Cohomotopy Groups for Fully Normal Spaces

By

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(Received November 1, 1952)

Recently E. H. Spanier [12] has made a detailed investigation for the cohomotopy groups which was defined originally by K. Borsuk [1]. Spanier has shown that with the induced homomorphisms and the coboundary operator, the cohomotopy groups satisfy all the Eilenberg-Steenrod axioms [5] for cohomology theory, and emphasized the importance of the cohomotopy groups. His investigation, however, has been restricted to the case of compact spaces. In view of the fact that fully normal spaces are taken up in recent researches of algebraic topology, it seems desirable to study the cohomotopy groups for the case of fully normal spaces. This is the purpose of the present note.

1. A topological space is called a countably fully normal space if every countable open covering has a star refinement. According to [9], a topological space is countably fully normal if and only if it is normal and countably paracompact (that is, every countable open covering has a locally finite refinement). Recently C. H. Dowker [3] has proved the following theorem.

(1.1) *Let X be a countably fully normal space and Y a compact metric space. Then the product space $X\times Y$ is countably fully normal.*

In a previous paper [11] we have obtained the following theorem.

(1.2) *Under the same assumption as in (1.1) the relation* $\dim(X\times Y)\leqq\dim X+\dim Y$ *holds.*

It is to be noted that fully normal spaces, compact Hausdorff spaces, perfectly normal spaces and metric spaces are all countably fully normal.

2. The theorems (1.1) and (1.2) admit the following arguments (§§ 6–8 in Spanier [12]).

Let S^n be an n-sphere and p its fixed point. Let X be a countably fully normal space and A a closed subset of X.

If $\dim(X-A)<2n$, then, for any two continuous maps α, β:

$(X, A) \to (S^n, p)$, there exist two closed sets M, N of X with $X = M \cup N$ and two homotopies F, G: $(X \times I,\ A \times I) \to (S^n, p)$ such that

$$F(x, 0) = \alpha(x) \text{ for } x \in X, \qquad F(x, 1) = p \text{ for } x \in N,$$
$$G(x, 0) = \beta(x) \text{ for } x \in X, \qquad G(x, 1) = p \text{ for } x \in M,$$

where I is the closed line interval consisting of non-negative real numbers not greater than unity. Moreover these homotopies F, G can be so chosen that $F(x, t) = \alpha(x)$, $G(x, t) = \beta(x)$ for each $t \in I$ if either $\alpha(x) = p$ or $\beta(x) = p$. Let us define γ : $(X, A) \to (S^n, p)$ by

$$\gamma(x) = \begin{cases} F(x, 1) & \text{for } x \in M \\ G(x, 1) & \text{for } x \in N . \end{cases}$$

In case $\dim(X - A) < 2n - 1$, the homotopy class $\{\gamma\}$ relative to A is uniquely determined by the homotopy classes $\{\alpha\}$ and $\{\beta\}$ (relative to A). Define $\{\alpha\} + \{\beta\} = \{\gamma\}$. Then we have the following theorem.

(2.1) *If X is a countably fully normal space with* $\dim(X - A) < 2n - 1$ [1]*, the homotopy classes $\{\alpha\}$ (relative to A) of continuous maps α of (X, A) into (S^n, p) form an abelian group with the law of addition defined above.*

This group is called the n-th *cohomotopy group* of X modulo A and is denoted by $\pi^n(X, A)$. We write $\pi^n(X)$ instead of $\pi^n(X, 0)$.

Let $f : (X, A) \to (Y, B)$ be a continuous map. If $\alpha : (Y, B) \to (S^n, p)$, then $\alpha f : (X, A) \to (S^n, p)$, and $\{\alpha\} = \{\beta\}$ implies $\{\alpha f\} = \{\beta f\}$. Hence f induces a mapping

$$f^{\#} : \pi^n(Y, B) \to \pi^n(X, A)$$

defined by $f^{\#}\{\alpha\} = \{\alpha f\}$ for $\{\alpha\} \in \pi^n(Y, B)$, and $f^{\#}$ is a homomorphism of $\pi^n(Y, B)$ into $\pi^n(X, A)$.

Let $\alpha : A \to S^n$. Then there is an extension $\tilde{\alpha}$ of α which maps (X, A) into (E^{n+1}, S^n) where E^{n+1} is an $(n+1)$-cell with S^n as its boundary. Let $\psi_1 : (E^{n+1}, S^n) \to (S^{n+1}, p)$ be a continuous map which takes $E^{n+1} - S^n$ homeomorphically onto $S^{n+1} - p$. Then $\{\psi_1 \alpha\} \in \pi^{n+1}(X, A)$ is uniquely determined by the homotopy class $\{\alpha\} \in \pi^n(A)$ where $\dim A < 2n - 1$, $\dim(X - A) < 2n + 1$. Define

$$\Delta : \pi^n(A) \to \pi^{n+1}(X, A)$$

by $\Delta\{\alpha\} = \{\psi_1 \tilde{\alpha}\}$ for $\{\alpha\} \in \pi^n(A)$. The mapping Δ is a homomorphism and is called the *coboundary operator*.

1) It is sufficient to assume that $\dim F < 2n - 1$ for any closed $F \subset X - A$. Cf. [12].

With the induced homomorphisms $f^\#$ and the coboundary operator, the cohomotopy groups of countably fully normal spaces satisfy the Eilenberg-Steenrod axioms [5] for cohomology theory, except the exactness axiom.

(2.2) *If* $f:(X, A)\to(X, A)$ *is the identity, then* $f^\#:\pi^n(X, A)\to\pi^n(X, A)$ *is the identity.*

(2.3) *If* $f:(X, A)\to(Y, B)$ *and* $g:(Y, B)\to(Z, C)$, *then* $(gf)^\#=f^\# g^\#$.

(2.4) *If* $f:(X, A)\to(Y, B)$, *then* $f^\#\Delta=\Delta(f|A)^\#:\pi^n(B)\to\pi^{n+1}(X,A)$.

(2.5) *If* $f, g:(X, A)\to(Y, B)$ *are homotopic, then* $f^\#=g^\#$.

(2.6) *If* V *is an open set contained in* A *and* $j:(X-V,\ A-V)\to(X, A)$ *is the inclusion map, then* $j^\#:\pi^n(X, A)\to\pi^n(X-V, A-V)$ *is an isomorphism onto.*

(2.7) *If* P *is a space consisting of a single point, then* $\pi^n(P)=0$ *for* $n\geqq 1$.

Here the suitable dimension restrictions are assumed in each case and X, Y, Z are countably fully normal spaces.

3. (2.6) is a special case of the following "map excision theorem" (cf. A. D. Wallace [13]).

(3.1) *If* $f:(X, A)\to(Y, B)$ *is a closed (continuous) map which takes* $X-A$ *homeomorphically onto* $Y-B$, *then* $f^\#:\pi^n(Y, B)\to\pi^n(X, A)$ *is an isomorphism onto.*

Proof. If we denote by X_A the space obtained from X by identifying the closed set A to a point q_A and $h:(X, A)\to(X_A, q_A)$ is the associated map, then X_A is a countably fully normal space and we can prove $h^\#:\pi^n(X_A, q_A)\approx\pi^n(X, A)$ similarly as in [12, Theorem 7.5]. The assumption that f is a closed map implies that f induces a homeomorphism of (X_A, q_A) onto (Y_B, q_B). Thus we have (3.1).

For the unrestricted Čech cohomology theory, the map excision theorem holds for fully normal spaces (but whether or not it holds for countably fully normal spaces is an open question). Thus in case X is a fully normal space, we have $H^n(X, A)\approx H^n(X_A, q_A)$. Further, if $\dim(X-A)<n$, then X_A is a fully normal space of dimension $<n$. Because every open covering of X_A has a refinement $\mathfrak{V}=\{V_1, \dots, V_s\}$ such that q_A is contained in V_1 but not in any V_j with $j>1$. Then $\{h^{-1}(V_1), \dots, h^{-1}(V_s)\}$ is an open covering of X. Since $A\subset h^{-1}(V_1)$, there is an open set W such that $A\subset W\subset\overline{W}\subset h^{-1}(V_1)$. Since $\dim(X-A)<n$ we have $\dim(X-W)<n$ and hence $\{h^{-1}(V_1)-W,$

$h^{-1}(V_2)-\overline{W}, \ldots, h^{-1}(V_s)-\overline{W}\}$ as an open covering of $X-W$ has a refinement $\{G_1, \ldots, G_s\}$ of order $<n+1$ such that $G_1 \subset h^{-1}(V_1)-W$ and $G_j \subset h^{-1}(V_j)-\overline{W}$ for $j>1$. The sets G_j for $j>1$ are open in X and there is an open set H_1 of X such that $G_1=H_1 \cap (X-W)$, $W \subset H_1 \subset h^{-1}(V_1)$. Hence $\{h(H_1), h(G_2), \ldots, h(G_s)\}$ is an open covering of X_A which is a refinement of $\mathfrak{V}$ and has order $<n+1$. This shows $\dim X_A<n$. Thus we have

$$H^n(X, A)=0 \qquad \text{for} \quad n>\dim(X-A).$$

The following theorem is the analogue of this fact.

(3.2) *If* $\dim(X-A)<n$, *then* $\pi^n(X, A)=0$.

Proof. According to Theorem 6.3 [2] of [10], if $\dim(X-A)<n$, then for any two maps $\alpha, \beta:(X, A)\to(S^n, p)$ there is a uniform homotopy $F:(X\times I, A\times I)\to(S^n, p)$ such that $F(x, 0)=\alpha(x)$, $F(x, 1)=\beta(x)$. This proves (3.2).

In case X is a normal space, if we denote by $\pi^n(X, A)$ the set of the homotopy classes $\{\alpha\}$ of maps α of (X, A) into (S^n, p), then (3.2) holds likewise as is seen from the above proof.

4. Let X be a countably fully normal space of dimension $<2n-1$ and A a closed set of X. Then the sequence of groups and homomorphisms

(4.1) $\pi^n(X, A)\xrightarrow{j^\#}\cdots\xrightarrow{j^\#}\pi^m(X)\xrightarrow{i^\#}\pi^m(A)\xrightarrow{\Delta}\pi^{m+1}(X, A)\xrightarrow{j^\#}\cdots$

is called the cohomotopy sequence of a pair (X, A), where $i:A\to X$ and $j:(X, 0)\to(X, A)$ are inclusion maps.

(4.2) *If X is a fully normal space of dimension $<2n-1$ the cohomotopy sequence of (X, A) is exact.*

This theorem will be proved later. Here we note that

(4.3) $i^\# j^\#=0$, $j^\#\Delta=0$, $\Delta i^\#=0$, *kernel of* $i^\#=$*image of* $j^\#$

are proved already for countably fully normal spaces.[3] Indeed the first two relations are proved similarly as in [12]. To prove that $\Delta i^\#=0$, let $\{\alpha\}\in\pi^n(X)$ and $\alpha:X\to S^n$. Then α considered as a map of (X, A) into (E^{n+1}, S^n) is an extension of the map $\alpha i:A\to S^n$, and hence we have $\Delta i^\#\{\alpha\}=\Delta\{\alpha i\}=\{\psi_1\alpha\}=0$.

2) This is an extension of Proposition B) of [6, p. 87] to the case of normal spaces. Propositions C) and D) of [6, p. 88] can be shown to hold for countably fully normal spaces by virtue of (1.2). They lead us to the following theorem: If $X=A_1\cup A_2$ and $\dim(A_1\cap A_2)\leq n-2$, then $\pi^n(X)\approx\pi^n(A_1)+\pi^n(A_2)$ (direct sum). This theorem has an analogue in cohomology theory and can also be deduced from the exactness axiom, (2.3), (2.6) and (3.2) for the case of fully normal spaces.

3) Whether (4.2) holds or not for the case of countably fully normal spaces is an open question.

The proof of the relation: kernel of $i^{\#}$=image of $j^{\#}$ runs as follows. Let $\{\alpha\}\in\pi^n(X)$, $\alpha: X\to S^n$, and $i^{\#}\{\alpha\}=0$. Then $\alpha i: A\to S^n$ is inessential. Hence by Borsuk's theorem [6, p. 86] which is valid for countably fully normal spaces, there exists a map $\beta:(X, A)\to(S^n, p)$ such that βj, $\alpha: X\to S^n$ are homotopic. Thus we have $\{\alpha\}=\{\beta j\}=j^{\#}\{\beta\}$. Since $i^{\#}j^{\#}=0$, the kernel of $i^{\#}$ is the image of $j^{\#}$.

5. Henceforth let X be a fully normal space of dimension less than $2n-1$ and A a closed set of X.

The family $\{\mathfrak{V}_\nu\}$ of all the locally finite open coverings of X whose nerves have dimension less than $2n-1$ forms a complete family of all open coverings. Let K_ν be the nerve of $\mathfrak{V}_\nu$ and L_ν the subcomplex of K_ν which corresponds to the nerve of the covering $\{V|V\cap A\neq 0,\ V\in\mathfrak{V}_\nu\}$ of the subspace A.

An infinite (or finite) complex K determines a topological space assigned with the Whitehead topology (cf. [14], [4]) which will be denoted by the same letter K. The complex K with this topology is a fully normal space and its topological dimension is identical with the combinatorial dimension (cf. [11]). Hence the cohomotopy groups $\pi^n(K_\nu, L_\nu)$ are defined by (2.1).

A continuous map $f:(X, A)\to(K_\nu, L_\nu)$ is said to be canonical if $f^{-1}(\text{star } v)\subset V$ holds for each vertex v corresponding to V of $\mathfrak{V}_\nu$. For each $\mathfrak{V}_\nu$ we choose a canonical map $h_\nu:(X, A)\to(K_\nu, L_\nu)$; such a map exists by [4], [11] and any other canonical map is shown to be homotopic to h_ν.

If $\mathfrak{V}_\mu$ is a refinement of $\mathfrak{V}_\nu$, then a correspondence

$$T_{\mu\nu}:(K_\mu, L_\mu)\to(K_\nu, L_\nu)$$

such that $T_{\mu\nu}(v_\mu^\sigma)=v_\nu^\tau$ implies $V_\mu^\sigma\subset V_\nu^\tau$ is a simplicial map and is called a projection. Any other projection $\overline{T}_{\mu\nu}$ is homotopic to $T_{\mu\nu}$. Hence $T_{\mu\nu}$ and $\overline{T}_{\mu\nu}$ induce the same homomorphism of $\pi^n(K_\nu, L_\nu)$ into $\pi^n(K_\mu, L_\mu)$.

The groups $\pi^n(K_\nu, L_\nu)$ and homomorphisms $T_{\mu\nu}^{\#}$ form a direct system of groups, whose limit group $\varinjlim \pi^n(K_\nu, L_\nu)$ will be denoted by $\overline{\pi}^n(X, A)$. The element of $\overline{\pi}^n(X, A)$ determined by $u_\nu\in\pi^n(K_\nu, L_\nu)$ will be denoted by $[u_\nu]$.

If $\mathfrak{V}_\mu$ is a refinement of $\mathfrak{V}_\nu$, then $T_{\mu\nu}h_\mu:(X, A)\to(K_\nu, L_\nu)$ is shown to be canonical (cf. [12, Lemma 13.2]) and hence is homotopic to h_ν. Therefore we can define a homomorphism

$$\overline{h}:\overline{\pi}^n(X, A)\to\pi^n(X, A)$$

by $\bar{h}[u_\nu]=h_\nu^\# u_\nu$ for $[u_\nu]\in\bar{\pi}^n(X, A)$, since $u_\mu\in[u_\nu]$ implies $h_\mu^\# u_\mu=h_\nu^\# u_\nu$.

(5.1) *The homomorphism* $\bar{h}:\bar{\pi}^n(X, A)\to\pi^n(X, A)$ *is an isomorphism onto.*

Proof. Let $\alpha:(X, A)\to(S^n, p)$. As in [12, p. 206] there can be found a (locally) finite open covering $\mathfrak{V}_\nu$ and a simplicial map $\beta:(K_\nu, L_\nu)\to(K, p)$ (where K is a simplicial subdivision of S^n with p as a vertex) such that βh_ν is homotopic to α. Then $\{\beta\}\in\pi^n(K_\nu, L_\nu)$ and $h_\nu^\#\{\beta\}=\{\alpha\}$. Hence $\bar{h}$ is onto.

To prove the isomorphism of $\bar{h}$, let $\bar{h}[u_\nu]=0$ and let $\alpha\in\{\alpha\}=u_\nu$. Then αh_ν is homotopic to a constant map $e:(X, A)\to(S^n, p)$ with $e(x)=p$ for every point x of X.

According to J. H. C. Whitehead [14, Theorem 36] there is a simplicial map

$$\alpha':(K'_\nu, L'_\nu)\to(K, p)$$

for a suitable simplicial subdivision (K'_ν, L'_ν) of (K_ν, L_ν) such that α' is homotopic to α.

Let

$$F:(X\times I,\ A\times I)\to(S^n, p)$$

be a homotopy between αh_ν and e such that $F(x, 0)=\alpha h_\nu(x)$, $F(x, 1)=p$ for $x\in X$. Let us put

$\mathfrak{U}=\{h_\nu^{-1}(\text{star } v')|v'$ extending over all vertices of $K'_\nu\}$,

$\mathfrak{W}=\{F^{-1}(\text{star } v)|v$ extending over all vertices of $K\}$,

where "star" means "open star". Then $\mathfrak{U}$ and $\mathfrak{W}$ are open coverings of X and $X\times I$ respectively.

Since X is fully normal, there exist a locally finite open covering $\mathfrak{V}_\mu=\{V_\mu^\sigma|\sigma\in\Omega\}$ of X and finite open coverings $\mathfrak{M}_\sigma$ of I $(\sigma\in\Omega)$ such that

(a) $\mathfrak{V}_\mu$ is a refinement of $\mathfrak{U}$,

(b) $\{V_\mu^\sigma\times M|\sigma\in\Omega, M\in\mathfrak{M}_\sigma\}$ is a refinement of $\mathfrak{W}$,

(c) the dimension of the nerve K_μ of $\mathfrak{V}_\mu$ is less than $2n-1$.

Each covering $\mathfrak{M}_\sigma$ has a refinement of the form: $\{(t_{j-1}^\sigma, t_{j+1}^\sigma)|j=0, 1, \cdots, k_\sigma\}$, where $0=t_0^\sigma<t_1^\sigma<\cdots<t_{k_\sigma}^\sigma=1$ and $(t_{-1}^\sigma, t_1^\sigma)$, $(t_{k_\sigma-1}^\sigma, t_{k_\sigma+1}^\sigma)$ means respectively $[0, t_1^\sigma)$, $(t_{k_\sigma-1}^\sigma, 1]$.

Let v_μ^σ be the vertex of K_μ corresponding to V_μ^σ. By the property (b) there is a vertex $v^{\sigma, j}$ of K such that

$$V_\mu^\sigma\times(t_{j-1}^\sigma, t_{j+1}^\sigma)\subset F^{-1}(\text{star } v^{\sigma, j}) .$$

We choose such a vertex $v^{\sigma, j}$ and put

$$\varphi(v_\mu^\sigma, t_j^\sigma)=v^{\sigma, j}\ .$$

Since

$$V_\mu^\sigma \times (t_{j-1}^\sigma, t_{j+1}^\sigma) \frown V_\mu^\sigma \times (t_j^\sigma, t_{j+2}^\sigma) \supset V_\mu^\sigma \times (t_j^\sigma, t_{j+1}^\sigma) \neq 0,$$

the vertices $v^{\sigma, j}$ and $v^{\sigma, j+1}$ are vertices of a simplex of K.

For $t_j^\sigma < t < t_{j+1}^\sigma$, let $\varphi(v_\mu^\sigma, t)$ be the point of K which divides the segment from $v^{\sigma, j}$ to $v^{\sigma, j+1}$ in the ratio $(t-t_j^\sigma):(t_{j+1}^\sigma - t)$.

If $v_\mu^{\sigma_0}, \cdots, v_\mu^{\sigma_p}$ is a simplex of K_μ, then we have $V_\mu^{\sigma_i} \times t \subset F^{-1}(\text{star } v^{\sigma_i, j})$ or $V_\mu^{\sigma_i} \times t \subset F^{-1}(\text{star } v^{\sigma_i, j}) \frown F^{-1}(\text{star } v^{\sigma_i, j+1})$ according as $t=t_j^{\sigma_i}$ or $t_j^{\sigma_i} < t < t_{j+1}^{\sigma_i}$. Hence $\varphi(v_\mu^\sigma, t)$ for $\sigma=\sigma_0, \cdots, \sigma_p$ are vertices or points on the 1-faces of a simplex of K. Therefore the mapping φ can be extended linearly over the simpexes of K_μ with t fixed, and we have (denoting by the same letter)

$$\varphi : K_\mu \times I \to K\ .$$

φ is clearly a continuous map by the property of the Whitehead topology.

Since $F(A\times I)=p$, $A\times I \subset F^{-1}(\text{star } p)$. Hence, if we take V_μ^σ so that $V_\mu^\sigma \frown A \neq 0$ implies $V_\mu^\sigma \times I \subset F^{-1}(\text{star } p)$ (this is possible ; see [6, p. 86]), we have $\varphi(v_\mu^\sigma, t)=p$ for v_μ^σ such that $V_\mu^\sigma \frown A \neq 0$ and for any $t \in I$. Thus we have

$$\varphi : (K_\mu \times I, L_\mu \times I) \to (K, p).$$

Since $F(X\times 1)=p$ and $V_\mu^\sigma \times (t_{k_\sigma - 1}^\sigma, 1] \subset F^{-1}(\text{star } v^{\sigma, k_\sigma})$, we have $p \in \text{star } v^{\sigma, k_\sigma}$ and hence $v^{\sigma, k_\sigma}=p$ for every $\sigma \in \Omega$. This shows that a map

$$\varphi_0 : (K_\mu, L_\mu) \to (K, p)$$

defined by $\varphi_0(x)=\varphi(x, 0)$ for $x \in K_\mu$ is homotopic to a constant map. Here we note that

$$V_\mu^\sigma \subset (\alpha h_\nu)^{-1}(\text{star } \varphi_0(v_\mu^\sigma))\ ,$$

since $F(x, 0)=\alpha h_\mu(x)$, and $V_\mu^\sigma \times [0, t_1^\sigma) \subset F^{-1}(\text{star } \varphi_0(v_\mu^\sigma))$.

By the property (a) there is a simplicial map

$$T : (K_\mu, L_\mu) \to (K'_\nu, L'_\nu)$$

such that $V_\mu^\sigma \subset h_\nu^{-1}(\text{star } T(v_\mu^\sigma))$ holds for any vertex v_μ^σ of K_μ. On the other hand, the simplicial approximation α' of α may be so chosen that star $v' \subset \alpha^{-1}(\text{star } \alpha'(v'))$ for each vertex v' of K'_ν. Hence we have

$$V_\mu^\sigma \subset h_\nu^{-1}(\text{star } T(v_\mu^\sigma)) \subset h_\nu^{-1}\alpha^{-1}(\text{star } \alpha' T(v_\mu^\sigma)).$$

Therefore $\alpha' T$ is homotopic to φ_0.

Since K'_ν is a simplicial subdivision of K_ν, there is a simplicial map T' of (K'_ν, L'_ν) into (K_ν, L_ν) such that each vertex v' of K'_ν is mapped into a vertex of the open simplex of K_ν containing v'. Therefore

$$V^\sigma_\mu \subset h_\nu^{-1}(\text{star } T(v^\sigma_\mu)) \subset h_\nu^{-1}(\text{star } T'T(v^\sigma_\mu)).$$

Since h_ν is canonical, if $T'T(v^\sigma_\mu)=v^\tau_\nu$ we have $V^\sigma_\mu \subset V^\tau_\nu$. Thus $T'T$ is a projection of (K_μ, L_μ) into (K_ν, L_ν) and hence $(T'T)^\# = T^\#_{\mu\nu}$.

Since T' is homotopic to the identity, we have

$$T^\#_{\mu\nu}\{\alpha\} = \{\alpha T'T\} = \{\alpha T\} = \{\alpha' T\} = \{\varphi_0\} = 0,$$

which shows that $[u_\nu]=[T^\#_{\mu\nu}u_\nu]=0$. Hence $\bar{h}$ is an isomorphism. This completes the proof of (5.1).

6. In view of the theorem (5.1) and a theorem of Kelley and Pitcher [7] that the direct limit of exact sequences is exact, for the proof of the exactness axiom (4.2) it is sufficient to verify this axiom for simplicial complexes. Since infinite simplicial complexes have also the homotopy extension property [14, Theorem 37], it is seen that Spanier's arguments remain valid for infinite simplicial complexes with slight modifications. Thus (4.2) is proved.[3a)]

As another application of (5.1) we note that the reduction and extension theorems of A. D. Wallace [13] hold also for the cohomotopy groups of fully normal spaces.[4)]

7. Under the same assumption as in § 5 we can define a homomorphism

$$\bar{\varphi} : \pi^n(X, A) \to H^n(X, A; \pi_n(S^n))$$

as in Spanier [12]. Here $H^n(X, A)$ denotes the unrestricted Čech cohomology group of X modulo A as before. Then the Hopf classification theorem [2] is stated as follows.

(7.1) *If* $\dim(X-A) \leqq n$, *the homomorphism* $\bar{\varphi} : \pi^n(X, A) \to H^n(X, A; \pi_n(S^n))$ *is an isomorphism onto for* $n>1$.

Indeed, by the results of § 3 this theorem can be reduced to the case of a pair (X_A, q_A) with $\dim X_A \leqq n$ and hence (7.1) follows.

3 a) Added in proof. Mr. H. Miyazaki has obtained the same result independently.

4) They are proved by using the following theorem, for the cohomotopy groups as well as for the unrestricted Cech cohomology groups: If A, X_0 are closed sets of a fully normal space X and $\{V_\lambda\}$ is a locally finite open covering of A, then there are an open set N containing A and a locally finite system $\{W_\lambda\}$ of open sets of X such that $A \subset \bigcup_\lambda W_\lambda$, $W_\lambda \cap A \subset V_\lambda$ for each λ and the systems of sets $\{W_\lambda, X_0 \cap N\}$, $\{W_\lambda \cap A, X_0 \cap A\}$ are similar. Cf. also S. Sakai's paper in this issue of our Science Reports.

The following theorem is deduced from the Hopf extension theorem [2] as is shown in [12].

(7.2) *If* $\dim(X-A) \leqq n+1$, *the homomorphism* $\bar{\varphi} : \pi^n(X, A) \to H^n(X, A; \pi_n(S^n))$ *is onto for* $n>2$.

8. C. H. Dowker [2] has introduced the notion of uniform homotopy. The spaces considered in this section will be normal. We define that $f, g : (X, A) \to (Y, B)$ are *uniformly homotopic* if there is a homotopy $h : (X \times I, A \times I) \to (Y, B)$ such that for any finite open covering $\mathfrak{V}$ of Y there exists a $\delta > 0$ so that $|t-t'| < \delta$ implies $h(x, t') \in S(h(x, t), \mathfrak{V})$ for every $x \in X$. Our definition is equivalent to that of Dowker in case Y is a compactum. Let $\beta(X)$, $\beta(Y)$ be the Cech compactifications of X, Y; the bar indicates the closure operations in $\beta(X)$, $\beta(Y)$.

(8.1) $f, g : (X, A) \to (Y, B)$ *are uniformly homotopic if and only if their extensions* $\tilde{f}, \tilde{g} : (\beta(X), \bar{A}) \to (\beta(Y), \bar{B})$ *are homotopic.*[5)]

Proof. It is sufficient to prove the "only if" part. Let f, g be uniformly homotopic and let $h(x, t)$ be a homotopy with the property mentioned in the above definition. If t is fixed, $h(x, t)$ defines a map $h_t : X \to Y$ and h_t has the unique extension $\tilde{h}_t : \beta(X) \to \beta(Y)$. We put $\tilde{h}(x, t) = \tilde{h}_t(x)$. We prove that $\tilde{h} : (\beta(X) \times I, \bar{A} \times I) \to (\beta(Y), \bar{B})$ is continuous. Let $(x_0, t_0) \in \beta(X) \times I$ and W_0 any neighbourhood of $\tilde{h}(x_0, t_0)$ in $\beta(Y)$. Then there are open sets W_1, W_2 of Y such that $\tilde{h}(x_0, t_0) \in \beta(Y) - \overline{Y - W_2}$, $\bar{W}_2 \frown Y \subset W_1$, $\bar{W}_1 \subset W_0$. Since $\tilde{h}_{t_0}$ is continuous, there is an open set V of X such that we have $x_0 \in \beta(X) - \overline{X - V}$ and $h_{t_0}(x) \in \beta(Y) - \overline{Y - W_2}$ for every $x \in \beta(X) - \overline{X - V}$. Since $\mathfrak{W} = \{W_1, Y - \bar{W}_2\}$ is an open covering of Y, there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies $h(x, t) \in S(h(x, t_0), \mathfrak{W})$ for every $x \in X$. Since $h(x, t_0) \in W_2$ for $x \in V$, we have $h(x, t) \in W_1$ for $x \in V$ and any t satisfying $|t - t_0| < \delta$. Consequently, if $x \in \beta(X) - \overline{X - V}$ and $|t - t_0| < \delta$, we have $\tilde{h}(x, t) \in \bar{W}_1 \subset W_0$. Thus (8.1) is proved.

(8.2) *If* $\dim(X-A) < 2n-1$, *then we have* $\dim F < 2n-1$ *for any closed set* F *of* $\beta(X)$ *such that* $F \subset \beta(X) - \bar{A}$.

Proof. Since $\beta(X)$ is normal, there is an open set G of $\beta(X)$ such that $F \subset G \subset \bar{G} \subset \beta(X) - \bar{A}$. Then we have $F \subset G \subset \beta(X) - \overline{X - G} \subset \overline{\bar{G} \frown X} \subset \bar{G}$ and $\dim F \leqq \dim \overline{\bar{G} \frown X} = \dim \bar{G} \frown X \leqq \dim(X - A)$.

Now let $\dim(X-A) < 2n-1$. By (8.1), (8.2) and (2.1) we see

5) Cf. [2], p. 229, where Dowker proved (8.1) for a compactum Y.

that the set of the uniform homotopy classes $\{\alpha\}_U$ of maps $\alpha:(X, A)\rightarrow(S^n, p)$ forms an abelian group with the addition defined similarly as in §2. This goup is called the n-th *uniform cohomotopy group* of X modulo A and is denoted by $\pi_U^n(X, A)$. Clearly we have

(8.3) *$\pi_U^n(X, A)\approx\pi^n(\beta(X), \overline{A})$ where X is a normal space and A is a closed set of X.*

If we replace homotopy by uniform homotopy, the results of §§2–4 hold for the uniform cohomotopy groups of normal spaces; in particular all the Eilenberg-Steenrod axioms for cohomology theory are satisfied. The analogous results of §§5 and 6 are also obtained; in this case we must replace locally finite coverings by finite open coverings. Thus we have

(8.4) *If $\dim X<2n-1$ and $\{\mathfrak{B}_\nu\}$ is a family of all finite open coverings of X of order$\leqq 2n-1$, then $\pi_U^n(X, A)$ is isomorphic to the limit group of the direct system $\{\pi^n(K_\nu, L_\nu); T_{\mu\nu}^{\#}\}$, where K_ν, L_ν, $T_{\mu\nu}$ are defined similarly as in §5.*

As is seen from (8.4) the uniform cohomotopy groups $\pi_U^n(X, A)$ correspond in some sense to the Cech cohomology groups $H_F^n(X, A)$ based on finite coverings; the latter satisfy also the uniform homotopy axiom. In fact, the Hopf classification theorem proved in Dowker [2] for normal spaces (cf. also [8] where homotopy is used in the sense of uniform homotopy) is stated as the analogue of (7.1):

(8.5) *If $\dim(X-A)\leqq n$ and $n>1$, then $\pi_U^n(X, A)\approx H_F^n(X, A; \pi_n(S^n))$.*

Similarly we have the analogue of (7.2) for the uniform cohomotopy groups.

Finally it is to be noted that in case X is a countably fully normal space there is a natural homomorphism of $\pi_U^n(X, A)$ onto $\pi^n(X, A)$.

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