Frobenius extensions and tilting complexes

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Dedicated to Takeshi Sumioka on the occasion of his 60-th birthday

Abstract

Let $n \geq 1$ be an integer and π a permutation of $I = \{1, \cdots, n\}$. For any ring R, we provide a systematic construction of rings A which contain R as a subring and enjoy the following properties: (a) $1 = \sum_{i \in I} e_i$ with the e_i orthogonal idempotents; (b) $e_i x = x e_i$ for all $i \in I$ and $x \in R$; (c) $e_i A e_j \neq 0$ for all $i, j \in I$; (d) $e_i A_A \not\cong e_j A_A$ unless $i = j$; (e) every $e_i A e_i$ is a local ring whenever so is R; (f) $e_i A_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)$ and ${}_A Ae_{\pi(i)} \cong {}_A\mathrm{Hom}_R(e_i A, {}_R R)$ for all $i \in I$; and (g) there exists a ring automorphism $\eta \in \mathrm{Aut}(A)$ such that $\eta(e_i) = e_{\pi(i)}$ for all $i \in I$. Furthermore, for any nonempty π-stable subset J of I, the mapping cone of the multiplication map $\bigoplus_{i \in J} Ae_i \otimes_R e_i A_A \to A_A$ is a tilting complex.

Let A be a ring and $e \in A$ an idempotent. Assume A contains a subring R such that $xe = ex$ for all $x \in R$, Ae_R is finitely generated and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule. Then A/AeA is finitely presented as a right A-module and $\mathrm{Hom}_A(A/AeA, eA) = 0$. Thus by [5, Proposition 1.2] there exists a tilting complex (see [13]) of the form

$$T^\bullet : \cdots \to 0 \to T^{-1} \to T^0 \to 0 \to \cdots$$

such that $T^0 \in \mathrm{add}((1-e)A_A)$, $T^{-1} \in \mathrm{add}(eA_A)$ and $eA[1] \in \mathrm{add}(T^\bullet)$. This type of tilting complex plays an important role in the theory of derived equivalences. For instance, Rickard [14] showed that the Brauer tree algebras over a field with the same numerical invariants are derived equivalent to each other and then Okuyama pointed out that such derived equivalences are given as iterations of derived equivalences induced by tilting complexes of the above type. Our aim is to provide a way to construct extensions A of a given ring R containing such an idempotent. To do so, we need the notion of Frobenius extensions of rings due to Nakayama-Tsuzuku [10, 11] (cf. also Kasch [6, 7]) which we modify as follows. Let A be a ring containing a ring R as a subring. Then A is said to be a Frobenius extension of R if the following conditions are satisfied: (F1) A_R

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and ${}_RA$ are finitely generated projective; and (F2) $A_A \cong \mathrm{Hom}_R(A, R_R)_A$ and ${}_AA \cong {}_A\mathrm{Hom}_R(A, {}_RR)$. We will see that Frobenius extensions preserve various homological properties (cf. [6], [7], [9], [10], [11] and so on). For instance, the following hold: inj dim $A_A \le$ inj dim R_R and inj dim ${}_AA \le$ inj dim ${}_RR$; if R is a noetherian ring satisfying the Auslander condition (see [1]) then so is A; and, if R is a quasi-Frobenius ring, i.e., a selfinjective artinian ring then so is A.

For any integer $n \ge 1$, any permutation π of $I = \{1, \cdots, n\}$ and any ring R, we will provide a way to construct a Frobenius extension A of R which enjoys the following properties: (a) $1 = \sum_{i \in I} e_i$ with the e_i orthogonal idempotents; (b) $e_i x = x e_i$ for all $i \in I$ and $x \in R$; (c) $e_i A e_j \ne 0$ for all $i, j \in I$; (d) $e_i A_A \not\cong e_j A_A$ unless $i = j$; (e) every $e_i A e_i$ is a local ring whenever so is R; (f) $e_i A_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A$ and ${}_AAe_{\pi(i)} \cong {}_A\mathrm{Hom}_R(e_iA, {}_RR)$ for all $i \in I$; and (g) there exists a ring automorphism $\eta \in \mathrm{Aut}(A)$ such that $\eta(e_i) = e_{\pi(i)}$ for all $i \in I$. In particular, for any nonempty π-stable subset J of I, we get a desired idempotent $e = \sum_{j \in J} e_j$. In case π is cyclic, we have constructed such a Frobenius extension in [4] (cf. also [8] and [12]). We generalize this construction. Namely, we define an appropriate multiplication on a free right R-module A with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I_0}$, where $I_0 = \{i \in I \mid \pi(i) = i\}$, and then set $e_i = e_{ii}$ for $i \in I$. To do so, we need a certain pair (t, ω) of an integer $t \ge 1$ and a mapping $\omega : I \times I \to \mathbb{Z}$ and a certain pair (c, σ) of a nonunit $c \in R$ and a ring automorphism $\sigma \in \mathrm{Aut}(R)$. Although the ring structure of A depends on the choice of (t, ω) and (c, σ), the properties (a)–(g) above are always enjoyed. Finally, consider the case where c is regular. Then we will see that if I_0 is empty then A can be embedded as a subring in the $n \times n$ full matrix ring $\mathrm{M}_n(R)$ over R, and that if $i \in I \setminus I_0$ then A is derived equivalent to a generalized triangular matrix ring

$$\begin{pmatrix} e_iAe_i & \mathrm{Ext}^1_A(A/Ae_iA, e_iA) \\ 0 & A/Ae_iA \end{pmatrix}$$

and $\mathrm{Ext}^1_A(A/Ae_iA, e_iA) \cong e_{\pi^{-1}(i)}(A/Ae_iA)$ as right (A/Ae_iA)-modules.

For a ring R, we denote by $\mathrm{Z}(R)$ the center of R, by $R^\times$ the set of units in R and by $\mathrm{Aut}(R)$ the group of ring automorphisms of R. We denote by Mod-R the category of right R-modules and sometimes consider left R-modules as right R^{op}-modules, where R^{op} denotes the opposite ring of R. We use the notation X_R (resp., ${}_RX$) to stress that the module X considered is a right (resp., left) R-module. For a module X, by an injective resolution of X we mean a cochain complex $I^\bullet$ of injective modules such that $I^i = 0$ for $i < 0$, $\mathrm{H}^i(I^\bullet) = 0$ for $i > 0$ and $\mathrm{H}^0(I^\bullet) \cong X$, where $\mathrm{H}^i(-)$ denotes the i^{th} cohomology. We refer to [2] for standard homological algebra in module categories.

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1 Definition and basic properties

In this note, a ring A is said to be an extension of a ring R if A contains R as a subring. We start by modifying the notion of Frobenius extensions of rings due to Nakayama-Tsuzuku [10, 11] (cf. also Kasch [6, 7]) as follows.

Definition 1.1. Let A be an extension of a ring R. Then A is said to be a Frobenius extension of R if the following conditions are satisfied:
(F1) A_R and ${}_RA$ are finitely generated projective; and
(F2) $A_A \cong \mathrm{Hom}_R(A, R_R)_A$ and ${}_AA \cong {}_A\mathrm{Hom}_R(A, {}_RR)$.

Remark 1.2. Let A be an extension of a ring R and assume there exists an isomorphism of right A-modules $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R(A, R_R)_A$. Then the following hold.

(1) There exists a ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. In particular, ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.

(2) Assume A_R is finitely generated projective. Then ${}_R\mathrm{Hom}_R(A, R_R)$ is finitely generated projective and we have an isomorphism of A-R-bimodules

$$ {}_AA_R \xrightarrow{\sim} {}_A\mathrm{Hom}_R(\mathrm{Hom}_R(A, R_R), {}_RR)_R, a \mapsto (h \mapsto h(a)). $$

Thus, if ϕ is an isomorphism of R-A-bimodules, then ${}_RA$ is finitely generated projective and we have an isomorphism of A-R-bimodules $\psi : {}_AA_R \xrightarrow{\sim} {}_A\mathrm{Hom}_R(A, {}_RR)_R$ such that $\psi(a)(b) = \phi(b)(a)$ for all $a, b \in A$.

Throughout the rest of this section, A is a Frobenius extension of R. We fix an isomorphism of right A-modules $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R(A, R_R)_A$. Then, as remarked above, there exists a ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. For a right (resp., left) A-module M_A (resp., ${}_AL$) we denote by $M_{\theta(R)}$ (resp., ${}_{\theta(R)}L$) the right (resp., left) R-module on which R operates via $\theta : R \to A$. Then ϕ yields an isomorphism of R-A-bimodules $\phi : {}_{\theta(R)}A_A \xrightarrow{\sim} {}_R\mathrm{Hom}_R(A, R_R)_A$. Similarly, we fix an isomorphism of left A-modules $\psi : {}_AA \xrightarrow{\sim} {}_A\mathrm{Hom}_R(A, {}_RR)$. Then there exists a ring homomorphism $\eta : R \to A$ such that $\psi(1)x = \eta(x)\psi(1)$ for all $x \in R$. For a right (resp., left) A-module M_A (resp., ${}_AL$) we denote by $M_{\eta(R)}$ (resp., ${}_{\eta(R)}L$) the right (resp., left) R-module on which R operates via $\eta : R \to A$. Then ψ yields an isomorphism of A-R-bimodules $\psi : {}_AA_{\eta(R)} \xrightarrow{\sim} {}_A\mathrm{Hom}_R(A, {}_RR)_R$. Note that ${}_{\theta(R)}A$ and $A_{\eta(R)}$ are finitely generated projective.

Recall that in [10, 11] A is said to be a Frobenius extension of second kind if θ induces a ring automorphism of R and to be a Frobenius extension of first kind if $\theta(x) = x$ for all $x \in R$. However, we will see in Section 3 that $\theta(R) \neq R$ in general. In the following, we collect several basic properties of Frobenius extensions (cf. [6], [7], [9], [10], [11] and so on). By symmetry, "right" and "left" can be exchanged in the following statements.

Remark 1.3. Let $X \in \text{Mod-}R$, $M \in \text{Mod-}A$ and $L \in \text{Mod-}A^{\mathrm{op}}$. Then we have the following bifunctorial isomorphisms:

(1) $\mathrm{Hom}_R(M, X \otimes_R A) \cong \mathrm{Hom}_R(M_{\eta(R)}, X)$;

(2) $\mathrm{Hom}_A(\mathrm{Hom}_R(A,X),M) \cong \mathrm{Hom}_R(X, M_{\theta(R)})$; and

(3) $\mathrm{Hom}_R(A,X) \otimes_A L \cong X \otimes_{R\ \theta(R)} L$.

Proof. Since ${}_RA$ and A_R are finitely generated projective, we have functorial isomorphisms in Mod-A

$$X \otimes_R A \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(A, {}_RR), X), x \otimes a \mapsto (h \mapsto xh(a)),$$

$$X \otimes_R \mathrm{Hom}_R(A, R_R) \xrightarrow{\sim} \mathrm{Hom}_R(A, X), x \otimes h \mapsto (a \mapsto xh(a))$$

which are special cases of Watt's theorem (cf. [15]). Since ${}_A\mathrm{Hom}_R(A, {}_RR)_R \cong {}_AA_{\eta(R)}$, we have bifunctorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_A(M, X \otimes_R A) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(A_{\eta(R)}, X)) \\ &\cong \mathrm{Hom}_R(M \otimes_A A_{\eta(R)}, X) \\ &\cong \mathrm{Hom}_R(M_{\eta(R)}, X). \end{aligned}$$

Similarly, since ${}_R\mathrm{Hom}_R(A, R_R)_A \cong {}_{\theta(R)}A_A$, we have bifunctorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_A(\mathrm{Hom}_R(A,X), M) &\cong \mathrm{Hom}_A(X \otimes_{R\ \theta(R)} A, M) \\ &\cong \mathrm{Hom}_R(X, \mathrm{Hom}_A({}_{\theta(R)}A, M)) \\ &\cong \mathrm{Hom}_R(X, M_{\theta(R)}), \end{aligned}$$

$$\mathrm{Hom}_R(A,X) \otimes_A L \cong X \otimes_{R\ \theta(R)} A \otimes_A L \cong X \otimes_{R\ \theta(R)} L.$$

□

The first two isomorphisms of the following preliminary lemma are known as Eckmann-Shapiro lemma.

Lemma 1.4. *Let $X \in$ Mod-R, $M \in$ Mod-A and $L \in$ Mod-A^{op}. Then for any $i \geq 0$ we have the following bifunctorial isomorphisms:*

(1) $\mathrm{Ext}^i_A(M, \mathrm{Hom}_R(A,X)) \cong \mathrm{Ext}^i_R(M,X)$;

(2) $\mathrm{Ext}^i_A(X \otimes_R A, M) \cong \mathrm{Ext}^i_R(X,M)$;

(3) $\mathrm{Tor}^A_i(X \otimes_R A, L) \cong \mathrm{Tor}^R_i(X,L)$;

(4) $\mathrm{Ext}^i_A(M, X \otimes_R A) \cong \mathrm{Ext}^i_R(M_{\eta(R)}, X)$;

(5) $\mathrm{Ext}^i_A(\mathrm{Hom}_R(A,X), M) \cong \mathrm{Ext}^i_R(X, M_{\theta(R)})$; *and*

(6) $\mathrm{Tor}^A_i(\mathrm{Hom}_R(A,X), L) \cong \mathrm{Tor}^R_i(X, {}_{\theta(R)}L)$.

Proof. See [2, Chapter VI, Section 4] for the first three isomorphisms; (1) follows by the projectivity of A_R and (2), (3) follow by the flatness of ${}_RA$.

Similarly, according to Remark 1.3, the last three isomorphisms follow by the exactness of $- \otimes_R A$ and $\mathrm{Hom}_R(A, -)$. □

Proposition 1.5. *The following hold.*

(1) If R is right noetherian (resp., artinian), so is A.

(2) $\operatorname{Ext}^i_A(M, A) \cong \operatorname{Ext}^i_R(M, R)$ *for all* $M \in \operatorname{Mod-}A$ *and* $i \geq 0$. *In particular,* $\operatorname{inj\,dim} A_A \leq \operatorname{inj\,dim} R_R$.

(3) If $I^\bullet$ is an injective resolution of R_R, then $\operatorname{Hom}_R(A, I^\bullet)$ *is an injective resolution of A_A with* $\operatorname{flat\,dim} \operatorname{Hom}_R(A, I^i)_A \leq \operatorname{flat\,dim} I^i_R$ *for all $i \geq 0$.*

Proof. (1) follows by the fact that A_R is finitely generated. Also, since $A_A \cong \operatorname{Hom}_R(A, R_R)_A$, (2) follows by Lemma 1.4(1). Finally, since $\operatorname{Hom}_R(A, -)$ is exact, and since $A_A \cong \operatorname{Hom}_R(A, R_R)_A$, (3) follows by (1), (6) of Lemma 1.4. □

Lemma 1.6. *Assume the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. Then for any $X \in \operatorname{Mod-}R$ the following hold.*

(1) $\operatorname{inj\,dim} \operatorname{Hom}_R(A, X)_A = \operatorname{inj\,dim} X \otimes_R A_A = \operatorname{inj\,dim} X_R$.

(2) $\operatorname{proj\,dim} \operatorname{Hom}_R(A, X)_A = \operatorname{proj\,dim} X \otimes_R A_A = \operatorname{proj\,dim} X_R$.

(3) $\operatorname{flat\,dim} \operatorname{Hom}_R(A, X)_A = \operatorname{flat\,dim} X \otimes_R A_A = \operatorname{flat\,dim} X_R$.

Proof. Note that every $X \in \operatorname{Mod-}R$ is a direct summand of both $\operatorname{Hom}_R(A, X)_R$ and $X \otimes_R A_R$.

(1) By Lemma 1.4(1) $\operatorname{inj\,dim} \operatorname{Hom}_R(A, X)_A \leq \operatorname{inj\,dim} X_R$. Conversely, assume $\operatorname{inj\,dim} \operatorname{Hom}_R(A, X)_A = d < \infty$. Then for any $Y \in \operatorname{Mod-}R$ and $i > d$ by Lemma 1.4(1) $\operatorname{Ext}^i_R(\operatorname{Hom}_R(A, Y), X) = 0$ and hence $\operatorname{Ext}^i_R(Y, X) = 0$. Thus $\operatorname{inj\,dim} X_R \leq d$.

Similarly, by Lemma 1.4(4) $\operatorname{inj\,dim} X \otimes_R A_A \leq \operatorname{inj\,dim} X_R$. Conversely, assume $\operatorname{inj\,dim} X \otimes_R A_A = d < \infty$. Then for any $Y \in \operatorname{Mod-}R$ and $i > d$ by Lemma 1.4(2) $\operatorname{Ext}^i_R(Y, X \otimes_R A) = 0$ and hence $\operatorname{Ext}^i_R(Y, X) = 0$. Thus $\operatorname{inj\,dim} X_R \leq d$.

(2) and (3) follow by the same arguments as in (1). □

Proposition 1.7. *Assume the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. Then the following hold.*

(1) If A is right noetherian (resp., artinian), so is R.

(2) $\operatorname{inj\,dim} A_A = \operatorname{inj\,dim} R_R$.

(3) If $I^\bullet$ is an injective resolution of R_R, then $\operatorname{Hom}_R(A, I^\bullet)$ *is an injective resolution of A_A with* $\operatorname{flat\,dim} \operatorname{Hom}_R(A, I^i)_A = \operatorname{flat\,dim} I^i_R$ *for all $i \geq 0$.*

Proof. (1) Take a homomorphism of R-R-bimodules $\gamma : A \to R$ such that $\gamma(x) = x$ for all $x \in R$. Then $\gamma(\mathfrak{a}A) = \mathfrak{a}$ for all right ideals $\mathfrak{a}$ of R and the assertion follows.

(2) Since $A_A \cong \operatorname{Hom}_R(A, R_R)_A$, this follows by Lemma 1.6(1).

(3) follows by Proposition 1.5(3) and Lemma 1.6(3). □

Definition 1.8. A Frobenius extension A of R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

2 Notation

To construct a desired Frobenius extension, we fix the following notation which will be kept throughout this and the next sections.

Let $n \geq 1$ be an integer, π a permutation of $I = \{1, \cdots, n\}$ and $I_0 = \{i \in I \mid \pi(i) = i\}$. Let $t \geq 1$ be an integer, let $\omega : I \times I \to \mathbb{Z}$ be a mapping and define a mapping $\chi : I \to \mathbb{Z}$ as follows:

$$\chi(i) = \begin{cases} t & \text{if } i \in I_0, \\ \omega(i, \pi(i)) & \text{if } i \in I \setminus I_0. \end{cases}$$

We assume the following conditions are satisfied:
(W1) $\omega(i,i) = 0$ for all $i \in I$;
(W2) $\omega(i,j) + \omega(j,k) \geq \omega(i,k)$ for all $i,j,k \in I$;
(W3) $\omega(i,j) + \omega(j,i) \geq 1$ unless $i = j$; and
(W4) $\omega(i,j) + \omega(j,\pi(i)) = \chi(i)$ unless $i = j \in I_0$.

Example 2.1. Let $t = 2$ and define $\omega : I \times I \to \mathbb{Z}$ as follows: $\omega(i,j) = 0$ if $i = j$, $\omega(i,j) = 2$ if $j = \pi(i) \neq i$ and $\omega(i,j) = 1$ otherwise. Then the conditions (W1)–(W4) are satisfied.

Lemma 2.2. *We have $\omega(\pi(i), \pi(j)) = \omega(i,j) - \chi(i) + \chi(j)$ for all $i, j \in I$.*

Proof. We may assume $i \neq j$. In case $j \neq \pi(i)$, by (W4) $\{\omega(i,j) - \chi(i)\} + \chi(j) = -\omega(j, \pi(i)) + \{\omega(j, \pi(i)) + \omega(\pi(i), \pi(j))\} = \omega(\pi(i), \pi(j))$. Assume $j = \pi(i)$. Then $i \in I \setminus I_0$ and $\omega(i,j) - \chi(i) + \chi(j) = \omega(i, \pi(i)) - \chi(i) + \chi(\pi(i)) = \chi(\pi(i))$. Note that by (W1) $\omega(\pi(j), \pi(\pi(i))) = 0$. Thus, since $\pi(i) \neq \pi(j)$, by (W4) $\chi(\pi(i)) = \omega(\pi(i), \pi(j)) + \omega(\pi(j), \pi(\pi(i))) = \omega(\pi(i), \pi(j))$. □

For the sake of convenience, we define a mapping $\lambda : I \times I \times I \to \mathbb{Z}$ as follows:

$$\lambda(i,j,k) = \omega(i,j) + \omega(j,k) - \omega(i,k)$$

for all $i,j,k \in I$. It is easy to see that the following hold:
(L1) $\lambda(i,j,k) \geq 0$ for all $i,j,k \in I$;
(L2) $\lambda(i,j,k) = 0$ if either $i = j$ or $j = k$;
(L3) $\lambda(i,j,i) = \lambda(j,i,j) \geq 1$ unless $i = j$;
(L4) $\lambda(i,j,\pi(i)) = 0$ for all $i \in I \setminus I_0$ and $j \in I$; and
(L5) $\lambda(i,j,i) = \chi(i)$ for all $i \in I_0$ and $j \in I \setminus \{i\}$.

Lemma 2.3. *The following hold.*

(1) $\omega(i,j) + \omega(j,k) = \lambda(i,j,k) + \omega(i,k)$ for all $i,j,k \in I$.

(2) $\lambda(i,j,k) + \lambda(i,k,l) = \lambda(i,j,l) + \lambda(j,k,l)$ for all $i,j,k,l \in I$.

(3) $\lambda(\pi(i), \pi(j), \pi(k)) = \lambda(i,j,k)$ for all $i,j,k \in I$.

(4) $\lambda(i,j,k) = \lambda(j,k,i)$ *for all* $i \in I_0$ *and* $j,k \in I \setminus \{i\}$.

Proof. (1) and (2) follow by the definition and (3) follows by Lemma 2.2.

(4) By (2) and (L5) $\lambda(i,j,k) - \lambda(j,k,i) = \lambda(i,j,i) - \lambda(i,k,i) = \chi(i) - \chi(i) = 0$. $\square$

Also, we fix a ring R together with a pair of a nonunit $c \in R \setminus R^\times$ and a ring automorphism $\sigma \in \mathrm{Aut}(R)$ satisfying the following condition:

$$(*) \quad \sigma(c) = c \text{ and } xc = c\sigma(x) \text{ for all } x \in R.$$

This is obviously satisfied if either $c = 0$ and σ is arbitrary, or $c \in \mathrm{Z}(R)$ and $\sigma = \mathrm{id}_R$. We provide a non-trivial example.

Example 2.4. Let $k[X]$ be a polynomial ring in one variable X over a commutative ring k and $\mathfrak{a} = (X^m)$ an ideal of $k[X]$ generated by X^m with $m \geq 3$. Set $R = k[X]/\mathfrak{a}$, $x = X + \mathfrak{a}$ and $c = x^r$ with $m > r \geq (m+1)/2$. Then there exists $\sigma \in \mathrm{Aut}(R)$ such that $\sigma(f(x)) = f(x+c)$ for all $f(X) \in k[X]$. It is easy to see that the condition $(*)$ is satisfied.

Here, we deal with the case of $n = 1$. Let S be a free right R-module with a basis $\{e, v\}$ and define the multiplication on S subject to the following axioms:
(S1) $e^2 = e$, $v^2 = -vc^t$ and $ev = v = ve$; and
(S2) $xe = ex$ and $xv = v\sigma^t(x)$ for all $x \in R$.

Lemma 2.5. *The following hold.*

(1) S *is an associative ring with* $1 = e$.

(2) S *is a split Frobenius extension of* R*, where* R *is considered as a subring of* S *via the injective ring homomorphism* $R \to S, x \mapsto ex$.

(3) If R *is local, so is* S.

Proof. (1) and (2) will be proved in the next section (see Theorem 3.1).

(3) Let $\mathfrak{m} = R \setminus R^\times$ and $\mathfrak{M} = e\mathfrak{m} + vR$. It is easy to see that $\mathfrak{M}$ is an ideal of S. We claim that $\mathfrak{M} = S \setminus S^\times$. Take a basis $\{\alpha, \rho\}$ for ${}_R\mathrm{Hom}_R(S, R_R)$ such that $a = e\alpha(a) + v\rho(a)$ for all $a \in S$. Then for any $a, b \in A$ we have $\alpha(ab) = \alpha(a)\alpha(b)$ and $\rho(ab) = \sigma^t(\alpha(a))\rho(b) + \rho(a)\alpha(b) - c^t\sigma^t(\rho(a))\rho(b)$. For any $a \in S^\times$ we have $\alpha(a) \in R^\times$ and $a \in S \setminus \mathfrak{M}$. Let $a \in S \setminus \mathfrak{M}$. Then $\alpha(a) \in R^\times$ and, since $c^t \in \mathfrak{m}$, $\alpha(a) - c^t\rho(a) \in R^\times$. Thus, by setting $x = \sigma^t(\alpha(a))^{-1}\rho(a)(c^t\rho(a) - \alpha(a))^{-1}$, we have $(e\alpha(a)^{-1} + vx)a = e$. Similarly, a has a right inverse too. Thus $a \in S^\times$. $\square$

3 Construction

Let A be a free right R-module with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I_0}$ and define the multiplication on A subject to the following axioms:
(A1) $e_{ij}e_{kl} = 0$ unless $j = k$;
(A2) $e_{ij}e_{jk} = e_{ik}c^{\lambda(i,j,k)}$ unless $i = k \in I_0$ and $j \in I \setminus \{i\}$;

(A3) $e_{ij}e_{ji} = v_i + e_{ii}c^{\chi(i)}$ for all $i \in I_0$ and $j \in I \setminus \{i\}$;
(A4) $v_iv_j = 0$ unless $i = j$ and $v_i^2 = -v_ic^{\chi(i)}$ for all $i \in I_0$;
(A5) $v_ie_{jk} = 0 = e_{ij}v_k$ unless $i = j = k$ and $v_ie_{ii} = v_i = e_{ii}v_i$ for all $i \in I_0$;
(A6) $xe_{ij} = e_{ij}\sigma^{\omega(i,j)}(x)$ for all $i, j \in I$ and $x \in R$; and
(A7) $xv_i = v_i\sigma^{\chi(i)}(x)$ for all $i \in I_0$ and $x \in R$.

As usual, we require $c^0 = 1$ even if $c = 0$. We fix a basis $\{\alpha_{ij}\}_{i,j\in I} \cup \{\rho_i\}_{i\in I_0}$ for ${}_R\mathrm{Hom}_R(A, R_R)$ such that

$$a = \sum_{i,j\in I} e_{ij}\alpha_{ij}(a) + \sum_{i\in I_0} v_i\rho_i(a)$$

for all $a \in A$. Recall that $\chi(i) = t$ for all $i \in I_0$. For any $a, b \in A$ we have

$$\begin{aligned} ab = &\sum_{i,j,k\in I} e_{ij}e_{jk}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b) \\ &\quad + \sum_{i\in I_0} v_i\{\sigma^t(\alpha_{ii}(a))\rho_i(b) + \rho_i(a)\alpha_{ii}(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)\} \\ = &\sum_{i,j,k\in I} e_{ik}c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b) + \sum_{i\in I_0, j\in I\setminus\{i\}} v_i\sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) \\ &\quad + \sum_{i\in I_0} v_i\{\sigma^t(\alpha_{ii}(a))\rho_i(b) + \rho_i(a)\alpha_{ii}(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)\} \end{aligned}$$

and hence the following hold:
(M1) $\alpha_{ik}(ab) = \sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b)$ for all $i, k \in I$; and
(M2) $\rho_i(ab) = \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) + \sigma^t(\alpha_{ii}(a))\rho_i(b) + \rho_i(a)\alpha_{ii}(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)$ for all $i \in I_0$.

In the following, we set $e_i = e_{ii}$ and $\alpha_i = \alpha_{ii}$ for $i \in I$. Note that by (W1), (A6) $xe_i = e_ix$ for all $i \in I$ and $x \in R$, and that by (L2), (A1), (A2) and (A5) $1 = \sum_{i\in I} e_i$ with the e_i orthogonal idempotents.

Theorem 3.1. *The following hold.*

(1) A is an associative ring with $1 = \sum_{i\in I} e_i$, where the e_i are orthogonal idempotents.

(2) $e_iAe_i = e_iR + v_iR$ for all $i \in I_0$ and $e_iAe_j = e_{ij}R$ unless $i = j \in I_0$. In particular, $e_iAe_i \cong S$ as rings for all $i \in I_0$ and $e_iAe_i \cong R$ as rings for all $i \in I \setminus I_0$.

(3) $e_iA_A \not\cong e_jA_A$ unless $i = j$.

(4) $e_iA_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A$ and ${}_AAe_{\pi(i)} \cong {}_A\mathrm{Hom}_R(e_iA, {}_RR)$ for all $i \in I$, so that for any nonempty π-stable subset J of I, by setting $e = \sum_{i\in J} e_i$, we have $eA_A \cong \mathrm{Hom}_R(Ae, R_R)_A$ and ${}_AAe \cong {}_A\mathrm{Hom}_R(eA, {}_RR)$.

(5) *A is a split Frobenius extension of R, where R is considered as a subring of A via the injective ring homomorphism $R \to A, x \mapsto \sum_{i \in I} e_i x$.*

Proof. (1) Let $a_1, a_2, a_3 \in A$. For any $i, l \in I$ by (M1) we have

$$
\begin{aligned}
&\alpha_{il}(a_1(a_2a_3)) \\
&= \sum_{j \in I} c^{\lambda(i,j,l)} \sigma^{\omega(j,l)}(\alpha_{ij}(a_1))\alpha_{jl}(a_2a_3) \\
&= \sum_{j \in I} c^{\lambda(i,j,l)} \sigma^{\omega(j,l)}(\alpha_{ij}(a_1))\{\sum_{k \in I} c^{\lambda(j,k,l)} \sigma^{\omega(k,l)}(\alpha_{jk}(a_2))\alpha_{kl}(a_3)\} \\
&= \sum_{j,k \in I} c^{\lambda(i,j,l)+\lambda(j,k,l)} \sigma^{\lambda(j,k,l)+\omega(j,l)}(\alpha_{ij}(a_1))\sigma^{\omega(k,l)}(\alpha_{jk}(a_2))\alpha_{kl}(a_3),
\end{aligned}
$$

$$
\begin{aligned}
&\alpha_{il}((a_1a_2)a_3) \\
&= \sum_{k \in I} c^{\lambda(i,k,l)} \sigma^{\omega(k,l)}(\alpha_{ik}(a_1a_2))\alpha_{kl}(a_3) \\
&= \sum_{k \in I} c^{\lambda(i,k,l)} \sigma^{\omega(k,l)}(\{\sum_{j \in I} c^{\lambda(i,j,k)} \sigma^{\omega(j,k)}(\alpha_{ij}(a_1))\alpha_{jk}(a_2)\})\alpha_{kl}(a_3) \\
&= \sum_{j,k \in I} c^{\lambda(i,k,l)+\lambda(i,j,k)} \sigma^{\omega(k,l)+\omega(j,k)}(\alpha_{ij}(a_1))\sigma^{\omega(k,l)}(\alpha_{jk}(a_2))\alpha_{kl}(a_3)
\end{aligned}
$$

and hence by (1), (2) of Lemma 2.3 $\alpha_{il}(a_1(a_2a_3)) = \alpha_{il}((a_1a_2)a_3)$. Similarly, for any $i \in I_0$ by (M1), (M2) we have

$$
\begin{aligned}
&\rho_i(a_1(a_2a_3)) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2a_3) \\
&\quad + \sigma^t(\alpha_i(a_1))\rho_i(a_2a_3) + \rho_i(a_1)\alpha_i(a_2a_3) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2a_3) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\{\sum_{k\in I} c^{\lambda(j,k,i)}\sigma^{\omega(k,i)}(\alpha_{jk}(a_2))\alpha_{ki}(a_3)\} \\
&\quad + \sigma^t(\alpha_i(a_1))\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad\qquad + \rho_i(a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_2))\rho_i(a_3)\} \\
&\quad + \rho_i(a_1)\{\sum_{j\in I} c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3)\} \\
&\quad - c^t\sigma^t(\rho_i(a_1))\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad\qquad + \rho_i(a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_2))\rho_i(a_3)\} \\
&= \sum_{j,k\in I\setminus\{i\}} c^{\lambda(j,k,i)}\sigma^{\lambda(j,k,i)+\omega(j,i)}(\alpha_{ij}(a_1))\sigma^{\omega(k,i)}(\alpha_{jk}(a_2))\alpha_{ki}(a_3) \\
&\qquad + \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2)\alpha_i(a_3) \\
&\quad + \sum_{j\in I\setminus\{i\}} \sigma^t(\alpha_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \sigma^t(\alpha_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad + \sigma^t(\alpha_i(a_1))\rho_i(a_2)\alpha_i(a_3) - c^t\sigma^{2t}(\alpha_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3) \\
&\quad + \sum_{j\in I\setminus\{i\}} c^t\sigma^t(\rho_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) + \rho_i(a_1)\alpha_i(a_2)\alpha_i(a_3) \\
&\quad - \sum_{j\in I\setminus\{i\}} c^t\sigma^t(\rho_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3) - c^t\sigma^t(\rho_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3) \\
&\qquad - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\alpha_i(a_3) + c^{2t}\sigma^{2t}(\rho_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3),
\end{aligned}
$$

$$
\begin{aligned}
&\rho_i((a_1a_2)a_3)\\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1a_2))\alpha_{ji}(a_3)\\
&\qquad + \sigma^t(\alpha_i(a_1a_2))\rho_i(a_3) + \rho_i(a_1a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_1a_2))\rho_i(a_3)\\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\{\sum_{k\in I} c^{\lambda(i,k,j)}\sigma^{\omega(k,j)}(\alpha_{ik}(a_1))\alpha_{kj}(a_2)\})\alpha_{ji}(a_3)\\
&\quad + \sigma^t(\{\sum_{j\in I} c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2)\})\rho_i(a_3)\\
&\quad + \{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2) + \sigma^t(\alpha_i(a_1))\rho_i(a_2)\\
&\qquad + \rho_i(a_1)\alpha_i(a_2) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\}\alpha_i(a_3)\\
&\quad - c^t\sigma^t(\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2) + \sigma^t(\alpha_i(a_1))\rho_i(a_2)\\
&\qquad\qquad + \rho_i(a_1)\alpha_i(a_2) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\})\rho_i(a_3)\\
&= \sum_{j,k\in I\setminus\{i\}} c^{\lambda(i,k,j)}\sigma^{\omega(j,i)+\omega(k,j)}(\alpha_{ik}(a_1))\sigma^{\omega(j,i)}(\alpha_{kj}(a_2))\alpha_{ji}(a_3)\\
&\qquad + \sum_{j\in I\setminus\{i\}} \sigma^t(\alpha_i(a_1))\sigma^{\omega(j,i)}(\alpha_{ij}(a_2))\alpha_{ji}(a_3)\\
&\quad + \sum_{j\in I\setminus\{i\}} c^t\sigma^{t+\omega(j,i)}(\alpha_{ij}(a_1))\sigma^t(\alpha_{ji}(a_2))\rho_i(a_3)\\
&\qquad + \sigma^t(\alpha_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3)\\
&\quad + \sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a_1))\alpha_{ji}(a_2)\alpha_i(a_3) + \sigma^t(\alpha_i(a_1))\rho_i(a_2)\alpha_i(a_3)\\
&\qquad + \rho_i(a_1)\alpha_i(a_2)\alpha_i(a_3) - c^t\sigma^t(\rho_i(a_1))\rho_i(a_2)\alpha_i(a_3)\\
&\quad - \sum_{j\in I\setminus\{i\}} c^t\sigma^{t+\omega(j,i)}(\alpha_{ij}(a_1))\sigma^t(\alpha_{ji}(a_2))\rho_i(a_3)\\
&\qquad - c^t\sigma^{2t}(\alpha_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3) - c^t\sigma^t(\rho_i(a_1))\sigma^t(\alpha_i(a_2))\rho_i(a_3)\\
&\qquad + c^{2t}\sigma^{2t}(\rho_i(a_1))\sigma^t(\rho_i(a_2))\rho_i(a_3)
\end{aligned}
$$

and hence by (1), (4) of Lemma 2.3 $\rho_i(a_1(a_2a_3)) = \rho_i((a_1a_2)a_3)$.

(2) Immediate by the construction.

(3) Let $i, j \in I$ and assume there exists an isomorphism $h : e_iA_A \xrightarrow{\sim} e_jA_A$. Let $a \in e_iA$ with $e_j = h(a) = h(e_i)a$. Since $h(ae_j) = h(a)e_j = e_j = h(a)$, $a = ae_j \in e_iAe_j$ and $e_j \in e_jAe_iAe_j$. Suppose to the contrary that $i \neq j$. Then by (2) $e_jAe_iAe_j = e_jAe_ie_iAe_j = e_{ji}Re_{ij}R = e_{ji}e_{ij}R$. If $j \in I \setminus I_0$, then $e_{ji}e_{ij}R = e_jc^{\lambda(j,i,j)}R$. Also, if $j \in I_0$, then $e_{ji}e_{ij}R = (v_j + e_jc^t)R$. In either case, we have $e_j \notin e_{ji}e_{ij}R$, a contradiction.

(4) Consider first the case $i \in I \setminus I_0$. We claim that the homomorphism

$$\phi_i : e_i A_A \to \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A, a \mapsto \alpha_{i,\pi(i)} a$$

is an isomorphism. For any $a, b \in A$ by (L4), (M1) we have

$$\begin{aligned}(\alpha_{i,\pi(i)} a)(b) &= \alpha_{i,\pi(i)}(ab) \\ &= \sum_{j \in I} c^{\lambda(i,j,\pi(i))} \sigma^{\omega(j,\pi(i))}(\alpha_{ij}(a)) \alpha_{j,\pi(i)}(b) \\ &= \sum_{j \in I} \sigma^{\omega(j,\pi(i))}(\alpha_{ij}(a)) \alpha_{j,\pi(i)}(b).\end{aligned}$$

Thus $\alpha_{i,\pi(i)} a = \sum_{j \in I} \sigma^{\omega(j,\pi(i))}(\alpha_{ij}(a)) \alpha_{j,\pi(i)}$ for all $a \in A$. In particular, $\alpha_{i,\pi(i)} e_{ij} = \alpha_{j,\pi(i)}$ for all $j \in I$ and ϕ_i is bijective. Next, let $i \in I_0$. We claim that the homomorphism

$$\phi_i : e_i A_A \to \mathrm{Hom}_R(Ae_i, R_R)_A, a \mapsto \rho_i a$$

is an isomorphism. For any $a, b \in A$ by (M2) we have

$$\begin{aligned}(\rho_i a) b &= \rho_i(ab) \\ &= \sum_{j \in I \setminus \{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a)) \alpha_{ji}(b) + \sigma^t(\alpha_i(a)) \rho_i(b) \\ &\quad + \rho_i(a) \alpha_i(b) - c^t \sigma^t(\rho_i(a)) \rho_i(b).\end{aligned}$$

Thus $\rho_i a = \sum_{j \in I \setminus \{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a)) \alpha_{ji} + \rho_i(a) \alpha_i + \sigma^t(\alpha_i(a) - c^t \rho_i(a)) \rho_i$ for all $a \in A$. For any $a \in e_i A$ with $\rho_i a = 0$, we have $\alpha_{ij}(a) = 0$ for all $j \in I \setminus \{i\}$, $\rho_i(a) = 0$ and $\alpha_i(a) - c^t \rho_i(a) = 0$, so that $a = 0$. Thus ϕ_i is monic. Also, we have $\rho_i e_{ij} = \alpha_{ji}$ for all $j \in I \setminus \{i\}$, $\rho_i e_i = \rho_i$ and $\rho_i(e_i c^t + v_i) = \alpha_i$, so that ϕ_i is epic.

(5) It follows by (1), (4) that $A_A \cong \mathrm{Hom}_R(A, R_R)_A$. Similarly, we have ${}_A A \cong {}_A\mathrm{Hom}_R(A, {}_R R)$. Finally, let $\varphi : R \to A, x \mapsto \sum_{i \in I} e_i x$. For any $i \in I$, α_i is R-R-bilinear and satisfies $\alpha_i \varphi = \mathrm{id}_R$. □

Recall that a ring R is said to be quasi-Frobenius if it is selfinjective and artinian on both sides. It follows by Propositions 1.5, 1.7 that A is quasi-Frobenius if and only if so is R.

Corollary 3.2. *Assume R is local. Then the following hold.*

(1) $e_i A e_i$ is local for all $i \in I$, so that A is semiperfect.

(2) A is connected, i.e., indecomposable as a ring.

(3) A is basic.

(4) If R is quasi-Frobenius, so is A with $\mathrm{soc}(e_i A_A) \cong e_{\pi(i)} A / e_{\pi(i)} \mathfrak{M}$ for all $i \in I$, where $\mathfrak{M}$ is the Jacobson radical of A.

Proof. (1) By Lemma 2.5(3) and Theorem 3.1(2).

(2) By Theorem 3.1(2).

(3) By Theorem 3.1(3).

(4) Let $\mathfrak{m} = R \setminus R^{\times}$. It is not difficult to see that $\mathfrak{M} = \sum_{i\in I} e_i\mathfrak{m} + \sum_{i\in I, j\in I\setminus\{i\}} e_{ij}R + \sum_{i\in I_0} v_iR$. Let $i \in I$. Note that e_iA_A is indecomposable by (1) and is injective by Proposition 1.5(2). Also, by Theorem 3.1(4) $e_iA_A \cong \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A$. Since $Ae_{\pi(i)}/\mathfrak{M}e_{\pi(i)} \cong R/\mathfrak{m}$ as right R-modules, there exists $0 \neq h \in \mathrm{Hom}_R(Ae_{\pi(i)}, R_R)$ with $h(\mathfrak{M}e_{\pi(i)}) = 0$. Then $h\mathfrak{M} = 0$ and $he_{\pi(i)} \neq 0$. Thus $\mathrm{soc}(\mathrm{Hom}_R(Ae_{\pi(i)}, R_R)_A) = hA \cong e_{\pi(i)}A/e_{\pi(i)}\mathfrak{M}$. □

The permutation π of I may be considered as a permutation of $\{e_i\}_{i\in I}$. We claim that this permutation can be extended to a ring automorphism of A. As an additive group, A has an automorphism η such that for any $a \in A$ the following hold:

(H1) $\alpha_{\pi(i),\pi(j)}(\eta(a)) = \sigma^{\chi(j)}(\alpha_{ij}(a))$ for all $i, j \in I$; and

(H2) $\rho_i(\eta(a)) = \sigma^{\chi(i)}(\rho_i(a))$ for all $i \in I_0$.

Proposition 3.3. *The mapping η is a ring automorphism of A satisfying the following conditions:*

(1) $\eta(e_{ij}) = e_{\pi(i),\pi(j)}$ for all $i, j \in I$;

(2) $\eta(v_i) = v_i$ for all $i \in I_0$; and

(3) $\eta(x) = \sum_{i\in I} e_{\pi(i)}\sigma^{\chi(i)}(x)$ for all $x \in R$.

Proof. It is easy to see that the required conditions are satisfied. In particular, we have $\eta(1) = 1$. Let $a, b \in A$. For any $i, k \in I$ by (H1), (M1) we have

$$\begin{aligned}
&\alpha_{\pi(i),\pi(k)}(\eta(ab)) \\
&= \sigma^{\chi(k)}(\alpha_{ik}(ab)) \\
&= \sigma^{\chi(k)}(\{\sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\omega(j,k)}(\alpha_{ij}(a))\alpha_{jk}(b)\}) \\
&= \sum_{j\in I} c^{\lambda(i,j,k)}\sigma^{\chi(k)+\omega(j,k)}(\alpha_{ij}(a))\sigma^{\chi(k)}(\alpha_{jk}(b)),
\end{aligned}$$

$$\begin{aligned}
&\alpha_{\pi(i),\pi(k)}(\eta(a)\eta(b)) \\
&= \sum_{j\in I} c^{\lambda(\pi(i),\pi(j),\pi(k))}\sigma^{\omega(\pi(j),\pi(k))}(\alpha_{\pi(i),\pi(j)}(\eta(a)))\alpha_{\pi(j),\pi(k)}(\eta(b)) \\
&= \sum_{j\in I} c^{\lambda(\pi(i),\pi(j),\pi(k))}\sigma^{\omega(\pi(j),\pi(k))}(\sigma^{\chi(j)}(\alpha_{ij}(a)))\sigma^{\chi(k)}(\alpha_{jk}(b)) \\
&= \sum_{j\in I} c^{\lambda(\pi(i),\pi(j),\pi(k))}\sigma^{\omega(\pi(j),\pi(k))+\chi(j)}(\alpha_{ij}(a)))\sigma^{\chi(k)}(\alpha_{jk}(b))
\end{aligned}$$

and hence by Lemmas 2.2, 2.3(3) $\alpha_{\pi(i),\pi(k)}(\eta(ab)) = \alpha_{\pi(i),\pi(k)}(\eta(a)\eta(b))$. Also, for any $i \in I_0$, since $\chi(i) = t$, by (H2), (M2) we have

$$\begin{aligned}
&\rho_i(\eta(ab)) \\
&= \sigma^t(\rho_i(ab)) \\
&= \sigma^t(\{\sum_{j\in I\setminus\{i\}} \sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}(b) + \sigma^t(\alpha_i(a))\rho_i(b) \\
&\qquad + \rho_i(a)\alpha_i(b) - c^t\sigma^t(\rho_i(a))\rho_i(b)\}) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{t+\omega(j,i)}(\alpha_{ij}(a))\sigma^t(\alpha_{ji}(b)) + \sigma^{2t}(\alpha_i(a))\sigma^t(\rho_i(b)) \\
&\qquad + \sigma^t(\rho_i(a))\sigma^t(\alpha_i(b)) - c^t\sigma^{2t}(\rho_i(a))\sigma^t(\rho_i(b)),
\end{aligned}$$

$$\begin{aligned}
&\rho_i(\eta(a)\eta(b)) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(\pi(j),\pi(i))}(\alpha_{\pi(i),\pi(j)}(\eta(a)))\alpha_{\pi(j),\pi(i)}(\eta(b)) \\
&\qquad + \sigma^t(\alpha_{\pi(i)}(\eta(a)))\rho_i(\eta(b)) + \rho_i(\eta(a))\alpha_{\pi(i)}(\eta(b)) \\
&\qquad - c^t\sigma^t(\rho_i(\eta(b)))\rho_i(\eta(b))) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(\pi(j),\pi(i))}(\sigma^{\chi(j)}(\alpha_{ij}(a)))\sigma^t(\alpha_{ji}(b)) \\
&\qquad + \sigma^t(\sigma^t(\alpha_i(a)))\sigma^t(\rho_i(b)) + \sigma^t(\rho_i(a))\sigma^t(\alpha_i(b)) \\
&\qquad - c^t\sigma^t(\sigma^t(\rho_i(a))\sigma^t(\rho_i(b))) \\
&= \sum_{j\in I\setminus\{i\}} \sigma^{\omega(\pi(j),\pi(i))+\chi(j)}(\alpha_{ij}(a))\sigma^t(\alpha_{ji}(b)) + \sigma^{2t}(\alpha_i(a))\sigma^t(\rho_i(b)) \\
&\qquad + \sigma^t(\rho_i(a))\sigma^t(\alpha_i(b)) - c^t\sigma^{2t}(\rho_i(a))\sigma^t(\rho_i(b))
\end{aligned}$$

and hence by Lemma 2.2 $\rho_i(\eta(ab)) = \rho_i(\eta(a)\eta(b))$. □

Remark 3.4. We have seen in the proof of Theorem 3.1(4) that there exists an isomorphism $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R(A, R_R)_A$ such that $\phi(1)(a) = \sum_{i\in I\setminus I_0} \alpha_{i,\pi(i)}(a) + \sum_{i\in I_0} \rho_i(a)$ for all $a \in A$. Set $\theta = \eta^{-1} \in \mathrm{Aut}(A)$. Then $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$ (cf. Remark 1.2(1)).

Remark 3.5. Set $w_i = v_i + e_i c^{\chi(i)}$ for $i \in I_0$ and $w_i = e_{i,\pi(i)}$ for $i \in I \setminus I_0$. Then the following hold.

(1) $\{e_{ij}\}_{i,j\in I} \cup \{w_i\}_{i\in I_0}$ is a basis for A_R and gives rise to another description of the multiplication of A.

(2) $\phi(w_i) = \alpha_{\pi(i)}$ for all $i \in I$, where ϕ is the same as in Remark 3.4.

(3) Set $w = \sum_{i\in I} w_i$. Then $\eta(w) = w$ and $aw = w\eta(a)$ for all $a \in A$, so that (R, c, σ) can be replaced by (A, w, η) in our construction.

In the following, we denote by $\mathrm{M}_n(R)$ the $n \times n$ full matrix ring over R. Recall that c is said to be regular if $cx \neq 0$ and $xc \neq 0$ for any $0 \neq x \in R$.

Proposition 3.6. *For any $i \in I$ there exists a ring homomorphism*

$$\xi_i : A \to \mathrm{M}_n(R), a \mapsto (c^{\lambda(i,j,k)}\sigma^{-\omega(i,k)}(\alpha_{jk}(a)))_{j,k\in I}.$$

Assume c is regular. Then $\mathrm{Ker}\ \xi_i = \sum_{j\in I_0} v_j R$. *In particular, if I_0 is empty, then ξ_i is injective.*

Proof. It is easy to see that $\xi_i(1)$ is the unit matrix. Let $a, b \in A$. Obviously, $\xi_i(a+b) = \xi_i(a) + \xi_i(b)$. Also, for any $j, l \in I$ by (M1) and (1), (2) of Lemma 2.3 we have

$$\begin{aligned}
&c^{\lambda(i,j,l)}\sigma^{-\omega(i,l)}(\alpha_{jl}(ab)) \\
&= c^{\lambda(i,j,l)}\sigma^{-\omega(i,l)}(\{\sum_{k\in I} c^{\lambda(j,k,l)}\sigma^{\omega(k,l)}(\alpha_{jk}(a))\alpha_{kl}(b)\}) \\
&= \sum_{k\in I} c^{\lambda(i,j,l)+\lambda(j,k,l)}\sigma^{\omega(k,l)-\omega(i,l)}(\alpha_{jk}(a))\sigma^{-\omega(i,l)}(\alpha_{kl}(b)) \\
&= \sum_{k\in I} c^{\lambda(i,j,k)+\lambda(i,k,l)}\sigma^{\lambda(i,k,l)-\omega(i,k)}(\alpha_{jk}(a))\sigma^{-\omega(i,l)}(\alpha_{kl}(b)) \\
&= \sum_{k\in I} \{c^{\lambda(i,j,k)}\sigma^{-\omega(i,k)}(\alpha_{jk}(a))\}\{c^{\lambda(i,k,l)}\sigma^{-\omega(i,l)}(\alpha_{kl}(b))\}
\end{aligned}$$

and hence $\xi_i(ab) = \xi_i(a)\xi_i(b)$. The last assertion is obvious. □

4 Tilting complexes

In this section, we provide a construction of two-term tilting complexes associated with a certain type of idempotent (cf. [5]).

For a ring A we denote by $\mathsf{K}(\mathrm{Mod}\text{-}A)$ (resp., $\mathsf{D}(\mathrm{Mod}\text{-}A)$) the homotopy (resp., derived) category of cochain complexes over Mod-A and consider modules as complexes concentrated in degree zero. We use the notation $(-)[m]$ to denote the m-shift of complexes. Also, we denote by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules and by $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathsf{K}(\mathrm{Mod}\text{-}A)$ consisting of bounded complexes over $\mathcal{P}_A$. Finally, for an object X in an additive category $\mathfrak{A}$ we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathfrak{A}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(m)}$ the direct sum of m copies of X. We refer to [13] for tilting complexes and derived equivalences and to [3], [16] for derived categories.

Let A be an extension of a ring R and $e \in A$ an idempotent. Assume $xe = ex$ for all $x \in R$, Ae_R is finitely generated projective and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule. Note first that we have a ring homomorphism $\varphi : R \to eAe, x \mapsto ex$. Let

$$\mu : Ae \otimes_R eA_A \to A_A, a \otimes b \mapsto ab$$

be the multiplication map and $S^\bullet$ its mapping cone. Set $T_1^\bullet = eA[1]$, $T_2^\bullet = (1-e)A \otimes_A S^\bullet$ and $T^\bullet = T_1^\bullet \oplus T_2^\bullet$. Note that $T_2^\bullet$ is the mapping cone of the multiplication map

$$(1-e)A \otimes_A \mu : (1-e)Ae \otimes_R eA_A \to (1-e)A_A.$$

Note also that $Ae \otimes_R eA_A \in \operatorname{add}(eA_A)$. Since the multiplication map

$$eA \otimes_A \mu : eAe \otimes_R eA_A \to eA_A$$

is a split epimorphism and its kernel belongs to $\operatorname{add}(eA_A)$, we have $eA \otimes_A S^\bullet \in \operatorname{add}(T_1^\bullet)$ and hence $S^\bullet \in \operatorname{add}(T^\bullet)$.

Proposition 4.1. *The following hold.*

(1) $T^\bullet$ is a tilting complex.

(2) Assume μ is monic. Then $\operatorname{End}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet)$ is isomorphic to the following generalized triangular matrix ring

$$\begin{pmatrix} eAe & \operatorname{Ext}^1_A(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Assume further that φ is an isomorphism and there exists an idempotent $f \in A$ such that $\operatorname{Hom}_R(Ae, R_R)_A \cong fA_A$. Then $\operatorname{Ext}^1_A(A/AeA, eA) \cong f(A/AeA)$ as right (A/AeA)-modules.

Proof. (1) Obviously, $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $\operatorname{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[m]) = 0$ unless $-1 \leq m \leq 1$. Since $e(A/AeA) = 0$, $A/AeA = (1-e)(A/AeA)$ and $\mathrm{H}^0(T^\bullet) \cong A/AeA$. Thus, since $\operatorname{Hom}_A(eA, A/AeA) \cong (A/AeA)e = 0$, it follows that $\operatorname{Hom}_A(T^{-1}, (1-e)A \otimes_A \mu)$ is epic and $\operatorname{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[1]) = 0$. Also,

$$\operatorname{Hom}_A(A/AeA, \operatorname{Hom}_R(Ae, R_R)) \cong \operatorname{Hom}_R((A/AeA)e, R_R) = 0$$

and hence $\operatorname{Hom}_A(A/AeA, eA) = 0$. Thus $\operatorname{Hom}_A(\mathrm{H}^0(T^\bullet), T^{-1}) = 0$ and hence $\operatorname{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, T^\bullet[-1]) = 0$. Next, we have a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of the form

$$A \to S^\bullet \to (Ae \otimes_R eA)[1] \to .$$

Since $S^\bullet \in \operatorname{add}(T^\bullet)$, and since $(Ae \otimes_R eA)[1] \in \operatorname{add}(T_1^\bullet)$, it follows that $\operatorname{add}(T^\bullet)$ generates $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category.

(2) We have $\operatorname{End}_{\mathsf{K}(\text{Mod-}A)}(T_1^\bullet) \cong \operatorname{End}_A(eA_A) \cong eAe$. Also, since $(1-e)A \otimes_A \mu$ is monic, we have $\operatorname{Hom}_{\mathsf{K}(\text{Mod-}A)}(T_1^\bullet, T_2^\bullet) = 0$. Furthermore,

$$\begin{aligned} \operatorname{End}_{\mathsf{K}(\text{Mod-}A)}(T_2^\bullet) &\cong \operatorname{End}_{\mathsf{D}(\text{Mod-}A)}(T_2^\bullet) \\ &\cong \operatorname{End}_{\mathsf{D}(\text{Mod-}A)}(A/AeA) \\ &\cong \operatorname{End}_A(A/AeA) \\ &\cong A/AeA, \end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet) &\cong \mathrm{Hom}_{\mathsf{D}(\text{Mod-}A)}(T_2^\bullet, T_1^\bullet)\\ &\cong \mathrm{Hom}_{\mathsf{D}(\text{Mod-}A)}(A/AeA, eA[1])\\ &\cong \mathrm{Ext}_A^1(A/AeA, eA).\end{aligned}$$

Consequently, $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet)$ is isomorphic to the desired generalized triangular matrix ring.

Next, assume φ is an isomorphism and there exists an idempotent $f \in A$ such that $\mathrm{Hom}_R(Ae, R_R)_A \cong fA_A$. For any $M \in \text{Mod-}A$ we have functorial isomorphisms

$$\mathrm{Hom}_A(M, fA) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(Ae, R_R)) \cong \mathrm{Hom}_R(M \otimes_A Ae, R_R).$$

Thus, since $\mu \otimes_A Ae$ is an isomorphism, so is $\mathrm{Hom}_A(\mu, fA)$. Then by applying $\mathrm{Hom}_A(-, fA)$ to the exact sequence

$$0 \to Ae \otimes_R eA \xrightarrow{\mu} A \to A/AeA \to 0,$$

we have $\mathrm{Hom}_A(A/AeA, fA) = 0$ and $\mathrm{Ext}_A^1(A/AeA, fA) = 0$. Note that $fAe \cong R$ as right R-modules. Thus, by applying $fA \otimes_A -$ to the above exact sequence, we get an exact sequence of the form

$$0 \to eA_A \to fA_A \to f(A/AeA)_A \to 0$$

to which we apply $\mathrm{Hom}_A(A/AeA, -)$ to conclude that

$$\begin{aligned}f(A/AeA) &\cong \mathrm{Hom}_A(A/AeA, f(A/AeA))\\ &\cong \mathrm{Ext}_A^1(A/AeA, eA)\end{aligned}$$

as right (A/AeA)-modules. $\square$

Remark 4.2. Let $K = \mathrm{Ker}(eA \otimes_A \mu)$ and assume $\mathrm{add}(K_A) = \mathrm{add}(eA_A)$. Then $\mathrm{add}(S^\bullet) = \mathrm{add}(T^\bullet)$ and $S^\bullet$ is a tilting complex.

Proof. Note that $eA \otimes_A S^\bullet \cong K[1]$ in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$. Since $eA_A \in \mathrm{add}(K_A)$, we have $T_1^\bullet \in \mathrm{add}(eA \otimes_A S^\bullet)$. Thus $T^\bullet \in \mathrm{add}(S^\bullet)$ and hence $\mathrm{add}(S^\bullet) = \mathrm{add}(T^\bullet)$. Then, since $T^\bullet$ is a tilting complex, so is $S^\bullet$. $\square$

In the following examples, A is the Frobenius extension of R constructed in the preceding section. We use the same notation as in the preceding section.

Example 4.3. Let J be a nonempty π-stable subset of I and set $e = \sum_{j \in J} e_j$. Then $xe = ex$ for all $x \in R$, Ae_R is finitely generated projective and $eA_A \cong \mathrm{Hom}_R(Ae, R_R)_A$. In this case, the mapping cone of the multiplication map

$$\bigoplus_{j \in J} Ae_j \otimes_R e_j A_A \to A_A$$

is a tilting complex.

Proof. We have seen in the preceding section that all the conditions are satisfied. Let $J_0 = J \cap I_0$ and d the number of elements of J. Set $d_j = d$ for $j \in J_0$ and $d_j = d - 1$ for $j \in J \setminus J_0$. Note that $d_j \geq 1$ for all $j \in J$. Since we have a split exact sequence in Mod-A of the form

$$0 \to \bigoplus_{j \in J} e_j A^{(d_j)} \to \bigoplus_{j \in J} eAe_j \otimes_R e_j A \to eA \to 0,$$

the last assertion follows by the same argument as in Remark 4.2. □

Example 4.4. Assume c is regular and $I \setminus I_0$ is not empty. Let $i \in I \setminus I_0$ and set $e = e_i$ and $f = e_{\pi^{-1}(i)}$. Then the following conditions are satisfied:

(1) $xe = ex$ for all $x \in R$, Ae_R is finitely generated projective and eA_A is embedded in $\mathrm{Hom}_R(Ae, R_R)_A$ as a submodule;

(2) the multiplication map $Ae \otimes_R eA \to A, a \otimes b \mapsto ab$ is monic;

(3) the ring homomorphism $R \to eAe, x \mapsto ex$ is an isomorphism; and

(4) $\mathrm{Hom}_R(Ae, R_R)_A \cong fA_A$.

Proof. We denote by $\mu : Ae_i \otimes_R e_i A \to A, a \otimes b \mapsto ab$ the multiplication map. Note that $Ae_i \otimes_R e_i A$ is a free right R-module with a basis $\{e_{ji} \otimes e_{il}\}_{j,l \in I}$, and that $e_{ji}e_{il} = e_{jl}c^{\lambda(j,i,l)}$ unless $j = l \in I_0$ and $e_{ji}e_{ij} = v_j + e_j c^{\chi(j)}$ for all $j \in I_0$. Thus, since c is regular, it is easy to see that μ is monic. Also, for any $a \in e_i A$ we have $(\alpha_i a)(e_{ji}) = \alpha_i(ae_{ji}) = c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a))$ for all $j \in I$ and hence $\alpha_i a = \sum_{j \in I} c^{\lambda(i,j,i)}\sigma^{\omega(j,i)}(\alpha_{ij}(a))\alpha_{ji}$, so that by the regularity of c the homomorphism

$$e_i A_A \to \mathrm{Hom}_R(Ae_i, R_R)_A, a \mapsto \alpha_i a$$

is monic. We have seen in the preceding section that the remaining conditions are satisfied. □

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