\tau}} : T_p M \rightarrow M$  which assigns  $\gamma_V(\bar{\tau})$ , if exists, to each tangent vector  $V \in T_p M$ . Here  $\gamma_V$  is the  $\mathcal{L}$ -geodesic determined by  $\gamma_V(0) = p$  and  $\lim_{\tau \rightarrow 0^+} \sqrt{\tau} \frac{d\gamma}{d\tau}(\tau) = V$ . In (2.10),  $dx_{g(0)}$  denotes the Lebesgue measure on the tangent space  $T_p M$  induced by the metric  $g(0)$ , and  $\mathcal{L}J_V(\bar{\tau})$  is called the  $\mathcal{L}$ -Jacobian. Remember that we are using the convention that  $\mathcal{L}J_V(\bar{\tau}) = 0$  unless  $V \in \Omega_{(p,0)}(\bar{\tau})$ . By  $V \in \Omega_{(p,0)}(\bar{\tau})$ , we mean that  $q := \mathcal{L}\exp_{\bar{\tau}}(V)$  exists and lies outside the  $\mathcal{L}$ -cut locus at time  $\bar{\tau}$ , i.e., there is a unique minimal  $\mathcal{L}$ -geodesic from  $(p, 0)$  to  $(q, \bar{\tau})$  and  $\mathcal{L}J_V(\bar{\tau}) > 0$ . It follows that  $\Omega_{(p,0)}(\bar{\tau})$  is an open set of  $T_p M$ , on which  $\mathcal{L}\exp_{\bar{\tau}}$  is a diffeomorphism, and that  $\Omega_{(p,0)}(\tau_2) \subset \Omega_{(p,0)}(\tau_1)$  for  $\tau_2 > \tau_1 > 0$ . The base point  $(p, 0)$  will often be suppressed.

*Remark 2.10.* A prototype of the reduced distance function appears in Li–Yau’s paper [LiYa].

In the next section, we will prove that the reduced volume is non-increasing for any super Ricci flow satisfying Assumption 2.4. Before going into the detail, we recall a heuristic argument given in [Pe, Section 6].

First, let us recall the Bishop–Gromov comparison theorem. We only state it for the case of non-negative Ricci curvature.

**Theorem 2.11** (Bishop–Gromov comparison theorem, e.g. Chavel [Cha]). *Let  $M^n$  be a complete Riemannian manifold of non-negative Ricci curvature. Then*

- (1)  $\text{Area } \partial B(p, r) / \alpha_{n-1} r^{n-1}$  is non-increasing in  $r > 0$ .
- (2)  $\text{Vol } B(p, r) / \omega_n r^n$  is non-increasing in  $r > 0$ .

Here  $\alpha_{n-1} := n\omega_n$  stands for the area of the unit sphere in the Euclidean space  $\mathbb{R}^n$ .

For a backward Ricci flow  $(M^n, g(\tau)), \tau \in [0, T)$ , Perelman equips the space-time  $\tilde{M} := M \times S^N \times (0, T_0]$ , for large  $N \gg 1$ , with a metric  $\tilde{g}$  written as

$$\tilde{g} = g(\tau) + \tau g_{S^N} + \left(R + \frac{N}{2\tau}\right) d\tau^2. \quad (2.11)$$

Here,  $(S^N, g_{S^N})$  is the  $N$ -sphere with constant curvature  $\frac{1}{2N}$ . He observed that  $(\tilde{M}, \tilde{g})$  has vanishing Ricci curvature up to mod  $N^{-1}$  (e.g. Wei [We]). An easy way to get a feeling of this is to regard  $\tilde{g}$  as a cone metric by setting  $\eta := \sqrt{2N\tau}$ . Recall that the metric cone  $(N \times (0, T), d\eta^2 + \eta^2 g_N)$  of  $(N, g_N)$  is Ricci-flat if and only if  $\text{Ric}_{g_N} = (\dim N - 1)g_N$ .

Then he applied the Bishop–Gromov inequality (Theorem 2.11.(1)) to  $(\tilde{M}, \tilde{g})$  formally so as to obtain an monotone quantity  $\tilde{V}_{(p,0)}(\tau)$  which he called the reduced volume;

$$\begin{aligned} & \frac{\text{Area } \partial B_{\tilde{M}}(\tilde{p}, r)}{\alpha_{n+N} r^{n+N}} \\ & \approx \frac{\alpha_N}{(2N\tau)^{n/2} \alpha_{n+N}} \int_M \left(1 - \frac{1}{N} \ell(\cdot, \tau) + O(N^{-2})\right)^N d\mu_{g(\tau)} \\ & \rightarrow \int_M (4\pi\tau)^{-n/2} e^{-\ell(\cdot, \tau)} d\mu_{g(\tau)} \quad \text{as } N \rightarrow \infty \end{aligned}$$

for  $\tilde{p} = (p, s, 0) \in M \times S^N \times \{0\}$  and  $r := \sqrt{2N\tau}$ .

As expected, it turns out that the reduced volume is non-increasing in  $\tau$  (Theorems 2.12 and 2.17).

## 2.3 Monotonicity of the reduced volume I

Next, we recall the computations performed in [Pe, Section 7].

Let  $q \in \mathcal{L} \exp_{\bar{\tau}}(\Omega_{(p,0)}(\bar{\tau}))$  and  $\gamma : [0, \bar{\tau}] \rightarrow M$  be the unique minimal  $\mathcal{L}$ -geodesic from  $(p, 0)$  to  $(q, \bar{\tau})$ . Take a tangent vector  $Y \in T_q M$  and extend it to the vector field along  $\gamma$  by solving

$$\nabla_X Y = -h(Y, \cdot) + \frac{Y}{2\tau}, \quad Y(\bar{\tau}) = Y$$

so that  $|Y|^2(\tau) = \frac{\tau}{\bar{\tau}}|Y|^2$ .

Then we have  $\nabla \ell(q, \bar{\tau}) = \frac{d\ell}{d\bar{\tau}}(\bar{\tau})$  and

$$\begin{aligned} \frac{\partial}{\partial \tau} \ell(q, \bar{\tau}) &= H(q, \bar{\tau}) - \frac{\ell(q, \bar{\tau})}{\bar{\tau}} + \frac{1}{2\bar{\tau}^{3/2}} K \\ |\nabla \ell|^2(q, \bar{\tau}) &= -H(q, \bar{\tau}) + \frac{\ell(q, \bar{\tau})}{\bar{\tau}} - \frac{1}{\bar{\tau}^{3/2}} K \\ \text{Hess } \ell(Y, Y)(q, \bar{\tau}) &\leq -h(Y, Y) + \frac{|Y|_{g(\bar{\tau})}^2}{2\bar{\tau}} - \frac{1}{2\sqrt{\bar{\tau}}} \int_0^{\bar{\tau}} \sqrt{\tau} \mathcal{H}(X, Y) d\tau \end{aligned} \quad (2.12)$$

$$\Delta \ell(q, \bar{\tau}) \leq -H(q, \bar{\tau}) + \frac{n}{2\bar{\tau}} - \frac{1}{2\bar{\tau}^{3/2}} K \quad (2.13)$$

$$\frac{\partial}{\partial \tau} \log \mathcal{L} J_V(\bar{\tau}) = \Delta \ell(q, \bar{\tau}) + H(q, \bar{\tau}) \leq \frac{n}{2\bar{\tau}} - \frac{1}{2\bar{\tau}^{3/2}} K.$$

We also have

$$\left[ \bar{\tau} (2\Delta \ell - |\nabla \ell|^2 + H) + \ell - n \right] (q, \bar{\tau}) \leq 0$$

and

$$\left[ \frac{\partial}{\partial \tau} \ell - \Delta \ell + |\nabla \ell|^2 - H + \frac{n}{2\bar{\tau}} \right] (q, \bar{\tau}) \geq 0. \quad (2.14)$$

Here, following [Pe, Section 7], we have put

$$\mathcal{H}(X) := -\frac{\partial H}{\partial \tau} - \frac{H}{\tau} - 2\langle \nabla H, X \rangle + 2h(X, X)$$

$$K := \int_0^{\bar{\tau}} \tau^{3/2} \mathcal{H}(X) d\tau$$

$$\begin{aligned} \mathcal{H}(X, Y) &:= -\langle \nabla_Y \nabla H, Y \rangle + 2\langle R(X, Y)Y, X \rangle + 4\nabla_Y h(X, Y) - 4\nabla_X h(Y, Y) \\ &\quad - 2\frac{\partial h}{\partial \tau}(Y, Y) + 2|h(Y, \cdot)|^2 - \frac{1}{\tau} h(Y, Y). \end{aligned}$$

The point where we have used Assumption 2.4 is the derivation of (2.13) from (2.12)

(cf. [Vol2-I, Lemma 7.42]):

$$\begin{aligned}
& \operatorname{tr} \mathcal{H}(X, \cdot) \\
&= -\Delta H + 2 \operatorname{Ric}(X, X) + 4 \operatorname{div} h(X) - 4 \langle \nabla H, X \rangle - 2 \frac{\partial H}{\partial \tau} - 2|h|^2 - \frac{H}{\tau} \\
&= \mathcal{H}(X) + 2 \left[ \operatorname{Ric}(X, X) - h(X, X) \right] + \left[ -\frac{\partial H}{\partial \tau} - \Delta H - 2|h|^2 \right] \\
&\quad + 2 \left[ 2 \operatorname{div} h(X) - \langle \nabla H, X \rangle \right] \\
&\geq \mathcal{H}(X).
\end{aligned}$$

The quantities corresponding to  $\mathcal{H}(X)$  and  $\operatorname{tr} \mathcal{H}(X, \cdot)$  appear in Theorems 8.1 and 8.2 below as the trace Harnack expressions of Hamilton [Ha3] and Chow–Hamilton [ChHa], respectively.

We now state the main theorem of this chapter (cf. [Pe, Vol2-I, KL, Ye2]).

**Theorem 2.12.** *Let  $(M^n, g(\tau)), \tau \in [0, T)$  be a complete super Ricci flow satisfying Assumption 2.4 with time derivative bounded below. Then for any  $p \in M$  and  $V \in T_p M$ ,*

$$(4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(\gamma_V(\tau), \tau)) \mathcal{L}J_V(\tau) \quad (2.15)$$

is non-increasing in  $\tau$  and

$$\lim_{\tau \rightarrow 0^+} \left[ (4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(\gamma_V(\tau), \tau)) \mathcal{L}J_V(\tau) \right] = \pi^{-n/2} e^{-|V|_{g(0)}^2}.$$

Moreover, (2.15) is constant on  $(0, \bar{\tau}]$  if and only if the shrinking soliton equation:

$$\left[ \frac{1}{2} \frac{\partial g}{\partial \tau} + \operatorname{Hess} \ell_{(p,0)} - \frac{1}{2\tau} g \right] (\gamma_V(\tau), \tau) = 0 \quad (2.16)$$

holds along the  $\mathcal{L}$ -geodesic  $\gamma_V(\tau)$  for  $\tau \in (0, \bar{\tau}]$ .

Hence,  $\tilde{V}_{(p,0)}(\tau)$  is non-increasing in  $\tau$ ,  $\lim_{\tau \rightarrow 0^+} \tilde{V}_{(p,0)}(\tau) = 1$  and  $\tilde{V}_{(p,0)}(\tau) \leq 1$ . Moreover,  $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$  for some  $\bar{\tau} > 0$  if and only if  $(M^n, g(\tau)), \tau \in [0, \bar{\tau}]$  is the Gaussian soliton.

We comment here that Theorem 2.12 is essentially the same as the main theorem of the recent paper [Mü], where Müller also considers the reduced volume in the context more general than that of the Ricci flow; but the rigidity case of Theorem 2.12 is not discussed there.

As for the proof of the theorem, we need to give a proof that  $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$  for some  $\bar{\tau} > 0$  implies that  $(M^n, g(\tau))$  is the Gaussian soliton on  $[0, \bar{\tau}]$ . The proofs of the other statements are minor modifications of those of Lemma 8.16 and Corollary 8.17 of [Vol2-I] for the Ricci flow. The first half of the proof of the rigidity case is identical to the one given in Morgan–Tian [MoTi]. It should be noted that we have no assumption on the curvature of  $g(\tau)$  other than the lower bound of  $\frac{\partial}{\partial \tau} g$  in contrast to [Vol2-I, Corollary 8.17].

*Proof of Theorem 2.12.* Suppose that  $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$ . This implies that  $M$  is simply connected. Otherwise, the reduced volume of the universal covering  $(\bar{M}, \bar{g}(\bar{\tau}))$  of  $(M, g(\bar{\tau}))$

must be greater than 1, which is a contradiction (see the proof of Lemma 3.3 below). Now we are going to show that  $(M^n, g(\tau))$  is flat; the only flat manifolds with finite fundamental group is the Euclidean space.

Fix some small  $\tau_\delta \in (0, \bar{\tau})$ . For any  $\tau \in (\tau_\delta, \bar{\tau}]$ , let  $\varphi_{\tau-\tau_\delta} : M \rightarrow M$  be the map which sends  $q \in M$  to  $\gamma(\tau)$ , where  $\gamma : [0, \bar{\tau}] \rightarrow M$  is the minimal  $\mathcal{L}$ -geodesic passing  $(q, \tau_\delta)$  with  $\gamma(0) = p$ .

Since

$$\frac{\partial}{\partial \tau} \varphi_{\tau-\tau_\delta}(q) = \frac{d\gamma}{d\tau}(\tau) = \nabla \ell_{(p,0)}(\gamma(\tau), \tau),$$

we deduce from (2.16) that

$$\frac{\partial}{\partial \tau} \frac{1}{\tau} (\varphi_{\tau-\tau_\delta})^* g(\tau) = \frac{1}{\tau} (\varphi_{\tau-\tau_\delta})^* \left[ -\frac{1}{\tau} g(\tau) + 2 \text{Hess } \ell_{(p,0)} + \frac{\partial g}{\partial \tau}(\tau) \right] = 0.$$

Hence,

$$\frac{1}{\tau} (\varphi_{\tau-\tau_\delta})^* g(\tau) = \frac{1}{\tau_\delta} g(\tau_\delta) \text{ or equivalently } g(\tau) = \frac{\tau}{\tau_\delta} (\varphi_{\tau-\tau_\delta}^{-1})^* g(\tau_\delta).$$

Since  $g(\tau)$  is smooth around  $(p, 0)$ , we have

$$\begin{aligned} |\text{Rm}|(q, \tau) &= \frac{\tau_\delta}{\tau} |\text{Rm}|(\varphi_{\tau-\tau_\delta}^{-1}(q), \tau_\delta) \\ &\leq \frac{\tau_\delta}{\tau} (|\text{Rm}|(p, 0) + \theta(\tau_\delta)) \rightarrow 0 \quad \text{as } \tau_\delta \rightarrow 0 \end{aligned}$$

where  $\text{Rm}$  denotes the Riemann curvature tensor of  $g(\tau)$  and  $\theta(\tau_\delta)$  is a function such that  $\theta(\tau_\delta) \rightarrow 0$  as  $\tau_\delta \rightarrow 0$ . Consequently,  $(M^n, g(\tau))$  is flat and hence isometric to  $(\mathbb{R}^n, g_E)$  for each  $\tau \in [0, \bar{\tau}]$ . Thus we can write  $g(\tau) = v(\tau)^{-2} g_E$  for some positive non-decreasing function  $v(\tau)$  with  $v(0) = 1$ . It remains to show that  $v(\tau) = 1$  for all  $\tau \in [0, \bar{\tau}]$ .

Introduce a new parameter  $\sigma := 2\sqrt{\tau}$  to write  $g(\sigma) = v(\sigma)^{-2} g_E$  for  $\sigma \in [0, \bar{\sigma}]$ , where  $\bar{\sigma} := 2\sqrt{\bar{\tau}}$ . In this case, it is easy to calculate the reduced distance and volume (cf. [Vol2-I, Lemma 7.67]).

**Sublemma 2.13.**

$$\ell_{(p,0)}(q, \bar{\sigma}) = \frac{v(\bar{\sigma})^2}{\bar{\sigma} \int_0^{\bar{\sigma}} v(\sigma)^2 d\sigma} d_{g(\bar{\sigma})}(p, q)^2 - \frac{n}{2} \log v(\bar{\sigma}) + \frac{n}{2\bar{\sigma}} \int_0^{\bar{\sigma}} \log v(\sigma) d\sigma \quad (2.17)$$

and

$$\tilde{V}_{(p,0)}(\bar{\sigma}) = \exp \left[ \frac{n}{2} \left( \log \frac{\int_0^{\bar{\sigma}} v^2 d\sigma}{\bar{\sigma} v(\bar{\sigma})} - \frac{1}{\bar{\sigma}} \int_0^{\bar{\sigma}} \log v d\sigma \right) \right]. \quad (2.18)$$

*Proof.* We know that  $H = -n \frac{d}{d\tau} \log v$ . Let  $\gamma : [0, \bar{\sigma}] \rightarrow M$  be a curve connecting  $p$  and  $q$ . Then

$$\begin{aligned} \mathcal{L}(\gamma) &= \int_0^{\bar{\sigma}} v(\sigma)^{-2} \left| \frac{d\gamma}{d\sigma} \right|_{g_E}^2 d\sigma - \frac{n}{2} \int_0^{\bar{\sigma}} \sigma \frac{d \log v(\sigma)}{d\sigma} d\sigma \\ &\geq \frac{\left( \int_0^{\bar{\sigma}} \left| \frac{d\gamma}{d\sigma} \right|_{g_E} d\sigma \right)^2}{\int_0^{\bar{\sigma}} v(\sigma)^2 d\sigma} - \frac{n}{2} \bar{\sigma} \log v(\bar{\sigma}) + \frac{n}{2} \int_0^{\bar{\sigma}} \log v d\sigma \\ &\geq \frac{d_E(p, q)^2}{\int_0^{\bar{\sigma}} v^2 d\sigma} - \frac{n}{2} \bar{\sigma} \log v(\bar{\sigma}) + \frac{n}{2} \int_0^{\bar{\sigma}} \log v d\sigma. \end{aligned}$$

in these inequalities, the equality is attained if  $\gamma$  is the minimal geodesic connecting  $p$  and  $q$  parametrized so that  $\left|\frac{d\gamma}{d\sigma}\right|_{g_E} = Cu(\sigma)$  on  $[0, \bar{\sigma}]$  for some constant  $C > 0$ . This proves equation (2.17).

Using (2.17),

$$\begin{aligned}\tilde{V}_{(p,0)}(\bar{\sigma}) &= \int_{\mathbb{R}^n} \left(\frac{v(\bar{\sigma})}{\pi\bar{\sigma}^2}\right)^{n/2} \exp\left[-\frac{v(\bar{\sigma})^2 d_{g(\bar{\sigma})}(p,q)^2}{\bar{\sigma} \int_0^{\bar{\sigma}} v^2 d\sigma} - \frac{n}{2\bar{\sigma}} \int_0^{\bar{\sigma}} \log v d\sigma\right] d\mu_{g(\bar{\sigma})} \\ &= \left(\frac{\int_0^{\bar{\sigma}} v^2 d\sigma}{\bar{\sigma} v(\bar{\sigma})}\right)^{n/2} \exp\left[-\frac{n}{2\bar{\sigma}} \int_0^{\bar{\sigma}} \log v d\sigma\right].\end{aligned}$$

□

Substituting (2.17) into the shrinking soliton equation (2.16) implies that

$$-\frac{v'(\sigma)}{v(\sigma)} + 2\frac{v(\sigma)^2}{\int_0^\sigma v^2 d\sigma} - \frac{2}{\sigma} = 0 \quad \text{for } \sigma \in (0, \bar{\sigma}).$$

From this, we obtain  $\int_0^\sigma v^2 d\sigma = \sigma v(\sigma)$  and  $v(\sigma) = 1$  for all  $\sigma \in [0, \bar{\sigma}]$ .

This completes the proof of Theorem 2.12. □

As an important example, let us look at a static super Ricci flow. Then we obtain an invariant which is called the *static reduced volume* in [Vol2-I]. Its relation to the volume ratio is given by

**Lemma 2.14** ([Vol2-I, Lemma 8.10]). *Let  $(M^n, g)$  be an  $n$ -dimensional complete Riemannian manifold of non-negative Ricci curvature regarded as a static super Ricci flow, i.e.,  $\frac{\partial}{\partial \tau} g = 0 \leq 2 \text{Ric}$ . Then for any  $p \in M$  and  $\tau > 0$ , we have*

$$\tilde{V}_{(p,0)}(\tau) = \int_M (4\pi\tau)^{-n/2} \exp\left(-\frac{d(p,q)^2}{4\tau}\right) d\mu(q) \leq 1, \quad (2.19)$$

and

$$\tilde{\mathcal{V}}(g) := \lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau) = \lim_{r \rightarrow \infty} \frac{\text{Vol } B(p,r)}{\omega_n r^n} =: \nu(g).$$

Furthermore, the equality holds in (2.19) for some  $\tau > 0$  if and only if  $(M^n, g)$  is isometric to  $(\mathbb{R}^n, g_E)$ .

By virtue of Lemma 2.14, we know that Theorem 1.1 generalizes Theorem 1.2.

One can easily compute how the reduced distance and reduced volume change under parabolic rescaling.

**Proposition 2.15** ([Vol2-I, Lemma 8.34]). *If  $g(\tau), \tau \in [0, T]$  is a super Ricci flow, then  $(Qg)(\tau) := Qg(Q^{-1}\tau), \tau \in [0, QT]$  is also a super Ricci flow for any  $Q > 0$ . Under this parabolic rescaling, we have*

$$\ell^{Qg}(q, \tau) = \ell^g(q, Q^{-1}\tau) \quad \text{and} \quad \tilde{V}^{Qg}(\tau) = \tilde{V}^g(Q^{-1}\tau).$$

In particular, the asymptotic reduced volume is invariant under the parabolic rescaling, i.e.,  $\tilde{\mathcal{V}}(g) = \tilde{\mathcal{V}}(Qg)$ , for any ancient super Ricci flow  $g(\tau), \tau \in [0, \infty)$ .

## 2.4 Monotonicity of the reduced volume II

In this section, we present an alternating proof of the monotonicity of the reduced volume along the super Ricci flow. We shall say that a super Ricci flow  $(M, g(\tau)), \tau \in [0, T]$  is  $C^1$ -controlled when we can find a positive function  $K(\tau) > 0$  of  $\tau$  such that

$$\sup_{M \times [0, \tau]} \{ |h| + |\nabla H|^2 \} \leq K(\tau) \quad \text{for each } \tau \in (0, T].$$

According to Shi's gradient estimate (Theorem 3.19), any Ricci flow with bounded curvature is  $C^1$ -controlled in this sense.

We let

$$K(q, \tau) = K_{(p,0)}(q, \tau) := (4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(q, \tau))$$

be the integrand of the reduced volume  $\tilde{V}_{(p,0)}(\tau)$ . For any non-negative function  $\varphi(\cdot, \tau) \geq 0$  on  $M \times [0, T]$ , we define a slight extension of Perelman's reduced volume  $\tilde{V}_{(p,0)}(\tau)$  by

$$\tilde{V}_{(p,0)}^\varphi(\tau) := \int_M K_{(p,0)}(\cdot, \tau) \varphi(\cdot, \tau) d\mu_{g(\tau)}. \quad (2.20)$$

The following proposition follows from inequality (2.14).

**Proposition 2.16.** *We have  $(\frac{\partial}{\partial\tau} - \Delta_{g(\tau)} + H)K \leq 0$  in the distributional sense. More precisely, for each  $\tau > 0$ ,*

$$\int_M \left[ -\langle \nabla \ell, \nabla \xi \rangle + \left( -|\nabla \ell|^2 + H + \frac{1}{K} \frac{\partial K}{\partial \tau} \right) \xi \right] d\mu_{g(\tau)} \leq 0 \quad (2.21)$$

holds for any non-negative Lipschitz function  $\xi \geq 0$  with compact support.

Morally, when two non-negative functions  $u \geq 0$  and  $v \geq 0$  satisfying

$$\left( \frac{\partial}{\partial\tau} - \Delta_{g(\tau)} + H \right) u \leq 0 \quad \text{and} \quad \left( \frac{\partial}{\partial\tau} + \Delta_{g(\tau)} \right) v \leq 0,$$

respectively, we are able to obtain a monotone quantity;

$$\begin{aligned} & \frac{d}{d\tau} \int_M uv d\mu_{g(\tau)} \\ &= \int_M \left[ \left( \frac{\partial}{\partial\tau} - \Delta_{g(\tau)} + H \right) u \right] v + u \left[ \left( \frac{\partial}{\partial\tau} + \Delta_{g(\tau)} \right) v \right] d\mu_{g(\tau)} \\ &\leq 0. \end{aligned} \quad (2.22)$$

(As a matter of course, these inequality needs justification.)

We now state a theorem asserting that the quantity introduced above is monotone.

**Theorem 2.17** (cf. [Pe], [Ye2], [Vol2-I]). *Let  $(M^n, g(\tau)), \tau \in [0, T]$  be a complete  $C^1$ -controlled super Ricci flow satisfying Assumption 2.4. Suppose that  $\varphi \geq 0$  satisfies  $(\frac{\partial}{\partial\tau} + \Delta_{g(\tau)})\varphi \leq 0$  in the distributional sense, namely,*

$$\int_M \left[ \xi \frac{\partial \varphi}{\partial \tau} - \langle \nabla \xi, \nabla \varphi \rangle \right] d\mu_{g(\tau)} \leq 0 \quad (2.23)$$

for any non-negative smooth function  $\xi \geq 0$  with compact support. Then for any  $p \in M$ ,  $\tilde{V}_{(p,0)}^\varphi(\tau)$  is non-increasing in  $\tau \in (0, T)$  and  $\lim_{\tau \rightarrow 0^+} \tilde{V}_{(p,0)}^\varphi(\tau) = \varphi(p, 0)$ .

*Proof.* The proof of the theorem is identical to the original one for

$$\frac{d}{d\tau} \tilde{V}_{(p,0)}(\tau) \leq 0;$$

the monotonicity of the reduced volume along the Ricci flow with bounded curvature (e.g. [Vol2-I, Theorem 8.20]). See the previous section for how the original proof is modified for the super Ricci flow satisfying Assumption 2.4. We leave the details to the interested reader.  $\square$

We close this section with several facts which will be required later in Chapter 7. They are also utilized in the proofs of Theorems 2.17. The proofs can be found in [Ye2] and [Vol2-I] etc.

**Proposition 2.18.** *If  $-K_1g(\tau) \leq \frac{\partial}{\partial\tau}g \leq K_2g(\tau)$  on  $M \times [0, \bar{\tau}]$  for some non-negative constants  $K_1, K_2 \geq 0$ , then for any  $q \in M$ ,*

$$e^{-K_1\bar{\tau}} \frac{d_{g(0)}(p, q)^2}{4\bar{\tau}} - \frac{nK_1\bar{\tau}}{3} \leq \ell_{(p,0)}(q, \bar{\tau}) \leq e^{K_2\bar{\tau}} \frac{d_{g(\bar{\tau})}(p, q)^2}{4\bar{\tau}} + \frac{nK_2\bar{\tau}}{3}.$$

**Proposition 2.19.** *Let  $(M, g(\tau)), \tau \in [0, T]$  be a complete  $C^1$ -controlled super Ricci flow. Then there exists a positive function  $K^*(\tau) > 0$  of  $\tau$  such that*

$$\max \left\{ |\nabla\ell|^2, \left| \frac{\partial\ell}{\partial\tau} \right| \right\} \leq \frac{K^*(\tau)}{\tau} (\ell + 1) \quad \text{a.e. on } M$$

for each  $\tau \in (0, T)$ . In particular,  $\int_M |\nabla\ell|^2 e^{-\ell} d\mu_{g(\tau)}$  and  $\int_M \frac{\partial\ell}{\partial\tau} e^{-\ell} d\mu_{g(\tau)}$  make sense.

# Chapter 3

## Three lemmas

In this chapter, we prove a series of lemmas which we will require in the proof of our main results.

### 3.1 Preliminary estimates

Given a super Ricci flow  $(M^n, g(\tau))$ ,  $\tau \in [0, T)$ , take  $p \in M$  and  $\tau \in (0, T)$ . Let us put

$$\mathcal{L}B_\tau(p, r) := \{ \mathcal{L} \exp_\tau(V) \mid V \in \Omega_{(p,0)}(\tau), |V|_{g(0)} < r \}.$$

This notation comes from the fact that a geodesic ball  $B(p, r)$  in a Riemannian manifold is the image of ball of the same radius in the tangent space under the exponential map.

In this section, we derive a few estimates of which we shall make frequent use in the remaining of this thesis. The first one is the following proposition, which says that the contribution of long tangent vectors to the value of the reduced volume can be ignored.

**Proposition 3.1.** *Let  $u(\cdot, \tau) := (4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(\cdot, \tau))$ .*

(1) *For all  $r > 0$  and  $\tau \in (0, T)$ , we have*

$$\tilde{V}_{(p,0)}(\tau) - \varepsilon(r) \leq \int_{\mathcal{L}B_\tau(p,r)} u(\cdot, \tau) d\mu_{g(\tau)}.$$

(2) *Given  $r > 0$  and  $\tau_0 \in (0, T)$ , we can find a family of subsets  $\mathcal{L}K_{\tau, \tau_0}(p, r)$  of  $M$  for  $\tau \in (0, T)$  satisfying the following properties:*

(a) *For all  $\tau \leq \tau_0$ ,  $\mathcal{L}K_{\tau, \tau_0}(p, r)$  is compact.*

(b) *For all  $\tau \leq \bar{\tau}$ ,  $\mathcal{L}K_{\tau, \tau_0}(p, r)$  contains all of the points  $\gamma(\tau)$  on any minimal  $\mathcal{L}$ -geodesics  $\gamma : [0, \bar{\tau}] \rightarrow M$  connecting  $(p, 0)$  and  $(q, \bar{\tau})$  with  $q \in \mathcal{L}K_{\bar{\tau}, \tau_0}(p, r)$ .*

(c) *For all  $\tau \geq \tau_0$  we have*

$$\tilde{V}_{(p,0)}(\tau) - 2\varepsilon(r) \leq \int_{\mathcal{L}K_{\tau, \tau_0}(p,r)} u(\cdot, \tau) d\mu_{g(\tau)}.$$

Here,  $\varepsilon(r)$  is a function of  $r > 0$  with  $\varepsilon(r) \leq e^{-r^2/2}$  for all  $r$  large enough. Clearly,  $\varepsilon(r)$  decays to 0 exponentially as  $r \rightarrow \infty$ .

*Proof.* (1) We deduce from (2.10) and Theorem 2.12 that

$$\begin{aligned} \int_{M \setminus \mathcal{L}B_\tau(p,r)} u(\cdot, \tau) d\mu_{g(\tau)} &= \int_{\Omega_{(p,0)}(\tau) \setminus B(0,r)} u(\mathcal{L} \exp_\tau(V), \tau) \mathcal{L}J_V(\tau) dx_{g(0)}(V) \\ &\leq \int_{T_p M \setminus B(0,r)} \pi^{-n/2} e^{-|V|_{g(0)}^2} dx_{g(0)}(V) =: \varepsilon(r). \end{aligned}$$

(2) Take a compact set  $K$  of  $T_p M$  so that  $K \subset B(0, r) \cap \Omega_{(p,0)}(\tau_0)$  and the Lebesgue measure of  $B(0, r) \cap \Omega_{(p,0)}(\tau_0) \setminus K$ , induced by  $g(0)$ , is less than  $\pi^{n/2} \varepsilon(r)$ . We show that  $\mathcal{L}K_{\tau, \tau_0}(p, r) := \mathcal{L} \exp_\tau(K \cap \Omega_{(p,0)}(\tau))$  has the desired properties. It is clear that (a) and (b) hold by construction, since  $\Omega_{(p,0)}(\tau_0) \subset \Omega_{(p,0)}(\tau)$  for  $\tau \leq \tau_0$ . Furthermore, by the same argument as in (1), we deduce that

$$\begin{aligned} &\int_{M \setminus \mathcal{L}K_{\tau, \tau_0}(p,r)} u(\cdot, \tau) d\mu_{g(\tau)} \\ &= \int_{M \setminus \mathcal{L}B_\tau(p,r)} u(\cdot, \tau) d\mu_{g(\tau)} + \int_{\mathcal{L}B_\tau(p,r) \setminus \mathcal{L}K_{\tau, \tau_0}(p,r)} u(\cdot, \tau) d\mu_{g(\tau)} \leq 2\varepsilon(r) \end{aligned}$$

for  $\tau \geq \tau_0$ .

Finally, we estimate  $\varepsilon(r)$  for  $r \geq r_0$  by

$$\varepsilon(r) = n\omega_n \pi^{-n/2} \int_r^\infty e^{-r^2} r^{n-1} dr \leq \int_r^\infty e^{-r^2/2} r dr = e^{-r^2/2}.$$

Here  $r_0 \gg 1$  is taken so that  $n\omega_n \pi^{-n/2} e^{-r^2/2} r^{n-2} \leq 1$  for all  $r \geq r_0$ .  $\square$

**Proposition 3.2** (cf. Kleiner–Lott [KL]). *Assume that  $h \geq -C_0 g(\tau)$  and  $|\nabla H| \leq D_0$  on  $\mathcal{K} \times [0, T_0]$  for some compact set  $\mathcal{K} \subset M$  containing a ball  $B_{g(0)}(p, r)$ . Consider the  $\mathcal{L}$ -geodesic  $\gamma_V : [0, \bar{\tau}] \rightarrow M$  with*

$$\gamma_V(0) = p \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} \sqrt{\tau} \frac{d\gamma_V}{d\tau} = V.$$

*Then we can find constants  $C = C(C_0 T_0)$ ,  $D = D(C_0 T_0, D_0 T_0^{3/2})$  and small  $\delta = \delta(C, D, |V|_{g(0)}) > 0$  such that*

$$d_{g(0)}(p, \gamma_V(\tau)) \leq (C|V|_{g(0)} + D)\sqrt{\tau} \tag{3.1}$$

*and hence  $\gamma_V(\tau) \in B_{g(0)}(p, r) \subset \mathcal{K}$  for all  $\tau \in [0, \min\{\delta r^2, T_0\}]$ .*

*Proof.* Let  $\tau' \in [0, T_0]$  be the maximal time such that  $\gamma_V([0, \tau']) \subset \mathcal{K}$ . For  $\tau \leq \tau'$ , we use the  $\mathcal{L}$ -geodesic equation (2.9) to obtain

$$\begin{aligned} \frac{d}{d\tau} |\sqrt{\tau} X|_{g(\tau)}^2 &= |X|_{g(\tau)}^2 + 2h(\sqrt{\tau} X, \sqrt{\tau} X) + 2\tau \langle \nabla_X X, X \rangle \\ &= -2h(\sqrt{\tau} X, \sqrt{\tau} X) + \tau \langle \nabla H, X \rangle \\ &\leq -2C_0 |\sqrt{\tau} X|_{g(\tau)}^2 + D_0 \sqrt{\tau} |\sqrt{\tau} X|_{g(\tau)}. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{d\tau} |\sqrt{\tau} X|_{g(\tau)} &= \frac{1}{2|\sqrt{\tau} X|_{g(\tau)}} \frac{d}{d\tau} |\sqrt{\tau} X|_{g(\tau)}^2 \\ &\leq C_0 |\sqrt{\tau} X|_{g(\tau)} + \frac{1}{2} D_0 \sqrt{T_0}. \end{aligned}$$

From this, we derive that

$$\begin{aligned} |\sqrt{\tau}X|_{g(\tau)} &\leq e^{C_0\tau}|V|_{g(0)} + \frac{D_0\sqrt{T_0}}{2C_0}(e^{C_0\tau} - 1) \\ &\leq C|V|_{g(0)} + D \end{aligned}$$

for  $C = C(C_0T_0)$  and  $D = D(C_0T_0, D_0T_0^{3/2})$ , and hence

$$\begin{aligned} d_{g(0)}(p, \gamma_V(\tau)) &\leq \int_0^\tau |X|_{g(0)} d\tau \leq \int_0^\tau e^{C_0\tau}|X|_{g(\tau)} d\tau \\ &\leq (C|V|_{g(0)} + D) \int_0^\tau \tau^{-1/2} d\tau = (C|V|_{g(0)} + D)\sqrt{\tau}. \end{aligned}$$

As a consequence, we can find  $\delta = \delta(C, D, |V|_{g(0)}) > 0$  such that

$$d_{g(0)}(p, \gamma_V(\tau)) < r \text{ holds for } \tau \in [0, \min\{\delta r^2, T_0\}].$$

This finishes the proof. □

### 3.2 Asymptotic reduced volume

Given an ancient super Ricci flow  $(M, g(\tau)), \tau \in [0, \infty)$ , it is natural to expect that the asymptotic reduced volume

$$\tilde{\mathcal{V}}(g) := \lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}^g(\tau)$$

is well-defined, namely it does not depend on  $p \in M$ , as the asymptotic volume ratio  $\nu(g)$  of a Riemmanian manifold  $(M^n, g)$  of non-negative Ricci curvature is. In this section, we prove the following

**Lemma 3.3.** *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient super Ricci flow satisfying Assumption 2.4 with time derivative bounded from below. Then for any  $(p_k, \tau_k) \in M \times [0, \infty)$  for  $k = 1, 2$  with  $\tau_2 \geq \tau_1$ , we have*

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p_2,0)}^{g_2}(\tau) \geq \lim_{\tau \rightarrow \infty} \tilde{V}_{(p_1,0)}^{g_1}(\tau)$$

where  $g_k(\tau) := g(\tau + \tau_k), \tau \in [0, \infty)$ .

**Corollary 3.4.** *In the setting of Lemma 3.3, we have*

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p_2,0)}^g(\tau) = \lim_{\tau \rightarrow \infty} \tilde{V}_{(p_1,0)}^g(\tau),$$

that is,  $\tilde{\mathcal{V}}(g)$  is well-defined.

The proof of Lemma 3.3 utilizes the following result of Chen [Che] (cf. [Yo2, Proposition A.3]).

**Proposition 3.5** (Chen [Che]). *Any complete ancient super Ricci flow  $(M^n, g(\tau)), \tau \in [0, \infty)$  satisfying*

$$-\frac{\partial}{\partial \tau} H \geq \Delta_{g(\tau)} H + \frac{2}{n} H^2 \tag{3.2}$$

has non-negative trace of time derivative  $2H := \text{tr}_{g(\tau)} \frac{\partial}{\partial \tau} g \geq 0$ . In particular, any complete ancient solution to the Ricci flow has non-negative scalar curvature.

To be precise, what Chen proved in [Che] is the second statement of Proposition 3.5. However, it is easy to see that his argument works well for the super Ricci flow satisfying (3.2) (cf. [Yo2, Proposition A.3]). Recall that the scalar curvature  $R$  of a backward Ricci flow  $(M^n, g(\tau))$  satisfies the following evolution equation:

$$-\frac{\partial}{\partial \tau} R = \Delta_{g(\tau)} R + 2|\text{Ric}|^2. \quad (3.3)$$

Then the Cauchy–Schwartz inequality implies that  $R$  satisfies inequality (3.2).

Note that we have no assumption on the bound of  $\frac{\partial}{\partial \tau} g$  in Proposition 3.5. If the ancient solution to the Ricci flow has bounded curvature, the second statement of Proposition 3.5 is shown by a standard maximum principle argument (e.g. [CLN, Lemma 2.18]). Chen’s proof of Proposition 3.5 makes effective use of the non-linear term  $\frac{2}{n}H^2$  in (3.2).

Chen [Che] also proves that *any 3-dimensional complete ancient solution to the Ricci flow has non-negative curvature operator*. In dimension 3, curvature operator being non-negative is equivalent to the non-negativity of the sectional curvature. The main result of Chen’s paper [Che] is the uniqueness of the complete Ricci flow with curvature not necessarily bounded and the Euclidean space as the initial metric in dimensions 2 and 3.

Subsequently, modifying Chen’s argument, Zhang [Zh] proved that *the scalar curvature  $R$  of any complete gradient shrinking Ricci soliton  $(M^n, g, f)$  is non-negative*. Then the inequality

$$\lambda|\nabla f|^2 \leq \lambda(R + |\nabla f|^2) = f$$

implies that  $|\nabla\sqrt{f}|^2 \leq (4\lambda)^{-1}$  and hence

$$\lambda \max\{R, |\nabla f|^2\}(x) \leq f(x) \leq \frac{1}{4\lambda} \left(d(x, p) + \sqrt{f(y)}\right)^2 \quad (3.4)$$

for any  $x$  and  $y \in M$  (cf. Theorem 5.4.(2)). His main result is

**Theorem 3.6** (Zhang [Zh]). *Let  $(M^n, g, f)$  be a gradient Ricci soliton which is not necessarily shrinking. Then the gradient vector field  $\nabla f$  is complete provided the metric  $g$  is complete.*

*Proof of Lemma 3.3.* Put  $\tau_\Delta := \tau_2 - \tau_1 \geq 0$  to notice that  $g_2(\tau - \tau_\Delta) = g_1(\tau)$ . We first verify

**Sublemma 3.7.** *For any  $(p, \tau_p), (q, \bar{\tau}) \in M \times [0, \infty)$  with  $\bar{\tau} > \tau_p \geq \tau_\Delta$ ,*

$$\frac{1}{2\sqrt{\bar{\tau} - \tau_\Delta}} L_{(p, \tau_p - \tau_\Delta)}^{g_2}(q, \bar{\tau} - \tau_\Delta) \leq \frac{1}{2\sqrt{\bar{\tau}}} L_{(p, \tau_p)}^{g_1}(q, \bar{\tau})$$

and

$$\frac{1}{2\sqrt{\bar{\tau} - \tau_\Delta}} L_{(p, \tau_p - \tau_\Delta)}^{g_2}(q, \bar{\tau} - \tau_\Delta) \geq \alpha(\tau_\Delta; \tau_p, \bar{\tau}) \frac{1}{2\sqrt{\bar{\tau}}} L_{(p, \tau_p)}^{g_1}(q, \bar{\tau}),$$

where  $\alpha(\tau_\Delta; \tau_p, \bar{\tau}) := \sqrt{\frac{\tau_p - \tau_\Delta}{\tau_p} \frac{\bar{\tau}}{\bar{\tau} - \tau_\Delta}} \geq \sqrt{1 - \frac{\tau_\Delta}{\tau_p}}$ .

*Proof.* We use the fact that  $H(\cdot, \tau) \geq 0$  for ancient super Ricci flows (Proposition 3.5) and the inequality

$$\frac{1}{2\sqrt{\bar{\tau}}} \sqrt{\bar{\tau}} \geq \frac{1}{2\sqrt{\bar{\tau} - \tau_\Delta}} \sqrt{\bar{\tau} - \tau_\Delta} \quad \text{for all } \tau_p \leq \tau \leq \bar{\tau}$$

to obtain

$$\begin{aligned}
\frac{1}{2\sqrt{\bar{\tau}}} L_{(p, \tau_p)}^{g_1}(q, \bar{\tau}) &= \frac{1}{2\sqrt{\bar{\tau}}} \inf_{\gamma} \left\{ \int_{\tau_p}^{\bar{\tau}} \sqrt{\tau} \left( |\gamma'|_{g_1(\tau)}^2 + H_{g_1(\tau)}(\gamma(\tau)) \right) d\tau \right\} \\
&\geq \frac{1}{2\sqrt{\bar{\tau} - \tau_{\Delta}}} \inf_{\gamma} \left\{ \int_{\tau_p}^{\bar{\tau}} \sqrt{\tau - \tau_{\Delta}} \left( |\gamma'|_{g_1(\tau)}^2 + H_{g_1(\tau)}(\gamma(\tau)) \right) d\tau \right\} \\
&= \frac{1}{2\sqrt{\bar{\tau} - \tau_{\Delta}}} L_{(p, \tau_p - \tau_{\Delta})}^{g_2}(q, \bar{\tau} - \tau_{\Delta}).
\end{aligned}$$

Here inf runs over all curves  $\gamma : [\tau_p, \bar{\tau}] \rightarrow M$  with  $\gamma(\tau_p) = p$  and  $\gamma(\bar{\tau}) = q$ .

To see the second inequality, we use instead

$$\alpha(\tau_{\Delta}; \tau_p, \bar{\tau}) \frac{1}{2\sqrt{\bar{\tau}}} \sqrt{\tau} \leq \frac{1}{2\sqrt{\bar{\tau} - \tau_{\Delta}}} \sqrt{\tau - \tau_{\Delta}} \text{ for all } \tau_p \leq \tau \leq \bar{\tau}.$$

□

We return to the proof of Lemma 3.3. Fix  $r > 0$  and  $\bar{\tau} \gg 1$ . Take  $q \in \mathcal{K}(\bar{\tau}) := \mathcal{L}^{g_1} K_{\bar{\tau}, 2\tau_{\Delta}}(p_1, r)$  and the point  $p_{\Delta} = \gamma(2\tau_{\Delta}) \in M$  on the minimal  $\mathcal{L}^{g_1}$ -geodesic  $\gamma : [0, \bar{\tau}] \rightarrow M$  from  $(p_1, 0)$  to  $(q, \bar{\tau})$  such that

$$\begin{aligned}
L_{(p_1, 0)}^{g_1}(q, \bar{\tau}) &= L_{(p_{\Delta}, 2\tau_{\Delta})}^{g_1}(q, \bar{\tau}) + L_{(p_1, 0)}^{g_1}(p_{\Delta}, 2\tau_{\Delta}) \\
&\geq L_{(p_{\Delta}, 2\tau_{\Delta})}^{g_1}(q, \bar{\tau}).
\end{aligned} \tag{3.5}$$

The inequality in (3.5) is due to the non-negativity of  $H$  (Proposition 3.5). Recall that  $\mathcal{K} := \mathcal{L}^{g_1} K_{2\tau_{\Delta}, 2\tau_{\Delta}}(p_1, r)$  is compact and  $p_{\Delta} \in \mathcal{K}$  by construction (Proposition 3.1). It follows from the combination of the triangle inequality for  $\mathcal{L}$ -distance, Sublemma 3.7 and (3.5) that

$$\begin{aligned}
\ell_{(p_2, 0)}^{g_2}(q, \bar{\tau} - \tau_{\Delta}) &\leq \frac{1}{2\sqrt{\bar{\tau} - \tau_{\Delta}}} \left( L_{(p_{\Delta}, \tau_{\Delta})}^{g_2}(q, \bar{\tau} - \tau_{\Delta}) + L_{(p_2, 0)}^{g_2}(p_{\Delta}, \tau_{\Delta}) \right) \\
&\leq \frac{1}{2\sqrt{\bar{\tau}}} L_{(p_{\Delta}, 2\tau_{\Delta})}^{g_1}(q, \bar{\tau}) + \frac{1}{2\sqrt{\bar{\tau} - \tau_{\Delta}}} \max_{\mathcal{K}} L_{(p_2, 0)}^{g_2}(\cdot, \tau_{\Delta}) \\
&\leq \ell_{(p_1, 0)}^{g_1}(q, \bar{\tau}) + C(r) \bar{\tau}^{-1/2}.
\end{aligned}$$

Thus, as  $\bar{\tau} > 0$  is large enough,

$$\begin{aligned}
\lim_{\tau \rightarrow \infty} \tilde{V}_{(p_2, 0)}^{g_2}(\tau) &\geq \tilde{V}_{(p_2, 0)}^{g_2}(\bar{\tau} - \tau_{\Delta}) - \varepsilon(r) \\
&\geq \int_{\mathcal{K}(\bar{\tau})} (4\pi\bar{\tau})^{-n/2} \exp\left(-\ell_{(p_2, 0)}^{g_2}(\cdot, \bar{\tau} - \tau_{\Delta})\right) d\mu_{g_2(\bar{\tau} - \tau_{\Delta})} - \varepsilon(r) \\
&\geq e^{-C(r)\bar{\tau}^{-1/2}} \int_{\mathcal{K}(\bar{\tau})} (4\pi\bar{\tau})^{-n/2} \exp\left(-\ell_{(p_1, 0)}^{g_1}(\cdot, \bar{\tau})\right) d\mu_{g_1(\bar{\tau})} - \varepsilon(r) \\
&\geq e^{-C(r)\bar{\tau}^{-1/2}} \tilde{V}_{(p_1, 0)}^{g_1}(\bar{\tau}) - 3\varepsilon(r) \\
&\geq e^{-C(r)\bar{\tau}^{-1/2}} \lim_{\tau \rightarrow \infty} \tilde{V}_{(p_1, 0)}^{g_1}(\tau) - 3\varepsilon(r).
\end{aligned}$$

We have used Proposition 3.1 to derive the fourth inequality. Since  $\bar{\tau} > 0$  and  $r > 0$  are arbitrary, the proof of Lemma 3.3 is now complete. □

### 3.3 Finiteness of fundamental group

Now we are ready to establish Theorem 1.3. Let us restate it here.

**Theorem 3.8.** *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient solution to the Ricci flow on  $M$  with Ricci curvature bounded below. If  $\tilde{\mathcal{V}}(g) > 0$ , then the fundamental group  $\pi_1(M)$  of  $M$  is finite.*

As mentioned in the introduction, we intend to prove this theorem for super Ricci flows.

**Lemma 3.9.** *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient super Ricci flow satisfying Assumption 2.4 with time derivative bounded below. We lift them to the universal covering  $\bar{M}$  of  $M$  to obtain the lifted flow  $(\bar{M}, \bar{g}(\tau))$ . Take  $p \in M$  and  $\bar{p} \in \pi^{-1}(p)$ , where  $\pi : \bar{M} \rightarrow M$  is the projection. Suppose that  $\tilde{\mathcal{V}}(g) := \lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}^g(\tau) > 0$ . Then we have*

$$|\pi_1(M)| = \tilde{\mathcal{V}}(\bar{g})\tilde{\mathcal{V}}(g)^{-1} < +\infty.$$

Before we begin the proof of Lemma 3.9, let us state the following immediate corollary, which follows from Lemma 3.9 combined with Lemma 2.14.

**Corollary 3.10** (Anderson [An2], Li [Li]). *Let  $(M, g)$  be a complete Riemannian manifold with non-negative Ricci curvature and  $(\bar{M}, \bar{g})$  be the universal covering of  $(M, g)$ . If  $(M, g)$  has Euclidean volume growth, i.e.,  $\nu(g) > 0$ , then we have*

$$|\pi_1(M)| = \nu(\bar{g})\nu(g)^{-1} < +\infty.$$

Here,  $\nu(g)$  denotes the asymptotic volume ratio as before.

*Proof of Lemma 3.9.* The proof is a modification of that of [An2, Theorem 1.1]. Fix large  $\bar{\tau} \in (0, \infty)$  and define

$$F := \bigcap_{\alpha \in \pi_1(M) \setminus \{e\}} \left\{ \bar{q} \in \bar{M} \mid L_{(\bar{p},0)}^{\bar{g}}(\bar{q}, \bar{\tau}) < L_{(\alpha\bar{p},0)}^{\bar{g}}(\bar{q}, \bar{\tau}) \right\}.$$

Then  $F$  is a fundamental domain of  $\pi : \bar{M} \rightarrow M$ , namely

$$F \cap \alpha F = \emptyset \quad \text{for } \alpha \in \pi_1(M) \setminus \{e\} \quad \text{and} \quad \bigcup_{\alpha \in \pi_1(M)} \alpha F = \bar{M}.$$

We claim that  $\pi : \bar{F} \rightarrow M$  is locally isometric and surjective. To see this, pick  $q \in M$  and connect  $(p, 0)$  and  $(q, \bar{\tau})$  by a minimal  $\mathcal{L}^g$ -geodesic  $\gamma : [0, \bar{\tau}] \rightarrow M$ . Then the lift  $\bar{\gamma}$  of  $\gamma$  with  $\bar{\gamma}(0) = \bar{p}$  is a minimal  $\mathcal{L}^{\bar{g}}$ -geodesic in  $\bar{M}$ . Let  $\bar{q} := \bar{\gamma}(\bar{\tau})$ . Then we have  $\bar{q} \in \bar{F}$  and  $\pi(\bar{q}) = q$ .

Furthermore,  $\bar{F} \setminus F$  has measure 0, since  $\pi(\bar{F} \setminus F)$  consists of the points in  $M$  such that minimal  $\mathcal{L}^g$ -geodesic from  $(p, 0)$  is not unique. The set of such points has measure 0 [Vol2-I, Lemma 7.99].

Fix any finite subset  $\Gamma \subset \pi_1(M)$  and set  $D_\Gamma := \max\{d_{\bar{g}(0)}(\bar{p}, \alpha\bar{p}) \mid \alpha \in \Gamma\}$ . Take  $C_0 < \infty$  such that

$$|h| \leq C_0 \quad \text{on } B_{\bar{g}(0)}(\bar{p}, D_\Gamma + 1) \times [0, 1]$$

and

$$|\nabla H| \leq C_0 \quad \text{on } B_{g(0)}(p, 1) \times [0, 1].$$

Fix a positive  $r > 0$ . Due Proposition 3.2, we can find  $\delta = \delta(C_0, r) > 0$  such that  $d_{\bar{g}(0)}(\bar{\gamma}_V(\tau), \alpha\bar{p}) \leq 1$  for any  $\mathcal{L}^{\bar{g}}$ -geodesic  $\bar{\gamma}_V$  starting from  $\alpha\bar{p}$  with  $|V|_{\bar{g}(0)} < r$  and  $\tau \in [0, \delta]$ .

For any  $\alpha \in \Gamma$  and  $\bar{q} \in \mathcal{L}B_{\bar{\tau}}(\alpha\bar{p}, r) \cap \alpha\bar{F}$ , let  $\bar{\gamma}$  be the minimal  $\mathcal{L}^{\bar{g}}$ -geodesic from  $(\alpha\bar{p}, 0)$  to  $(\bar{q}, \bar{\tau})$  in  $\bar{M}$  and connect  $\bar{p}$  and  $\bar{\gamma}(\delta)$  by a minimal  $\bar{g}(0)$ -geodesic  $\xi_{\bar{p}, \bar{\gamma}(\delta)} : [0, \delta] \rightarrow \bar{M}$ . Define a curve  $\hat{\gamma} : [0, \bar{\tau}] \rightarrow \bar{M}$  by

$$\hat{\gamma}(\tau) := \begin{cases} \xi_{\bar{p}, \bar{\gamma}(\delta)}(\tau) & \text{on } [0, \delta] \\ \bar{\gamma}(\tau) & \text{on } [\delta, \bar{\tau}] \end{cases}$$

Then, letting  $q := \pi(\bar{q})$ ,

$$\begin{aligned} \ell_{(\bar{p}, 0)}^{\bar{g}}(\bar{q}, \bar{\tau}) &\leq \frac{1}{2\sqrt{\bar{\tau}}} \mathcal{L}^{\bar{g}}(\hat{\gamma}) \\ &= \frac{1}{2\sqrt{\bar{\tau}}} \left( \mathcal{L}^{\bar{g}}(\bar{\gamma}) - \mathcal{L}^{\bar{g}}(\bar{\gamma}|_{[0, \delta]}) + \mathcal{L}^{\bar{g}}(\xi_{\bar{p}, \bar{\gamma}(\delta)}) \right) \\ &\leq \ell_{(\alpha\bar{p}, 0)}^{\bar{g}}(\bar{q}, \bar{\tau}) + \frac{1}{3\sqrt{\bar{\tau}}} \delta^{3/2} \left( e^{2C_0\delta} \left( \frac{D_\Gamma + 1}{\delta} \right)^2 + 2nC_0 \right) \\ &= \ell_{(p, 0)}^g(q, \bar{\tau}) + C(\delta, \Gamma)\bar{\tau}^{-1/2} \end{aligned}$$

where we have used that

$$\ell_{(\alpha\bar{p}, 0)}^{\bar{g}}(\bar{q}, \bar{\tau}) = \ell_{(p, 0)}^g(q, \bar{\tau}) \quad \text{for any } \bar{q} \in \mathcal{L}B_{\bar{\tau}}(\alpha\bar{p}, r) \cap \alpha\bar{F}.$$

We apply Proposition 3.1 to obtain

$$\begin{aligned} \tilde{V}_{(\bar{p}, 0)}^{\bar{g}}(\bar{\tau}) &\geq \sum_{\alpha \in \Gamma} \int_{\mathcal{L}B_{\bar{\tau}}(\alpha\bar{p}, r) \cap \alpha\bar{F}} (4\pi\bar{\tau})^{-n/2} \exp\left(-\ell_{(\bar{p}, 0)}^{\bar{g}}(\cdot, \bar{\tau})\right) d\mu_{\bar{g}(\bar{\tau})} \\ &\geq |\Gamma| \int_{\mathcal{L}B_{\bar{\tau}}(p, r)} (4\pi\bar{\tau})^{-n/2} \exp\left(-\ell_{(p, 0)}^g(\cdot, \bar{\tau}) - C(\delta, \Gamma)\bar{\tau}^{-1/2}\right) d\mu_{g(\bar{\tau})} \\ &\geq e^{-C(\delta, \Gamma)\bar{\tau}^{-1/2}} |\Gamma| \left( \tilde{V}_{(p, 0)}^g(\bar{\tau}) - \varepsilon(r) \right) \end{aligned}$$

and taking  $\bar{\tau} \rightarrow \infty$  and  $r \rightarrow \infty$  yields that

$$\tilde{\mathcal{V}}(\bar{g}) \geq |\Gamma| \tilde{\mathcal{V}}(g) \quad \text{for any finite subset } \Gamma \subset \pi_1(M). \quad (3.6)$$

Thus,  $\pi_1(M)$  is finite and (3.6) holds for  $\Gamma = \pi_1(M)$ .

On the other hand, since

$$\ell_{(\bar{p}, 0)}^{\bar{g}}(\bar{q}, \tau) \geq \ell_{(p, 0)}^g(\pi(\bar{q}), \tau) \quad \text{for any } (\bar{q}, \tau) \in \bar{M} \times (0, \infty)$$

we have

$$\tilde{V}_{(\bar{p}, 0)}^{\bar{g}}(\tau) \leq |\pi_1(M)| \tilde{V}_{(p, 0)}^g(\tau)$$

and hence  $\tilde{\mathcal{V}}(\bar{g}) \leq |\pi_1(M)| \tilde{\mathcal{V}}(g)$ . This finishes the proof of the lemma.  $\square$

Let us give a corollary of Theorem 3.8, which was pointed out by Professor Lei Ni.

**Corollary 3.11** (Finiteness of fundamental groups of singularity models). *Let  $(M^n, g(t))$ ,  $t \in [0, T)$  be a complete Ricci flow with bounded curvature and positive injectivity radius at  $t = 0$  which develops singularity at finite time  $t = T < \infty$ . Then any singularity model of  $(M^n, g(t))$  has finite fundamental group.*

The singularity model is a Ricci flow which arises as a limit of dilations of a Ricci flow  $(M^n, g(t))$  around a singular point (see [CLN, Chapter 8] for the precise definition). We can take such a blow-up limit in the corollary by virtue of Perelman's no local collapsing theorem (Theorem 2.1) and Hamilton's compactness theorem (Theorem 3.20). The corollary immediately follows from the fact that such a singularity model is an ancient solution with positive asymptotic reduced volume. This is verified by combining Lemma 3.22 below with the following fact.

**Lemma 3.12** ([Vol2-I, Lemma 8.22]). *Let  $(M^n, \tilde{g}(t))$ ,  $t \in [0, T)$  be a complete Ricci flow with Ricci curvature bounded below and  $T < \infty$  such that*

$$\inf \{ \text{Vol}_{\tilde{g}(0)} B_{\tilde{g}(0)}(p, r_1) \mid p \in M \} \geq v_1 > 0$$

and

$$\sup \{ |\text{Ric}|(p, t) \mid (p, t) \in M \times [0, T/2] \} \leq C_0 < \infty.$$

For any  $T_0 \in [T/2, T)$ , let  $g(\tau) := g(T_0 - \tau)$ ,  $\tau \in [0, T_0]$  be the backward Ricci flow. Then there exists a constant  $C = C(r_1, v_1, n, T, C_0) > 0$  such that

$$\tilde{V}_{(p,0)}(\tau) \geq C \quad \text{for any } p \in M \text{ and } \tau \in [0, T_0].$$

We will be able to use this corollary in order to understand the singularities of the Ricci flow further. For example, we can prove the following: for any ancient solution  $(N^{n-1}, g_N(t))$ ,  $t \in (-\infty, \alpha)$ , the canonical ancient solution on  $S^1 \times N^{n-1}$  cannot occur as a blow-up limit of the Ricci flow as in Corollary 3.11. In the case where  $N$  is a sphere, this result was conjectured by Hamilton [Ha5, Section 26] and proved by Ilmanen–Knopf [IlKn] with a different method.

We should mention that Naber [Na] proved that any blow-up limit of Type I singularity of the Ricci flow is a gradient shrinking Ricci soliton. (See also Zhang [Zh2] for a relevant result.) Although we already know that *any gradient shrinking Ricci soliton has finite fundamental group* (e.g. Wylie [Wy]), Corollary 3.11 is applicable even to a blow-up limit of Type IIa singularity as well.

Here we briefly recall the classification of types of finite time singularities of the Ricci flow.

**Definition 3.13.** Let  $(M^n, g(t))$ ,  $t \in [0, T)$  be a Ricci flow. If  $T < \infty$  is the finite singular time,

$$\begin{aligned} (M^n, g(t)) \text{ is of Type I} &\iff \sup_{M \times [0, T)} |\text{Rm}|(T - t) < \infty \\ (M^n, g(t)) \text{ is of Type IIa} &\iff \sup_{M \times [0, T)} |\text{Rm}|(T - t) = \infty. \end{aligned}$$

Typical examples of the Ricci flow of Type I singularity are provided by shrinking Ricci solitons (cf. (5.3)). The existence of Type IIa singularity of the Ricci flow was confirmed by Gu–Zhu [GuZh].

The result of Naber [Na] mentioned above can be regarded as a Ricci flow analogue of the following theorem of Cheeger–Colding [ChCo2, Theorem 5.2].

**Theorem 3.14** (Tangent cone is a metric cone [ChCo2], also Cheeger [Chee]). *Let a complete metric space  $(X, x)$  be a pointed Gromov–Hausdorff limit of a non-collapsing sequence  $\{(M_k^n, p_k)\}_{k \in \mathbb{Z}^+}$  of complete Riemannian manifolds whose Ricci curvature is uniformly bounded below. Then for any  $p \in X$ , any tangent cone  $C_p$  at  $p$  is a metric cone.*

In the statement above, a *tangent cone*  $C_p$  at a point  $p$  of a metric space  $X$  is a pointed Gromov–Hausdorff limit of the sequence  $\{(r_i X, p)\}$  for some sequence  $r_i \rightarrow \infty$  as  $i \rightarrow \infty$ . See Burago–Burago–Ivanov [BBI] for the definition of a *metric cone*.

We close this section giving another corollary of Theorem 3.8.

**Corollary 3.15** (Finiteness of fundamental groups of  $\kappa$ -solutions). *Any ancient  $\kappa$ -solution to the Ricci flow has finite fundamental group.*

We have to give a definition of ancient  $\kappa$ -solutions (cf. the statement of Theorem 2.1).

**Definition 3.16.** Let  $(M^n, g(\tau)), \tau \in [0, T)$  be a backward Ricci flow and  $\kappa > 0$  be a positive constant. We say that the backward Ricci flow is (weakly)  $\kappa$ -noncollapsing at  $(p_*, \tau_*) \in M \times [0, T)$  on scale  $r > 0$  if

$$|\text{Rm}| \leq r^{-2} \quad \text{on } B_{g(\tau_*)}(p_*, r) \times [\tau_*, \tau_* + r^2]$$

implies that

$$\text{Vol}_{g(\tau_*)} B_{g(\tau_*)}(p, r) \geq \kappa r^n.$$

An ancient solution  $(M^n, g(\tau)), \tau \in [0, \infty)$  is an *ancient  $\kappa$ -solution*, by definition, if it satisfies that

- (1)  $(M^n, g(\tau))$  is complete and has uniformly bounded non-negative curvature operator, and
- (2)  $(M^n, g(\tau))$  is  $\kappa$ -noncollapsing at all  $(p_*, \tau_*) \in M \times [0, \infty)$  and on all scale  $r > 0$ .

According to Hamilton’s Harnack inequality (Theorem 8.1), the scalar curvature  $R$  of any ancient solution to the Ricci flow with bounded non-negative curvature operator satisfies that  $\frac{\partial}{\partial \tau} R(\cdot, \tau) \leq 0$ . If the curvature operator is non-negative, the scalar curvature controls the curvature operator. Hence it turns out that any ancient solution to the Ricci flow with non-negative bounded curvature operator has uniformly bounded curvature operator.

The importance of ancient  $\kappa$ -solutions comes from the fact that any blow-up limit of a singularity of the Ricci flow of dimension 3 is an ancient  $\kappa$ -solution. This is a consequence of the Hamilton–Ivey estimate (e.g. [Ha5, Theorem 24.4]) and Perelman’s no local collapsing theorem (Theorem 2.1).

The proof of Corollary 3.15 is immediate since any ancient  $\kappa$ -solution has positive asymptotic reduced volume (e.g. [Vol2-I, Lemma 8.38]). Meanwhile, Perelman [Pe, Proposition 11.4] has shown the following.

**Proposition 3.17** (Perelman [Pe]). *Any ancient  $\kappa$ -solution has zero asymptotic volume ratio.*

This is why Corollary 3.15 does not follow from Corollary 3.10, but from Theorem 3.8.

*Remark 3.18.* Carrillo–Ni [CaNi] proved that any complete gradient shrinking Ricci soliton of non-negative Ricci curvature has zero asymptotic volume ratio. This generalizes Perelman’s result mentioned above, since any ancient  $\kappa$ -solution has asymptotic soliton.

On the other hand, any complete gradient expanding Ricci soliton with non-negative Ricci curvature has Euclidean volume growth (Hamilton [CLN, Proposition 9.46], Carrillo–Ni [CaNi]).

### 3.4 Reduced volume under Cheeger–Gromov convergence

Although we have considered the super Ricci flow so far, Theorem 4.1 is not true for them. In this section, we concentrate on the Ricci flow. To begin with, let us recall Shi’s gradient estimate. Shi’s derivative estimate was also employed in the proof of the compactness theorem for the Ricci flow (Theorem 3.20), which we will use later.

**Theorem 3.19** (Shi’s local gradient estimate, e.g. [Ha5, Theorem 13.1], also [CLN]). *There exists a constant  $C(n) < \infty$  satisfying the following: let  $(M^n, g(\tau)), \tau \in [0, T_0]$  be a complete backward Ricci flow on an  $n$ -manifold  $M$ . Assume that the ball  $B_{g(T_0)}(p, r)$  is contained in  $\mathcal{K}$  and  $|\text{Rm}| \leq C_0$  on  $\mathcal{K} \times [0, T_0]$  for some compact set  $\mathcal{K} \subset M$ . Then for  $\tau \in [0, T_0]$ ,*

$$|\nabla \text{Rm}|^2(p, \tau) \leq C(n)C_0^2 \left( \frac{1}{r^2} + \frac{1}{T_0 - \tau} + \frac{1}{C_0} \right). \quad (3.7)$$

Recall that we say that a sequence of pointed backward Ricci flows

$$\{(M_k^n, g_k(\tau), p_k)\}_{k \in \mathbb{Z}^+, \tau \in [0, T]}$$

converges to a backward Ricci flow  $(M_\infty^n, g_\infty(\tau), p_\infty), \tau \in [0, T]$  in the  $C^\infty$  Cheeger–Gromov sense if there exist open sets  $U_k$  of  $M_\infty$  with  $p_\infty \in U_k$  and  $\cup_{k \in \mathbb{Z}^+} U_k = M_\infty$  and diffeomorphisms  $\Phi_k : U_k \rightarrow V_k := \Phi_k(U_k) \subset M_k$  with  $\Phi_k(p_\infty) = p_k$  so that  $\{(U_k, \Phi_k^* g_k(\tau))\}_{k \in \mathbb{Z}^+}$  converges to  $(M_\infty^n, g_\infty(\tau))$  in the  $C^\infty$  topology on each compact set of  $M_\infty^n \times [0, T]$ .

**Theorem 3.20** (Compactness theorem for the Ricci flow, Hamilton [Ha6], also [Vol2-I]). *Let  $\{(M_k^n, g_k(t), p_k)\}_{k \in \mathbb{Z}^+, t \in (0, T]}$  be a pointed sequence of complete Ricci flows such that*

- (1)  $(M_k^n, g_k(t))$  has bounded curvature and bounds are independent of  $k \in \mathbb{Z}^+$ , and
- (2) the injectivity radii  $\text{inj}_{g_k(T)}(p_k)$  at the base points  $(p_k, T) \in M_k \times \{T\}$  are uniformly bounded from below.

*Then there exists a subsequence of  $\{(M_k^n, g_k(t), p_k)\}_{k \in \mathbb{Z}^+}$  converging to the limit Ricci flow  $(M_\infty^n, g_\infty(t), p_\infty), t \in (0, T]$  in the  $C^\infty$  Cheeger–Gromov sense.*

By carefully investigating the proof of [Vol2-I, Lemma 7.66], where curvature is assumed to be bounded on the whole of  $M_k \times [0, T]$ , one can show the following lemma without modification (cf. [Vol2-I, Lemma 7.66]).

**Lemma 3.21.** *Let  $\{(M_k^n, g_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}, \tau \in [0, T)$  be a converging sequence of pointed backward Ricci flows in the sense of  $C^\infty$  Cheeger–Gromov and  $(M_\infty^n, g_\infty(\tau), p_\infty), \tau \in [0, T)$  be the limit. Then we have*

$$\limsup_{k \rightarrow \infty} \ell_{(p_k, 0)}^{g_k}(\Phi_k(q), \tau) \leq \ell_{(p_\infty, 0)}^{g_\infty}(q, \tau) \quad (3.8)$$

for  $\tau \in (0, T)$ . The equality is achieved in (3.8), with  $\limsup$  replaced by  $\lim$ , provided  $(\Phi_k(q), \tau)$  can be joined to  $(p_k, 0)$  by a minimal  $\mathcal{L}^{g_k}$ -geodesic within the image  $\Phi_k(\mathcal{K}) \subset M_k$  of some compact set  $\mathcal{K} \subset M_\infty$  for all large  $k \in \mathbb{Z}^+$ .

Now we verify the convergence of reduced volumes.

**Lemma 3.22.** *Let  $\{(M_k^n, g_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}, \tau \in [0, T)$  be a sequence of pointed backward Ricci flows converging to  $(M_\infty^n, g_\infty(\tau), p_\infty)$ . Assume that*

$$|\text{Rm}| \leq C_0 \text{ on } V_k \times [0, T) \quad \text{and} \quad \bigcup_{k \in \mathbb{Z}^+} U_k = M_\infty.$$

Then for any  $\tau \in (0, T)$ ,

$$\lim_{k \rightarrow \infty} \tilde{V}_{(p_k, 0)}^{g_k}(\tau) = \tilde{V}_{(p_\infty, 0)}^{g_\infty}(\tau). \quad (3.9)$$

*Proof.* Let us put  $u_\star(q, \tau) := (4\pi\tau)^{-n/2} \exp\left(-\ell_{(p_\star, 0)}^{g_\star}(q, \tau)\right)$  for  $\star \in \mathbb{Z}^+ \cup \{\infty\}$ , and fix  $\bar{\tau} \in (0, T)$  and  $T_0 \in (\bar{\tau}, T)$ . Set  $V_\infty := M_\infty$ .

We invoke Shi's gradient estimate (Theorem 3.19):

$$|\nabla R|^2(\cdot, \tau) \leq \frac{C(n)C_0^2}{\min\{C_0, T_0 - \tau\}} \text{ on } B_{[0, T_0]}(V_\star, -\sqrt{C_0}) \text{ for } \tau \in [0, T_0]$$

where

$$B_{[0, T_0]}(V_\star, -\sqrt{C_0}) := \left\{ x \in V_\star \mid B_{g_\star(\tau)}(x, \sqrt{C_0}) \subset V_\star \text{ for all } \tau \in [0, T_0] \right\}.$$

Fix  $r > 0$ . Then by Proposition 3.2, we can find  $C(r) < \infty$  such that any  $\mathcal{L}^{g_\star}$ -geodesic  $\gamma_V([0, \bar{\tau}])$  in  $M_\star$  with  $\gamma_V(0) = p_\star$  and  $|V|_{g_\star(0)} \leq r$  can not escape from  $B_0(p_\star, C(r))$  when  $\star$  is sufficiently large or  $= \infty$ .

Define  $\hat{u}_k(\cdot, \tau) : M_k \rightarrow [0, \infty)$  by

$$\hat{u}_k(q, \tau) := \begin{cases} u_k(q, \tau) & \text{if } q \in \mathcal{L}B_\tau(p_k, r) \\ 0 & \text{otherwise} \end{cases}$$

Then each  $\hat{u}_k(\cdot, \bar{\tau})$  has a compact support contained in  $B_{g_k(0)}(p_k, C(r))$  and it follows from Lemma 3.21 that

$$\limsup_{k \rightarrow \infty} \hat{u}_k(\Phi_k(q), \bar{\tau}) \in \{u_\infty(q, \bar{\tau}), 0\}. \quad (3.10)$$

Therefore, noting that  $\hat{u}_k(\cdot, \bar{\tau}) \leq (4\pi\bar{\tau})^{-n/2} \exp\left(\frac{1}{3}n(n-1)C_0\bar{\tau}\right)$ , we derive from Proposition 3.1, Fatou's lemma and (3.10) that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \tilde{V}_{(p_k, 0)}^{g_k}(\bar{\tau}) - \varepsilon(r) &\leq \limsup_{k \rightarrow \infty} \int_{\mathcal{L}B_{\bar{\tau}}(p_k, r)} u_k(\cdot, \bar{\tau}) d\mu_{g_k(\bar{\tau})} \\ &= \limsup_{k \rightarrow \infty} \int_{B_0(p_\infty, C(r))} \hat{u}_k(\Phi_k(\cdot), \bar{\tau}) d\mu_{\Phi_k^\star g_k(\bar{\tau})} \\ &\leq \int_{B_0(p_\infty, C(r))} \limsup_{k \rightarrow \infty} \hat{u}_k(\Phi_k(\cdot), \bar{\tau}) d\mu_{\Phi_k^\star g_k(\bar{\tau})} \\ &\leq \tilde{V}_{(p_\infty, 0)}^{g_\infty}(\bar{\tau}). \end{aligned}$$

On the other hand, by combining Fatou's lemma and (3.8), we obtain

$$\begin{aligned}
\liminf_{k \rightarrow \infty} \tilde{V}_{(p_k, 0)}^{g_k}(\bar{\tau}) &\geq \liminf_{k \rightarrow \infty} \int_{\mathcal{L}B_{\bar{\tau}}(p_\infty, r)} u_k(\Phi_k(\cdot), \bar{\tau}) d\mu_{\Phi_k^* g_k}(\bar{\tau}) \\
&\geq \int_{\mathcal{L}B_{\bar{\tau}}(p_\infty, r)} \liminf_{k \rightarrow \infty} u_k(\Phi_k(\cdot), \bar{\tau}) d\mu_{\Phi_k^* g_k}(\bar{\tau}) \\
&\geq \int_{\mathcal{L}B_{\bar{\tau}}(p_\infty, r)} u_\infty(\cdot, \bar{\tau}) d\mu_{g_\infty}(\bar{\tau}) \\
&\geq \tilde{V}_{(p_\infty, 0)}^{g_\infty}(\bar{\tau}) - \varepsilon(r).
\end{aligned}$$

We also used Proposition 3.1 to get the last inequality. Since  $r > 0$  and  $\bar{\tau} \in (0, T)$  are chosen arbitrarily, we conclude that

$$\lim_{k \rightarrow \infty} \tilde{V}_{(p_k, 0)}^{g_k}(\tau) = \tilde{V}_{(p_\infty, 0)}^{g_\infty}(\tau)$$

for any  $\tau \in (0, T)$ . This completes the proof of Lemma 3.22.  $\square$

Equipped with these three lemmas, we will give the proof of Theorem 1.1 in the next chapter.

## Chapter 4

# A gap theorem for ancient solutions

In the present chapter, we present the proof of the main theorem of this thesis (Theorem 1.1). Now that we have seen that the asymptotic reduced volume  $\tilde{\mathcal{V}}(g) := \lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau)$  of an ancient solution  $(M^n, g(\tau)), \tau \in [0, \infty)$  is independent of the choice of the base point  $p \in M$  in Corollary 3.4 in the previous section, our main theorem (Theorem 1.1) can be restated as follows.

**Theorem 4.1** (Gap theorem for ancient solutions). *For any  $n \geq 2$ , there exists  $\varepsilon_n > 0$  which depends only on  $n$  and satisfies the following: let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be an  $n$ -dimensional complete ancient solution to the Ricci flow with Ricci curvature bounded below. Suppose that the asymptotic reduced volume  $\tilde{\mathcal{V}}(g)$  is greater than  $1 - \varepsilon_n$ . Then  $(M^n, g(\tau)), \tau \in [0, \infty)$  is the Gaussian soliton  $(\mathbb{R}^n, g_E)$ .*

In the following section, we begin by showing the technical lemma. Then Theorem 4.1 will be established in Section 4.2.

### 4.1 Technical lemma

Before proceeding to the proof of Theorem 4.1, we first establish the following technical lemma.

**Lemma 4.2.** *For any  $\alpha > 0$  and  $\bar{\tau} > 0$  with  $\alpha\bar{\tau}^{-1} > 2$ , we can find  $\varepsilon_n(\alpha\bar{\tau}^{-1}) > 0$  depending on  $\alpha\bar{\tau}^{-1}$  and  $n \geq 2$  which satisfies the following: let  $(M^n, g(\tau)), \tau \in [0, T), T < \infty$  be a complete backward Ricci flow with Ricci curvature bounded below. Put*

$$M(\alpha) := \{(p, s) \in M \times [0, T) \mid |\mathrm{Rm}|(p, s) > \alpha(T - s)^{-1}\}.$$

*Suppose that the reduced volume based at  $(p, s)$  satisfies*

$$\tilde{V}_{(p,s)}(Q_{(p,s)}^{-1}\bar{\tau}) > 1 - \varepsilon_n(\alpha\bar{\tau}^{-1}) \quad \text{at all } (p, s) \in M(\alpha)$$

*with  $Q_{(p,s)} := |\mathrm{Rm}|(p, s)$ . Here we define  $\tilde{V}_{(p,s)}(\bar{\tau})$  as  $\tilde{V}_{(p,0)}^{g_s}(\bar{\tau})$  for  $g_s(\tau) := g(\tau + s), \tau \in [0, T - s)$ . Then  $M(\alpha) = \emptyset$ , that is,*

$$|\mathrm{Rm}|(\cdot, \tau) \leq \alpha(T - \tau)^{-1} \quad \text{on } M \times [0, T).$$

One might notice the similarity of the statement of Lemma 4.2 to those of Perelman's pseudolocality theorem [Pe, Theorem 10.1] and Ni's  $\varepsilon$ -regularity theorem [Ni3, Theorem 4.4]. In fact, there is a close relation between gap and local regularity theorems. The proof of Lemma 4.2 follows the same line as those of the theorems mentioned above.

**Theorem 4.3** (Pseudolocality theorem, Perelman [Pe, Theorem 10.1]). *For every  $\alpha > 0$ , we can find  $\delta > 0$  and  $\varepsilon > 0$  with the following property: let  $(M^n, g(t)), t \in [0, (\varepsilon r_0)^2]$  be a Ricci flow on a closed manifolds  $M^n$  with*

$$R(x, 0) \geq -r_0^{-2} \quad \text{and} \quad \text{Vol}_{g(0)}(\partial\Omega)^n \geq (1 - \delta)c_n \text{Vol}_{g(0)}(\Omega)^{n-1}$$

for any  $x$  and  $\Omega$  in  $B_{g(0)}(x_0, r_0)$ , where  $c_n$  is the isoperimetric constant for  $\mathbb{R}^n$ . Then we have

$$|\text{Rm}|(x, t) \leq \alpha t^{-1} + (\varepsilon r_0)^{-2}$$

for all  $(x, t)$  with  $0 < t \leq (\varepsilon r_0)^2$  and  $d_{g(t)}(x, x_0) < \varepsilon r_0$ .

*Remark 4.4.* Later, Perelman's pseudolocality theorem was extended to the complete Ricci flow with bounded curvature by Chau–Tam–Yu [CTY].

*Proof of Lemma 4.2.* We prove by contradiction. Fix  $\alpha > 0$  and  $\bar{\tau} > 0$  with  $\alpha\bar{\tau} > 2$ . Assume that we have a sequence  $\{(M_k^n, g_k(\tau))\}_{k \in \mathbb{Z}^+}$ ,  $\tau \in [0, T_k]$  of complete backward Ricci flows with Ricci curvature bounded below such that

- $M_k(\alpha) := \{(p, \tau) \in M_k \times [0, T_k] \mid |\text{Rm}|(p, \tau)(T_k - \tau) > \alpha\} \neq \emptyset$  and
- $\tilde{V}_{(p, \tau)}^{g_k}(Q_{(p, \tau)}^{-1}\bar{\tau}) > 1 - k^{-1}$  for any  $(p, \tau) \in M_k(\alpha)$ , where  $Q_{(p, \tau)} := |\text{Rm}|(p, \tau)$ .

Applying Perelman's point picking lemma (Lemma A.2) for  $(A, B) = (k, \alpha)$ , we can find a point  $(p_k, \tau_k) \in M_k(\alpha)$  such that  $\tilde{V}_{(p_k, \tau_k)}^{g_k}(Q_k^{-1}\bar{\tau}) > 1 - k^{-1}$  and that

$$|\text{Rm}|(x, \tau) \leq 2Q_k := 2|\text{Rm}|(p_k, \tau_k)$$

for all  $(x, \tau) \in B_{g_k(\tau_k)}(p_k, kQ_k^{-1/2}) \times [\tau_k, \tau_k + \frac{1}{2}Q_k^{-1}\alpha]$ .

Consider the sequence  $\{(M_k^n, \tilde{g}_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}$  of rescaled Ricci flows

$$\tilde{g}_k(\tau) := Q_k g_k(Q_k^{-1}\tau + \tau_k), \quad \tau \in [0, \alpha/2].$$

Then every  $\tilde{g}_k(\tau)$  has  $|\text{Rm}|(p_k, 0) = 1$ ,  $|\text{Rm}| \leq 2$  on  $B_{\tilde{g}_k(0)}(p_k, k) \times [0, \alpha/2]$ , and  $\tilde{V}_{(p_k, 0)}^{\tilde{g}_k}(\bar{\tau}) > 1 - k^{-1}$  by Proposition 2.15.

Now we observe that the injectivity radius of  $(M_k, \tilde{g}_k(0))$  at  $p_k$  is uniformly bounded from below. To see this, we use Proposition 3.2 to get small  $\delta = \delta(r) > 0$  so that  $\mathcal{L} \exp_\delta(p_k, r) \subset B_{\tilde{g}_k(0)}(p_k, 1)$  for some large  $r > 0$  and all large  $k$ . Then

$$\begin{aligned} 1 - k^{-1} &< \tilde{V}_{(p_k, 0)}^{\tilde{g}_k}(\bar{\tau}) \leq \tilde{V}_{(p_k, 0)}^{\tilde{g}_k}(\delta) \\ &\leq (4\pi\delta)^{-n/2} e^{n(n-1)\delta} \text{Vol}_{\tilde{g}_k(0)} B_{\tilde{g}_k(0)}(p_k, 1) + \varepsilon(r) \end{aligned}$$

from which we obtain a uniform lower bound for  $\text{Vol}_{\tilde{g}_k(0)} B_{\tilde{g}_k(0)}(p_k, 1)$ . The lower bound for the injectivity radius is equivalent to the bound of the volume of a metric ball if the sectional curvature is bounded in absolute value.

Since each  $(M_k^n, \tilde{g}_k(\tau))$  has a uniform curvature bound and lower bound for the injectivity radius at  $(p_k, 0)$ , according to Hamilton's compactness theorem (Theorem 3.20), we can take a subsequence of  $\{(M_k^n, \tilde{g}_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}$  converging to the limit Ricci flow  $(M_\infty^n, g_\infty(\tau), p_\infty)$ ,  $\tau \in [0, \alpha/2)$ . From Lemma 3.22, we infer that  $\tilde{V}_{(p_\infty, 0)}^{g_\infty}(\bar{\tau}) = 1$ , which implies that the limit  $(M_\infty^n, g_\infty(0))$  is isometric to the Euclidean space by Theorem 2.12. This is in conflict with that  $|\text{Rm}|(p_\infty, 0) = 1$ . The proof of Lemma 4.2 is now complete.  $\square$

## 4.2 Proof of Theorem 4.1

Now we present the proof of Theorem 4.1.

*Proof of Theorem 4.1.* Take  $\varepsilon_n := \varepsilon_n(3) > 0$  from Lemma 4.2. Suppose that  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  is a complete ancient solution to the Ricci flow with Ricci curvature bounded from below satisfying that

$$\tilde{\mathcal{V}}(g) > 1 - \varepsilon_n.$$

Due to Lemma 3.3 in the previous chapter and the monotonicity of the reduced volume, we know that

$$\tilde{V}_{(p, \tau)}(\bar{\tau}) > 1 - \varepsilon_n \quad \text{for all } (p, \tau) \in M \times [0, \infty) \text{ and } \bar{\tau} > 0.$$

By Lemma 3.9, we know that  $\pi_1(M)$  is finite, and applying Lemma 4.2 for all  $T > 0$  yields that  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  is flat. The only flat manifold with finite fundamental group is the Euclidean space. Thus  $(M^n, g(\tau))$  is isometric to  $(\mathbb{R}^n, g_E)$  for all  $\tau \in [0, \infty)$ , i.e.,  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  is the Gaussian soliton. This concludes the proof of Theorem 4.1.  $\square$

*Remark 4.5.* Theorem 4.1 may have several variations. (See the questions in [Ni] for instance.) The two chapters following the next chapter are devoted to such an issue.

## Chapter 5

# A gap theorem for gradient shrinkers

In this chapter, we present the proof of the gap theorem for gradient shrinking Ricci solitons (Corollary 1.4), which is obtained by applying Theorem 4.1 to them.

Recall that a triple  $(M^n, g, f)$  is called a *gradient shrinking Ricci soliton*, or *gradient shrinker*, if

$$\operatorname{Ric}(g) + \operatorname{Hess} f - \frac{1}{2\lambda}g = 0, \quad (5.1)$$

for some positive constant  $\lambda > 0$ . The tensor  $\operatorname{Ric} + \operatorname{Hess} f$  is called the *Bakry–Emery Ricci tensor*. There has been intense study of the comparison geometry of the Bakry–Emery tensor; see Morgan [Mo, Mo2] and Wei–Wylie [WeWy] for instance.

The potential function  $f \in C^\infty(M)$  of a gradient Ricci soliton  $(M^n, g, f)$  is normalized so that

$$\lambda(|\nabla f|^2 + R) = f, \quad (5.2)$$

where  $R$  is the scalar curvature of  $(M^n, g)$ . Due to Zhang [Zh],  $R$  is non-negative and  $\nabla f$  is complete for any complete gradient shrinking Ricci soliton (see the paragraph preceding Theorem 3.6).

The object of this chapter is to prove the following.

**Corollary 5.1** (Gap theorem for gradient shrinkers). *Let  $(M^n, g, f)$  be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below. Then*

- (1) *the Gaussian density  $\Theta(M) := \int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g$  does not exceed 1.*
- (2) *Any gradient shrinking Ricci soliton  $(M^n, g, f)$  satisfying that  $\Theta(M) > 1 - \varepsilon_n$  is, up to scaling, the Gaussian soliton  $(\mathbb{R}^n, g_E, \frac{|\cdot|^2}{4})$ . Here the constant  $\varepsilon_n$  comes from Theorem 4.1.*

We will do this in the first section. In the second section, we discuss the case of expanding Ricci solitons.

### 5.1 Shrinking Ricci solitons

In the present section, we describe the proof of Corollary 5.1.

For the proof of the corollary, we first construct an ancient solution to the Ricci flow. (Recall the proof of Theorem 2.12. See also [CLN, Theorem 4.1].) Define a one-parameter family of diffeomorphisms  $\varphi_\tau : M \rightarrow M$ ,  $\tau \in (0, \infty)$  by

$$\frac{d}{d\tau}\varphi_\tau = \frac{\lambda}{\tau}\nabla f \circ \varphi_\tau \quad \text{and} \quad \varphi_\lambda = \text{id}_M.$$

The result of Zhang [Zh] (Theorem 3.6) says that the gradient vector field  $\nabla f$  is complete; however, it is easy to see that  $\nabla f$  is complete in our situation, thanks to the assumption on the lower bound for Ric.

Then we pull back  $g$  by  $\psi_\tau := \varphi_\tau^{-1}$  so as to obtain a backward Ricci flow  $(M^n, g_0(\tau))$  determined by

$$g_0(\tau) := \frac{\tau}{\lambda}(\psi_\tau)^*g, \tau \in (0, \infty) \quad \text{with} \quad g_0(\lambda) = g. \quad (5.3)$$

Put  $g_S(\tau) := g(\tau+s)$ ,  $\tau \in [0, \infty)$  for some fixed  $s > 0$  and fix some point  $p \in M$ . In what follows, abusing the terminology, we call both of  $(M^n, g, f)$  and  $(M^n, g_S(\tau))$ ,  $\tau \in [0, \infty)$  a gradient Ricci soliton. It suffices to show that the Gaussian density  $\Theta(M)$  and the asymptotic reduced volume  $\tilde{\mathcal{V}}(g_S)$  satisfies

$$\tilde{\mathcal{V}}(g_S) \geq \Theta(M) \quad (5.4)$$

since the left-hand side of (5.4) is  $\leq 1$ .

Let us first give a heuristic argument. It seems reasonable to hold that

$$\tilde{V}_{(p,0)}^{g_0}(\tau) = \tilde{\mathcal{V}}(g_0) = \Theta(M) \quad \text{for all } \tau > 0$$

(cf. Cao–Hamilton–Ilmanen [CHI]). Then inequality (5.4) will follow from Lemma 3.3, if it is applicable to this case. Of course, the problem arises from the fact that  $\tau = 0$  is the singular time for  $g_0(\tau)$ .

Now we give a rigorous proof of inequality (5.4).

*Proof of Corollary 5.1.* Recall that we have normalized  $f$  in (1.3) so that

$$R_{g_0(\tau)} + |\nabla f_\tau|_{g_0(\tau)}^2 - \frac{f_\tau}{\tau} = 0 \quad \text{for } \tau > 0 \quad (5.5)$$

where  $f_\tau = f(\cdot, \tau) := (\psi_\tau)^*f = f \circ \psi_\tau$ . Since  $R_{g_0(\tau)}$  is non-negative, so is  $f_\tau$ . Put  $x_1 := \varphi_{\tau_1}(x)$  and  $x_2 := \varphi_{\tau_2}(x)$  for some  $x \in M$ . Then the argument in [Vol2-I, p. 344] yields the following proposition.

**Proposition 5.2.** *In these notation,  $\gamma(\tau) := \varphi_\tau \circ \varphi_{\tau_1}^{-1}(x_1)$  is the unique  $\mathcal{L}^{g_0}$ -minimal geodesic from  $(x_1, \tau_1)$  to  $(x_2, \tau_2)$  and*

$$\frac{1}{2\sqrt{\tau_2}}L_{(x_1, \tau_1)}^{g_0}(x_2, \tau_2) = f(x_2, \tau_1) - \sqrt{\frac{\tau_1}{\tau_2}}f(x_1, \tau_1). \quad (5.6)$$

*Proof.* Let  $\eta : [\tau_1, \tau_2] \rightarrow M$  be a curve from  $x_1$  to  $x_2$ . Then, using (5.5), we obtain

$$\begin{aligned} & 2\sqrt{\tau_2}f(\eta(\tau_2), \tau_2) - 2\sqrt{\tau_1}f(\eta(\tau_1), \tau_1) \\ &= \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left( \frac{f_\tau}{\tau} + 2\frac{\partial f}{\partial \tau} + 2\langle \eta', \nabla f \rangle \right) d\tau \\ &= \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left( |\eta'|^2 + R_{g_0(\tau)}(\eta(\tau)) - |\eta' - \nabla f|^2 \right) d\tau \\ &= \mathcal{L}(\eta) - \int_{\tau_1}^{\tau_2} \sqrt{\tau} |\eta' - \nabla f|^2 d\tau. \end{aligned}$$

Hence

$$2\sqrt{\tau_2}f(\eta(\tau_2), \tau_2) - 2\sqrt{\tau_1}f(\eta(\tau_1), \tau_1) \leq \mathcal{L}(\eta), \quad (5.7)$$

and the equality is attained in (5.7) for and only for the curve  $\eta(\tau) = \gamma(\tau)$ . This proves the proposition.  $\square$

Fix a compact set  $\mathcal{K} \subset M$ ,  $\varepsilon > 0$  and  $\bar{\tau} \gg 1$ . Take  $q \in \varphi_{\bar{\tau}}(\mathcal{K})$  and  $p_2 \in \varphi_{2s}(\mathcal{K})$  with  $q = \varphi_{\bar{\tau}} \circ \varphi_{2s}^{-1}(p_2)$ . From the triangle inequality for  $\mathcal{L}$ -distance, Sublemma 3.7 and (5.6), it follows that

$$\begin{aligned} \ell_{(p,0)}^{g_S}(q, \bar{\tau} - s) &\leq \frac{1}{2\sqrt{\bar{\tau} - s}} \left( L_{(p_2,s)}^{g_S}(q, \bar{\tau} - s) + L_{(p,0)}^{g_S}(p_2, s) \right) \\ &\leq \frac{1}{2\sqrt{\bar{\tau}}} L_{(p_2,2s)}^{g_0}(q, \bar{\tau}) + \frac{1}{2\sqrt{\bar{\tau} - s}} \max_{\varphi_{2s}(\mathcal{K})} L_{(p,0)}^{g_S}(\cdot, s) \\ &\leq f(q, \bar{\tau}) - \sqrt{\frac{2s}{\bar{\tau}}} f(p_2, 2s) + C(\mathcal{K})\bar{\tau}^{-1/2} \\ &\leq f(q, \bar{\tau}) + C(\mathcal{K})\bar{\tau}^{-1/2}. \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \tilde{\mathcal{V}}(g_S) &\geq \tilde{V}_{(p,0)}^{g_S}(\bar{\tau} - s) - \varepsilon \\ &\geq e^{-C(\mathcal{K})\bar{\tau}^{-1/2}} \int_{\varphi_{\bar{\tau}}(\mathcal{K})} (4\pi\bar{\tau})^{-n/2} e^{-f(q,\bar{\tau})} d\mu_{g_0(\bar{\tau})}(q) - \varepsilon \\ &= e^{-C(\mathcal{K})\bar{\tau}^{-1/2}} \int_{\mathcal{K}} (4\pi\lambda)^{-n/2} e^{-f} d\mu_g - \varepsilon. \end{aligned}$$

We have used the equation

$$\int_M h \circ \psi_{\bar{\tau}} d\mu_{(\psi_{\bar{\tau}})^*g} = \int_M h d\mu_g \quad \text{for any } h \in L^1(d\mu_g)$$

which follows from the definition of pull back. Inequality (5.4) then follows from the arbitrariness of  $\bar{\tau} > 0, \varepsilon > 0$  and  $\mathcal{K} \subset M$ .

By using (5.4), Theorem 4.1 immediately implies Corollary 5.1. As for (2) of Corollary 5.1, it is easy to see that the only way to regard a Ricci-flat space as a shrinking soliton is the Gaussian soliton up to scaling (e.g. Tashiro [Ta], Naber [Na]). This concludes the proof of Corollary 5.1.  $\square$

In the above proof, inequality (5.4) was enough for our purpose, however, we can actually show that the equality holds in (5.4) in this situation. Here we describe the proof of this for future applications.

**Proposition 5.3.** *Let  $(M^n, g, f)$  be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below. Assume that the potential function  $f \in C^\infty(M)$  is normalized so that (1.3) holds. Then, with notation as in the proof of Corollary 5.1, we have*

$$\tilde{\mathcal{V}}(g_S) = \Theta(M) := \int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g. \quad (5.8)$$

For the proof of Proposition 5.3, we will require the following estimates.

**Theorem 5.4** (Cao–Zhou [CaZho]). *Let  $(M^n, g, f)$  be a complete gradient shrinking Ricci soliton. Fix a point  $p \in M$ . Then*

- (1) *it has at most Euclidean volume growth, i.e.,  $\text{Vol} B(p, r) \leq Cr^n$  for some constant  $C > 0$ .*
- (2) *There are constants  $c_1, c_2 > 0$  such that*

$$\frac{1}{4\lambda} (d(x, p)^2 - c_1) \leq f(x) \leq \frac{1}{4\lambda} (d(x, p)^2 + c_2) \quad \text{for any } x \in M^n.$$

From this theorem, we deduce that the integral  $\int_M e^{-\alpha f} d\mu_g$  makes sense for any positive constant  $\alpha > 0$ .

*Proof of Proposition 5.3.* Take a sequence  $\{\tau_i\}_{i \in \mathbb{Z}^+}$  with  $\tau_i \rightarrow \infty$  as  $i \rightarrow \infty$  and put  $\alpha_i := \sqrt{1 - \frac{s}{\tau_i}}$ . Fix  $r > 0$  and  $\bar{\tau} > 0$  sufficiently large. We let

$$\mathcal{K}_i(\bar{\tau} - s) := \mathcal{L}^{g_S} K_{\bar{\tau}-s, \tau_i-s}(p, r) \quad \text{and} \quad \mathcal{K}_i := \mathcal{L}^{g_S} K_{\tau_i-s, \tau_i-s}(p, r).$$

For any  $q \in \mathcal{K}_i(\bar{\tau} - s)$ , take  $p_i := \gamma(\tau_i - s) \in \mathcal{K}_i$ , where  $\gamma$  is the minimal  $\mathcal{L}^{g_S}$ -geodesic from  $(p, 0)$  to  $(q, \bar{\tau} - s)$ .

It follows from the combination of Sublemma 3.7 and (5.6) that

$$\begin{aligned} \ell_{(p,0)}^{g_S}(q, \bar{\tau} - s) &= \frac{1}{2\sqrt{\bar{\tau} - s}} \left( L_{(p_i, \tau_i-s)}^{g_S}(q, \bar{\tau} - s) + L_{(p,0)}^{g_S}(p_i, \tau_i - s) \right) \\ &\geq \alpha_i \frac{1}{2\sqrt{\bar{\tau}}} L_{(p_i, \tau_i)}^{g_0}(q, \bar{\tau}) \\ &= \alpha_i \left( f(q, \bar{\tau}) - \sqrt{\frac{\tau_i}{\bar{\tau}}} f(p_i, \tau_i) \right) \\ &\geq \alpha_i \left( f(q, \bar{\tau}) - \sqrt{\frac{\tau_i}{\bar{\tau}}} \max_{\mathcal{K}_i} f(\cdot, \tau_i) \right) \\ &= \alpha_i f(q, \bar{\tau}) - C(\tau_i) \bar{\tau}^{-1/2}. \end{aligned}$$

Recall that  $L_{(p,0)}^{g_S}(\cdot, \cdot) \geq 0$ , which follows from the non-negativity of the scalar curvature of  $g_S(\tau)$  (see the paragraph before Theorem 3.6), and that  $\mathcal{K}_i$  is compact.

Thus, by Proposition 3.1,

$$\begin{aligned} \tilde{\mathcal{V}}(g_S) &\leq \left(1 - \frac{s}{\bar{\tau}}\right)^{n/2} \tilde{V}_{(p,0)}^{g_S}(\bar{\tau} - s) + \varepsilon(r) \\ &\leq \int_{\mathcal{K}_i(\bar{\tau}-s)} (4\pi\bar{\tau})^{-n/2} \exp\left(-\ell_{(p,0)}^{g_S}(\cdot, \bar{\tau} - s)\right) d\mu_{g_0(\bar{\tau})} + 3\varepsilon(r) \\ &\leq e^{C(\tau_i)\bar{\tau}^{-1/2}} \int_{\mathcal{K}_i(\bar{\tau}-s)} (4\pi\bar{\tau})^{-n/2} e^{-\alpha_i f(\cdot, \bar{\tau})} d\mu_{g_0(\bar{\tau})} + 3\varepsilon(r) \\ &\leq e^{C(\tau_i)\bar{\tau}^{-1/2}} \int_M (4\pi\lambda)^{-n/2} e^{-\alpha_i f} d\mu_g + 3\varepsilon(r). \end{aligned}$$

Since  $r > 0$  and  $\bar{\tau} > 0$  are arbitrary, we have obtained that

$$\tilde{\mathcal{V}}(g_S) \leq \int_M (4\pi\lambda)^{-n/2} e^{-\alpha_i f} d\mu_g \quad (< \infty) \tag{5.9}$$

and the right-hand side of (5.9) converges to the Gaussian density  $\Theta(M)$  as  $i \rightarrow \infty$ . Combined with (5.4), this completes the proof of the proposition.  $\square$

We close this section by giving an application of Proposition 5.3 (cf. Naber [Na]). Recall the definition of being  $\kappa$ -noncollapsing from Definition 3.16.

**Proposition 5.5** (Gradient shrinking Ricci solitons are  $\kappa$ -noncollapsing). *Let  $(M^n, g_S(\tau)), \tau \in [0, \infty)$  be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below. Then we can find a constant  $\kappa > 0$  depending only on  $n$  and its Gaussian density  $\Theta(M)$  such that  $(M^n, g_S(\tau)), \tau \in [0, \infty)$  is  $\kappa$ -noncollapsing on all scale  $r > 0$ .*

This is an immediate corollary of Proposition 5.3 and the following (cf. [Vol2-I, Theorem 8.24]).

**Proposition 5.6** ( $\tilde{\mathcal{V}}(g) > 0$  implies  $\kappa$ -noncollapsing). *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient solution to the Ricci flow with Ricci curvature bounded below. Suppose that the asymptotic reduced volume  $\tilde{\mathcal{V}}(g)$  of the ancient solution is strictly positive. Then we can find  $\kappa > 0$  depending only on  $n$  and  $\tilde{\mathcal{V}}(g) > 0$  such that  $(M^n, g(\tau)), \tau \in [0, \infty)$  is  $\kappa$ -noncollapsing on all scale  $r > 0$ .*

## 5.2 Expanding Ricci solitons

Next, we consider gradient expanders of non-negative Ricci curvature and prove the result corresponding to Corollary 5.1 for them. A *gradient expanding Ricci soliton*, or *gradient expander*, is a triple  $(M^n, g, f)$  satisfying

$$\text{Ric}(g) - \text{Hess } f + \frac{1}{2\lambda}g = 0 \quad (5.10)$$

for some positive constant  $\lambda > 0$ . We normalize the potential function  $f \in C^\infty(M)$  so that  $\lambda(R + |\nabla f|^2) - f = 0$  on  $M$  for the expander  $(M, g, f)$  too. Beware that our definition of gradient expanding Ricci solitons differs from the traditional one. The use of equation (5.10) as the definition makes the statement of the following proposition compatible with that of Corollary 5.1.

**Proposition 5.7** (Carrillo–Ni [CaNi]). *Let  $(M^n, g, f)$  be a complete expanding Ricci soliton with non-negative Ricci curvature. Then*

(1)  *$M^n$  is diffeomorphic to  $\mathbb{R}^n$ .*

(2) *We have*

$$\int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g \leq 1 \quad (5.11)$$

*and the equality holds if and only if  $(M^n, g, f)$  is, up to scaling, the expanding Gaussian soliton  $(\mathbb{R}^n, g_E, \frac{|\cdot|^2}{4})$ .*

As in the case of gradient shrinking Ricci solitons, we will also refer to the left-hand side of inequality (5.11) as the Gaussian density.

We remark that the proposition is a restatement of a result of Carrillo–Ni [CaNi]. Because our proof is simple and purely geometric in contrast to the one in [CaNi], we decided to include it here.

*Proof of Proposition 5.7.* First, we note that the potential function  $f \in C^\infty$  is bounded below and  $\frac{1}{2\lambda}$ -convex, i.e.,  $\text{Hess } f \geq \frac{1}{2\lambda}g > 0$ . Therefore,  $f$  has the unique critical point  $p \in M$  where the minimum value of  $f$  is attained. Part (1) of the proposition follows from this and a Morse theoretic argument.

Next, as in the proof of Corollary 5.1, we construct a self-similar solution to the (forward) Ricci flow  $g_0(t) := \frac{t}{\lambda}(\psi_t)^*g, t \in (0, \infty)$  and put  $g_1(t) := g_0(t+1), t \in [0, \infty)$ .

Define the *forward reduced distance* at  $(q, \bar{t}) \in M \times (0, \infty)$  by

$$\ell_{(p,0)}^+(q, \bar{t}) := \frac{1}{2\sqrt{\bar{t}}} \inf \{ \mathcal{L}^+(\gamma) \mid \gamma(0) = p, \gamma(\bar{t}) = q \}$$

where we defined the *forward  $\mathcal{L}$ -length*  $\mathcal{L}^+(\gamma)$  of  $\gamma : [0, \bar{t}] \rightarrow M$  by

$$\mathcal{L}^+(\gamma) := \int_0^{\bar{t}} \sqrt{t} \left( \left| \frac{d\gamma}{dt} \right|_{g_1(t)}^2 + R_{g_1(t)}(\gamma(t)) \right) dt.$$

Then we consider the formal reduced volume defined by

$$\widehat{V}_{(p,0)}^{g_1}(t) := \int_M (4\pi t)^{-n/2} \exp \left( -\ell_{(p,0)}^+(\cdot, t) \right) d\mu_{g_1(t)}. \quad (5.12)$$

We do not care whether  $\widehat{V}_{(p,0)}^{g_1}(t)$  is monotone. (This is the case when  $g_1(t)$  has bounded non-negative curvature operator or non-negative bisectional curvature in the Kähler case [Ni4].)

Since  $(M^n, g_1(t))$  has non-negative Ricci curvature, we have

$$\begin{aligned} \ell_{(p,0)}^+(q, \bar{t}-1) &\geq \frac{1}{2\sqrt{\bar{t}-1}} \inf_{\gamma} \int_0^{\bar{t}-1} \sqrt{t} \left| \frac{d\gamma}{dt} \right|_{g_1(\bar{t}-1)}^2 dt \\ &= \frac{1}{4(\bar{t}-1)} d_{g_1(\bar{t}-1)}(p, q)^2 \geq \frac{1}{4\lambda} d_g(p, \psi_{\bar{t}}(q))^2 \end{aligned}$$

and by Lemma 2.14,

$$\begin{aligned} \left( \frac{\bar{t}-1}{\bar{t}} \right)^{n/2} \widehat{V}_{(p,0)}^{g_1}(\bar{t}-1) &\leq \int_M (4\pi \bar{t})^{-n/2} \exp \left( -\frac{1}{4\lambda} d_g(p, \psi_{\bar{t}}(\cdot))^2 \right) d\mu_{g_0(\bar{t})} \\ &= \int_M (4\pi \lambda)^{-n/2} \exp \left( -\frac{d_g(p, \cdot)^2}{4\lambda} \right) d\mu_g \\ &\leq 1. \end{aligned}$$

Then, from the same argument as in the derivation of (5.4) in the proof of Corollary 5.1, we derive that

$$\int_M (4\pi \lambda)^{-n/2} e^{-f} d\mu_g \leq \liminf_{t \rightarrow \infty} \widehat{V}_{(p,0)}^{g_1}(t)$$

and hence

$$\int_M (4\pi \lambda)^{-n/2} e^{-f} d\mu_g \leq \int_M (4\pi \lambda)^{-n/2} \exp \left( -\frac{d_g(p, \cdot)^2}{4\lambda} \right) d\mu_g \leq 1 \quad (5.13)$$

which yields (5.11).

When the Gaussiand density is 1, we have equalities in (5.13). Then we know from the equality case of Lemma 2.14 that  $(M^n, g)$  is isometric to the Euclidean space. The only way to regard  $(\mathbb{R}^n, g_E)$  as a gradient expanding Ricci soliton is the Gaussian soliton, up to rescaling. This finishes the proof.  $\square$

### 5.3 Concluding remarks

In this section, we collect some remarks on the gap theorems we established in Chapters 4 and 5.

*Remark 5.8.* There is the optimal value  $\varepsilon_n$  of the constant obtained in Theorem 4.1, namely  $\varepsilon_n := 1 - \max \tilde{\mathcal{V}}(g) > 0$ . We take the maximum over all the complete  $n$ -dimensional non-Gaussian ancient solutions to the Ricci flow with Ricci curvature bounded below. The maximum is achieved, as is seen by the limit argument used in the proof of Lemma 4.2. Then it is easy to see that  $\{\varepsilon_n\}_{n=2}^\infty$  is a non-increasing sequence. It seems interesting to determine the exact value of  $\lim_{n \rightarrow \infty} \varepsilon_n$ .

**Proposition 5.9.** *In the above notation, there exists an  $n$ -dimensional complete ancient solution  $(M^n, g_{\max}(\tau)), \tau \in [0, \infty)$  to the Ricci flow with uniformly bounded curvature such that  $\tilde{\mathcal{V}}(g_{\max}) = 1 - \varepsilon_n$ .*

*Furthermore, for  $n = 2, 3$ ,  $(M^n, g_{\max}(\tau)), \tau \in [0, \infty)$  is the shrinking round sphere.*

*Proof.* The second statement of this proposition is a consequence of the facts that in dimensions 2 and 3, any ancient solution with bounded curvature and positive asymptotic reduced volume is an ancient  $\kappa$ -solution (Proposition 5.6), and there is an asymptotic soliton for any ancient  $\kappa$ -solution.

An *asymptotic soliton* of an ancient  $\kappa$ -solution  $(M^n, g(\tau)), \tau \in [0, \infty)$  is a blow-down limit of it around points  $\{(q_i, \tau_i)\}$  of  $M^n \times [0, \infty)$  where  $\ell_{(p,0)}(q_i, \tau_i) \leq n/2$  with  $\tau_i \rightarrow \infty$  as  $i \rightarrow \infty$ ; the existence of such  $(q_i, \tau_i)$ 's is guaranteed by the maximum principle applied to inequality (7.5). An asymptotic soliton of any ancient  $\kappa$ -solution exists and is a gradient shrinking Ricci soliton whose Gaussian density equals the asymptotic reduced volume of the original ancient  $\kappa$ -solution ([Pe, Proposition 11.2]).

Then we can appeal to the classification results (Theorem 5.10) of gradient shrinkers in dimensions 2 and 3, according to which, the asymptotic soliton of  $(M^n, g_{\max}(\tau))$  is the round sphere. Hamilton's results on 3-dimensional Ricci flows with positive Ricci curvature [Ha] and on 2-dimensional Ricci flows with positive curvature [Ha4] tells us that an ancient  $\kappa$ -solution with the round sphere as an asymptotic soliton cannot be anything but the shrinking round sphere. This proves the second statement of the proposition.  $\square$

**Theorem 5.10** (Classification of gradient shrinking Ricci solitons). *Let  $(M^n, g, f)$  be a non-flat complete gradient shrinking Ricci soliton.*

- (1) *If  $n = 2$ ,  $(M^n, g, f)$  is the round sphere  $\mathbb{S}^2$  or its quotient (e.g. Hamilton [Ha4], Petersen–Wylie [PeWy]).*
- (2) *If  $n = 3$ ,  $(M^n, g, f)$  is the round sphere  $\mathbb{S}^3$ , the round cylinder  $\mathbb{S}^2 \times \mathbb{R}$  or their quotient (e.g. Perelman [Pe2], Ni–Wallach [NiWa]).*

The construction of an asymptotic soliton has in common with that of an asymptotic cone of a manifold of non-negative Ricci curvature. We recall the following theorem of Cheeger–Colding [ChCo, Theorem 7.6].

**Theorem 5.11** (Asymptotic cone is a metric cone [ChCo], also Cheeger [Chee]). *Let  $M^n$  be a complete Riemannian manifold of non-negative Ricci curvature with Euclidean*

volume growth, i.e.,  $\nu(g) > 0$ . Then any asymptotic cone, i.e., a pointed Gromov–Hausdorff limit of  $(r_i^{-2}M^n, o)$  with  $r_i \rightarrow \infty$  for some fixed point  $o \in M$ , is a metric cone.

The definition of metric cones is in Burago–Burago–Ivanov [BBI] for example.

*Remark 5.12.* Now we calculate an asymptotic reduced volume (or the Gaussian density) for the round  $n$ -sphere  $(S^n, g_{S^n})$  with constant Ricci curvature  $\text{Ric}(g_{S^n}) = \frac{1}{2}g_{S^n}$ . Then  $g(\tau) := (1 + \tau)g_{S^n}$ ,  $\tau \in [0, \infty)$  is an ancient solution to the Ricci flow, while  $(S^n, g_{S^n}, f)$  with  $f \equiv \frac{n}{2}$  is a gradient shrinking Ricci soliton. By Proposition 5.3,

$$\begin{aligned}\tilde{\mathcal{V}}(g) &= \int_{S^n} (4\pi)^{-n/2} e^{-n/2} d\mu_{g_{S^n}} \\ &= \frac{\sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m}}{\Gamma(m+1)} \sqrt{\frac{2}{e}} \nearrow \sqrt{\frac{2}{e}} \quad \text{as } n \nearrow \infty.\end{aligned}$$

Here we have put  $n = 2m+1$ ,  $m \in \frac{1}{2}\mathbb{Z}^+$  and used that  $\text{Vol}(S^n, \frac{1}{2(n-1)}g_{S^n}) = 2\pi^{m+1}/\Gamma(m+1)$  and Stirling’s formula:

$$\Gamma(m+1) = \sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m} e^{\theta(m)} \quad \text{for } m > 0,$$

where  $\theta(m) \searrow 0$  as  $m \nearrow \infty$ . This gives an upper bound for the constant  $\varepsilon_n$  obtained in Theorem 4.1:

$$\varepsilon_n \leq 1 - e^{-\theta(m)} \sqrt{2e^{-1}} \searrow 1 - \sqrt{2e^{-1}} \quad \text{as } n \nearrow \infty.$$

*Remark 5.13.* (1) Feldman–Ilmanen–Ni [FIN] have discovered the *forward reduced volume*  $\tilde{V}_{(p,0)}^+(t)$  for the (forward) Ricci flow  $(M^n, g(t))$ ,  $t \in [0, T)$  which is non-increasing in  $t$ . However, its definition is given by

$$\tilde{V}_{(p,0)}^+(t) := \int_M (4\pi t)^{-n/2} \exp\left(\ell_{(p,0)}^+(\cdot, t)\right) d\mu_{g(t)}$$

(cf. with (5.12)) and it is not well-defined for general non-compact manifolds. It is not likely that Theorem 4.1 has an analogue for the forward reduced volume  $\tilde{V}_{(p,0)}^+(t)$ .

(2) One can also easily generalize the monotonicity of  $\tilde{V}_{(p,0)}^+(t)$  to the forward super Ricci flows  $\frac{\partial}{\partial t}g \geq -2\text{Ric}(g(t))$ , if the condition corresponding to Assumption 2.4 is imposed; see also Müller [Mü].

*Remark 5.14.* In Carrillo–Ni’s preprint [CaNi], the potential function  $f \in C^\infty$  of the gradient Ricci soliton  $(M^n, g, f)$  is normalized so that

$$\int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g = 1.$$

Then their main result is the logarithmic Sobolev inequality for gradient Ricci solitons with the constant  $\mu(g, f) := \lambda(R + |\nabla f|^2) - f$  as the best constant.

*Theorem 5.15* (Logarithmic Sobolev inequality for gradient shrinkers [CaNi]). *Let  $(M^n, g, f)$  be a complete gradient shrinking Ricci soliton. Then*

$$\int_M \left[ \lambda(|\nabla\psi|^2 + R) + \psi - n \right] \rho d\mu_g \geq -\mu(g, f) \quad (5.14)$$

for any non-negative function  $\rho := (4\pi\lambda)^{-n/2} e^{-\psi} \geq 0$  with compact support and  $\int_M \rho d\mu_g = 1$ .

They also showed that  $\mu(g, f) \geq 0$  for gradient shrinking Ricci solitons under the curvature condition stronger than ours and conjectured that  $\mu(g, f) = 0$  implies that it is the Gaussian soliton. It is easily checked that  $\mu(g, f) = -\log \Theta(M)$ , where  $\Theta(M)$  is the Gaussian density of  $(M^n, g, f)$  with  $f$  being normalized in our sense as in (1.3). Hence, Corollary 5.1.(2) gives an affirmative answer to the conjecture in [CaNi].

The Logarithmic Sobolev inequality of Carrillo–Ni was utilized by Cao–Zhu in order to prove that *any complete non-compact gradient shrinking Ricci soliton has infinite volume* (see Cao [Ca]).

*Remark 5.16.* Recall that, in the proof of Corollary 5.1, we have used the assumption that  $\text{Ric} \geq -K$  for some  $K \in \mathbb{R}$  only to ensure the existence of minimal  $\mathcal{L}$ -geodesics connecting any point in space-time to the base point. A natural question is whether the assumption on  $\text{Ric}$  in the statement of Corollary 5.1 is superfluous.

## Chapter 6

# Asymptotic volume ratio under the Ricci flow

In this short chapter, we consider the behavior of the asymptotic volume ratio under the Ricci flow. Recall that the asymptotic volume ratio, or shortly AVR,  $\nu(g)$  of a complete Riemannian manifold  $(M^n, g)$  with non-negative Ricci curvature is, by definition,  $\nu(g) := \lim_{r \rightarrow \infty} \text{Vol } B(p, r) / \omega_n r^n$ . Here  $\omega_n$  represents the volume of the unit ball in the Euclidean space  $(\mathbb{R}^n, g_E)$ .

The purpose of this chapter is to prove the following.

**Proposition 6.1.** *Let  $(M^n, g(t)), t \in [0, T]$  be a complete Ricci flow with bounded non-negative Ricci curvature. Then its asymptotic volume ratio  $\nu(g(t))$  is independent of  $t$ .*

Hamilton [Ha5, Theorem 18.3] proved this proposition under the assumption that the curvature operator is non-negative and decays to 0 at the spacial infinity, i.e.,  $|\text{Rm}|(x_i) \rightarrow 0$  as  $x_i \rightarrow \infty$ .

We invoke the following theorem of Cheeger–Colding [ChCo2, Theorem 5.9] to prove Proposition 6.1.

**Theorem 6.2** (Volume convergence theorem [ChCo2], also Cheeger [Chee]). *Let  $\{(M_i, p_i)\}$  be a sequence of pointed  $n$ -dimensional complete Riemannian manifolds with uniform lower Ricci curvature bound  $\text{Ric}(g_i) \geq -(n-1)K$  for some constant  $K \geq 0$ . Suppose that there is a positive constant  $v > 0$  such that*

$$\text{Vol}(B_{M_i}(p_i, 1)) \geq v \quad \text{for all } i = 1, 2, \dots,$$

*and  $\{(M_i, p_i)\}$  converges to a metric space  $(X, x)$  in the pointed Gromov–Hausdorff topology. Then, for any  $r > 0$ ,*

$$\lim_{i \rightarrow \infty} \text{Vol}(B_{M_i}(p_i, r)) = \mathcal{H}^n(B_X(x, r)), \quad (6.1)$$

*where  $\mathcal{H}^n$  denotes the  $n$ -dimensional Hausdorff measure which is normalized to agree with the  $n$ -dimensional Lebesgue measure.*

We need to give the definition of pointed Gromov–Hausdorff convergence. At first, recall that a metric space is said to be *proper* if any closed metric ball in it is compact.

**Definition 6.3.** We say that a sequence  $\{(X_i, x_i)\}$  of pointed proper metric spaces converges to  $(X, x)$  in the pointed Gromov–Hausdorff topology if for any  $\varepsilon > 0$ , there exists an  $\varepsilon$ -approximation map from  $(X, x)$  to  $(X_i, x_i)$  for all large  $i$ . A map  $f : (X, p) \rightarrow (Y, q)$  with  $f(p) = q$  is called an  $\varepsilon$ -approximation map if

$$\text{the } \varepsilon\text{-neighborhood of } f(B_X(p, 1/\varepsilon)) \text{ contains } B_Y(q, 1/\varepsilon), \quad (6.2)$$

and

$$|d(x, y) - d(f(x), f(y))| < \varepsilon \quad \text{for any } x, y \in B_X(p, 1/\varepsilon). \quad (6.3)$$

An  $\varepsilon$ -approximation map does not need to be continuous.

*Proof of Proposition 6.1.* We fix  $0 \leq t_1 < t_2 \leq T$  and  $K < \infty$  such that  $0 \leq \text{Ric} \leq K$  on  $M \times [t_1, t_2]$ . Since it follows easily from

$$e^{2K(t_2-t_1)}g(t_2) \geq g(t_1) \geq g(t_2)$$

that  $\nu(g(t_1)) = 0$  if and only if  $\nu(g(t_2)) = 0$ , we may assume that  $\nu(g(t)) > 0$ .

We recall the following lemma (e.g. [Ha5, Theorem 17.2]) which can be deduced from the second variational formula.

**Lemma 6.4** (Hamilton [Ha5], cf. Lemma A.1). *Let  $(M^n, g(t)), t \in [0, T]$  be a Ricci flow which is complete with Ricci curvature bounded above  $\text{Ric} \leq K$  by  $K \geq 0$ . Then, for any two points  $x, y \in M$ ,*

$$\frac{d^-}{dt}d_{g(t)}(x, y) \geq -\text{Const.}\sqrt{K}. \quad (6.4)$$

This lemma tells us that when the Ricci flow shrinks the metric, the distance does not shrink so much. From this lemma and the non-negativity of the Ricci curvature, we have

$$d_{g(t_1)}(x, y) \geq d_{g(t_2)}(x, y) \geq d_{g(t_1)}(x, y) - \text{Const.}\sqrt{K}(t_2 - t_1), \quad (6.5)$$

for any  $x, y \in M$ .

By Gromov’s pre-compactness theorem [Gr], there exists a sequence  $r_i \rightarrow \infty$  such that  $\{(M, r_i^{-2}g(t_1), p)\}$  converges to a metric space  $(X, x)$  in the pointed Gromov–Hausdorff topology. This  $(X, x)$  is called an *asymptotic cone* of  $(M, g(t_1), p)$ . In our case, due to (6.5), we know that  $\{(M, r_i^{-2}g(t_2), p)\}$  also converges to  $(X, x)$ , because an  $\varepsilon$ -approximation map of  $(X, x)$  into  $(M, r_i^{-2}g(t_1), p)$  is a  $2\varepsilon$ -approximation map of  $(X, x)$  into  $(M, r_i^{-2}g(t_2), p)$  as well for sufficiently large  $i$ . Namely, both of  $(M, g(t_1))$  and  $(M, g(t_2))$  have  $(X, x)$  as an asymptotic cone.

Under this setting, Proposition 6.1 follows from the volume convergence theorem quoted above. Indeed,

$$\begin{aligned} \nu(g(t_1)) &= \lim_{i \rightarrow \infty} \text{Vol } B(p, 1; r_i^{-2}g(t_1)) \\ &= \mathcal{H}^n(B_X(x, 1)) \\ &= \lim_{i \rightarrow \infty} \text{Vol } B(p, 1; r_i^{-2}g(t_2)) = \nu(g(t_2)). \end{aligned}$$

This proves Proposition 6.1. □

As a first application of Proposition 6.1, we are able to state a variant of the gap theorem we obtained in Chapter 4. Notice that Theorem 6.5 also generalizes Theorem 1.2 in the introduction which is a gap theorem for Ricci-flat manifolds.

**Theorem 6.5** (Gap theorem II for ancient solutions). *For any  $n \geq 2$ , there exists a constant  $\varepsilon'_n > 0$  satisfying the following: let  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  be an  $n$ -dimensional complete ancient solution to the Ricci flow with bounded non-negative Ricci curvature. Suppose that the asymptotic volume ratio  $\nu(g(\tau_*))$  of  $g(t_*)$  is greater than  $1 - \varepsilon'_n$  at some time  $\tau_* \in [0, \infty)$ . Then  $(M^n, g(\tau))$ ,  $\tau \in [0, \infty)$  is the Gaussian soliton.*

It follows from Proposition 6.1 that the asymptotic volume ratio  $\nu(g(\tau))$  of a Ricci flow with bounded non-negative Ricci curvature is constant in  $\tau$ . The proof of Theorem 6.5 is essentially the same as that of Theorem 4.1 and we leave it to the interested reader.

We also comment here that Theorem 6.5 is not true when the ancient solution  $g(\tau)$  in the statement is replaced with an *immortal solution*  $g(t)$ ,  $t \in [0, \infty)$  to the (forward) Ricci flow. In fact, one can show that any Ricci flow  $g(t)$ ,  $t \in [0, T)$  which has bounded non-negative curvature operator and the initial metric  $g(0) = g_0$  with positive  $\nu(g_0) > 0$  extends to the immortal solution  $g(t)$ ,  $t \in [0, \infty)$  (see Proposition 6.6 below). (See also the example in [CLN, Chapter 4, Section 5].)

The following proposition is another useful application of Proposition 6.1.

**Proposition 6.6.** *Let  $(M^n, g(t))$ ,  $t \in [0, T)$  be a complete Ricci flow with non-negative bounded curvature operator whose initial metric  $g(0) = g_0$  has Euclidean volume growth, i.e.,  $\nu(g(0)) > 0$ . Then we can extend  $(M, g(t))$  to the immortal solution  $(M, g(t))$ ,  $t \in [0, \infty)$  to the Ricci flow.*

*Proof.* Let  $(M^n, g(t))$ ,  $t \in [0, T)$  be a Ricci flow as in the proposition. Proposition 6.1 says that  $\nu(g(t)) = \nu(g(0))$  is constant in  $t$  and positive. Assume that it develops a singularity in finite time  $T < \infty$ . Then we know that  $\sup_{M \times [0, T)} |\text{Rm}| = \infty$  (Shi [Sh]) and take a blow-up limit  $(M_\infty^n, g_\infty(t))$ ,  $\tau \in (-\infty, 0]$  of  $(M^n, g(t))$  around the singular time  $t = T$ . Then the asymptotic volume ratio of  $(M_\infty^n, g_\infty(t))$  is not less than  $\nu(g(0)) > 0$  and hence  $(M_\infty^n, g_\infty(t))$  is a  $\kappa$ -solution with  $\nu(g_\infty(t)) \geq \nu(g(0)) > 0$ . This is in conflict with Proposition 3.17.  $\square$

Proposition 6.6 is closely related to the result in Ni [Ni2] for the Kähler Ricci flow.

# Chapter 7

## Asymptotic reduced volume

In the study of the Ricci and other geometric flows, monotone quantities have been playing significant roles. One of Perelman's achievements in his seminal paper [Pe] is the monotonicity of the reduced volume given by

$$\tilde{V}_{(p,0)}(\tau) := \int_M (4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(\cdot, \tau)) d\mu_{g(\tau)} \quad \text{for } \tau > 0$$

(Theorems 2.12 and 2.17). Here,  $\ell(\cdot, \tau) = \ell_{(p,0)}(\cdot, \tau)$  is the reduced distance from the base point  $(p, 0)$  (Definition 2.9). An application of the monotonicity of the reduced volume is to rule out local collapsing for the Ricci flow (Theorem 2.1), and even for the Ricci flow with surgery (see [Pe2]).

Subsequently, another monotone quantity of the form

$$I_{(p,0)}(r) := \frac{1}{r^n} \iint_{E_r} \left[ |\nabla \ell|^2 + R \left( n \log \frac{r}{\sqrt{4\pi\tau}} - \ell \right) \right] d\mu d\tau \quad \text{for } r > 0$$

was also discovered by Ecker–Knopf–Ni–Topping [EKNT]. Here  $E_r$  is a certain subset, called ‘pseudo heat ball’, of the space-time. The precise definition is given in the following section.

At this point, it is natural to ask the following question ([EKNT]).

*How is the local monotone quantity  $I_{(p,0)}(r)$  related to the global one  $\tilde{V}_{(p,0)}(\tau)$ ?*

Partially motivated by this question, we now state the main result of this chapter as follows:

**Theorem 7.1.** *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a complete ancient solution to the Ricci flow with bounded curvature. Then for any  $p \in M$ , we have*

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau) = \lim_{r \rightarrow \infty} I_{(p,0)}(r). \quad (7.1)$$

Our Theorem 7.1 can be thought of as a general answer to the question quoted above, as well as a Ricci flow analogue of the following (cf. Lemma 2.14): *for any Riemannian manifold  $(M^n, g)$  with non-negative Ricci curvature,*

$$\lim_{\tau \rightarrow \infty} \int_M (4\pi\tau)^{-n/2} \exp\left(-\frac{d(\cdot, p)^2}{4\tau}\right) d\mu = \lim_{r \rightarrow \infty} \frac{\text{Vol } B(p, r)}{\omega_n r^n}. \quad (7.2)$$

## 7.1 Definition

We first give a definition of the monotone quantity  $I_{(p,0)}(r)$ . Suppose that we have a super Ricci flow  $(M^n, g(\tau)), \tau \in [0, T]$  which is complete and whose time-derivative  $\frac{\partial}{\partial \tau} g$  is bounded below by a constant on  $M \times [0, T]$ . Let  $K = K_{(p,0)}$  be a reduced volume density:

$$K(q, \tau) = K_{(p,0)}(q, \tau) := (4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(q, \tau)). \quad (7.3)$$

Recall that if  $(M^n, g(\tau))$  is  $C^1$ -controlled, we have seen in Theorem 2.17 that the quantity defined by

$$\tilde{V}_{(p,0)}^\varphi(\tau) := \int_M K(\cdot, \tau) \varphi(\cdot, \tau) d\mu_{g(\tau)}$$

is non-increasing in  $\tau$  for any non-negative function  $\varphi(\cdot, \tau) \geq 0$  on  $M \times [0, T]$  satisfying  $(\frac{\partial}{\partial \tau} + \Delta_{g(\tau)}) \varphi \leq 0$  in the distributional sense, i.e.,

$$\int_M \left[ \xi \frac{\partial \varphi}{\partial \tau} - \langle \nabla \xi, \nabla \varphi \rangle \right] d\mu_{g(\tau)} \leq 0 \quad (7.4)$$

for any non-negative smooth function  $\xi \geq 0$  on  $M \times [0, T]$  with compact support.

**Definition 7.2** ([EKNT], [Ni3]). For any  $r > 0$ , we let  $E_r$  denote the ‘pseudo heat ball’ given by

$$E_r := \{(q, \tau) \in M \times (0, T] \mid K(q, \tau) > r^{-n}\}.$$

Then for a non-negative function  $\varphi \geq 0$  on  $M \times [0, T]$ , we define

$$I_{(p,0)}^\varphi(r) := \frac{1}{r^n} \iint_{E_r} \left( |\nabla \log(Kr^n)|^2 + H \log(Kr^n) \right) \varphi d\mu d\tau$$

and

$$J_{(p,0)}^\varphi(r) := \iint_{\partial E_r} \frac{|\nabla K|^2}{\sqrt{|\nabla K|^2 + \left| \frac{\partial}{\partial \tau} K \right|^2}} \varphi d\tilde{A} + \frac{1}{r^n} \iint_{E_r} H \varphi d\mu d\tau.$$

Here  $d\tilde{A}$  is the area element induced by the product metric  $\tilde{g} := g(\tau) + d\tau^2$  on  $M \times (0, T)$ .

The following proposition stated in [Ni3] gives the relation between  $I_{(p,0)}^\varphi(r)$  and  $J_{(p,0)}^\varphi(r)$ .

**Proposition 7.3** (Ni [Ni3]). *For any  $r > 0$ ,*

$$I_{(p,0)}^\varphi(r) = \frac{n}{r^n} \int_0^r \eta^{n-1} J_{(p,0)}^\varphi(\eta) d\eta.$$

We state a theorem stating that the quantity defined above is monotone.

**Theorem 7.4** (Monotonicity of  $I_{(p,0)}(r)$  [EKNT]). *Let  $(M^n, g(\tau)), \tau \in [0, T]$  be a complete super Ricci flow satisfying Assumption 2.4 with time-derivative  $\frac{\partial}{\partial \tau} g$  bounded below. Suppose that  $\varphi \geq 0$  satisfies  $(\frac{\partial}{\partial \tau} + \Delta_{g(\tau)}) \varphi \leq 0$  in the distributional sense. For any  $p \in M$ , find  $r_* > 0$  such that  $r \in (0, r_*)$  implies that  $E_r \subset M \times [0, T - \varepsilon]$  for some  $\varepsilon = \varepsilon(r) > 0$ . Then  $I_{(p,0)}^\varphi(r)$  is non-increasing in  $r \in (0, r_*)$  and  $\lim_{r \rightarrow 0^+} I_{(p,0)}^\varphi(r) = \varphi(p, 0)$ .*

It was shown by Ni [Ni3] that  $J_{(p,0)}^\varphi(r)$  is non-increasing in  $r$  as well for any smooth  $\varphi \geq 0$  and sufficiently small  $r > 0$  so that  $K$  is also smooth on  $E_r$ . His point is that the monotonicity of  $I_{(p,0)}^\varphi(r)$  is a consequence of that of  $J_{(p,0)}^\varphi(r)$  and Proposition 7.3. The following fact is well-known:

$$\text{If } \frac{f(r)}{g(r)} \text{ is non-increasing in } r > 0, \text{ then so is } \frac{\int_0^r f(\eta)d\eta}{\int_0^r g(\eta)d\eta}.$$

An example of a locally Lipschitz function  $\varphi \geq 0$  on the space-time as in the above theorems, other than the constant function  $\varphi \equiv 1$ , is the function

$$\varphi(q, \tau) := \max \left\{ 0, \frac{\bar{L}_{(p_*,0)}(q, \tau) - 2n\tau}{\rho^2} \right\},$$

where  $\bar{L}_{(p_*,0)}(q, \tau) := 4\tau\ell_{(p_*,0)}(q, \tau)$  with  $(p_*, 0) \in M \times \{0\}$  and  $\rho > 0$  is a positive constant. This is a consequence of the inequality ([Pe, (7.15)]):

$$\left( \frac{\partial}{\partial \tau} + \Delta_{g(\tau)} \right) \bar{L} \leq 2n, \quad (7.5)$$

which follows from the computations in Section 2.3. A function which is similar but has compact support was utilized by Ni [Ni4] in order to localize the *forward Reduced volume* of Feldman–Ilmanen–Ni [FIN] and Perelman’s  $\mathcal{W}$ -functional (cf. Proposition 2.5).

See Ecker [Ec] for earlier works on (local) monotonicity formulae and their applications for mean curvature flow.

Let us look at examples. In our terminology, Watson’s mean value formula for heat equations ([Wa], also Evans [Ev]) can be stated as follows:

**Example 7.5.** Let  $(\mathbb{R}^n, g(\tau) \equiv g_E), \tau \in [0, \infty)$  be the Gaussian soliton and  $\varphi = \varphi(\cdot, \tau)$  be a smooth solution to the heat equation  $(\frac{\partial}{\partial \tau} + \Delta)\varphi = 0$ . (Recall that  $\tau$  is the backward time.) Take any  $p \in \mathbb{R}^n$ . Then  $K_{(p,0)}(\cdot, \tau)$  is the heat kernel and

$$I_{(p,0)}^\varphi(r) = \frac{1}{r^n} \iint_{E_r} |\nabla \log K|^2 \varphi \, d\mu d\tau = \varphi(p, 0) \quad \text{for all } r > 0.$$

As a special case, we consider the static super Ricci flow.

**Proposition 7.6.** *Let  $(M^n, g)$  be a complete Riemannian manifold of non-negative Ricci curvature regarded as a static super Ricci flow, i.e.,  $g(\tau) \equiv g$ . Then for any  $p \in M$ ,*

$$\lim_{r \rightarrow \infty} I_{(p,0)}(r) = \nu(g) = \lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau). \quad (7.6)$$

Here  $\nu(g)$  denotes the asymptotic volume ratio of  $(M^n, g)$  as before.

*Proof.* Because the second equality of (7.6) is a consequence of Lemma 2.14, it suffices to show the first equality.

As was shown in [EKNT, Lemma 9], with  $\psi := \log K$ ,

$$I_{(p,0)}(r) = \frac{1}{r^n} \iint_{E_r} \left[ |\nabla \psi|^2 - \frac{\partial \psi}{\partial \tau} \right] d\mu d\tau \quad \text{for all } r > 0. \quad (7.7)$$

Using (7.7) and that  $\ell_{(p,0)}(\cdot, \tau) = \frac{1}{4\tau}d(\cdot, p)^2$ , we know

$$\begin{aligned} I_{(p,0)}(r) &= \frac{1}{r^n} \iint_{E_r} \frac{n}{2\tau} d\mu d\tau = \frac{1}{r^n} \int_0^{\frac{r^2}{4\pi}} \frac{n}{2\tau} \text{Vol } B \left( p, \sqrt{2n\tau \log \frac{r^2}{4\pi\tau}} \right) d\tau \\ &= \frac{1}{r^n} \int_0^\infty \text{Vol } B \left( p, \sqrt{\frac{r^2}{\pi} u \exp\left(-\frac{2}{n}u\right)} \right) du, \end{aligned}$$

where we change the variable by  $u := \frac{n}{2} \log\left(\frac{r^2}{4\pi\tau}\right)$ .

Since, by the Bishop–Gromov inequality,

$$\omega_n r^n \geq \text{Vol } B(p, r) \geq \nu(g) \omega_n r^n \quad \text{for any } r > 0,$$

we have

$$I_{(p,0)}(r) \geq \nu(g) \frac{\omega_n}{\pi^{n/2}} \Gamma\left(\frac{n}{2} + 1\right) = \nu(g).$$

On the other hand, for any  $\varepsilon > 0$ , find  $R(\varepsilon) > 0$  such that

$$\text{Vol } B(p, r) \leq (\nu(g) + \varepsilon) \omega_n r^n \quad \text{for } r \geq R(\varepsilon),$$

and for some fixed  $r > 0$ , let  $[u_1, u_2]$  be the maximal interval such that

$$\frac{r^2}{\pi} u \exp\left(-\frac{2}{n}u\right) \geq R(\varepsilon)^2 \quad \text{for } u \in [u_1, u_2].$$

Notice that  $[u_1, u_2] \rightarrow [0, \infty)$  as  $r \rightarrow \infty$ . Letting  $\Gamma(a, b) := \int_a^b u^{n/2} e^{-u} du$ , we obtain

$$\begin{aligned} I_{(p,0)}(r) &\leq \frac{\omega_n}{\pi^{n/2}} \left[ (\nu(g) + \varepsilon) \Gamma(u_1, u_2) + \Gamma(0, u_1) + \Gamma(u_2, \infty) \right] \\ &\rightarrow \nu(g) + \varepsilon \quad \text{as } r \rightarrow \infty. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, this proves the proposition.  $\square$

Now, let us go back to the question at the beginning of this chapter. In order to answer the question, Ecker et al. [EKNT] focused on gradient shrinking Ricci solitons. Recall that a gradient shrinking Ricci soliton is a triple  $(M^n, g, f)$  satisfying

$$\text{Ric}(g) + \text{Hess } f = \frac{1}{2\lambda} g \quad \text{for some constant } \lambda > 0.$$

A shrinking Ricci soliton naturally gives rise to an ancient solution  $(M, g_0(\tau)), \tau \in (0, \infty)$  (see (5.3)). An important geometrical quantity associated to a gradient shrinking Ricci soliton  $(M^n, g, f)$  is the Gaussian density

$$\Theta(M) := \int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g.$$

In this setting, Ecker et al. prove

**Proposition 7.7** ([EKNT, Corollary 18]). *Let  $(M, g_0(\tau)), \tau \in (0, \infty)$  be the ancient solution to the Ricci flow determined by a compact gradient shrinking Ricci soliton  $(M, g, f)$ . Fix any  $p \in M$ . Then for any  $\tau > 0$  and  $r > 0$ ,*

$$\tilde{V}_{(p,0)}(\tau) = I_{(p,0)}(r) = \Theta(M).$$

Note that in the situation of Proposition 7.7,  $(M, g_0(\tau))$  shrinks to a point as  $\tau \rightarrow 0+$  and what they are dealing with is the reduced distance and volume from the *singular* point. Hence the results like

- $\Theta(M) \leq 1$  and
- $\Theta(M) = 1 \iff (M^n, g)$  is isometric to  $(\mathbb{R}^n, g_E)$

does not follow from Proposition 7.7 (immediately, at least). It was [CaNi] and [Yo2] where such comparison geometric results were established under respective curvature conditions. This is how our Theorem 7.1 differs from Proposition 7.7.

We should mention Hamilton–Sesum [HaSe], Naber [Na] and Enders [Ed] where the reduced distance/volume based at a singular point was studied under some singularity assumptions. Among them, Naber [Na] used the reduced distance function from the Type I singular point to prove that any blow-up limit of Type I singularity of the Ricci flow is a gradient shrinking Ricci soliton (see the remark following Corollary 3.11).

Now we are in a position to formulate the main result of this chapter. Recall that we say that a super Ricci flow  $(M, g(\tau)), \tau \in [0, T]$  is  $C^1$ -controlled (section 2.4) when we can find a positive function  $K(\tau) > 0$  of  $\tau$  such that

$$\sup_{M \times [0, \tau]} \{ |h| + |\nabla H|^2 \} \leq K(\tau) \quad \text{for each } \tau \in (0, T].$$

**Theorem 7.8.** *Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be a super Ricci flow which is complete, ancient and  $C^1$ -controlled. Then for any  $p \in M$  and a non-negative locally-Lipschitz function  $\varphi \geq 0$  on  $M \times [0, \infty)$  with  $(\frac{\partial}{\partial \tau} + \Delta_{g(\tau)})\varphi \leq 0$  in the distributional sense, we have*

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}^\varphi(\tau) = \lim_{r \rightarrow \infty} I_{(p,0)}^\varphi(r). \quad (7.8)$$

Clearly, Theorem 7.8 contains Theorem 7.1 as a special case.

## 7.2 Proof of Theorem 7.1

In this section, we describe the proof of Theorem 7.8 which generalizes Theorem 7.1.

*Proof of Theorem 7.8.* In order to establish (7.8), we follow the same line as in the proof of Theorem 7.4 given in [EKNT].

First of all, we fix small  $\varepsilon > 0$ . Take a  $C^\infty$ -function  $\eta : (-\infty, \infty) \rightarrow [0, \infty)$  such that the support is contained in  $[0, \varepsilon]$  and  $\int_0^\infty \eta(y) dy = 1$ . We define

$$\zeta(x) := \int_0^x \eta(y) dy \quad \text{and} \quad Z(x) := \int_0^x \zeta(y) dy.$$

Notice that  $\eta$  is a smooth approximation of the Delta function, and hence,  $\zeta$  and  $Z$  approach to the Heviside function  $\chi$  and the function  $x \mapsto [x]_+ := \max\{x, 0\}$  as  $\varepsilon \rightarrow 0+$ , respectively. More precisely, for any  $x \in (-\infty, \infty)$ , we have

$$\chi(x - \varepsilon) \leq \zeta(x) \leq \chi(x)$$

and hence

$$[x - \varepsilon]_+ \leq Z(x) \leq [x]_+.$$

Set

$$Q(s, r) := \iint_{M \times [s, \infty)} \left( |\nabla \log K|^2 \zeta(\log(Kr^n)) + HZ(\log(Kr^n)) \right) \varphi \, d\mu d\tau$$

for all  $s \geq 0$  and  $r > 0$ . We also put  $\mathcal{I}(s, r) := Q(s, r)/r^n$  and  $\mathcal{I}(r) := \mathcal{I}(0, r)$ . Then we know

$$e^{-\varepsilon} I_{(p,0)}^\varphi(e^{-\varepsilon/n} r) \leq \mathcal{I}(r) \leq I_{(p,0)}^\varphi(r) \quad \text{for all } r > 0. \quad (7.9)$$

Here we used the fact (Proposition 3.5, cf. Chen [Che]) that: *For any complete ancient super Ricci flow  $(M, g(\tau))$  satisfying (2.3),  $H$  is non-negative on  $M \times [0, \infty)$ .*

Next, we have

$$\begin{aligned} & \frac{d}{dr} \mathcal{I}(s, r) \\ &= \frac{n}{r^{n+1}} \left( \frac{r}{n} \frac{d}{dr} Q(s, r) - Q(s, r) \right) \\ &= \frac{n}{r^{n+1}} \iint_{M \times [s, \infty)} \left( |\nabla \log K|^2 \zeta' + HZ' - |\nabla \log K|^2 \zeta - HZ \right) \varphi \, d\mu d\tau. \end{aligned}$$

We use the well-known formula  $\frac{\partial}{\partial \tau} d\mu_{g(\tau)} = H d\mu_{g(\tau)}$  to get

$$\frac{d}{d\tau} \int_M Z(\log(Kr^n)) \varphi \, d\mu_{g(\tau)} = \int_M \left[ \left( Z' \frac{1}{K} \frac{\partial K}{\partial \tau} + ZH \right) \varphi + Z \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)}.$$

Since  $\int_{M \times \{s'\}} Z \varphi \, d\mu = 0$  for all  $s' \gg 1$  with  $M \times [0, s') \supset E_r$ , we obtain

$$- \int_{M \times \{s\}} Z(\log(Kr^n)) \varphi \, d\mu = \iint_{M \times [s, \infty)} \left[ \left( Z' \frac{1}{K} \frac{\partial K}{\partial \tau} + ZH \right) \varphi + Z \frac{\partial \varphi}{\partial \tau} \right] d\mu d\tau.$$

Thus

$$\frac{d}{dr} \mathcal{I}(s, r) = \frac{n}{r^{n+1}} \iint_{M \times [s, \infty)} \mathcal{A} \, d\mu d\tau + \int_{M \times \{s\}} Z(\log(Kr^n)) \varphi \, d\mu, \quad (7.10)$$

where we let

$$\mathcal{A} := \left( |\nabla \log K|^2 \zeta' + HZ' - |\nabla \log K|^2 \zeta + Z' \frac{1}{K} \frac{\partial K}{\partial \tau} \right) \varphi + Z \frac{\partial \varphi}{\partial \tau}.$$

We also let

$$\begin{aligned} \mathcal{A}_* &:= - \langle \nabla \ell, \nabla(\zeta \varphi) \rangle + \left( -|\nabla \ell|^2 + H + \frac{1}{K} \frac{\partial K}{\partial \tau} \right) (\zeta \varphi) \\ &\quad + \left( Z \frac{\partial \varphi}{\partial \tau} - \langle \nabla Z, \nabla \varphi \rangle \right) \end{aligned}$$

to observe that

$$\int_M \mathcal{A} \, d\mu_{g(\tau)} = \int_M \mathcal{A}_* \, d\mu_{g(\tau)} \leq 0 \quad \text{for all } \tau > 0. \quad (7.11)$$

To show (7.11), we have used

$$\begin{aligned} \int_M |\nabla \log K|^2 \zeta' \varphi \, d\mu &= \int_M \langle \nabla \log K, \nabla \zeta \rangle \varphi \, d\mu \\ &= \int_M \left[ \langle \nabla \log K, \nabla(\zeta \varphi) \rangle - \zeta \langle \nabla \log K, \nabla \varphi \rangle \right] d\mu \\ &= \int_M \left[ \langle \nabla \log K, \nabla(\zeta \varphi) \rangle - \langle \nabla Z, \nabla \varphi \rangle \right] d\mu. \end{aligned}$$

The last inequality in (7.11) follows from (7.4) and (2.21).

Integrating (7.10) for  $r_1 \leq r \leq r_2$  yields

$$\begin{aligned} & \mathcal{I}(s, r_2) - \mathcal{I}(s, r_1) \\ &= \int_{r_1}^{r_2} \frac{n}{r^{n+1}} dr \int_s^\infty d\tau \int_M \mathcal{A} d\mu_{g(\tau)} + E(s; r_1, r_2) \\ &= \int_s^\infty d\tau \int_{r_1}^{r_2} \frac{n}{r^{n+1}} dr \int_M \mathcal{A}_* d\mu_{g(\tau)} + E(s; r_1, r_2), \end{aligned}$$

where  $E(s; r_1, r_2)$  denotes the extra term:

$$E(s; r_1, r_2) := \int_{r_1}^{r_2} dr \int_{M \times \{s\}} Z(\log(Kr^n)) \varphi d\mu.$$

We have applied Fubini's theorem. This can be done freely due to (7.11).

Now we see that  $E(s; r_1, r_2) \rightarrow 0$  as  $s \rightarrow 0+$ . Indeed,

$$\begin{aligned} & \limsup_{s \rightarrow 0+} E(s; r_1, r_2) \\ & \leq \limsup_{s \rightarrow 0+} (r_2 - r_1) \left( n \log \frac{r_2}{\sqrt{4\pi s}} \right) \text{Vol}_{g(s)} \left\{ \ell(\cdot, s) < n \log \frac{r_2}{\sqrt{4\pi s}} \right\} \\ & \leq \limsup_{s \rightarrow 0+} (r_2 - r_1) \left( n \log \frac{r_2}{\sqrt{4\pi s}} \right) \omega_n \left( 4ns \log \frac{r_2}{\sqrt{4\pi s}} \right)^{n/2} \\ & = 0. \end{aligned}$$

With this observation, letting  $s \rightarrow 0+$  yields

$$\mathcal{I}(r_2) - \mathcal{I}(r_1) = \int_0^\infty d\tau \int_{r_1}^{r_2} \frac{n}{r^{n+1}} dr \int_M \mathcal{A}_* d\mu_{g(\tau)} \leq 0, \quad (7.12)$$

which implies the monotonicity of  $\mathcal{I}(r)$  in  $r > 0$ . (We can now let  $\varepsilon \rightarrow 0+$  to see that  $I_{(p,0)}^\varphi(r)$  is non-increasing in  $r > 0$ .)

Here, for any positive  $K > 0$ , we set

$$K_\eta := \int_0^\infty \frac{n}{r^{n+1}} \eta(\log(Kr^n)) dr,$$

and  $K_\zeta$  and  $K_Z$  are also defined similarly. It is easy to check by using the integration by parts that

$$K_\eta = K_\zeta = K_Z = e^{\delta(\eta)} K, \text{ where } e^{\delta(\eta)} := \int_0^\infty \eta(y) e^{-y} dy.$$

Notice that  $\delta(\eta) \leq 0$  and  $\delta(\eta) \rightarrow 0$  as  $\varepsilon \rightarrow 0+$ .

Then for any  $\tau > 0$ ,

$$\begin{aligned} & \int_0^\infty \frac{n}{r^{n+1}} dr \int_M \mathcal{A} d\mu_{g(\tau)} \\ &= \int_M \left[ \left( |\nabla \ell|^2 K_\eta + HK_\zeta - |\nabla \ell|^2 K_\zeta + \frac{K_\zeta}{K} \frac{\partial K}{\partial \tau} \right) \varphi + K_Z \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)} \\ &= e^{\delta(\eta)} \int_M \left[ \left( |\nabla \ell|^2 K + HK - |\nabla \ell|^2 K + \frac{\partial K}{\partial \tau} \right) \varphi + K \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)} \\ &= e^{\delta(\eta)} \int_M \left[ \left( HK + \frac{\partial K}{\partial \tau} \right) \varphi + K \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)}. \end{aligned}$$

We are implicitly using the integrability of each term (Proposition 2.19) to derive the equations above.

By letting  $r_2 \rightarrow \infty$  and  $r_1 \rightarrow 0$  in (7.12), we then get

$$\begin{aligned} & \lim_{r \rightarrow \infty} \mathcal{I}(r) - \lim_{r \rightarrow 0} \mathcal{I}(r) \\ &= \int_0^\infty d\tau \int_0^\infty \frac{n}{r^{n+1}} dr \int_M \mathcal{A} d\mu_{g(\tau)} \\ &= e^{\delta(n)} \int_0^\infty d\tau \int_M \left[ \left( HK + \frac{\partial K}{\partial \tau} \right) \varphi + K \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)}. \end{aligned}$$

Finally, we take  $\varepsilon \rightarrow 0+$  and use (7.9) to conclude that

$$\lim_{r \rightarrow \infty} I_{(p,0)}^\varphi(r) - \varphi(p,0) = \int_0^\infty d\tau \int_M \left[ \left( HK + \frac{\partial K}{\partial \tau} \right) \varphi + K \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)}. \quad (7.13)$$

On the other hand, we know that

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}^\varphi(\tau) - \varphi(p,0) = \int_0^\infty d\tau \int_M \left[ \left( \frac{\partial K}{\partial \tau} + KH \right) \varphi + K \frac{\partial \varphi}{\partial \tau} \right] d\mu_{g(\tau)}. \quad (7.14)$$

Combining (7.13) and (7.14) completes the proof of Theorem 7.8.  $\square$

Now, with the help of Theorem 7.1, we are able to restate Theorem 4.1 as follows:

**Theorem 7.9** (Gap theorem III for ancient solutions). *The constant  $\varepsilon_n > 0$  we obtained in Theorem 4.1 satisfies the following property as well: Let  $(M^n, g(\tau)), \tau \in [0, \infty)$  be an  $n$ -dimensional complete ancient solution to the Ricci flow with bounded curvature such that*

$$\lim_{r \rightarrow \infty} I_{(p,0)}(r) > 1 - \varepsilon_n \quad \text{for some } p \in M.$$

*Then  $(M^n, g(\tau)), \tau \in [0, \infty)$  must be the Gaussian soliton  $(\mathbb{R}^n, g_E)$ .*

It is interesting to compare Theorem 7.9 with the following local regularity theorem for the Ricci flow, which is originally due to Ni [Ni3, Theorem 4.4]. See [Ec] and [Si] for local regularity theorems for mean curvature flow and harmonic maps, respectively.

**Theorem 7.10** ( $\varepsilon$ -regularity theorem, cf. [Ni3]). *For any  $n \geq 2$ , there exists constants  $\varepsilon = \varepsilon(n) > 0$ ,  $C < \infty$  and  $\rho_0 > 0$  such that: Let  $(M^n, g(\tau)), \tau \in [0, r_0^2)$  be a complete backward Ricci flow and  $o \in M$  be a fixed point. Suppose that  $(M^n, g(\tau))$  satisfies that*

$$I_{(p,\tau)}(\rho_0) > 1 - \varepsilon$$

*for all  $(p, \tau)$  with  $|\text{Rm}|(p, \tau) > C(r_0^2 - \tau)^{-1}$ ,  $d_{g(\tau)}(o, p) < 2r_0$ ,  $\tau \in [0, r_0^2)$  and*

$$|\text{Rm}| \leq 2|\text{Rm}|(p, \tau) =: 2Q \quad \text{on } B_{g(\tau)}(p, r_0 Q^{-1/2}) \times [\tau, \tau + CQ^{-1}].$$

*Here  $I_{(p,\tau)}(\rho_0)$  denotes  $I_{(p,0)}^{g_\tau}(\rho_0)$  for  $g_\tau(s) := g(s - \tau)$ ,  $s \in [0, r_0^2 - \tau)$ , as in Lemma 4.2.*

*Then we have*

$$|\text{Rm}|(p, \tau) \leq C(r_0^2 - \tau)^{-1}$$

*for all  $(p, \tau)$  with  $d_{g(\tau)}(o, p) < r_0$  and  $\tau \in [0, r_0^2)$ .*

Let us reconsider gradient shrinking Ricci solitons. We let  $g_S(\tau) := g_0(\tau + s)$ ,  $\tau \in [0, \infty)$  for some fixed  $s > 0$ , where  $g_0(\tau)$  is the ancient solution defined in (5.3). Notice that  $\tau = 0$  is no longer the singular time for  $(M, g_S(\tau))$ .

Combined with Proposition 5.3, Theorem 7.1 implies the following corollary (compare with Proposition 7.7).

**Corollary 7.11.** *Let  $(M^n, g_S(\tau))$ ,  $\tau \in [0, \infty)$  be the ancient solution to the Ricci flow determined by a complete gradient shrinking Ricci soliton  $(M^n, g, f)$  with bounded curvature. Then for any  $p \in M$ ,*

$$\lim_{\tau \rightarrow \infty} \tilde{V}_{(p,0)}(\tau) = \lim_{r \rightarrow \infty} I_{(p,0)}(r) = \Theta(M).$$

We conclude this chapter with a few remarks.

*Remark 7.12.* (1) The author wonders whether Theorem 7.1 still holds under the assumption of Theorem 4.1, i.e., only the Ricci curvature is bounded from below. He believes that a more understanding of Perelman's reduced geometry from a geometric viewpoint is required to attack the problem. The reduced volume  $\tilde{V}_{(p,0)}(\tau)$  and  $I_{(p,0)}(r)$  are well-defined as long as  $\frac{\partial}{\partial \tau} g$  is bounded from below (see [Ye2], [EKNT]).

(2) The reader is referred to Ni's paper [Ni] where he raises an interesting question closely related to our Theorem 7.1.

## Chapter 8

# Harnack implies bounded curvature in dimension 2

In this chapter, we consider the Ricci flow  $(M^n, g(t)), t \in [0, T]$  with non-negative curvature operator. A remarkable achievement of Hamilton [Ha3] is the following Harnack inequality for the Ricci flow with non-negative bounded curvature operator, by which we are able to compare the curvatures at different points in space-time. We state its trace version.

**Theorem 8.1** (Trace Harnack inequality, Hamilton [Ha3]). *Let  $(M^n, g(t)), t \in [0, T]$  be a complete Ricci flow with bounded non-negative curvature operator. Then, we have*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2 \langle \nabla R, V \rangle + 2 \operatorname{Ric}(V, V) \geq 0 \quad (8.1)$$

for any vector field  $V$  on  $M$ .

Hamilton's Harnack inequality has its root in Li–Yau's paper [LiYa]. This inequality was generalized by Chow–Hamilton [ChHa] as follows. We use the Einstein convention to state it.

**Theorem 8.2** (Linear trace Harnack inequality, Chow–Hamilton [ChHa]). *Let  $(M^n, g(t)), t \in [0, T]$  be a complete Ricci flow with bounded non-negative curvature operator. Suppose that we have a weakly positive definite symmetric 2-tensor  $h_{ij} \geq 0$ , with appropriate bound on the growth order, evolving the heat equation:*

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} + 2 R_{pijq} h_{pq} - R_{ip} h_{jp} - R_{jp} h_{ip}. \quad (8.2)$$

Then, letting  $H := \operatorname{tr}_{g(t)} h$  be the trace of  $h_{ij}$ , we have

$$Z := \operatorname{div}(\operatorname{div}(h)) + \langle \operatorname{Ric}, h \rangle + \langle 2 \operatorname{div}(h), V \rangle + h(V, V) + \frac{H}{2t} \geq 0 \quad (8.3)$$

for any vector field  $V$  on  $M$ .

Recall that the Ricci tensor  $R_{ij}$  of the Ricci flow  $(M^n, g(t))$  evolves along the heat equation (e.g. [Vol1, Lemma 6.9]):

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2 R_{pijq} R_{pq} - 2 R_{ip} R_{jp}, \quad (8.4)$$

while the evolution equation for the scalar curvature  $R$  is given by

$$\frac{\partial}{\partial t}R = \Delta R + 2|R_{ij}|^2 \quad (8.5)$$

(e.g. [Vol1, Lemma 6.7]). From these equations and the second Bianchi identity:  $2\nabla_j R_{ij} = \nabla_i R$ , we know that Theorem 8.2 indeed generalizes Theorem 8.1.

The proofs of Theorems 8.1 and 8.2 rely on the argument based on the maximum principle. Hence the assumptions of bounded curvature are essential. Now, here is a natural question: *Is it possible to show Hamilton's Harnack estimate for the Ricci flow if we do not assume that the curvature is bounded* (e.g. [CLN, Problem 10.45]).

The goal of this chapter is to prove the following proposition, which can be regarded as a negative answer to the above question. In dimension 2, the Ricci curvature is the Gauss curvature as well as the half of the scalar curvature.

**Proposition 8.3** (Harnack implies bounded curvature). *Let  $(M^2, g(t)), t \in [0, T]$  be a complete Ricci flow with non-negative curvature on a surface  $M^2$ . Suppose that it satisfies the trace Harnack inequality:*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\langle \nabla R, V \rangle + Rg(V, V) \geq 0. \quad (8.6)$$

*Then  $(M^2, g(t))$  has bounded curvature. More precisely, we can find a positive constant  $C > 0$  such that*

$$R \leq CTt^{-1} \quad \text{on } M^2 \times (0, T].$$

**Corollary 8.4.** *Any two-dimensional complete ancient solution with Harnack  $(M^2, g(t)), t \in (-\infty, 0]$ , i.e., ancient solution satisfying*

$$\frac{\partial R}{\partial t} + 2\langle \nabla R, V \rangle + Rg(V, V) \geq 0, \quad (8.7)$$

*has uniformly bounded non-negative curvature. More precisely, we can find a positive constant  $C > 0$  such that*

$$0 \leq R \leq C \quad \text{on } M^2 \times (-\infty, 0].$$

These results are rephrased as follows: *In dimension 2, any Ricci flow with non-negative unbounded curvature, if exists, does not enjoy the Harnack inequality.*

In the following section, we will prove Hamilton's point picking lemma for Riemannian manifolds whose scalar curvature are not necessarily assumed to be bounded. After that, we turn to the proof of Proposition 8.3 in the second section.

## 8.1 Hamilton's point picking lemma

In order to prove Proposition 8.3 by contradiction, we need a sort of point picking argument. For this purpose, we prove the following lemma, which was shown by Hamilton [Ha5, Lemma 22.2] for manifolds with bounded scalar curvature.

**Lemma 8.5** (Hamilton’s point picking lemma, cf. [Ha5]). *Let  $M^n$  be a complete non-compact Riemannian manifold with non-negative scalar curvature  $R \geq 0$ . Fix a base point  $o \in M$  and suppose that we have a sequence  $\{p_{0,j}\}$  of points in  $M$  such that*

$$R(p_{0,j})s_{0,j}^2 \rightarrow \infty \quad \text{as } j \rightarrow \infty, \quad (8.8)$$

where  $s_{0,j} := d(p_{0,j}, o)$  is the distance from the base point.

Then we can find sequences  $\{p_j\}$  of points in  $M$ ,  $\{r_j\}$  of radii and  $\{\delta_j\}$  of positive constants with  $\delta_j \rightarrow 0$  as  $j \rightarrow \infty$  such that

(a)  $R(p) \leq (1 + \delta_j)R(p_j)$  for all  $p$  in  $B(p_j, r_j)$ .

(b)  $R(p_j)r_j^2 \rightarrow \infty$  as  $j \rightarrow \infty$ .

(c)  $s_j/r_j \rightarrow \infty$  as  $j \rightarrow \infty$ , where  $s_j := d(p_j, o)$ .

(d)  $R(p_j) \geq R(p_{0,j})$  for each  $j = 1, 2, \dots$ .

The assumption (8.8) implies that the *asymptotic scalar curvature ratio*, often abbreviated to ASCR, defined by

$$A(M) := \limsup_{x \rightarrow \infty} R(x)d(o, x)^2 \quad (8.9)$$

is equal to  $\infty$ . The value of ASCR is independent of the choice of the base point  $o \in M$ . The invariant, which appears in the paper of Petrunin–Tuschmann [PeTu], obtained by replacing the scalar curvature  $R(x)$  in (8.9) with the maximum absolute value of the sectional curvature at  $x \in M$  also contains important geometric information of the manifold  $M$ .

*Proof of Lemma 8.5.* First, set  $A_j := R(p_{0,j})s_{0,j}^2$  and find a sequence  $\{\varepsilon_j\}$  of positive numbers with

$$A_j\varepsilon_j^2 \rightarrow 0 \quad \text{and} \quad \varepsilon_j \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Define  $\delta_j > 0$  by  $(1 + \varepsilon_j)^2 = 1 + \delta_j$ . We then take the smallest  $\sigma_j > 0$  such that

$$\max \{R(q)d(q, o)^2 \mid d(q, o) \leq \sigma_j\} \geq A_j.$$

Then there exists a point  $q_j$  in  $M$  with

$$R(q_j)\sigma_j^2 = A_j \quad \text{and} \quad d(q_j, o) = \sigma_j.$$

By construction, we know that

$$\sigma_j \leq s_{0,j} \quad \text{and} \quad R(q_j) \geq R(p_{0,j}).$$

Now we can find a point  $p_j$  in  $M \setminus B(o, \sigma_j)$  such that  $R(p_j) \geq R(q_j)$  and

$$R(p) \leq (1 + \delta_j)R(p_j) \quad \text{for all } p \in B(p_j, r_j),$$

where  $r_j := \varepsilon_j\sigma_j\sqrt{R(q_j)/R(p_j)}$ . Indeed, if this is not true for some  $j$ , we can find a sequence  $\{q_{j,k}\}$  of points in  $M \setminus B(o, \sigma_j)$  with  $q_{j,0} = q_j$  satisfying that

$$d(q_{j,k+1}, q_{j,k}) < \varepsilon_j\sigma_j\sqrt{\frac{R(q_j)}{R(q_{j,k})}} \quad \text{and} \quad R(q_{j,k+1}) > (1 + \delta_j)R(q_{j,k})$$

for  $k = 1, 2, \dots$ . Then we obtain

$$R(q_{j,k}) > (1 + \delta_j)^k R(q_j) \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

which contradicts to

$$d(q_{j,k}, o) \leq d(q_{j,k}, q_{j,k-1}) + \dots + d(q_{j,0}, o) < (1 + \varepsilon_j)\sigma_j + \sigma_j.$$

Now it is easy to see that the sequence  $\{p_j\}$  satisfies the desired properties.  $\square$

## 8.2 Proof of Proposition 8.3

In this section, we give a proof of Proposition 8.3. Since we are dealing with the Ricci flow possibly of unbounded curvature, we cannot appeal to the following injectivity radius estimate, which seems to be originally due to Toponogov [Topo].

**Theorem 8.6** (e.g. [Vol1, Appendix B]). *Let  $M^n$  be a complete non-compact Riemannian manifold with strictly positive sectional curvature. Suppose that the sectional curvature is less than or equal to  $K$  for some positive  $K > 0$ . Then the injectivity radius  $\text{inj}(M)$  of  $M$  satisfies*

$$\text{inj}(M) \geq \pi/\sqrt{K}.$$

Instead, we invoke the following theorem of Hamilton [Ha5, Theorem 21.5].

**Theorem 8.7** (Finite bump theorem, Hamilton [Ha5], also [CLN]). *For every positive  $\beta > 0$ , there exists a constant  $\lambda > 0$  such that any complete Riemannian manifold  $M^n$  of non-negative curvature can contain at most finite number of disjoint  $\lambda$ -remote  $\beta$ -bumps.*

**Definition 8.8.** Let  $B(p, r)$  be a metric ball of radius  $r > 0$  in a Riemannian manifold  $M$  with a fixed point  $o \in M$ . We say that  $B(p, r)$  is a  $\beta$ -bump if the sectional curvature  $K$  satisfies  $K \geq \beta r^{-2}$  on the ball. The ball  $B(p, r)$  is  $\lambda$ -remote if  $d(p, o) \geq \lambda r$ .

*Proof of Proposition 8.3.* First of all, because we know from the Harnack inequality (8.6) that  $tR(\cdot, t)$  is pointwise non-decreasing, it suffices to show that  $(M^2, g(T))$  has bounded curvature. More strongly, we will show that the curvature of  $(M^2, g(T))$  decays to 0 at the spacial infinity, i.e.,  $R(x_j, T) \rightarrow 0$  as  $x_j \rightarrow \infty$ . (This was shown by Chu [CLN, Lemma 9.7] in the bounded curvature setting.)

Fix a point  $o$  in  $M^2$ . Suppose that there exists a sequence  $\{p_{0,j}\}$  of points of  $M^2$  such that  $R(p_{0,j}, T) \geq \beta > 0$  and  $d_{g(T)}(p_{0,j}, o) \rightarrow \infty$  as  $j$  tends to infinity. By applying Hamilton's point picking lemma (Lemma 8.5), we can find a small constant  $r_0 > 0$  and a sequence  $\{p_j\}$  of points such that  $R(p_j, T) \geq \beta > 0$  and

$$R(\cdot, T) \leq r_0^{-2}R(p_j, T) \quad \text{on } B_{g(T)}(p_j, 2r_0).$$

Harnack inequality (8.6) implies that

$$R(\cdot, t) \leq 2r_0^{-2}R(p_j, T) \quad \text{on } B_{g(t)}(p_j, r_0) \text{ for all } t \in [T - r_0^2, T].$$

Then, by Shi's gradient estimate (Theorem 3.19),

$$|\nabla R|(\cdot, T) \leq Cr_0^{-3}R(p_j, T) \quad \text{on } B_{g(T)}(p_j, r_0).$$

Using this, for some sufficiently small  $\varepsilon_0 \ll r_0^2$ ,

$$R(\cdot, T) \geq R(p_j, T) - C\varepsilon_0 r_0^{-2} R(p_j, T) \geq \varepsilon_0 \beta r_0^{-2} \quad \text{on } B_{g(T)}(p_j, \varepsilon_0 r_0).$$

Taking a subsequence, we may assume that the balls  $B_{g(T)}(p_j, r_0)$  are pairwise disjoint. This contradicts to the finite bump theorem quoted above.  $\square$

# Chapter A

## Appendix

In this appendix, we present a very detailed proof to the fact on the super Ricci flow used in the proof of main theorem, i.e., Perelman's point picking lemma (Lemma A.2), which is found in [Pe, Section 10]. The proof rely on the following lemma whose proof in [Pe] works as well for the super Ricci flow.

**Lemma A.1** (Perelman [Pe, Lemma 8.3]). *Let  $(M^n, g(\tau))$  be an  $n$ -dimensional complete super Ricci flow.*

- (a) *Assume that  $\text{Ric}(\cdot, \tau_0) \leq (n-1)K$  on the ball  $B_{g(\tau_0)}(x_0, r_0)$ . Then outside of  $B_{g(\tau_0)}(x_0, r_0)$ ,*

$$\left( \frac{\partial}{\partial \tau} + \Delta_{g(\tau_0)} \right) d_{g(\tau_0)}(\cdot, x_0) \leq (n-1) \left( \frac{2}{3} K r_0 + r_0^{-1} \right).$$

*The inequality is understood in the barrier sense.*

- (b) *Assume that  $\text{Ric}(\cdot, \tau_0) \leq (n-1)K$  on the union of the balls  $B_{g(\tau_0)}(x_0, r_0)$  and  $B_{g(\tau_0)}(x_1, r_0)$ . Then*

$$\left. \frac{d^+}{d\tau} d_{g(\tau)}(x_0, x_1) \right|_{\tau=\tau_0} \leq 2(n-1) \left( \frac{2}{3} K r_0 + r_0^{-1} \right).$$

*Here,  $\frac{d^+}{d\tau} f(\tau) := \limsup_{\varepsilon \rightarrow 0^+} \frac{f(\tau+\varepsilon) - f(\tau)}{\varepsilon}$  denotes the upper Dini derivative.*

**Lemma A.2** (Perelman's Point picking lemma [Pe], also Kleiner–Lott [KL]). *Let  $(M^n, g(\tau))$ ,  $\tau \in [0, T)$  be a complete super Ricci flow and  $A, B > 0$  are arbitrary numbers. Fix  $x_0 \in M$ . Assume that there exists a point  $(x_1, \tau_1) \in M(B)$  with  $Q_1 := |\text{Rm}|(x_1, \tau_1)$ , where*

$$M(B) := \{(x, \tau) \in M \times [0, T) \mid |\text{Rm}|(x, \tau)(T - \tau) > B\}.$$

*Then we can find a point  $(p_*, \tau_*) \in M(B)$  with*

$$d_{g(\tau_*)}(x_0, p_*) < d_{g(\tau_1)}(x_0, x_1) + 2 \max\{3A, 4(n-1)B\} Q_1^{-1/2}$$

*such that*

$$|\text{Rm}|(x, \tau) \leq 2|\text{Rm}|(p_*, \tau_*) =: 2Q \tag{A.1}$$

*for all  $(x, \tau)$  with  $d_{g(\tau_*)}(x, p_*) < A Q^{-1/2}$  and  $\tau_* \leq \tau \leq \tau_* + \frac{1}{2} B Q^{-1}$ .*

The proof is divided into two steps as in [Pe].

**Claim 1.** Take  $A' > 0$  satisfying that

$$4(n-1)B\varepsilon \leq 1 \text{ and } (\varepsilon A')^2 \geq 3/4 \text{ for some small } \varepsilon > 0 \text{ and } A' \geq 3A.$$

Then we can find a point  $(p_*, \tau_*) \in M(B)$  such that (A.1) holds for all  $(x, \tau)$  with

$$d_{g(\tau)}(x, x_0) < d_{g(\tau_*)}(p_*, x_0) + A'Q^{-1/2} \quad \text{and} \quad \tau_* \leq \tau \leq \tau_* + \frac{1}{2}BQ^{-1}.$$

*Proof.* If not, we can construct a sequence  $\{(x_i, \tau_i)\}_{i \in \mathbb{Z}^+}$  of points of  $M \times [0, T)$  starting from  $(x_1, \tau_1) \in M(B)$  satisfying that

$$Q_{i+1} > 2Q_i, \quad d_{i+1} < d_i + A'Q_i^{-1/2} \quad \text{and} \quad \tau_i \leq \tau_{i+1} \leq \tau_i + \frac{1}{2}BQ_i^{-1}$$

where we put  $Q_i := |\text{Rm}|(x_i, \tau_i)$  and  $d_i := d_{g(\tau_i)}(x_i, x_0)$ . We see that  $(x_{i+1}, \tau_{i+1})$  lies in  $M(B)$  if  $(x_i, \tau_i)$  does. Indeed,

$$\begin{aligned} Q_{i+1}(T - \tau_{i+1}) - B &> 2Q_i \left( T - \tau_i - \frac{1}{2}BQ_i^{-1} \right) - B \\ &= 2(Q_i(T - \tau_i) - B) > 0. \end{aligned}$$

This implies that  $\{(x_i, \tau_i)\}_{i \in \mathbb{Z}^+} \subset M(B)$ .

Then  $Q_i > 2^{i-1}Q_1 \rightarrow \infty$  as  $i \rightarrow \infty$ , which contradicts to that

$$\tau_i < \tau_1 + BQ_1^{-1} < T - \varepsilon_1 < T \quad \text{and} \quad d_i < d_1 + 2A'Q_1^{-1/2}.$$

Here  $\varepsilon_1 > 0$  is taken so that  $|\text{Rm}|(x_1, \tau_1)(T - \tau_1 - \varepsilon_1) > B$ . Hence the sequence  $\{(x_i, \tau_i)\}$  stops at finite steps and the terminal one is the desired point  $(p_*, \tau_*)$ .  $\square$

**Claim 2.** The point  $(p_*, \tau_*)$  just obtained satisfies the desired property.

*Proof.* Take  $x \in B_{g(\tau_*)}(p_*, AQ^{-1/2})$  and put  $r_0 := \varepsilon A'Q^{-1/2}$ . Let  $\tau' \in [\tau_*, \tau_* + \frac{1}{2}BQ^{-1}]$  be the supremum of  $\tau''$  such that

$$|\text{Rm}|(\cdot, \tau) \leq 2Q \text{ on } B_{g(\tau)}(x_0, r_0) \cup B_{g(\tau)}(x, r_0) \quad \text{for all } \tau \in [\tau_0, \tau''].$$

It follows easily from the choice of  $(p_*, \tau_*)$  that  $\tau' > \tau_*$  and  $|\text{Rm}| \leq 2Q$  on  $B_{g(\tau)}(x_0, r_0)$  for  $\tau \in [\tau_*, \tau_* + \frac{1}{2}BQ^{-1}]$ .

Applying Lemma A.1.(b) for  $r_0 = \varepsilon A'Q^{-1/2}$ ,

$$\begin{aligned} d_{g(\tau')}(x, x_0) - d_{g(\tau_*)}(x, x_0) &\leq 2(n-1) \left( \frac{4}{3}\varepsilon A'Q^{1/2} + (\varepsilon A')^{-1}Q^{1/2} \right) (\tau' - \tau_*) \\ &\leq \frac{8}{3}(n-1)\varepsilon A'BQ^{-1/2} \\ &\leq \frac{2}{3}A'Q^{-1/2}. \end{aligned}$$

Therefore, we have

$$d_{g(\tau')}(x, x_0) \leq d_{g(\tau_*)}(x, p_*) + d_{g(\tau_*)}(p_*, x_0) + \frac{2}{3}A'Q^{-1/2} < d_{g(\tau_*)}(p_*, x_0) + A'Q^{-1/2}$$

and  $\tau' = \tau_* + \frac{1}{2}BQ^{-1}$ . As  $x \in B_{g(\tau_*)}(p_*, AQ^{-1/2})$  is arbitrary, we conclude that

$$|\text{Rm}| \leq 2Q \quad \text{on } B_{g(\tau_*)}(p_*, AQ^{-1/2}) \times [\tau_*, \tau_* + \frac{1}{2}BQ^{-1}].$$

This completes the proof of the lemma.  $\square$

# Bibliography

- [An] M. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, *Invent. Math.* **102**(2) (1990), 429–445.
- [An2] M. Anderson, *On the topology of complete manifolds of nonnegative Ricci curvature*, *Topology* **29**(1) (1990), 41–55.
- [BöWi] C. Böhm and B. Wilking, *Manifolds with positive curvature operators are space forms*, *Ann. of Math. (2)* **167**(3) (2008), 1079–1097.
- [BrSc] S. Brendle and R. Schoen, *Manifolds with 1/4-pinched curvature are space forms*, *J. Amer. Math. Soc.* **22**(1) (2009), 287–307.
- [BBI] D. Burago, Y. Burago and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics **33**. American Mathematical Society, Providence, RI, 2001.
- [Ca] H.-D. Cao, *Geometry of Complete Gradient Shrinking Ricci Solitons*, Preprint, arXiv:0903.3927.
- [CHI] H.-D. Cao, R. Hamilton and T. Ilmanen, *Gaussian densities and stability for some Ricci solitons*, Preprint, arXiv:math/0404165.
- [CaZho] H.-D. Cao and D. Zhou, *On complete gradient shrinking Ricci solitons*, Preprint, arXiv:0903.3932.
- [CaZhu] H.-D. Cao and X.-P. Zhu, *A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow*, *Asian J. Math.* **10**(2) (2006), 165–492.
- [CaNi] J. Carrillo and L. Ni, *Sharp logarithmic Sobolev inequalities on gradient solitons and applications*, Preprint, arXiv:0806.2417.
- [CTY] A. Chau, L.-F. Tam and C. Yu, *Pseudolocality for the Ricci flow and applications*, Preprint, arXiv:math/0701153.
- [Cha] I. Chavel, *Riemannian geometry. A modern introduction. Second edition*, Cambridge Studies in Advanced Mathematics **98** (2006).
- [Chee] J. Cheeger, *Degeneration of Riemannian metrics under Ricci curvature bounds*, *Lezioni Fermiane. Scuola Normale Superiore, Pisa*, 2001.

- [ChCo] J. Cheeger and T. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144**(1) (1996), 189–237.
- [ChCo2] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46**(3) (1997), 406–480.
- [Che] B.-L. Chen, *Strong uniqueness of the Ricci flow*, Preprint, J. Differential Geom. **82**(2) (2009), 363–382.
- [ChZh] B.-L. Chen and X.-P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Differential Geom. **74**(1) (2006), 119–154.
- [CLT] X. Chen, P. Lu and G. Tian *A note on uniformization of Riemann surfaces by Ricci flow*, Proc. Amer. Math. Soc. **134** (2006), 3391–3393.
- [Vol2-I] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, *The Ricci flow: techniques and applications. Part I. Geometric aspects*, Mathematical Surveys and Monographs, **135**, American Mathematical Society, Providence, RI, 2007.
- [ChHa] B. Chow and R. Hamilton, *Constrained and linear Harnack inequalities for parabolic equations*, Invent. Math. **129**(2) (1997), 213–238.
- [Vol1] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, **110**, American Mathematical Society, Providence, RI, 2004.
- [CLN] B. Chow, P. Lu and L. Ni, *Hamilton’s Ricci flow*, Graduate Studies in Mathematics, **77**, American Mathematical Society, Providence, RI; Science Press, New York, 2006.
- [Chu] S.-C. Chu, *Type II ancient solutions to the Ricci flow on surfaces*, Comm. Anal. Geom. **15**(1) (2007), 195–215.
- [DaSe] P. Daskalopoulos and N. Sesum, *Eternal solutions to the Ricci flow on  $\mathbb{R}^2$* , Int. Math. Res. Not. 2006, Art. ID 83610, 20 pp.
- [DHS] P. Daskalopoulos, R. Hamilton and N. Sesum, *Classification of compact ancient solutions to the Ricci flow on surfaces*, Preprint, arXiv:0902.1158.
- [De] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18**(1) (1983), 157–162.
- [Ec] K. Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, **57**, Birkhäuser Boston, Inc., Boston, MA, 2004.
- [EKNT] K. Ecker, D. Knopf, L. Ni and P. Topping, *Local monotonicity and mean value formulas for evolving Riemannian manifolds*, J. Reine Angew. Math. **616** (2008), 89–130.
- [Eh] P. Ehrlich, *Metric deformations of curvature. II. Compact 3-manifolds*, Geometriae Dedicata **5**(2) (1976), 147–161.

- [Ed] J. Enders, *Reduced distance based at singular time in the Ricci flow*, Preprint, arXiv:0711.0558.
- [Ev] L. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 1998.
- [FZZ] F. Fang, Y. Zhang and Z. Zhang, *Non-singular solutions to the normalized Ricci flow equation*, Math. Ann. **340**(3) (2008), 647–674.
- [FIN] M. Feldman, T. Ilmanen and L. Ni, *Entropy and reduced distance for Ricci expanders*, J. Geom. Anal. **15**(1) (2005), 49–62.
- [Gr] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces. With appendices by M. Katz, P. Pansu and S. Semmes*, Progress in Mathematics, **152**, Birkhäuser Boston, Inc., Boston, MA, 1999.
- [GuZh] H.-L. Gu and X.-P. Zhu, *The existence of type II singularities for the Ricci flow on  $S^{n+1}$* , Comm. Anal. Geom. **16**(3) (2008), 467–494.
- [Ha] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17**(2) (1982), 255–306.
- [Ha2] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24**(2) (1986), 153–179.
- [Ha3] R. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geom. **37**(1) (1993), 225–243.
- [Ha4] R. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math. **71**, Amer. Math. Soc., Providence, RI, 1988.
- [Ha5] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry, **II**, Cambridge, MA, 1993, 7–136, Int. Press, Cambridge, MA, 1995.
- [Ha6] R. Hamilton, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. **117**(3) (1995), 545–572.
- [Ha7] R. Hamilton, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom. **7**(4) (1999), 695–729.
- [HaSe] R. Hamilton and N. Sesum, *Properties of the solutions of the conjugate heat equations*, Amer. J. Math. **131**(1) (2009), 153–169.
- [IlKn] T. Ilmanen and D. Knopf, *A lower bound for the diameter of solutions to the Ricci flow with nonzero  $H^1(M^n; \mathbb{R})$* , Math. Res. Lett. **10**(2–3) (2003), 161–168.
- [Is] M. Ishida, *Einstein metrics, Ricci flow and exotic smooth structures*, Sūgaku **59**(4) (2007), 399–416.
- [KL] B. Kleiner and J. Lott, *Notes on Perelman’s papers*, Geom. Topol. **12**(5) (2008), 2587–2855.

- [Li] P. Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*, Ann. Math. (2) **124**(1) (1986), 1–21.
- [LiYa] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156**(3–4) (1986), 153–201.
- [Lo] J. Lott, *Optimal transport and Perelman’s reduced volume*, Calc. Var. Partial Differential Equations **36**(1) (2009), 49–84.
- [McTo] R. McCann and P. Topping, *Ricci flow, entropy and optimal transportation*, Amer. J. Math, to appear.
- [Mo] F. Morgan, *Manifolds with density*, Notices Amer. Math. Soc. **52**(8) (2005), 853–858.
- [Mo2] F. Morgan, *Geometric measure theory. A beginner’s guide. Fourth edition*, Elsevier/Academic Press, Amsterdam, 2009.
- [MoTi] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs **3**, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.
- [Mü] R. Müller, *Differential Harnack inequalities and the Ricci flow*, EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2006.
- [Mü] R. Müller, *Monotone Volume Formulas for Geometric Flows*, J. Reine Ang. Math, to appear.
- [Na] A. Naber, *Noncompact Shrinking 4-Solitons with Nonnegative Curvature*, Preprint, arXiv:0710.5579.
- [Ni] L. Ni, *The entropy formula for linear heat equation*, J. Geom. Anal. **14**(1) (2004), 87–100; Addenda: J. Geom. Anal. **14**(2) (2004), 369–374.
- [Ni2] L. Ni, *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. **12**(5–6) (2005), 633–653.
- [Ni3] L. Ni, *Mean value theorems on manifolds*, Asian J. Math. **11**(2) (2007), 277–304.
- [Ni4] L. Ni, *A matrix Li–Yau–Hamilton estimate for Kähler-Ricci flow*, J. Differential Geom. **75**(2) (2007), 303–358.
- [NiWa] L. Ni and N. Wallach, *On a classification of gradient shrinking solitons*, Math. Res. Lett. **15**(5) (2008), 941–955.
- [Pe] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, arXiv:math/0211159.
- [Pe2] G. Perelman, *Ricci flow with surgery on three-manifolds*, Preprint, arXiv:math/0303109.

- [Pet] P. Petersen, *Convergence theorems in Riemannian geometry*, Comparison geometry (Berkeley, CA, 1993–94), 167–202, Math. Sci. Res. Inst. Publ. **30**, Cambridge Univ. Press, Cambridge, 1997.
- [PeTu] A. Petrunin and W. Tuschmann, *Asymptotical flatness and cone structure at infinity*, Math. Ann. **321**(4) (2001), 775–788.
- [PeWy] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Preprint, arXiv:0712.1298.
- [ReSt] M.-K. von Renesse and K.-T. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math. **58**(7) (2005), 923–940.
- [Se] N. Sesum, *Curvature tensor under the Ricci flow*, Amer. J. Math. **127**(6) (2005), 1315–1324.
- [Sh] X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30**(1) (1989), 223–301.
- [Si] L. Simon, *Theorems on regularity and singularity of energy minimizing maps*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996.
- [Ta] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.
- [Topo] V. Toponogov, *Theorems on minimizing paths in noncompact Riemannian spaces of positive curvature*, English translation, Soviet Math. Dokl. **11** (1970), 412–414.
- [Topp] P. Topping, *Diameter control under Ricci flow*, Comm. Anal. Geom. **13**(5) (2005), 1039–1055.
- [Topp2] P. Topping, *Lectures on the Ricci flow*, London Mathematical Society Lecture Note Series **325**, Cambridge University Press, Cambridge, 2006.
- [Topp3] P. Topping,  *$\mathcal{L}$ -optimal transportation for Ricci flow*, J. Reine Angew. Math, to appear.
- [Vi] C. Villani, *Optimal transport. Old and new*, Grundlehren der Mathematischen Wissenschaften **338**, Springer-Verlag, Berlin, 2009.
- [Wa] M.-A. Watson, *A theory of subtemperatures in several variables*, Proc. London Math. Soc. **26**(3) (1973), 385–417.
- [We] G. Wei, *Curvature formulas in Section 6.1 of Perelman’s paper (math.DG/0211159)*, Preprint.
- [WeWy] G. Wei and W. Wylie, *Comparison geometry for the Bakry–Emery Ricci tensor*, Preprint, arXiv:0706.1120.
- [Wy] W. Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. **136**(5) (2008), 1803–1806.

- [Ye] R. Ye *The logarithmic Sobolev inequality along the Ricci flow*, Preprint, arXiv:0707.2424.
- [Ye2] R. Ye, *On the  $l$ -function and the reduced volume of Perelman I*, Trans. Amer. Math. Soc. **360**(1) (2008), 507–531.
- [Yo] T. Yokota, *Curvature integrals under the Ricci flow on surfaces*, Geom. Dedicata. **133**(1) (2008), 169–179.
- [Yo2] T. Yokota, *Perelman’s reduced volume and a gap theorem for the Ricci flow*, Comm. Anal. Geom. **17**(2) (2009), 227–263.
- [Yo3] T. Yokota, *A gap theorem for ancient solutions to the Ricci flow*, Proceedings of the 1st MSJ-SI, “Probabilistic Approach to Geometry”, to appear.
- [Yo4] T. Yokota, *On the asymptotic reduced volume of the Ricci flow*, Submitted.
- [Zh] Z.-H. Zhang, *On the completeness of gradient Ricci solitons*, Proc. Amer. Math. Soc. **37**(8) (2009), 2755–2759.
- [Zh2] Z.-I. Zhang, *Compact blow-up limits of finite time singularities of Ricci flow are shrinking Ricci solitons*, C. R. Math. Acad. Sci. Paris **345**(9) (2007), 503–506.