The denominators of Lagrangian surfaces in complex Euclidean plane

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Abstract. A quotient of two linearly independent quaternionic holomorphic sections of a quaternionic holomorphic line bundle over a Riemann surface is a conformal branched immersion from a Riemann surface to four-dimensional Euclidean space. On the assumption that a quaternionic holomorphic line bundle is associated with a Lagrangian branched immersion from a Riemann surface to complex Euclidean plane, we shall classify the denominators of Lagrangian branched immersion from a Riemann surface to complex Euclidean plane.

Keywords: Lagrangian surface, quaternionic holomorphic vector bundle, the Carleman-Bers-Vekua system.

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1. Introduction

In this paper, we shall discuss a similarity between a Lagrangian branched immersion from a Riemann surface to complex Euclidean plane and a complex holomorphic function on a Riemann surface by the quaternionic theory of surfaces.

The quaternionic theory of surfaces in four-dimensional Euclidean space $\mathbb{R}^4$ is developed by Pedit and Pinkall [8], Burstall, Ferus, Leschke, Pedit, and Pinkall [1], and Ferus, Leschke, Pedit, and Pinkall [2]. This theory presents many new points of view on conformal geometry of surfaces in $\mathbb{R}^4$, where $\mathbb{R}^4$ is identified with the set $\mathbb{H}$ of quaternions.

In this theory, a *right normal vector* is defined for a conformal immersion from a Riemann surface M to $\mathbb{H}$. A right normal vector is a quaternionic-valued function on M whose square is -1. It coincides with a part of the generalized Gauss map of the conformal immersion by taking a suitable decomposition of the Grassmanian manifold of two-planes in $\mathbb{H}$ into a direct product of two spheres of dimension two. The tangent space of the immersion is preserved by the right multiplication of the right normal vector. Then a vector bundle endomorphism of the trivial (right) quaternionic line bundle $\underline{\mathbb{H}}$ over M is defined by the right normal vector. This endomorphism is called a *complex structure* of $\underline{\mathbb{H}}$.

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For a smooth mapping from M to $\mathbb{H}$, a right normal vector is defined uniquely on the set where the mapping is a conformal immersion. If the domain of a right normal vector can be extended to every point where the differential of the smooth mapping is not injective, then the smooth mapping is a conformal branched immersion by Lemma 1 in Section 2.

A complex structure of $\underline{\mathbb{H}}$ plays a similar role to the complex structure of the space $\mathbb{C}$ of complex numbers. By a complex structure of $\underline{\mathbb{H}}$, a *quaternionic holomorphic structure* of $\underline{\mathbb{H}}$ is defined in a similar way to define the complex holomorphic structure of $\mathbb{C}$. A quaternionic holomorphic structure is a zero-order perturbation of a complex holomorphic structure of a complex Euclidean plane $\mathbb{C}^2$ identified with $\mathbb{H}$ (see p. 520 in [2]). This quaternionic holomorphic structure is called a *Euclidean holomorphic structure* in Peters [9]. A smooth section of $\underline{\mathbb{H}}$ in the kernel of a quaternionic holomorphic structure is called a *quaternionic holomorphic section*. When we consider a smooth section of $\underline{\mathbb{H}}$ as a smooth mapping from M to $\mathbb{H}$, a non-constant quaternionic holomorphic section is a conformal branched immersion with a right normal vector. Hence a conformal branched immersion is a natural generalization of a complex holomorphic function on M. In the same way as a quotient of two complex holomorphic sections of a complex line trivial bundle is a complex holomorphic function except at the zeros of its denominator, a quotient of two linearly independent quaternionic holomorphic sections of $\underline{\mathbb{H}}$ with a complex structure is a conformal branched immersion with a right normal vector except at the zeros of its denominator by Example on p. 395 in [8].

We will look for a set of conformal branched immersions with a right normal vector satisfying a geometric property such that it is similar to a set of complex holomorphic function. Then it is expected that a set of *Lagrangian branched immersion* from M to complex Euclidean plane $\mathbb{C}^2$ with a right normal vector is a candidate, where $\mathbb{C}^2$ is identified with $\mathbb{H}$. Indeed, we shall characterize a Lagrangian immersion by its right normal vector in Section 3. We define a complex structure by a right normal vector of a Lagrangian branched immersion. Then every quaternionic conjugate of non-constant quaternionic holomorphic section of $\underline{\mathbb{H}}$ is a Lagrangian branched immersion with the same right normal vector by the discussion in Section 2.

We will consider the problem that whether the quotient of two Lagrangian branched immersions is a Lagrangian branched immersion. We should take a quotient of two linearly independent quaternionic holomorphic sections of $\underline{\mathbb{H}}$ with a complex structure defined by a right normal vector of a Lagrangian branched immersion. Then their quotient is not necessarily a Lagrangian branched immersion. Hence it is an interesting problem to classify the pairs of two quaternionic holomorphic sections of $\underline{\mathbb{H}}$ such that their quotient is a Lagrangian branched immersion.

We shall devote this paper to classify quaternionic holomorphic sections of $\underline{\mathbb{H}}$ vanishing nowhere which are the denominators of Lagrangian branched immersions from M to $\mathbb{C}^2$ with their right normal vector. This paper is organized as follows.

In Section 2, we shall review the quaternionic theory of conformal branched immersions from M to $\mathbb{H}$ and rewrite Example on p. 395 in [8] to make it convenient for our use.

In Section 3, we shall characterize a Lagrangian immersion and a *Hamiltonian-minimal Lagrangian immersion* in terms of the quaternionic formulation. The notions of Hamiltonian-minimality is introduced by Oh [7].

In Section 4, we shall assume that a quaternionic holomorphic line bundle is associated with a Lagrangian branched immersion with a right normal vector. We shall classify the quaternionic holomorphic sections vanishing nowhere which are the denominators of Lagrangian branched immersions. In the case where M is closed, the image of M by a denominator is a torus (Theorem 1). In the case where M is open, a complex-valued function is defined locally as a function of a complex holomorphic function on M and *Lagrangian angle mappings* of a Lagrangian branched immersion and its denominator so that it is a solution to a differential equation called the Carleman-Bers-Vekua system in Rodin [10] (cf. Vekua [12]). A denominator is a mapping of this complex holomorphic function on M and these Lagrangian angle mappings (Theorem 2).

In Section 5, we discuss the case where a Lagrangian branched immersion or its denominator is a Hamiltonian-minimal Lagrangian branched immersion. If both of them are Hamiltonian-minimal Lagrangian branched immersions, then the image of M by a denominator is a plane or a torus (Theorem 3). If one is a Hamiltonian-minimal Lagrangian branched immersion and another is not a Hamiltonian-minimal Lagrangian branched immersion, then we have a formula for the denominator as a mapping of a holomorphic function (Theorem 4 and Theorem 5).

In Section 6, we construct a numerator and obtain a Lagrangian branched immersion by Theorem 4 and Theorem 5.

2. Quaternionic holomorphic line bundles

We shall recall the quaternionic theory of surfaces by Pedit and Pinkall [8], Burstall, Ferus, Leschke, Pedit, and Pinkall [1], and Ferus, Leschke, Pedit, and Pinkall [2].

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{H}$ the set of quaternions $\{a_0 + a_1 i + a_2 j + a_3 k \,|\, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, where i, j, and k are elements of $\mathbb{H}$ such that

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = -ji = k, \;\; jk = -kj = i, \;\; ki = -ik = j. \end{aligned}$$

For a quaternion $a_0 + a_1 i + a_2 j + a_3 k$ such that a_0, a_1, a_2, and $a_3 \in \mathbb{R}$, the quaternionic conjugate $\hat{a}$ of a, the real part $\operatorname{Re} a$ of a, and the imaginary part $\operatorname{Im} a$ of a are defined by $\hat{a} = a_0 - a_1 i - a_2 j - a_3 k$, $\operatorname{Re} a = a_0$, and $\operatorname{Im} a = a_1 i + a_2 j + a_3 k$ respectively. We denote by $\operatorname{Im} \mathbb{H}$ the set of imaginary parts of quaternions. The set of quaternions $\mathbb{H}$ is considered as the set of quadruplets of real numbers $\mathbb{R}^4$ by the identification of a quaternion $a_0 + a_1 i + a_2 j + a_3 k$ such that a_0, a_1, a_2, and $a_3 \in \mathbb{R}$ with a quadruplet (a_0, a_1, a_2, a_3) of real numbers. Let q be a quaternionic sesquilinear product on $\mathbb{H}$ by $q(x, y) = \hat{x}y$ for every pair (x, y) of quaternions. We define real-valued quadratic forms ω_0, ω_1, ω_2, and ω_3 by $q(x, y) = \omega_0(x, y) + \omega_1(x, y)i + \omega_2(x, y)j + \omega_3(x, y)k$. Then the quadratic form ω_0 is the standard Euclidean inner product of $\mathbb{R}^4$. Let $(\mathbb{H}, \omega_0)$ be four-dimensional Euclidean space and $|a| = (\omega_0(a, a))^{1/2}$ Euclidean norm of $a \in \mathbb{H}$.

The set $\{a_0 + a_1 i \,|\, a_0, a_1 \in \mathbb{R}\}$ is considered as the set $\mathbb{C}$ of complex numbers. Then the set of quaternions $\mathbb{H}$ is considered as the set of pairs of complex numbers $\mathbb{C}^2$ by the identification of a quaternion $a_0 + a_1 i + a_2 j + a_3 k$ such that a_0, a_1, a_2, and $a_3 \in \mathbb{R}$ with a pair of complex numbers $(a_0 + a_1 i, a_2 - a_3 i)$. Then the quadratic ω_1 is the standard symplectic form of $\mathbb{C}^2$ and $\omega_0 + \omega_1 i$ is the standard Hermitian inner product on $\mathbb{C}^2$.

Euclidean inner product ω_0 induces the standard Riemannian metric of $\mathbb{R}^4$. We use the same notation ω_0 for this Riemannian metric. Similarly, we use the same notation ω_1 for the standard symplectic structure of $\mathbb{C}^2$ induced by the symplectic form ω_1 on $\mathbb{C}^2$. Then $\omega_0 + \omega_1 i$ is the standard Hermitian metric of $\mathbb{C}^2$.

Let (M, g) be a two-dimensional oriented connected Riemannian manifold M with a Riemannian metric g, TM its tangent bundle, and T^*M its cotangent bundle. Then there exists a complex structure J^{TM} of (M, g) such that the ordered pair $(\mathbf{e}, J^{TM}\mathbf{e})$ is a positive orthonormal basis of T_pM for every point p in M and every unit vector $\mathbf{e}$ in the fiber T_pM of TM at p.

For a smooth vector bundle V over M, we denote by $\Gamma(V)$ the set of smooth sections of V and $\Omega^n(V)$ the set of smooth differential n-forms on M with coefficients in V $(n = 0, 1, 2)$. We define a mapping $*\colon \Omega^1(V) \to \Omega^1(V)$ by $*\,\omega = \omega \circ J^{TM}$ for every $\omega \in \Omega^1(V)$.

Let $\underline{\mathbb{H}}$ be the trivial (right) quaternionic line bundle $\underline{\mathbb{H}}$ over M. A smooth mapping $\phi\colon M \to \mathbb{H}$ is considered as a smooth section $\underline{\phi}$ of $\underline{\mathbb{H}}$. Let L be a pair $(\underline{\mathbb{H}}, J^L)$ with a quaternionic vector bundle endomorphism J^L of $\underline{\mathbb{H}}$. The endomorphism J^L is called a *complex structure* of L in [1].

Let $T^*M \otimes_{\mathbb{R}} \underline{\mathbb{H}}$ be the tensor bundle of T^*M and $\underline{\mathbb{H}}$ over $\mathbb{R}$ and $\zeta\underline{\phi}$ an element of $T^*M \otimes_{\mathbb{R}} \underline{\mathbb{H}}$ such that $\zeta \in T^*M$ and $\underline{\phi} \in \underline{\mathbb{H}}$. A quaternionic-valued one-form on M is a section of $T^*M \otimes_{\mathbb{R}} \underline{\mathbb{H}}$. We define a vector bundle endomorphism J of $T^*M \otimes_{\mathbb{R}} \underline{\mathbb{H}}$ by the equation $J\zeta\underline{\phi} = \zeta J^L \underline{\phi}$. A quaternionic vector bundle $\bar{K}\underline{\mathbb{H}}$ is defined by

$$\bar{K}\underline{\mathbb{H}} = \{\omega \in T^*M \otimes_{\mathbb{R}} \underline{\mathbb{H}} \mid *\,\omega = -J\omega\}.$$

We define a quaternionic homomorphism $D\colon \Gamma(\underline{\mathbb{H}}) \to \Gamma(\bar{K}\underline{\mathbb{H}})$ by

$$D(\underline{\phi}) = \frac{1}{2}\{(\mathrm{d}\phi) + J * (\mathrm{d}\phi)\}.$$

for every smooth mapping ϕ from M to $\mathbb{H}$. Following Peters [9], we call the quaternionic homomorphism D the *Euclidean quaternionic holomorphic structure* of L and the pair $L = (\underline{\mathbb{H}}, J^L)$ with its Euclidean quaternionic holomorphic structure D a *Euclidean quaternionic holomorphic line bundle*. A smooth section $\underline{\phi}$ of L is called a *quaternionic holomorphic section* of L if $D(\underline{\phi}) = 0$. We see that a constant section is a quaternionic holomorphic section.

A smooth mapping $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ is called a *conformal immersion* on M if f is an immersion and there exists a pair (N^f, R^f) of smooth mappings from M to $S^2(1) \subset \operatorname{Im}\mathbb{H}$ such that

$$\begin{aligned} &(N^f)^2 = (R^f)^2 = -1, \\ &*(\mathrm{d}f) = N^f(\mathrm{d}f) = (\mathrm{d}f)(-R^f). \end{aligned} \tag{2.1}$$

The smooth mappings N^f and R^f defined by the equation (2.1) are called the *left normal vector* of f and the *right normal vector* of f respectively (Definition 2 in [1]).

A point $p \in M$ is called a branch point of a smooth mapping $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ if the differential mapping $(\mathrm{d}f)_p$ of f at p is the zero mapping. A non-constant smooth mapping $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ is called a *conformal branched immersion* if every point $p \in M$ such that $(\mathrm{d}f)_p$ is not injective is a branch point and f is a conformal immersion on M except branch points.

A right normal vector is not defined by the equation (2.1) at a point $p \in M$ such that $(\mathrm{d}f)_p$ is not injective.

LEMMA 1. *Let* $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ *be a non-constant smooth mapping. If there exists a mapping* $R^f\colon M \to S^2(1) \subset \operatorname{Im}\mathbb{H}$ *such that*

$$(R^f)^2 = -1, \quad *(\mathrm{d}f) = (\mathrm{d}f)(-R^f),$$

then f is a conformal branched immersion.

Proof. It is indicated on p. 8 in [1] that if $*(\mathrm{d}f) = (\mathrm{d}f)(-R^f)$, then f is conformal at every point $p \in M$ such that $(\mathrm{d}f)_p$ is injective. Let $p \in M$ be a point such that $(\mathrm{d}f)_p$ is not injective and (u_1, u_2) is an isothermal coordinate around p such that $J^{TM}(\partial/\partial u_1) = \partial/\partial u_2$. If $*(\mathrm{d}f) = (\mathrm{d}f)(-R)$, then

$$\frac{\partial f}{\partial u_2}(p) = \frac{\partial f}{\partial u_1}(p)(-R^f(p)).$$

Since $(\mathrm{d}f)_p$ is not injective and R^f is a mapping from M to $S^2(1) \subset \operatorname{Im}\mathbb{H}$, we have

$$\frac{\partial f}{\partial u_2}(p) = \frac{\partial f}{\partial u_1}(p) = 0.$$

Hence $(\mathrm{d}f)_p = 0$. $\square$

We call the mapping f with a smooth mapping $R^f\colon M \to S^2(1) \subset \operatorname{Im}\mathbb{H}$ such that $*(\mathrm{d}f) = (\mathrm{d}f)(-R^f)$ on M a *conformal branched immersion with a right normal vector* R^f.

Let $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ be a conformal branched immersion with its right normal vector R^f. We define a complex structure J^f of $\underline{\mathbb{H}}$ by $J^f \underline{1} = \underline{R^f}$. Let D^f be the Euclidean quaternionic holomorphic structure of $L^f = (\underline{\mathbb{H}}, J^f)$ and $\underline{\hat{\phi}}$ a smooth section of L^f. Since $D^f(\underline{\hat{\phi}}) = \{(\mathrm{d}\hat{\phi}) + R^f * (\mathrm{d}\hat{\phi})\}/2$, a section $\underline{\hat{\phi}}$ of L^f is a non-constant quaternionic holomorphic section if and only if ϕ is a conformal branched immersion with its right normal vector R^f. Hence the section $\underline{\hat{f}}$ is a non-constant quaternionic holomorphic section of L^f.

Let L be a Euclidean quaternionic holomorphic line bundle over M with its complex structure J^L defined by $J^L \underline{1} = \underline{R}$ for a smooth mapping $R\colon M \to S^2(1) \subset \operatorname{Im}\mathbb{H}$. The following Lemma 2 is a variant of Example on p. 395 in [8].

LEMMA 2. *We assume that $\underline{\hat{\nu}}$ is a non-zero quaternionic holomorphic section of L and $\underline{\hat{\mu}}$ is a smooth section vanishing nowhere of L. A smooth mapping* $\lambda\colon (M, g) \to (\mathbb{H}, \omega_0)$ *defined by the equation* $\underline{\hat{\nu}} = \underline{\hat{\mu}}\hat{\lambda}$ *is* *a conformal branched immersion with its right normal vector* $\mu R \mu^{-1}$ *if and only if* $\underline{\hat{\mu}}$ *is linearly independent of* $\underline{\hat{\nu}}$ *and it is a quaternionic holomorphic section of* L.

Proof. Evaluating the both side of the equation $\underline{\hat{\nu}} = \underline{\hat{\mu}}\hat{\lambda}$ by the Euclidean quaternionic holomorphic structure D of L, we have

$$\underline{0} = D(\underline{\hat{\mu}})\hat{\lambda} + \frac{1}{2}\hat{\mu}\{(\mathrm{d}\hat{\lambda}) + \hat{\mu}^{-1}R\hat{\mu} * (\mathrm{d}\hat{\lambda})\}.$$

Hence Lemma 2 holds. □

3. Lagrangian surfaces

We shall describe a conformal Lagrangian immersion from (M, g) to $(\mathbb{C}^2, \omega_0)$ in terms of quaternions.

We identify $\mathbb{H}$ with $\mathbb{C}^2$ by the identification of a quaternion $a_0 + a_1 i + a_2 j + a_3 k$ such that a_0, a_1, a_2, and $a_3 \in \mathbb{R}$ with a pair of complex numbers $(a_0 + a_1 i, a_2 - a_3 i)$. A conformal immersion $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ is called a *Lagrangian immersion* if

$$\omega_0 \left(\{(\mathrm{d}f)_p(X)\}i, (\mathrm{d}f)_p(Y)\right) = 0, \tag{3.1}$$

for every point $p \in M$ and every pair (X, Y) of vectors X and $Y \in T_pM$. A conformal branched immersion $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ is called a *Lagrangian branched immersion* if f is a Lagrangian immersion on M except at branch points.

We shall rephrase this definition in terms of quaternions. Let $\mathbb{Z}$ be the set of integers and $\mathbb{R}/2\pi\mathbb{Z}$ the quotient space of $\mathbb{R}$ by $2\pi\mathbb{Z} = \{2\pi n \,|\, n \in \mathbb{Z}\}$. Let $f\colon (M, g) \to (\mathbb{H}, \omega_0)$ be a conformal immersion. We make another identification of $\mathbb{C}^2$ with $\mathbb{H}$ by the identification of $(z_0, z_1) \in \mathbb{C}^2$ with $\tau(z_0 + jz_1)\tau^{-1}$, where $\tau = i + j$. Under this identification, Hélein and Romon [5] showed that a conformal immersion $\tilde{f} = \tau f \tau^{-1}$ is a Lagrangian immersion if and only if $(\mathrm{d}\tilde{f}) = r(\mathrm{d}z)e^{\theta j/2}$ for a local complex holomorphic coordinate z of M, a quaternionic-valued function r, and a smooth mapping $\theta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$. The mapping θ is called the *Lagrangian angle mapping* of f. If the Lagrangian angle mapping is constant, then $f(M)$ is a Lagrangian plane. Let h be the Riemannian metric of $\mathbb{R}/2\pi\mathbb{Z}$ induced by the standard Riemannian metric of $\mathbb{R}$. If the map $\theta\colon (M, g) \to (\mathbb{R}/2\pi\mathbb{Z}, h)$ is harmonic, then f is called *Hamiltonian-minimal Lagrangian immersion* (see Hélein and Romon [6]). We see that

$$*(\mathrm{d}\tilde{f}) = r(\mathrm{d}z)ie^{\theta j/2} = (\mathrm{d}\tilde{f})e^{-\theta j/2}ie^{\theta j/2} = (\mathrm{d}\tilde{f})ie^{\theta j}.$$

Coming back to the identification of $\mathbb{C}^2$ with $\mathbb{H}$ by the identification $(z_0, z_1) \in \mathbb{C}^2$ with $z_0 + jz_1 \in \mathbb{H}$, we have

$$*(\mathrm{d}f) = (\mathrm{d}f)\tau^{-1} i e^{\theta j} \tau = (\mathrm{d}f) j e^{\theta i}.$$

Hence the right normal vector of f is $-je^{\theta i}$. We define a mapping $\beta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$ by $\beta = \theta + \pi$. Then the right normal vector of f is $je^{\beta i}$ and f is Hamiltonian-minimal if and only if β is harmonic.

4. Lagrangian line bundles

We shall classify the denominators of Lagrangian branched immersions from (M, g) to $(\mathbb{H}, \omega_0)$.

Let L be a Euclidean quaternionic holomorphic line bundle L over a Riemann surface M with complex structure J^L. We call L a *Lagrangian line bundle* if J^L is defined by $J^L \underline{1} = \underline{je^{\beta i}}$ with a smooth mapping $\beta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$. A non-constant quaternionic holomorphic section of a Lagrangian line bundle with its complex structure defined by $J^L \underline{1} = \underline{je^{\beta i}}$ is identified with a Lagrangian branched immersion with a right normal vector $je^{\beta i}$.

LEMMA 3. *We assume that $\underline{\hat{\nu}}$ is a non-zero quaternionic holomorphic section of a Lagrangian line bundle L with its complex structure J^L defined by $J^L \underline{1} = \underline{je^{\beta i}}$ and that $\underline{\hat{\mu}}$ is a nowhere-vanishing smooth section of L. A mapping $\lambda\colon (M, g) \to (\mathbb{H}, \omega_0)$ defined by the equation $\hat{\nu} = \hat{\mu}\hat{\lambda}$ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and*

$$\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}, \tag{4.1}$$

$$\mu_0(- * (\mathrm{d}\beta) + *(\mathrm{d}\gamma)) = \mu_1((\mathrm{d}\beta) + (\mathrm{d}\gamma)), \tag{4.2}$$

where μ_0 and μ_1 are real-valued functions on M such that $\mu_0 - \mu_1 i$ is a complex holomorphic function vanishing nowhere on M.

Proof. It is an immediate consequence of Lemma 2 that a mapping λ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$, $\underline{\hat{\mu}}$ is a nowhere-vanishing, quaternionic holomorphic section of L satisfying $\mu j e^{\beta i}\mu^{-1} = je^{\gamma i}$. We rewrite the last equation.

Let F_0 and F_1 be complex-valued functions on M such that $\mu = F_0 + jF_1$. Then the equation $\mu j e^{\beta i}\mu^{-1} = je^{\gamma i}$ is equivalent to the equation

$$-\bar{F}_1 e^{\beta i} + j\bar{F}_0 e^{\beta i} = -F_1 e^{-\gamma i} + jF_0 e^{\gamma i}.$$

Then the mapping $\mu\colon M \to \mathbb{H}$ given by the equation (4.1) is the solution to this equation.

A section $\underline{\hat{\mu}}$ of L defined by (4.1) is a quaternionic holomorphic section if and only if

$$\begin{aligned}
&-e^{(\beta-\gamma)i/2} * (\mathrm{d}\mu_1) + \frac{1}{2}\mu_1 i e^{(\beta-\gamma)i/2}(*(\mathrm{d}\beta) + *(\mathrm{d}\gamma)) \\
&+j\left\{e^{(\beta+\gamma)i/2} * (\mathrm{d}\mu_0) - \frac{1}{2}\mu_0 i e^{(\beta+\gamma)i/2}(*(\mathrm{d}\beta) - *(\mathrm{d}\gamma))\right\} \\
&= e^{(\beta-\gamma)i/2}(\mathrm{d}\mu_0) + \frac{1}{2}\mu_0 i e^{(\beta-\gamma)i/2}((\mathrm{d}\beta) - (\mathrm{d}\gamma)) \\
&+j\left\{e^{(\beta+\gamma)i/2}(\mathrm{d}\mu_1) + \frac{1}{2}\mu_1 i e^{(\beta+\gamma)i/2}((\mathrm{d}\beta) + (\mathrm{d}\gamma))\right\}.
\end{aligned}$$

This equation is equivalent to the system of equations (4.2) and

$$*(\mathrm{d}\mu_0) = (\mathrm{d}\mu_1).$$

Since this equation is equivalent to the equation

$$(\mathrm{d}(\mu_0 - \mu_1 i)) + i * (\mathrm{d}(\mu_0 - \mu_1 i)) = 0,$$

$\mu_0 - \mu_1 i$ is a complex holomorphic function. Since the section $\underline{\hat{\mu}}$ vanishes nowhere on M by the assumption, the function $\mu_0 - \mu_1 i$ vanishes nowhere on M. □

We shall classify the branch points of a smooth mapping μ defined by (4.1) and (4.2) with real-valued functions μ_0 and μ_1 on M such that $\mu_0 - \mu_1 i$ is a nowhere vanishing complex holomorphic function. Since

$$\begin{aligned}
(\mathrm{d}\mu) \;=\; & (\mathrm{d}\mu_0)e^{(\beta-\gamma)i/2} + \frac{1}{2}\mu_0 i e^{(\beta-\gamma)i/2}((\mathrm{d}\beta) - (\mathrm{d}\gamma)) \\
+ \; & j\left\{(\mathrm{d}\mu_1)e^{(\beta+\gamma)i/2} + \frac{1}{2}\mu_1 i e^{(\beta+\gamma)i/2}((\mathrm{d}\beta) + (\mathrm{d}\gamma))\right\},
\end{aligned}$$

a point $p \in M$ is a branch point of μ if and only if

$$\begin{aligned}
&(\mathrm{d}\mu_0)_p = 0, \;\; (\mathrm{d}\mu_1)_p = 0, \\
&\mu_0(p)((\mathrm{d}\beta)_p - (\mathrm{d}\gamma)_p) = 0, \;\; \mu_1(p)((\mathrm{d}\beta)_p + (\mathrm{d}\gamma)_p) = 0.
\end{aligned}$$

Hence a point $p \in M$ is a branch point of μ if and only if a point p is a branch point of $\mu_0 - \mu_1 i$ and

$$\mu_0(p) = 0 \text{ and } (\mathrm{d}\beta)_p + (\mathrm{d}\gamma)_p = 0, \tag{4.3}$$

$$\mu_1(p) = 0 \text{ and } (\mathrm{d}\beta)_p - (\mathrm{d}\gamma)_p = 0, \tag{4.4}$$

or

$$(\mathrm{d}\beta)_p = (\mathrm{d}\gamma)_p = 0. \tag{4.5}$$

We shall classify the denominators of Lagrangian branched immersions with a right normal vector. Let $\underline{\hat{\nu}}$ be a non-zero quaternionic holomorphic section of a Lagrangian line bundle L with its complex structure J^L defined by $J^L\underline{1} = \underline{je^{\beta i}}$ and $\underline{\hat{\mu}}$ a nowhere-vanishing smooth section of L.

THEOREM 1. *We assume that M is a closed Riemann surface. The mapping $\lambda\colon (M, g) \to (\mathbb{H}, \omega_0)$ defined by the equation $\hat{\nu} = \hat{\mu}\hat{\lambda}$ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real constants μ_0 and μ_1 such that $(\mu_0)^2 + (\mu_1)^2 \neq 0$ and that $\Psi = \mu_0(\beta-\gamma) + \mu_1(\beta+\gamma)i$ is a complex holomorphic mapping from M to the torus $\mathbb{C}/\Lambda$ with $\Lambda = \{2\pi\mu_0 n + 2\pi\mu_1 m\, i \mid n, m \in \mathbb{Z}\}$.*

Proof. By Lemma 3, the mapping λ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and μ is defined by (4.1) and (4.2) with real-valued function μ_0 and μ_1 on M such that $\mu_0 - \mu_1 i$ is a complex holomorphic function vanishing nowhere on M. Hence μ_0 and μ_1 are real constants. Since μ vanishes nowhere, $(\mu_0)^2 + (\mu_1)^2 \neq 0$. Then the mapping $\Psi = \mu_0(\beta - \gamma) + \mu_1(\beta + \gamma)i$ is a non-constant complex holomorphic mapping from M to $\mathbb{C}/\Lambda$. Indeed, the equation (4.2) is equivalent to the equation

$$*(\mathrm{d}\{\mu_0(-\beta + \gamma)\}) = (\mathrm{d}\{\mu_1(\beta + \gamma)\}).$$

This is equivalent to Ψ being a complex holomorphic mapping from M to $\mathbb{C}/\Lambda$. □

We see that the Lagrangian branched immersions μ, ν, and λ in the above theorem are Hamiltonian-minimal and that $\mu(M)$ is a torus. If Ψ is non-constant, then the total branching order of Ψ is two times the genus of M by the Riemann-Hurwitz formula on p.140 in [3].

Next, we discuss the case where M is an open Riemann surface. Let $\bar{\partial}$ be a mapping from the set of smooth complex-valued functions on M to the set of smooth complex-valued one-forms of $(0, 1)$-type on M defined by $\bar{\partial} = 2^{-1}(\mathrm{d} + i * \mathrm{d})$. Then a differential equation

$$\bar{\partial}\psi = \psi a + \bar{\psi} b,$$

with complex-valued one-forms a and b of $(0, 1)$-type for a complex-valued function ψ on M is called the Carleman-Bers-Vekua system and a solution ψ to the equation is called a generalized analytic function in Rodin [12] (cf, Vekua [10]).

On a sufficiently small open set of M, we may consider the mapping $\Psi = \mu_0(\beta - \gamma) + \mu_1(\beta + \gamma)i$ as a complex-valued function.

THEOREM 2. *We assume that M is an open Riemann surface. The mapping $\lambda\colon (M,g) \to (\mathbb{H},\omega_0)$ defined by the equation $\hat{\nu} = \hat{\mu}\hat{\lambda}$ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real-valued functions μ_0 and μ_1 on M such that*

- *the function $\mu_0 - \mu_1 i$ is a complex holomorphic function on M vanishing nowhere,*
- *the equation $(\mathrm{d}\beta) + (\mathrm{d}\gamma) = 0$ holds on $\{p \in M \mid \mu_0(p) = 0\}$,*
- *the equation $(\mathrm{d}\beta) - (\mathrm{d}\gamma) = 0$ holds on $\{p \in M \mid \mu_1(p) = 0\}$,*

and that

- *a mapping $\Psi = \mu_0(\beta-\gamma) + \mu_1(\beta+\gamma)i$ is a generalized analytic function for the Carleman-Bers-Vekua system*

$$\bar{\partial}\Psi = \Psi\frac{\bar{\partial}\log(\mu_0\mu_1)}{2} + \bar{\Psi}\frac{\bar{\partial}\log(\mu_0\mu_1^{-1})}{2}, \tag{4.6}$$

on every sufficiently small open set of $\{p \in M \mid \mu_0(p)\mu_1(p) \neq 0\}$.

Proof. On the set $\{p \in M \mid \mu_0(p) = 0\}$, the equation (4.2) is equivalent to the equation $\mu_1((\mathrm{d}\beta) + (\mathrm{d}\gamma)) = 0$. Since $(\mu_0)^2 + (\mu_1)^2 \neq 0$, the equation (4.2) is equivalent to the equation $(\mathrm{d}\beta) + (\mathrm{d}\gamma) = 0$. Similarly, the equation (4.2) is equivalent to $(\mathrm{d}\beta) - (\mathrm{d}\gamma) = 0$ on $\{p \in M \mid \mu_1(p) = 0\}$.

On a sufficiently small open set of a point p with $\mu_0(p)\mu_1(p) \neq 0$, we define local real-valued functions η and ξ by $\eta = \beta - \gamma$ and $\xi = \beta + \gamma$. Then $\Psi = \mu_0\eta + \mu_1\xi i_1$ and the equation (4.2) is equivalent to the equation $\mu_0 * (\mathrm{d}\eta) = -\mu_1(\mathrm{d}\xi)$. Since

$$(\mathrm{d}(\mu_0\eta)) = \eta(\mathrm{d}\mu_0) + \mu_0(\mathrm{d}\eta), \quad (\mathrm{d}(\mu_1\xi)) = \xi(\mathrm{d}\mu_1) + \mu_1(\mathrm{d}\xi),$$

the equation (4.2) is equivalent to the equation

$$*(\mathrm{d}(\mu_0\eta)) - \mu_0\eta * (\mathrm{d}\log\mu_0) = -(\mathrm{d}(\mu_1\xi)) + \mu_1\xi(\mathrm{d}\log\mu_1).$$

Then the equation (4.2) is equivalent to

$$\begin{aligned}
2\bar{\partial}\Psi &= (\mathrm{d}\Psi) + i * (\mathrm{d}\Psi) \\
&= (\mathrm{d}(\mu_0\eta)) + i(\mathrm{d}(\mu_1\xi)) + i * (\mathrm{d}(\mu_0\eta)) - *(\mathrm{d}(\mu_1\xi)) \\
&= \mu_0\eta(\mathrm{d}\log\mu_0) - \mu_1\xi * (\mathrm{d}\log\mu_1) \\
&\quad + i\{\mu_0\eta * (\mathrm{d}\log\mu_0) + \mu_1\xi(\mathrm{d}\log\mu_1)\} \\
&= \mu_0\eta((\mathrm{d}\log\mu_0) + i * (\mathrm{d}\log\mu_0))
\end{aligned}$$

$$
\begin{aligned}
&+\mu_1\xi(-*(\mathrm{d}\log\mu_1)+i(\mathrm{d}\log\mu_1))\\
=&\ (\Psi+\bar{\Psi})(\bar{\partial}\log\mu_0)-(\Psi-\bar{\Psi})(-\bar{\partial}\log\mu_1)\\
=&\ \Psi(\bar{\partial}\log\mu_0+\bar{\partial}\log\mu_1)+\bar{\Psi}(\bar{\partial}\log\mu_0-\bar{\partial}\log\mu_1)\\
=&\ \Psi\{\bar{\partial}\log(\mu_0\mu_1)\}+\bar{\Psi}\{\bar{\partial}\log(\mu_0\mu_1^{-1})\}.
\end{aligned}
$$

Then Theorem 2 follows from Lemma 3. □

5. Formulae for denominators

We shall discuss the case where λ or its denominator μ is a Hamiltonian-minimal Lagrangian branched immersion with a right normal vector. Throughout this section, we assume that M is an open Riemann surface. We call a Lagrangian line bundle L with its Lagrangian angle β a *Hamiltonian-minimal* Lagrangian line bundle if β is a harmonic mapping.

We shall rewrite the equation (4.2) in another way. Let μ_0 and μ_1 be real-valued functions on M such that $\mu_0-\mu_1 i$ is a complex holomorphic function vanishing nowhere on M and let M' be the set of branch points of $\mu_0-\mu_1 i$. Then the functions μ_0 and μ_1 are constant if and only if $M'=M$ and not constant if and only if every element of M' is an isolated point. We assume that μ_0 and μ_1 are not constant. Then (μ_0,μ_1) is an isothermal coordinate on $M\setminus M'$. We define real-valued functions β_{μ_k}, γ_{μ_k}, $\beta_{\mu_k\mu_l}$ and $\gamma_{\mu_k\mu_l}$ on $M\setminus M'$ by the equations

$$
\begin{aligned}
(\mathrm{d}\beta) &= \beta_{\mu_0}(\mathrm{d}\mu_0)+\beta_{\mu_1}(\mathrm{d}\mu_1),\\
(\mathrm{d}\gamma) &= \gamma_{\mu_0}(\mathrm{d}\mu_0)+\gamma_{\mu_1}(\mathrm{d}\mu_1),\\
(\mathrm{d}\beta_{\mu_k}) &= \beta_{\mu_k\mu_0}(\mathrm{d}\mu_0)+\beta_{\mu_k\mu_l}(\mathrm{d}\mu_1),\\
(\mathrm{d}\gamma_{\mu_k}) &= \gamma_{\mu_k\mu_0}(\mathrm{d}\mu_0)+\gamma_{\mu_k\mu_1}(\mathrm{d}\mu_1),\quad (k,l=0,1).
\end{aligned}
$$

Then $\beta_{\mu_0\mu_1}=\beta_{\mu_1\mu_0}$ and $\gamma_{\mu_0\mu_1}=\gamma_{\mu_1\mu_0}$. The equation (4.2) on $M\setminus M'$ is equivalent to the equation

$$
\begin{pmatrix}\mu_1 & \mu_0\\ \mu_0 & -\mu_1\end{pmatrix}\begin{pmatrix}\gamma_{\mu_0}\\ \gamma_{\mu_1}\end{pmatrix}=\begin{pmatrix}-\mu_1 & \mu_0\\ \mu_0 & \mu_1\end{pmatrix}\begin{pmatrix}\beta_{\mu_0}\\ \beta_{\mu_1}\end{pmatrix}, \tag{5.1}
$$

on $M\setminus M'$.

LEMMA 4. *If the equation (5.1) holds on $M\setminus M'$, then the system of equations*

$$
\begin{aligned}
2\gamma_{\mu_1}+\mu_1(\gamma_{\mu_0\mu_0}+\gamma_{\mu_1\mu_1}) &= -\mu_1(\beta_{\mu_0\mu_0}+\beta_{\mu_1\mu_1}), &(5.2)\\
2\gamma_{\mu_0}+\mu_0(\gamma_{\mu_0\mu_0}+\gamma_{\mu_1\mu_1}) &= \mu_0(\beta_{\mu_0\mu_0}+\beta_{\mu_1\mu_1}), &(5.3)
\end{aligned}
$$

holds on $M\setminus M'$.

Proof. By the differentiation of the both side of the equation (5.1), we have a system of equations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \gamma_{\mu_0} \\ \gamma_{\mu_1} \end{pmatrix} + \begin{pmatrix} \mu_1 & \mu_0 \\ \mu_0 & -\mu_1 \end{pmatrix}\begin{pmatrix} \gamma_{\mu_0\mu_0} \\ \gamma_{\mu_1\mu_0} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta_{\mu_0} \\ \beta_{\mu_1} \end{pmatrix} + \begin{pmatrix} -\mu_1 & \mu_0 \\ \mu_0 & \mu_1 \end{pmatrix}\begin{pmatrix} \beta_{\mu_0\mu_0} \\ \beta_{\mu_1\mu_0} \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma_{\mu_0} \\ \gamma_{\mu_1} \end{pmatrix} + \begin{pmatrix} \mu_1 & \mu_0 \\ \mu_0 & -\mu_1 \end{pmatrix}\begin{pmatrix} \gamma_{\mu_0\mu_1} \\ \gamma_{\mu_1\mu_1} \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta_{\mu_0} \\ \beta_{\mu_1} \end{pmatrix} + \begin{pmatrix} -\mu_1 & \mu_0 \\ \mu_0 & \mu_1 \end{pmatrix}\begin{pmatrix} \beta_{\mu_0\mu_1} \\ \beta_{\mu_1\mu_1} \end{pmatrix}.$$

This system of equations is equivalent to the system of equations

$$\begin{aligned} \gamma_{\mu_1} + \mu_1\gamma_{\mu_0\mu_0} + \mu_0\gamma_{\mu_1\mu_0} &= \beta_{\mu_1} - \mu_1\beta_{\mu_0\mu_0} + \mu_0\beta_{\mu_1\mu_0}, \\ \gamma_{\mu_0} + \mu_0\gamma_{\mu_0\mu_0} - \mu_1\gamma_{\mu_1\mu_0} &= \beta_{\mu_0} + \mu_0\beta_{\mu_0\mu_0} + \mu_1\beta_{\mu_1\mu_0}, \\ \gamma_{\mu_0} + \mu_1\gamma_{\mu_0\mu_1} + \mu_0\gamma_{\mu_1\mu_1} &= -\beta_{\mu_0} - \mu_1\beta_{\mu_0\mu_1} + \mu_0\beta_{\mu_1\mu_1}, \\ -\gamma_{\mu_1} + \mu_0\gamma_{\mu_0\mu_1} - \mu_1\gamma_{\mu_1\mu_1} &= \beta_{\mu_1} + \mu_0\beta_{\mu_0\mu_1} + \mu_1\beta_{\mu_1\mu_1}. \end{aligned}$$

Lemma 4 follows from this system of equations. □

We shall discuss the case where μ, ν and λ are Hamiltonian-minimal Lagrangian branched immersions. Let $\underline{\hat{\nu}}$ be a non-zero quaternionic holomorphic section of a Hamiltonian-minimal Lagrangian line bundle L with its complex structure J^L defined by $J^L\underline{1} = \underline{je^{\beta i}}$ and $\underline{\hat{\mu}}$ a nowhere-vanishing quaternionic holomorphic sections of $\underline{L}$.

THEOREM 3. *The mapping $\lambda\colon (M, g) \to (\mathbb{H}, \omega_0)$ defined by the equation $\hat{\nu} = \hat{\mu}\hat{\lambda}$ is a Hamiltonian-minimal Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real-valued functions μ_0 and μ_1 on M such that*

- *the functions μ_0 and μ_1 are constants with $(\mu_0)^2 + (\mu_1)^2 \neq 0$ and $\Psi = \mu_0(\beta-\gamma) + \mu_1(\beta+\gamma)i$ is a complex holomorphic mapping from M to $\mathbb{C}/\Lambda$ with $\Lambda = \{2\pi\mu_0 n + 2\pi\mu_1 m\, i \mid n, m \in \mathbb{Z}\}$,*

or

- *the function $\mu_0 - \mu_1 i$ is a non-constant complex holomorphic function vanishing nowhere and β and γ are constant mappings.*

Proof. Let λ be a Hamiltonian-minimal Lagrangian branched immersion. If μ_0 and μ_1 are constants, then Ψ is a complex holomorphic mapping from M to $\mathbb{C}/\Lambda$ by (4.2) in the same way as the proof of Theorem 1. We assume that $\mu_0 - \mu_1 i$ is a non-constant complex holomorphic function. By Lemma 4, we have

$$2\gamma_{\mu_1} = 0, \quad 2\gamma_{\mu_0} = 0,$$

on $M \setminus M'$ since β and γ are harmonic mappings. Hence γ is a constant mapping on $M \setminus M'$. Then $-\mu_0 * (\mathrm{d}\beta) = \mu_1(\mathrm{d}\beta)$ by the equation (4.2) on $M \setminus M'$. Since $\mu_0(\mathrm{d}\beta) = \mu_1 * (\mathrm{d}\beta)$, we have $\{(\mu_0)^2 + (\mu_1)^2\}(\mathrm{d}\beta) = 0$ on $M \setminus M'$. Hence $(\mathrm{d}\beta) = 0$ and β is a constant mapping on $M \setminus M'$. Since every element of M' is an isolated point and β and γ are smooth on M, both β and γ are constant mappings on M.

It is easy to see that the converse holds, $\square$

We shall discuss the case where μ and ν are Hamiltonian-minimal Lagrangian branched immersion and λ is a Lagrangian branched immersion with its right normal vector $je^{i\gamma}$ which is not Hamiltonian-minimal. Let $\underline{\hat{\nu}}$ be a non-zero quaternionic holomorphic section of a Hamiltonian-minimal Lagrangian line bundle L with its complex structure J^L defined by $J^L\underline{1} = \underline{je^{\beta i}}$ and $\underline{\hat{\mu}}$ a nowhere-vanishing quaternionic holomorphic sections of L.

THEOREM 4. *The mapping $\lambda\colon (M, g) \to (\mathbb{H}, \omega_0)$ defined by the equation $\hat{\nu} = \hat{\mu}\hat{\lambda}$ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ which is not Hamiltonian-minimal if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real-valued functions μ_0 and μ_1 on M such that $\mu_0 - \mu_1 i$ is a non-constant complex holomorphic function vanishing nowhere on M and mappings β and γ are given by the equations*

$$\beta(\mu_0, \mu_1) = A\frac{(\mu_0^2 - \mu_1^2)}{(\mu_0^2 + \mu_1^2)^2} + B, \tag{5.4}$$

$$\gamma(\mu_0, \mu_1) = \frac{A}{\mu_0^2 + \mu_1^2} + C, \tag{5.5}$$

on M for an arbitrary non-zero real number A and arbitrary real numbers B and C.

Proof. We assume that λ is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ which is not a Hamiltonian-minimal Lagrangian branched immersion. If μ_0 and μ_1 are constant functions, then $\Psi = \mu_0(\beta - \gamma) + \mu_1(\beta + \gamma)i$ is a complex holomorphic mapping in the same way as the proof of Theorem 1. Then β and γ are harmonic mappings. Since γ is not a harmonic mapping by the assumption, the functions μ_0 and μ_1 are not constant functions.

Since β is a harmonic mapping, we have

$$2\gamma_{\mu_1} + \mu_1(\gamma_{\mu_0\mu_0} + \gamma_{\mu_1\mu_1}) = 0, \tag{5.6}$$

$$2\gamma_{\mu_0} + \mu_0(\gamma_{\mu_0\mu_0} + \gamma_{\mu_1\mu_1}) = 0, \tag{5.7}$$

on $M \setminus M'$ by Lemma 4. Then $\mu_1\gamma_{\mu_0} - \mu_0\gamma_{\mu_1} = 0$. Hence $\gamma(\mu_0, \mu_1) = \phi(\mu_0^2+\mu_1^2)$ on $M\setminus M'$ for a smooth real-valued function ϕ on $\mathbb{R}\setminus\{\mu_0^2(p)+\mu_1^2(p) \,|\, p \in M \setminus M'\}$.

Since

$$\begin{aligned}\gamma_{\mu_0\mu_0}(\mu_0, \mu_1) &= 4\mu_0^2\phi''(\mu_0^2 + \mu_1^2) + 2\phi'(\mu_0^2 + \mu_1^2),\\ \gamma_{\mu_1\mu_1}(\mu_0, \mu_1) &= 4\mu_1^2\phi''(\mu_0^2 + \mu_1^2) + 2\phi'(\mu_0^2 + \mu_1^2),\end{aligned}$$

the equations (5.6) and (5.7) is equivalent to the equation

$$t\phi''(t) + 2\phi'(t) = 0, \ \ t = \mu_0^2 + \mu_1^2.$$

The solution to this equation is $\phi'(t) = -At^{-2}$ for a real number A. Since γ is not a harmonic mapping, it is not a constant mapping. Then we obtain the equation (5.5) with a non-zero real number A and a real number C on $M \setminus M'$. Since every element of M' is an isolated point and γ is smooth on M, the equation (5.5) holds on M.

Since

$$\begin{aligned}\gamma_{\mu_0}(\mu_0, \mu_1) &= -\frac{2A\mu_0}{(\mu_0^2 + \mu_1^2)^2},\\ \gamma_{\mu_1}(\mu_0, \mu_1) &= -\frac{2A\mu_1}{(\mu_0^2 + \mu_1^2)^2},\end{aligned}$$

we have

$$\begin{aligned}\begin{pmatrix}\beta_{\mu_0}\\ \beta_{\mu_1}\end{pmatrix} &= \begin{pmatrix}-\mu_1 & \mu_0\\ \mu_0 & \mu_1\end{pmatrix}^{-1}\begin{pmatrix}\mu_1 & \mu_0\\ \mu_0 & -\mu_1\end{pmatrix}\begin{pmatrix}-2A\mu_0(\mu_0^2 + \mu_1^2)^{-2}\\ -2A\mu_1(\mu_0^2 + \mu_1^2)^{-2}\end{pmatrix}\\ &= \frac{-2A}{(\mu_0^2 + \mu_1^2)^3}\begin{pmatrix}\mu_0^3 - 3\mu_0\mu_1^2\\ 3\mu_0^2\mu_1 - \mu_1^3\end{pmatrix},\end{aligned}$$

on $M \setminus M'$ by the equation (5.1). Since

$$\begin{aligned}&\frac{\mu_0^3 - 3\mu_0\mu_1^2}{(\mu_0^2 + \mu_1^2)^3} = \frac{\mu_0(\mu_0^2 + \mu_1^2 - 4\mu_1^2)}{(\mu_0^2 + \mu_1^2)^3} = \frac{\mu_0}{(\mu_0^2 + \mu_1^2)^2} + \frac{\mu_0(-4\mu_1^2)}{(\mu_0^2 + \mu_1^2)^3}\\ &= \frac{\partial}{\partial\mu_0}\frac{-1}{2(\mu_0^2 + \mu_1^2)} + \frac{\partial}{\partial\mu_0}\frac{\mu_1^2}{(\mu_0^2 + \mu_1^2)^2},\end{aligned}$$

we have

$$\begin{aligned}&\beta(\mu_0, \mu_1) = -2A\left(\frac{-1}{2(\mu_0^2 + \mu_1^2)} + \frac{\mu_1^2}{(\mu_0^2 + \mu_1^2)^2}\right) + E(\mu_1)\\ &= -2A\left(\frac{1}{2(\mu_0^2 + \mu_1^2)} - \frac{\mu_0^2}{(\mu_0^2 + \mu_1^2)^2}\right) + E(\mu_1),\end{aligned}$$

where $E(\mu_1)$ is a differentiable function of μ_1. Then

$$\beta_{\mu_1}(\mu_0,\mu_1) = -2A\left(\frac{-\mu_1}{(\mu_0^2+\mu_1^2)^2} + \frac{4\mu_0^2\mu_1}{(\mu_0^2+\mu_1^2)^3}\right) + \frac{\partial}{\partial\mu_1}E(\mu_1)$$
$$= \frac{-2A(3\mu_0^2\mu_1 - \mu_1^3)}{(\mu_0^2+\mu_1^2)^3} + \frac{\partial}{\partial\mu_1}E(\mu_1).$$

Hence $E(\mu_1)$ is a constant and the equation (5.4) is satisfied on $M\setminus M'$ for a non-zero real number A and a real number B. Since every element of M' is an isolated point and β is smooth, the equation (5.4) holds on M.

Conversely, we assume that β and γ satisfies the equations (5.4) and (5.5). Then we see that β is a harmonic mapping and that the equation (4.2) holds by a direct calculation. □

We discuss the case where μ and ν are Lagrangian branched immersions which are not Hamiltonian-minimal and λ is a Hamiltonian-minimal Lagrangian branched immersion. Let $\underline{\hat{\nu}}$ be a non-zero quaternionic holomorphic section of a Lagrangian line bundle L with its complex structure J^L defined by $J^L\underline{1} = \underline{je^{\beta i}}$ which is not Hamiltonian-minimal and $\underline{\hat{\mu}}$ a nowhere-vanishing quaternionic holomorphic sections of L.

THEOREM 5. *The mapping $\lambda\colon(M,g)\to(\mathbb{H},\omega_0)$ defined by the equation $\hat{\nu}=\hat{\mu}\hat{\lambda}$ is a Hamiltonian-minimal Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ if and only if $\underline{\hat{\mu}}$ is linearly independent of $\underline{\hat{\nu}}$ and $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real-valued functions μ_0 and μ_1 on M such that $\mu_0-\mu_1 i$ is non-constant complex holomorphic function vanishing nowhere and mappings β and γ are given by the equations*

$$\beta(\mu_0,\mu_1) = A(\mu_0^2+\mu_1^2)+B, \tag{5.8}$$
$$\gamma(\mu_0,\mu_1) = A(\mu_0^2-\mu_1^2)+C, \tag{5.9}$$

on M with a non-zero real number A and real numbers B and C.

Proof. We assume that λ is a Hamiltonian-minimal Lagrangian immersion with its right normal vector $je^{\gamma i}$. Since β is not a harmonic mapping, we see that the functions μ_0 and μ_1 are not constant functions in a similar way as the proof of Theorem 4.

Since γ is a harmonic mapping, we have the equations

$$2\gamma_{\mu_0} = \mu_0(\beta_{\mu_0\mu_0}+\beta_{\mu_1\mu_1}),$$
$$2\gamma_{\mu_1} = -\mu_1(\beta_{\mu_0\mu_0}+\beta_{\mu_1\mu_1}),$$

on $M \setminus M'$ by Lemma 4. Then $\mu_1\gamma_{\mu_0} + \mu_0\gamma_{\mu_1} = 0$. Hence $\gamma(\mu_0, \mu_1) = \phi(\mu_0^2 - \mu_1^2)$ for a smooth real-valued function ϕ on $\mathbb{R}\setminus\{\mu_0^2(p) - \mu_1^2(p) \mid p \in M \setminus M'\}$. Since

$$\begin{aligned}
\gamma_{\mu_0}(\mu_0, \mu_1) &= 2\mu_0\phi'(\mu_0^2 - \mu_1^2),\\
\gamma_{\mu_1}(\mu_0, \mu_1) &= -2\mu_1\phi'(\mu_0^2 - \mu_1^2),\\
\gamma_{\mu_0\mu_0}(\mu_0, \mu_1) &= 4\mu_0^2\phi''(\mu_0^2 - \mu_1^2) + 2\phi'(\mu_0^2 - \mu_1^2),\\
\gamma_{\mu_1\mu_1}(\mu_0, \mu_1) &= 4\mu_1^2\phi''(\mu_0^2 - \mu_1^2) - 2\phi'(\mu_0^2 - \mu_1^2),
\end{aligned}$$

we have

$$\gamma_{\mu_0\mu_0}(\mu_0, \mu_1) + \gamma_{\mu_1\mu_1}(\mu_0, \mu_1) = 4(\mu_0^2 + \mu_1^2)\phi''(\mu_0^2 - \mu_1^2) = 0.$$

Hence the equation (5.9) holds on $M \setminus M'$ for a non-zero real number A and a real number C. Since every element of M' is an isolated point and γ is smooth on M, the equation (5.9) holds on M.

By the equation (5.1), we have the equation

$$\begin{pmatrix} \beta_{\mu_0} \\ \beta_{\mu_1} \end{pmatrix} = 2A \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix}.$$

Hence the equation (5.8) holds for a non-zero real number A and a real number B on $M \setminus M'$. Since every element of M' is an isolated point and β is smooth, the equation (5.8) holds on M.

Conversely, we assume that β and γ are given by the equations (5.8) and (5.9) respectively. Then we see that the equation (4.2) holds and γ is a harmonic mapping by a direct calculation.

6. Examples

We apply Theorem 4 and Theorem 5 to obtain examples of Lagrangian branched immersions. We calculate left normal vectors of the examples to see that there are examples with both conformal Maslov forms (see [4]) and non-conformal Maslov forms.

Let $f: M \to \mathbb{C}^2$ be a Lagrangian immersion with its left normal vector N and its right normal vector $je^{\beta i}$. The map $(N, je^{\beta i}): M \to S^2(1) \times S^1(1)$ is a decomposition of the generalized Gauss map of f, where $S^1(1)$ is a circle in $\{j(u+vi) \mid u, v \in \mathbb{R}\}$ with radius one centered at origin. Let ω_1 is the symplectic form of $\mathbb{C}^2$ and H the mean curvature vector of f. The one-form ϖ on M defined by $\varpi(X) = \omega_1(X, H)/\pi$ is called the *Maslov form* of f. A Maslov form ϖ is said to be *conformal* if the tension field of the left normal vector N of f vanishes, or equivalently $\mathrm{d} * (\mathrm{d}N) = hN$ with a real-valued function h on M. Locally, this equation is equivalent to the equation

$$N_{xx} + N_{yy} = kN, \tag{6.1}$$

where (x, y) is a local coordinate of M such that $x+yi$ is a local complex holomorphic coordinate and k is a local real-valued function on M (see [11] for example).

We use the following coordinate transformation. Let μ_0 and μ_1 be real-functions on M such that $\mu_0 - \mu_1 i$ is a complex holomorphic function on M. Then (μ_0, μ_1) is a coordinate of M and $\mu_0 - \mu_1 i$ is a complex holomorphic coordinate except branch points of $\mu_0 - \mu_1 i$. Let $x = \mu_0(\mu_0^2 + \mu_1^2)^{-1}$ and $y = \mu_1(\mu_0^2 + \mu_1^2)^{-1}$. Then (x, y) is a coordinate on M such that $x+yi$ is a complex holomorphic coordinate of M except branch point and zeros of $\mu_0 - \mu_1 i$. We see that $x_0^2 + x_1^2 = (\mu_0^2 + \mu_1^2)^{-1}$.

Example 1. Let $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real-valued functions μ_0 and μ_1 on M such that $\mu_0 - \mu_1 i$ is a non-constant complex holomorphic function vanishing nowhere on M. We assume that the mappings β and γ are given by the equations (5.4) and (5.5) with non-zero real number A and $B = C = 0$. Then

$$\mu(\mu_0, \mu_1) = \mu_0 e^{-A\mu_1^2(\mu_0^2+\mu_1^2)^{-2}i} + j\mu_1 e^{A\mu_0^2(\mu_0^2+\mu_1^2)^{-2}i}$$

is a Hamiltonian-minimal Lagrangian branched immersion with its right normal vector $je^{\beta i}$ by Theorem 4.

Let (x, y) be a coordinate of M such that $x = \mu_0(\mu_0^2 + \mu_1^2)^{-1}$ and $y = \mu_1(\mu_0^2 + \mu_1^2)^{-1}$. Since

$$\begin{aligned}
\mu &= \frac{x}{x^2+y^2} e^{-Ay^2 i} + j\frac{y}{x^2+y^2} e^{Ax^2 i},\\
\mu_x &= \frac{-x^2+y^2}{(x^2+y^2)^2} e^{-Ay^2 i} + j\frac{2xy\{-1+A(x^2+y^2)i\}}{(x^2+y^2)^2} e^{Ax^2 i},\\
\mu_y &= \frac{2xy\{-1-A(x^2+y^2)i\}}{(x^2+y^2)^2} e^{-Ay^2 i} + j\frac{x^2-y^2}{(x^2+y^2)^2} e^{Ax^2 i},
\end{aligned}$$

the left normal vector of μ is

$$\mu_y \mu_x^{-1} = -je^{-A(x^2-y^2)i}.$$

By the equation (6.1), we see that the Maslov form of μ is conformal. The section $\underline{1}$ of a Hamiltonian-minimal Lagrangian line bundle L with its complex structure J^L defined by $J^L\underline{1} = \underline{je^{\beta i}}$ is a non-zero quaternionic holomorphic section. We define a smooth mapping λ by $\underline{1} = \underline{\hat{\mu}}\hat{\lambda}$.

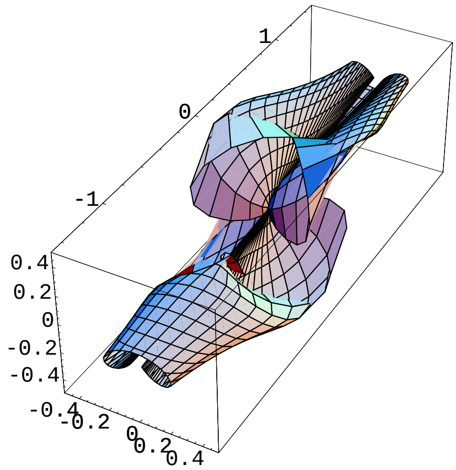

Figure 1. (Example 1) $U = \{x + yi \in \mathbb{C} \,|\, 0.5^2 \leq x^2 + y^2 \leq 4^2\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\mathrm{Im}\,\mu\colon U \to \mathrm{Im}\,\mathbb{H}$.

Then

$$\begin{aligned}\lambda(\mu_0, \mu_1) &= \frac{1}{\mu_0^2 + \mu_1^2} \\ &\times \left(\mu_0 e^{A\mu_1^2(\mu_0^2+\mu_1^2)^{-2}i} - j\mu_1 e^{A\mu_0^2(\mu_0^2+\mu_1^2)^{-2}i}\right)\end{aligned}$$

is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ which is not Hamiltonian-minimal by Theorem 4.

Let $x = \mu_0(\mu_0^2 + \mu_1^2)^{-1}$ and $y = \mu_1(\mu_0^2 + \mu_1^2)^{-1}$. Then (x, y) is a coordinate of M such that $x + yi$ is a complex holomorphic coordinate except branch point of $\mu_0 - \mu_1 i$. Since

$$\begin{aligned}&\lambda = xe^{Ay^2 i} - jye^{Ax^2}, \\ &\lambda_x = e^{Ay^2 i} - j2Axyie^{Ax^2 i}, \;\; \lambda_y = 2Axyie^{Ay^2 i} - je^{Ax^2 i},\end{aligned}$$

the left normal vector of λ is

$$\lambda_y \lambda_x^{-1} = \frac{4Axyi}{1 + 4A^2x^2y^2} + j\frac{(4A^2x^2y^2 - 1)e^{A(x^2-y^2)i}}{1 + 4A^2x^2y^2}.$$

After long computation, we see that the Maslov form of λ is not conformal since the equation (6.1) does not hold.

Example 2. Let μ be the Hamiltonian-minimal Lagrangian branched immersion with its right normal vector $je^{\beta i}$ defined in the same way as Example 1. The function

$$\alpha = \frac{2A\mu_0\mu_1}{(\mu_0^2 + \mu_1^2)^2}$$

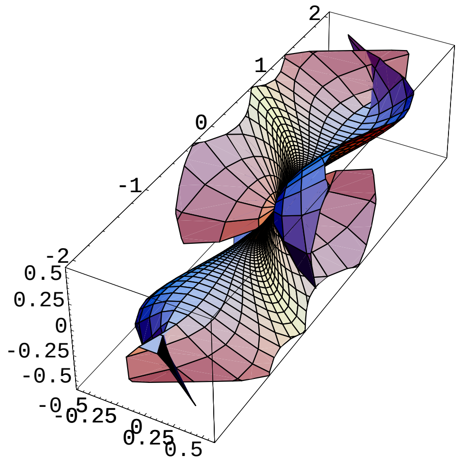

Figure 2. (Example 1) $U = \{x + yi \in \mathbb{C} \,|\, 0.5^2 \leq x^2 + y^2 \leq 4^2\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\operatorname{Im}\lambda \colon U \to \operatorname{Im}\mathbb{H}$.

satisfies the equation $*(\mathrm{d}\alpha) = (\mathrm{d}\beta)$. Then

$$\nu = \alpha - jie^{\beta i} = \frac{2A\mu_0\mu_1}{(\mu_0^2 + \mu_1^2)^2} - jie^{\{A(\mu_0^2-\mu_1^2)(\mu_0^2+\mu_1^2)^{-2}\}i}$$

is a Hamiltonian-minimal Lagrangian branched immersion with its right normal vector $je^{\beta i}$. Indeed,

$$\begin{aligned} (\mathrm{d}\nu) &= (\mathrm{d}\alpha) + je^{\beta i}(\mathrm{d}\beta), \\ *(\mathrm{d}\nu) &= (\mathrm{d}\beta) - je^{\beta i}(\mathrm{d}\alpha) = (\mathrm{d}\nu)(-je^{\beta i}). \end{aligned}$$

Since the left normal vector of ν is $-je^{\beta i}$, we see that the Maslov form of ν is conformal by the equation (6.1). The image $\nu(M)$ is a part of a circular cylinder.

We define a smooth mapping λ by $\underline{\hat{\nu}} = \underline{\hat{\mu}}\hat{\lambda}$. Then

$$\begin{aligned} &\lambda(\mu_0, \mu_1) = \frac{1}{(\mu_0^2 + \mu_1^2)^3} \\ &\times \left[\left\{2A\mu_0^2\mu_1 e^{A\mu_1^2(\mu_0^2+\mu_1^2)^{-2}i} + \mu_1(\mu_0^2 + \mu_1^2)^2 i e^{A\mu_1^2(\mu_0^2+\mu_1^2)^{-2}i}\right\}\right. \\ &\left. + j\left\{-2A\mu_0\mu_1^2 e^{A\mu_0^2(\mu_0^2+\mu_1^2)^{-2}i} - \mu_0(\mu_0^2 + \mu_1^2)^2 i e^{A\mu_0^2(\mu_0^2+\mu_1^2)^{-2}i}\right\}\right] \end{aligned}$$

is a Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ which is not Hamiltonian-minimal by Theorem 4.

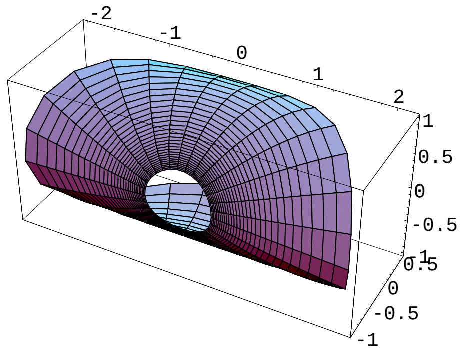

Figure 3. (Example 2) $U = \{x + yi \in \mathbb{C} \,|\, 0.68^2 \leq x^2 + y^2 \leq 1.5^2\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\mathrm{Im}(i\nu)\colon U \to \mathrm{Im}\,\mathbb{H}$.

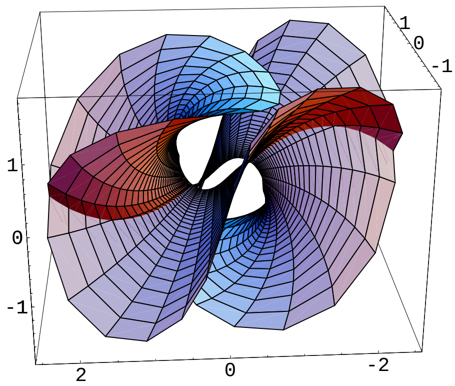

Figure 4. (Example 2) $U = \{x + yi \in \mathbb{C} \,|\, 0.68^2 \leq x^2 + y^2 \leq 1.5^2\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\mathrm{Im}(i\lambda)\colon U \to \mathrm{Im}\,\mathbb{H}$.

Let $x = \mu_0(\mu_0^2 + \mu_1^2)^{-1}$ and $y = \mu_1(\mu_0^2 + \mu_1^2)^{-1}$. Then

$$\begin{aligned}
\lambda &= (2Ax^2 + i)ye^{Ay^2 i} + jx(-2Ay^2 - i)e^{Ax^2 i}, \\
\lambda_x &= 4Axye^{Ay^2 i} + j(1 + 2Ax^2 i)(-2Ay^2 - i)e^{Ax^2 i}, \\
\lambda_y &= (2Ax^2 + i)(1 + 2Ay^2 i)e^{Ay^2 i} + j(-4Axy)e^{Ax^2 i}.
\end{aligned}$$

Hence the left normal vector of λ is

$$\begin{aligned}
\lambda_y\lambda_x^{-1} &= \frac{8Axy(1 + 4A^2x^2y^2)}{1 + 4A^2x^4 + 16A^2x^2y^2 + 4A^2y^4 + 16A^4x^4y^4} i \\
&+ j\left\{\frac{1 + 16A^4x^4y^4 - 4A^2(x^4 + y^4)}{1 + 4A^2x^4 + 16A^2x^2y^2 + 4A^2y^4 + 16A^4x^4y^4}\right.
\end{aligned}$$

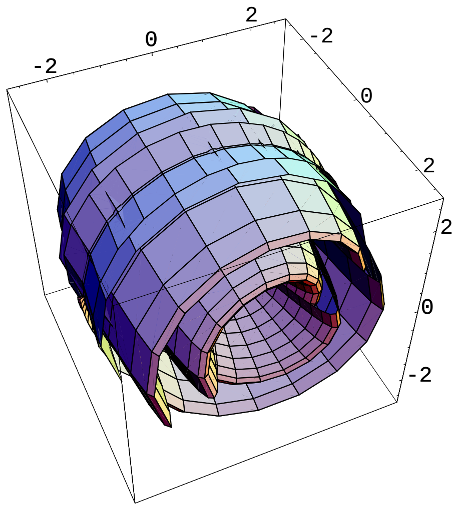

Figure 5. (Example 3) $U = \{x + yi \in \mathbb{C} \,|\, 0.1 \le |x| \le 2.6, |y| \le 2.6\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\mathrm{Im}\,\mu\colon U \to \mathrm{Im}\,\mathbb{H}$.

$$+\left.\frac{4A(x^2-y^2)(1+4A^2x^2y^2)}{1+4A^2x^4+16A^2x^2y^2+4A^2y^4+16A^4x^4y^4}i\right\} e^{A(x^2-y^2)}.$$

After long computation, we see that the Maslov form of λ is not conformal by the equation (6.1).

Example 3. Let $\mu = \mu_0 e^{(\beta-\gamma)i/2} + j\mu_1 e^{(\beta+\gamma)i/2}$ with real-valued functions μ_0 and μ_1 on M such that $\mu_0 - \mu_1 i$ is a non-constant complex holomorphic function vanishing nowhere on M. We assume that the mappings β and γ are given by the equations (5.8) and (5.9) with non-zero real number A and $B = C = 0$. Then

$$\mu = \mu_0 e^{A\mu_1^2 i} + j\mu_1 e^{A\mu_0^2 i}$$

is a Lagrangian branched immersion which is not Hamiltonian minimal with its right normal vector $je^{\beta i}$ by Theorem 5. We see that the Maslov form of μ is not conformal in a similar way as Example 1.

Let p be a point in M and

$$\nu(\mu_0, \mu_1) = \int_{\mu_0(p)}^{\mu_0} e^{At^2 i}\mathrm{d}t + j\int_{\mu_1(p)}^{\mu_1} e^{At^2 i}\mathrm{d}t.$$

Then ν is a Lagrangian branched immersion with its right normal vector $je^{\beta i}$ which is not Hamiltonian-minimal. Indeed,

$$\begin{aligned} (\mathrm{d}\nu) &= e^{A\mu_0^2 i}(\mathrm{d}\mu_0) + je^{A\mu_1^2 i}(\mathrm{d}\mu_1), \\ *(\mathrm{d}\nu) &= e^{A\mu_0^2 i}(\mathrm{d}\mu_1) - je^{A\mu_1^2 i}(\mathrm{d}\mu_0) = (\mathrm{d}\nu)\left(-je^{A(\mu_0^2+\mu_1^2)i}\right). \end{aligned}$$

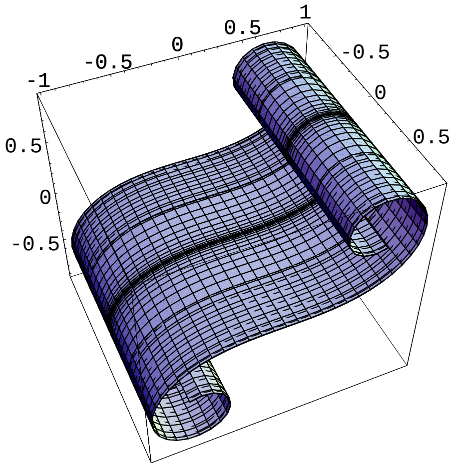

Figure 6. (Example 3) $U = \{x + yi \in \mathbb{C} \,|\, 0.1 \leq |x| \leq 2.6, |y| \leq 2.6\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\operatorname{Im}\nu\colon U \to \operatorname{Im}\mathbb{H}$.

Since the left normal vector of ν is $-je^{A(-\mu_0^2+\mu_1^2)i}$, we see that the Maslov form of ν is conformal by the equation (6.1). In the case where $M = \{x + yi \,|\, x, y \in \mathbb{R}\} = \mathbb{C}$, $A = 1$, $\mu_0 = x$, $\mu_1 = -y$, and $p = 0$, the map ν is a flat Lagrangian embedding given in [4].

Let us define a smooth mapping λ by $\underline{\hat{\nu}} = \underline{\hat{\mu}}\hat{\lambda}$. Then

$$\begin{aligned}\lambda(\mu_0, \mu_1) &= \frac{1}{\mu_0^2 + \mu_1^2} \\ &\times \left[\left\{\mu_0 e^{-A\mu_1^2 i} \int_{\mu_0(p)}^{\mu_0} e^{At^2 i}\mathrm{d}t + \mu_1 e^{A\mu_0^2 i} \int_{\mu_1(p)}^{\mu_1} e^{-At^2 i}\mathrm{d}t\right\}\right. \\ &+ j\left.\left\{\mu_0 e^{-A\mu_1^2 i} \int_{\mu_1(p)}^{\mu_1} e^{At^2 i}\mathrm{d}t - \mu_1 e^{A\mu_0^2 i} \int_{\mu_0(p)}^{\mu_0} e^{-At^2 i}\mathrm{d}t\right\}\right].\end{aligned}$$

is a Hamiltonian-minimal Lagrangian branched immersion with its right normal vector $je^{\gamma i}$ by Theorem 5. We should seek an alternative method to the equation (6.1) to conclude whether the Maslov form of λ is conformal since the computation becomes very long.

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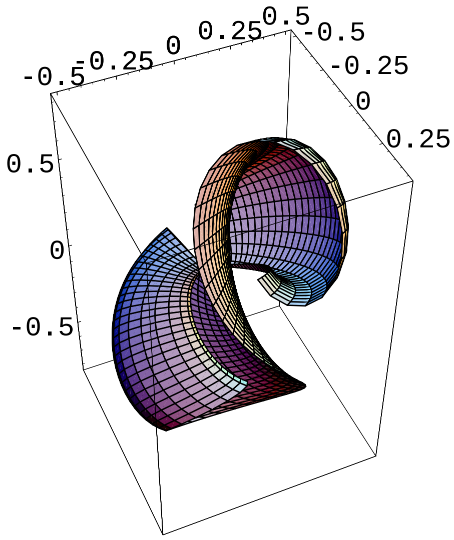

Figure 7. (Example 3) $U = \{x + yi \in \mathbb{C} \,|\, 0.1 \leq |x| \leq 2.6, |y| \leq 2.6\}$, $\mu_0 = x$, $\mu_1 = -y$, $A = 1$, $\operatorname{Im}\lambda\colon U \to \operatorname{Im}\mathbb{H}$.

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