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SEPARABLE FUNCTORS IN COALGEBRAS. APPLICATIONS

By

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Introduction

The notion of separable functor was introduced in [5], where some applications in the framework of group-graded rings where done. This notion fits satisfactorily to the classical notion of separable algebra over a commutative ring. The concept of coseparable coalgebra over a field appears in [1] to prove a result of Sullivan [7]. A more complete study of the separability of coalgebras was performed in [2]. In this last paper, an analysis of the relationship between coseparability and the cohomology theory for coalgebras is developed.

Our aim is to study the separability, in the sense of [5], of some canonical functors stemming from a morphism of coalgebras.

In Section 1 we fix some notation and we prove a preliminary characterization of the bicomodules.

The Section 2 contains the theoretical body of the paper. For a morphism of coalgebras $\varphi : C \to D$, we characterize the separability of the corestriction functor $(-)_{\varphi}$ (Theorem 2.4) and of the coinduction functor $(-)^{\varphi}$ (Theorem 2.7). The reader can find the definitions of these functors in Section 1. For the particular case of the coalgebra morphism $\varepsilon : C \to k$ given by the counit of the k-coalgebra C, the separability of the corestriction functor gives precisely the notion of coseparable coalgebra. We finish the section with Theorem 2.9, that entails that a coseparable coalgebra need not to be necessarily of finite dimension (Theorem 3.4).

Section 3 is devoted to study the relationship between coseparability and cosemisimiplicity for coalgebras. As a consequence, we obtain that a k-coalgebra

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C is coseparable if and only if the coalgebra induced by any field extension of k is co-semi-simple.

1. Notation and Preliminaries

Let k be a commutative field. Any tensor product $\otimes_k$ over k will be simply denoted by $\otimes$. The identity map on a set X will be denoted by 1_X or even by 1. A coalgebra over k is a k-vector space C together with two k-linear maps $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to k$ such that $(1 \otimes \Delta_C) \circ \Delta_C = (\Delta_C \otimes 1) \circ \Delta_C$ and $(\varepsilon_C \otimes 1) \circ \Delta_C = (1 \otimes \varepsilon_C) \circ \Delta_C = 1$. We shall refer to [8] for details. The dual space $C^* = \mathrm{Hom}_k(C, k)$ can be canonically endowed with structure of k-algebra. A right C-comodule is a k-vector space M together with a structure k-linear map $\rho_M : M \to M \otimes C$ such that $(1_M \otimes \varepsilon) \circ \rho_M = 1_M$ and $(\rho_M \otimes 1_C) \circ \rho_M = (1_M \otimes \Delta_C) \circ \rho_M$. The coalgebra C can be considered as a right C-comodule with structure map $\rho_C = \Delta_C$. A k-linear map $f : M \to N$ between right C-comodules is said to be *C-colinear* or a *morphism of right C-comodules* if $(f \otimes 1) \circ \rho_M = \rho_N \circ f$. The right C-comodules together with the C-colinear maps between them form a Grothendieck category M^C. In fact, M^C is isomorphic to a closed subcategory of the category C^*—Mod of all left modules over C^*. In particular, the C-colinear maps between C-comodules are precisely the C^*-linear maps between them. For the notion of closed subcategory we shall refer to [3, p. 395]. The notation $\mathrm{Com}_C(M, N)$ stands for the k-vector space of all the C-colinear maps between two C-comodules M, N. The category of left C-comodules will be denoted by ${}^C M$. We will use Sweedler's Σ-notation. For example, if M is a right C-comodule, then $\rho_M(m) = \sum_{(m)} m_0 \otimes m_1 \in M \otimes C$ for $m \in M$. The structure of left C^*-module is given by $fm = \sum_{(m)} m_0 f(m_1)$, for $f \in C^*$.

It is not difficult to see that if W is a k-vector space and X is a right C-comodule, then $W \otimes X$ is a right C-comodule with structure map $1_W \otimes \rho_X : W \otimes X \to W \otimes X \otimes C$. Moreover, if W is a right C-comodule, then the structure map $\rho_W : W \to W \otimes C$ becomes C-colinear. Consider coalgebras C and D. Following [9], a C–D-bicomodule M is a left C-comodule and a right D-comodule such that the C-comodule structure map $\rho_M^- : M \to C \otimes M$ is D-colinear or, equivalently, that the D-comodule structure map $\rho_M^+ : M \to M \otimes D$ is C-colinear. Equivalently, if $\rho_M^-(m) = \sum_{(m)} m_{-1} \otimes m_0$ and $\rho_m^+(m) = \sum_{(m)} m_0 \otimes m_1$, then

$$\sum_{(m)} m_{-1} \otimes (m_0)_0 \otimes (m_0)_1 = \sum_{(m)} (m_0)_{-1} \otimes (m_0)_0 \otimes m_1.$$

Furthermore, given a k-coalgebra D and a k-algebra R we can consider the category $\mathcal{M}_R^D$ consisting of the right D-comodules and right R-modules M satisfying the compatibility condition

$$\sum_{(mr)} (mr)_0 \otimes (mr)_1 = \sum_{(m)} m_0 r \otimes m_1$$

for every $m \in M$ and $r \in R$ or, equivalently, that the homothety $h_r : M \to M$, $m \mapsto mr$ is a D-comodule map on M for every $r \in R$. The morphisms in this category are the right D-colinear and right R-linear maps.

The following characterization of the bicomodules will be useful in this paper.

Proposition 1.1. *Let C, D be two coalgebras and consider a k-vector space M such that M is a left C-comodule and a right D-comodule. The following statements are equivalent*

(i) *M is a C–D-bicomodule.*
(ii) *M is a D^*–C^*-bimodule.*
(iii) $M \in \mathcal{M}_{C^*}^D$.
(iv) $M \in {}_{D^*}^{C}\mathcal{M}$.

Proof. (i) $\Rightarrow$ (ii) Let $f \in C^*$, $g \in D^*$ and $m \in M$.

$$(gm)f = \left(\sum_{(m)} m_0 g(m_1)\right) f = \left(\sum_{(m)} m_0 f\right) g(m_1) = \sum_{(m)} f((m_0)_{-1})(m_0)_0 g(m_1)$$

Moreover

$$g(mf) = g\left(\sum_{(m)} f(m_{-1}) m_0\right) = \sum_{(m)} f(m_{-1}) g m_0 = \sum_{(m)} f(m_{-1})(m_0)_0 g((m_0)_1)$$

Since

$$\sum_{(m)} m_{-1} \otimes (m_0)_0 \otimes (m_0)_1 = \sum_{(m)} (m_0)_{-1} \otimes (m_0)_0 \otimes m_1$$

we conclude that $(gm)f = g(mf)$.

(ii) $\Rightarrow$ (i) Let $m \in M$. The k-subspace mC^* of M is finite-dimensional and, thus, $D^*(mC^*)$ is finite-dimensional. Since M is a bimodule, $D^*(mC^*) = (D^*m)C^*$. Let

$\{e_1, \dots, e_n\}$ be a k-basis of this vector space. We will prove that

$$(1 \otimes \rho_M^+) \circ \rho_M^-(e_i) = (\rho_M^- \otimes 1) \circ \rho_M^+(e_i) \tag{I}$$

for every $i = 1, \dots, n$. Put

$$\rho_M^+(e_i) = \sum_j e_j \otimes d_i^j \qquad \rho_M^-(e_i) = \sum_k c_i^k \otimes e_k$$

for $d_i^j \in D$ and $c_i^k \in C$. Choose a k-basis $\{c_1, \dots, c_r\}$ of the k-vector subspace of C spanned by the c_i^k's, for $i, k = 1, \dots, n$. Analogously, let $\{d_1, \dots, d_s\}$ be a k-basis of the k-vector subspace of D spanned by the d_i^j's. After some computations, we obtain

$$(\rho_M^- \otimes 1) \circ \rho_M^-(e_i) = \sum_{h,l} c_h \otimes m_{h,l} \otimes d_l$$

$$(1 \otimes \rho_M^+) \circ \rho_M^-(e_i) = \sum_{h,l} c_h \otimes m'_{h,l} \otimes d_l$$

where $m_{h,l}, m'_{h,l} \in D^* m C^*$. Moreover, for $f \in C^*$ and $g \in D^*$, we can check that

$$(ge_i)f = \sum_{h,l} f(c_h) m_{h,l} g(d_l)$$

$$g(e_i f) = \sum_{h,l} f(c_h) m'_{h,l} g(d_l)$$

It is evident that certain particular choices of $f \in C^*$, $g \in D^*$ give rise to $m_{h,l} = m'_{h,l}$ for every $h = 1, \dots, s$; $l = 1, \dots, r$. Thus, the identity (I) holds.
(ii) $\Rightarrow$ (iii). Let $f \in C^*$. We have to prove that the homothety $h_f : M \to M$ is a morphism of right D-comodules, i.e., that h_f is a morphism of left D^*-modules. But this is true because M is a D^*–C^*-bimodule.
(iii) $\Rightarrow$ (ii) Since $h_f : M \to M$ is a D-comodule map we have

$$\sum_{(m)} f((m_0)_{-1})(m_0)_0 \otimes m_1 = \sum_{(m)} f(m_{-1})(m_0)_0 \otimes (m_0)_1$$

Therefore, for every $g \in D^*$, the equality

$$\sum_{(m)} f((m_0)_{-1})(m_0)_0 g(m_1) = \sum_{(m)} f(m_{-1})(m_0)_0 g((m_0)_1)$$

holds, that is, $(gm)f = g(mf)$.

The proof of (ii)$\Longleftrightarrow$(iv) is similar to that of (ii)$\Longleftrightarrow$(iii). $\square$

We shall recall the concept of cotensor product from [9, 2]. If M is a right C-comodule and N is a left C-comodule, then the *cotensor product* $M\square_C N$ is the kernel of the k-linear map

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \to M \otimes C \otimes N$$

Let C and D be two coalgebras. A *morphism of coalgebras* is a k-linear map $\varphi : C \to D$ such that $\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C$. The morphism of coalgebras φ induces a morphism of k-algebras $\varphi^* : D^* \to C^*$. Let $\varphi : C \to D$ be a morphism of k-coalgebras. Every right C-comodule M with structure map $\rho_M : M \to M \otimes C$ can be considered as a right D-comodule with structure map

$$M \xrightarrow{\rho_M} M \otimes C \xrightarrow{1_M \otimes \varphi} M \otimes D$$

This gives an exact functor $(-)_\varphi : \mathcal{M}^C \to \mathcal{M}^D$ called *co-restriction* functor. In particular, C can be viewed as D-bicomodule and we can also consider the coinduction functor $(-)^\varphi : \mathcal{M}^D \to \mathcal{M}^C$ where $N^\varphi = N\square_D C$ for every right D-comodule N. In fact, $N\square_D C$ is a D-subcomodule of the right D-comodule $N \otimes C$ whose structure map is

$$N \otimes C \xrightarrow{1_N \otimes \Delta_C} N \otimes C \otimes C \xrightarrow{1_N \otimes 1_C \otimes \varphi} N \otimes C \otimes D$$

It is proved in [4] that if M is a right C-comodule then the structure map ρ_M induces a C-colinear map $\overline{\rho_M} : M \to M_\varphi \square_D C$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "\rho_M"] \arrow[dr, "\overline{\rho_M}"'] & & M \otimes C \\ & M_\varphi \square_D C \arrow[ur, "i"'] & \end{tikzcd}$$

is commutative. Taking $M = C$ we obtain a C-bicomodule map $\overline{\Delta}_C : C \to C\square_D C$.

2. Separability of functors over comodules

DEFINITION 2.1. Consider abelian categories $\mathscr{C}$ and $\mathscr{D}$. A covariant functor $F : \mathscr{C} \to \mathscr{D}$ is said to be a *separable functor* (see [5]) if for all objects $M, N \in \mathscr{C}$ there are maps

$$\nu^F_{M,N} : \mathrm{Hom}_{\mathscr{D}}(F(M), F(N)) \to \mathrm{Hom}_{\mathscr{C}}(M, N)$$

satisfying the following *separability conditions*

1. For every $\alpha \in \mathrm{Hom}_{\mathscr{C}}(M,N)$ we have $\nu^F_{M,N}(F(\alpha)) = \alpha$.
2. For $M', N' \in \mathscr{C}$, $f \in \mathrm{Hom}_{\mathscr{D}}(F(M), F(N))$, $g \in \mathrm{Hom}_{\mathscr{D}}(F(M'), F(N'))$, $\alpha \in \mathrm{Hom}_{\mathscr{C}}(M, M')$ and $\beta \in \mathrm{Hom}_{\mathscr{C}}(N, N')$, such that the following diagram is commutative

$$\begin{tikzcd} F(M) \arrow[r, "f"] \arrow[d, "F(\alpha)"] & F(N) \arrow[d, "F(\beta)"] \\ F(M') \arrow[r, "g"] & F(N') \end{tikzcd}$$

then the following diagram is also commutative

$$\begin{tikzcd} M \arrow[r, "\nu^F_{M,N}(f)"] \arrow[d, "\alpha"] & N \arrow[d, "\beta"] \\ M' \arrow[r, "\nu^F_{M',N'}(g)"] & N' \end{tikzcd}$$

In this section we will characterize the separability of the corestriction and coinduction functors defined by a morphism of coalgebras.

Let C, D be k-coalgebras. Let $F : M^C \to M^D$ be a *k-functor*, i.e., the induced map $\mathrm{Com}_C(M,N) \to \mathrm{Com}_D(F(M), F(N))$ is assumed to be a k-linear map. If M is a C-bicomodule, then $M \in M^C_{C^*}$ by Proposition 1.1. For $z \in F(M)$ and $f \in C^*$, define $z \cdot f = F(h_f)(z)$, where $h_f : M \to M$, $m \mapsto h_f(m) = mf$ is a morphism of C-comodules. This implies that $F(h_f)$ is a morphism of D-comodules. Thus, $F(M) \in M^D_{C^*}$.

PROPOSITION 2.2. *Let $F : M^C \to M^D$ be a separable k-functor. Assume that M, N are C-bicomodules and let*

$$\nu_{M,N} : \mathrm{Com}_D(F(M), F(N)) \to \mathrm{Com}_C(M,N)$$

be the map given by the separability conditions. If $\alpha \in \mathrm{Com}_D(F(M), F(N))$ is also a morphism of right C^*-modules, then $\nu_{M,N}(\alpha)$ is a morphism of C-bicomodules.

PROOF. Since α is C^*-linear, the following square is commutative

$$\begin{tikzcd} F(M) \arrow[r, "\alpha"] \arrow[d, "F(h_f)"] & F(N) \arrow[d, "F(h_f)"] \\ F(M) \arrow[r, "\alpha"] & F(N) \end{tikzcd}$$

The separability of F implies that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\nu_{M,N}(\alpha)} & N \\ \Big\downarrow{\scriptstyle h_f} & & \Big\downarrow{\scriptstyle h_f} \\ M & \xrightarrow{\nu_{M,N}(\alpha)} & N \end{array}$$

This means that $\nu_{N,M}(\alpha)$ is a morphism of right C^*-modules, that is, it is a morphism of left C-comodules and, thus, of C-bicomodules. □

We will denote by $f - M^C$ the full subcategory of M^C consisting of the comodules of finite dimension.

PROPOSITION 2.3. *Let $F : M^C \to M^D$ be a left exact k-functor that commutes with direct limits. Assume that $F(f - M^C) \subseteq f - M^D$. The functor F is separable if and only if its restriction $F' : f - M^C \to f - M^D$ is separable.*

PROOF. It is clear that if F is separable then its restriction $F' : f - M^C \to f - M^D$ is separable. Conversely, assume that this last functor is separable. Take $M, N \in M^C$ and $\alpha \in \mathrm{Com}_D(F(M), F(N))$. Write $M = \cup_{i\in I} M_i$, $N = \cup_{j\in J} N_j$, as direct unions of finite-dimensional subcomodules. It is clear that $F(M) = \cup_{i\in I} F(M_i) = \cup_{i\in I} F'(M_i)$ and analogously $F(N) = \cup_{j\in J} F(N_j) = \cup_{j\in J} F'(N_j)$. For every $i \in I$, there is $j \in J$ such that $\alpha(F(M_i)) \subseteq F(N_j)$. Put $\alpha_i = \alpha_{i|F(M_i)}$. Since F' is separable, there is a map

$$\nu_{M_i,N_j} : \mathrm{Com}_D(F'(M_i), F'(N_j)) \to \mathrm{Com}_C(M_i, N_j)$$

which satisfies the separability conditions. Put $\beta_i = \nu_{M_i,N_j}(\alpha_i) : M_i \to N_j$. Consider $i \le i'$ with $i, i' \in I$ and let $\iota_{i,i'}$ denote the inclusion $M_i \le M_{i'}$. There are $j, j' \in J$ with $j \le j'$ such that $\alpha(F(M_i)) \subseteq F(N_j)$ and $\alpha(F(M_{i'})) \subseteq F(N_{j'})$. In other words, the diagram

$$\begin{array}{ccc} F'(M_i) & \xrightarrow{\alpha_i} & F'(N_j) \\ \Big\downarrow{\scriptstyle F'(\iota_{i,i'})} & & \Big\downarrow{\scriptstyle F(\iota_{j,j'})} \\ F'(M_{i'}) & \xrightarrow{\alpha_{i'}} & F'(N_{j'}) \end{array}$$

is commutative, where $\iota_{j,j'}$ denotes the inclusion $N_j \le N_{j'}$. Since F' is separable, we have that the following diagram is commutative

$$\begin{tikzcd} M_i \arrow[r, "\beta_i"] \arrow[d, "\iota_{i,i'}"] & N_j \arrow[d, "\iota_{j,j'}"] \\ M_{i'} \arrow[r, "\beta_{i'}"] & N_{j'} \end{tikzcd}$$

Therefore, we can define $\nu_{M,N}(\alpha) = \varinjlim \beta_i$. Thus, we have defined a map $\nu_{M,N} : \mathrm{Com}_D(F(M), F(N)) \to \mathrm{Com}_C(M, N)$. Now it is a routine matter to check that these maps satisfy the separability conditions, i.e., F is separable. □

Let $r : X \to Y$, $s : Y \to X$ be morphisms of bicomodules such that $r \circ s = 1_Y$. We will say that s is a *splitting monomorphism* of bicomodules and that r is a *splitting epimorphism* of bicomodules. The proof of the following Theorem was performed after [5, Proposition 1.3.(1)].

THEOREM 2.4. *Let $\varphi : C \to D$ be a morphism of coalgebras. The functor $(-)_\varphi : M^C \to M^D$ is separable if and only if the canonical morphism $\overline{\Delta_C} : C \to C\square_D C$ is a splitting monomorphism of C-bicomodules.*

PROOF. Assume that $(-)_\varphi$ is separable and consider the map $p : C\square_D C \to C$ defined as the restriction of the map $C \otimes C \to C$, $c_1 \otimes c_2 \mapsto c_1\varepsilon_C(c_2)$. This p is a morphism of right D-comodules and of right C^*-modules. Let $\phi = \nu_{C\square_D C, C}(p)$, where the map

$$\nu_{C\square_D C, C} : \mathrm{Com}_D((C\square_D C)_\varphi, C_\varphi) \to \mathrm{Com}_C(C\square_D C, C)$$

is given by the separability of $(-)_\varphi$. By Proposition 2.2, ϕ is a morphism of C-bicomodules. Write $\bar{\Delta} = \overline{\Delta_C}$. Now, the diagram

$$\begin{tikzcd} (C\square_D C)_\varphi \arrow[r, "p"] & C_\varphi \\ C_\varphi \arrow[u, "\bar{\Delta}_\varphi"] \arrow[ur, "1"'] & \end{tikzcd}$$

is commutative. Since $(-)_\varphi$ is separable, the diagram

$$\begin{tikzcd} C\square_D C_\varphi \arrow[r, "\phi"] & C \\ C \arrow[u, "\bar{\Delta}"] \arrow[ur, "1"'] & \end{tikzcd}$$

is commutative. Thus, $\bar{\Delta}$ is a splitting monomorphism of C-bicomodules. Assume that there is a morphism of C-bicomodules $\phi : C\square_D C \to C$ such that $\phi \circ \bar{\Delta} = 1_C$. Let $M, N \in M^C$ and $f \in \mathrm{Com}_D(M_\varphi, N_\varphi)$. Define $\tilde{f}$ by the following commutative diagram of right C-comodule maps:

$$\begin{tikzcd}
M\square_D C \arrow[rr, "f\square_D 1"] & & N\square_D C \arrow[d, "\gamma\square_D 1"] \\
M\square_C C\square_D C \arrow[u, "\cong"] & & N\square_C C_{\square} DC \arrow[d, "1\square_D\phi"] \\
M\square_C C \arrow[u, "1\square_D\bar{\Delta}"] & & N\square_C C \arrow[d, "\cong"] \\
M \arrow[u, "\cong"] \arrow[rr, "\tilde{f}"] & & N
\end{tikzcd}$$

where γ denotes the isomorphism $N \cong N\square_C C$. Let $u_M : M \to M\square_D C$, $v_N : N\square_D C \to N$ be the compositions of the vertical maps on the left and on the right in the diagram, respectively. From the condition $\phi \circ \bar{\Delta} = 1_C$ it follows easily that $v_M \circ u_M = 1_M$. Given C-comodules M', N' and $\alpha \in \mathrm{Com}_C(M, M')$, $\beta \in \mathrm{Com}_C(N, N')$ and $g \in \mathrm{Com}_D(M'_\varphi, N'_\varphi)$, consider the diagram

$$\begin{tikzcd}
M\square_D C \arrow[rrr, "f\square_D 1"] \arrow[ddd, "\alpha\square_D 1"'] \arrow[dr, shift left, "v_M"] & & & N\square_D C \arrow[ddd, "\beta\square_D 1"] \arrow[dl, shift left, "v_N"] \\
& M \arrow[ul, shift left, "u_M"] \arrow[r, "\tilde{f}"] \arrow[d, "\alpha"] & N \arrow[ur, shift left, "u_N"] \arrow[d, "\beta"] & \\
& M' \arrow[r, "\tilde{g}"'] \arrow[dl, shift left, "u_{M'}"] & N' \arrow[dr, shift left, "u_{N'}"] & \\
M'\square_D C \arrow[rrr, "g\square_D 1"'] \arrow[ur, shift left, "v_{M'}"] & & & N'\square_D C \arrow[ul, shift left, "v_{N'}"]
\end{tikzcd}$$

Now, it is easy to see that if the outer square is commutative then the inner square is commutative. Moreover, if f is a morphism of right C-comodules, then

$\tilde{f} = f$. Therefore, if we define

$$\nu_{M,N} : \mathrm{Com}_D(M_\varphi, N_\varphi) \to \mathrm{Com}_C(M, N)$$

by $\nu_{M,N}(f) = \tilde{f}$, then the functor $(-)_\varphi$ is separable. □

Following [4], we will say that a morphism of coalgebras φ is a *monomorphism of coalgebras* provided that $\varphi \circ u = \varphi \circ v$, for u, v morphisms of coalgebras, it follows that $u = v$. Although every injective morphism of coalgebras is a monomorphism of coalgebras, both notions are not equivalent.

Corollary 2.5. *If $\varphi : C \to D$ is a monomorphism of coalgebras, then the functor $(-)_\varphi : M^C \to M^D$ is separable.*

Proof. By [4, Theorem 3.5], the map $\overline{\Delta_C} : C \to C \square_D C$ is an isomorphism. By Theorem 2.4, $(-)_\varphi$ is a separable functor. □

Corollary 2.6. *Let $\varphi : C \to D$ be a morphism of coalgebras. If the functor $(-)_\varphi : M^C \to M^D$ is separable and D is a co-semi-simple coalgebras, then C is a co-semi-simple coalgebra.*

Proof. Let M be any right C-comodule. Since D is co-semi-simple, M_φ is completely reducible. By [5, Proposition 1.2.(2)], M is completely reducible and, thus, C is co-semi-simple. □

The proof of the following Theorem was performed after [5, Proposition 1.3.(2)].

Theorem 2.7. *Let $\varphi : C \to D$ be a morphism of k-coalgebras. The functor $(-)^\varphi = - \square_D C : M^D \to M^C$ is separable if and only if φ is a splitting epimorphism of D-bicomodules.*

Proof. Assume that $- \square_D C$ is separable. For $M, N \in M^D$, there exists the map

$$\nu_{M,N} : \mathrm{Com}_C(M \square_D C, N \square_D C) \to \mathrm{Com}_D(M, N)$$

satisfying the separability conditions. Taking $M = D$ and $N = C$, we have $\nu_{D,C}$: $\mathrm{Com}_C(D \square_D C, C \square_D C) \to \mathrm{Com}_D(D, C)$. Now, consider the canonical C-bicolinear map $\overline{\Delta} : C \to C \square_D C$ and define $\overline{\Delta}' : D \square_D C \to C \square_D C$ as $\overline{\Delta}' = \overline{\delta} \circ (\delta_C)^{-1}$, where

$\delta_C : C \to D\Box_D C$ is the canonical isomorphism. Put $\psi = \nu_{D,C}(\bar{\Delta}')$. Since $\bar{\Delta}'$ is a morphism of right D-comodules and of right D^*-comodules, we can apply Proposition 2.2 to obtain that ψ is a morphism of D-bicomodules. On the other hand, the following triangle is commutative

$$\begin{tikzcd} C\Box_D C \arrow[r, "\varphi\Box_D 1"] & D\Box_D C \\ D\Box_D C \arrow[u, "\bar{\Delta}'"] \arrow[ur, "1"'] & \end{tikzcd}$$

By the separability conditions, we deduce that the diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] & D \\ D \arrow[u, "\psi"] \arrow[ur, "1"'] & \end{tikzcd}$$

is commutative. Hence, φ is a splitting epimorphism of D-bicomodules. Conversely, assume that there is a D-bicolinear map $\vartheta : D \to C$ such that $\varphi \circ \vartheta = 1_D$. If $f \in \mathrm{Com}_C(M\Box_D C, N\Box_D C)$, then we define them map $\nu_{M,N} : \mathrm{Com}_C(M\Box_D C, N\Box_D C) \to \mathrm{Com}_D(M,N)$ by putting $\nu_{M,N}(f) = \tilde{f}$, where $\tilde{f}$ makes the following diagram commutative.

$$\begin{tikzcd} M\Box_D C \arrow[r, "f"] & N\Box_D C \arrow[d, "1_N\Box_D\varphi"] \\ M\Box_D D \arrow[u, "1_M\Box_D\vartheta"] & N\Box_D D \arrow[d, "\cong"] \\ M \arrow[u, "\cong"] \arrow[r, "\tilde{f}"] & N \end{tikzcd}$$

A verification shows (as in Theorem 2.4) that the functor $-\Box_D C$ is then separable. $\square$

Lemma 2.8. *Let A, B be subcoalgebras of C. If $\varphi : C \to D$ is a morphism of coalgebras such that $\varphi(A) \cap \varphi(B) = 0$, then $A\Box_D B = 0$.*

Proof. We have the following equalizer

$$A\Box_D B \to A \otimes B \underset{1_A\otimes\rho_B}{\overset{\rho_A\otimes 1_B}{\rightrightarrows}} A \otimes D \otimes B$$

where $\rho_A = (1_A \otimes \varphi) \circ \Delta_A$ and $\rho_B = (\varphi \otimes 1_B) \circ \Delta_B$. If $z \in A \Box_D B$, then $(\rho_A \otimes 1_B)(z) = (1_A \otimes \rho_B)(z)$. Since $\mathrm{Im}(\rho_A \otimes 1_B) \subseteq A \otimes \varphi(A) \otimes B$ and $\mathrm{Im}(1_A \otimes \rho_B) \subseteq A \otimes \varphi(B) \otimes B$, we have that $(\rho_A \otimes 1_B)(z) = (1_A \otimes \rho_B)(z) \subseteq (A \otimes \varphi(A) \otimes B) \cap (A \otimes \varphi(B) \otimes B) = 0$. Thus, $(\rho_A \otimes 1_B)(z) = 0$ and, since $\rho_A \otimes 1_B$ is a monomorphism, $z = 0$. □

Let $\{C_i : i \in I\}$ be a set of coalgebras with structure maps Δ_i, ε_i. The vector space $\oplus C_i$ can be canonically endowed with structure of coalgebra (see e.g. [8, page 50]). Moreover, from a set of morphisms of coalgebras $\{\varphi_i : C_i \to D_i : i \in I\}$, we obtain the coalgebra morphism $\bigoplus \varphi_i : \bigoplus C_i \to \bigoplus D_i$.

THEOREM 2.9. *Let $\varphi_i : C_i \to D_i$ be morphisms of k-coalgebras, $i \in I$.*

1. *If the functor $(-)_{\varphi_i} : M^{C_i} \to M^{D_i}$ is separable for every $i \in I$, then the functor $(-)_{\bigoplus \varphi_i} : M^{\bigoplus C_i} \to M^{\bigoplus D_i}$ is separable.*
2. *If the functor $(-)^{\varphi_i} : M^{D_i} \to M^{C_i}$ is separable for every $i \in I$, then the functor $(-)^{\bigoplus \varphi_i} : M^{\bigoplus D_i} \to M^{\bigoplus C_i}$ is separable.*

PROOF. (1) By Theorem 2.4, for every $i \in I$, there is a morphism of C_i-bicomodules $\phi_i : C_i \Box_{D_i} C_i \to C_i$ such that $\phi \circ \overline{\Delta_i} = 1_{C_i}$. The map $\bigoplus \phi_i : \bigoplus C_i \Box_{D_i} C_i \to \bigoplus C_i$ is a morphism of $\bigoplus C_i$-bicomodules and $\bigoplus \phi_i \circ \bigoplus \overline{\Delta_i} = 1_{\bigoplus C_i}$. If we prove that there is an isomorphism of $\bigoplus C_i$-bicomodules

$$(\bigoplus C_i) \Box_{\bigoplus D_i} (\bigoplus C_i) \cong \bigoplus (C_i \Box_{D_i} C_i),$$

then we can deduce from Theorem 2.4 that the functor $(-)_{\bigoplus \varphi_i}$ is separable. The Lemma 2.8 assures that $C_i \Box_{\bigoplus D_i} C_j = 0$ if $i \neq j$. Therefore,

$$(\bigoplus C_i) \Box_{\bigoplus D_i} (\bigoplus C_i) \cong \bigoplus_{i,j} (C_i \Box_{\bigoplus D_i} C_j) \cong \bigoplus (C_i \Box_{D_i} C_i)$$

(2) By Theorem 2.7, for every $i \in I$ there is a morphism of D_i-bicomodules $\psi_i : D_i \to C_i$ such that $\varphi_i \circ \psi_i = 1_{D_i}$. It is clear that $\bigoplus \varphi_i \circ \bigoplus \psi_i = 1_{\bigoplus D_i}$. Moreover, it is not difficult to see that $\bigoplus \psi_i$ is a morphism of $\bigoplus D_i$-bicomodules. By Theorem 2.7, the functor $(-)^{\bigoplus \varphi_i}$ is separable. □

REMARK 2.10. Theorem 2.9 can be used to construct coalgebra morphisms between infinite-dimensional coalgebras such that the corestriction and the coinduction functors are separable.

3. Applications

Recall that a k-algebra A is said to be separable if the canonical map $A \otimes A \to A$ is a splitting epimorphism of A-bimodules. By [5, Proposition 1.3] this is equivalent to say that the restriction functor $A - \mathrm{Mod} \to k - \mathrm{Mod}$ is separable. In this section we investigate the coseparable coalgebras.

A morphism of k-coalgebras $\varphi : C \to D$ induces a morphism of k-algebras $\varphi^* : D^* \to C^*$. Let us denote by $(-)_{\varphi^*} : C^* - \mathrm{Mod} \to D^* - \mathrm{Mod}$ the functor restriction of scalars. Recall that if C is a finite-dimensional coalgebra, then there is an isomorphism of categories $M^C \cong C^* - \mathrm{Mod}$.

PROPOSITION 3.1. *Let $\varphi : C \to D$ be a morphism of coalgebras. Assume that C and D are finite-dimensional. The following statements are equivalent.*

(i) *The functor $(-)_\varphi : M^C \to M^D$ is separable.*
(ii) *The functor $(-)_{\varphi^*} : C^* - \mathrm{Mod} \to D^* - \mathrm{Mod}$ is separable.*

PROOF. (i)$\Leftrightarrow$(ii) The functorial diagram

$$\begin{array}{ccc} \mathrm{M}^C & \xrightarrow{(-)_\varphi} & \mathrm{M}^D \\ \downarrow & & \downarrow \\ C^* - \mathrm{Mod} & \xrightarrow{(-)_{\varphi^*}} & D^* - \mathrm{Mod} \end{array}$$

where the vertical arrows represent canonical isomorphisms of categories, commutes. This entails that $(-)_\varphi$ is separable if and only if $(-)_{\varphi^*}$ is separable. $\square$

PROPOSITION 3.2. *Let $\varphi : C \to D$ be a morphism of coalgebras.*

1. *The functor $(-)_\varphi : M^C \to M^D$ is separable if and only if the restriction $(-)_\varphi : f - M^C \to f - M^D$ is separable.*

2. *Let $C' \leq C$, $D' \leq D$ be subcoalgebras such that $\varphi(C') \leq D'$, and let us denote by $\varphi' : C' \to D'$ the induced coalgebra map. If the functor $(-)_\varphi$ is separable then the functor $(-)_{\varphi'} : M^{C'} \to M^{D'}$ is separable.*

3. *If C is cosemisimple then the functor $(-)_\varphi$ is separable if and only if for any finite-dimensional subcoalgebras $C' \leq C$ and $D' \leq D$ such that $\varphi(C') \leq D'$, the functor $(-)_{\varphi'}$ is separable.*

PROOF. (1) This follows from Proposition 2.3.

(2) Let C' be a subcoalgebra of C and let us denote by $i : C' \to C$ the inclusion coalgebra map. By [4, Theorem 3.5] the functor $(-)_i : M^{C'} \to M^C$ is separable. For any subcoalgebra D' of D with $\varphi(C') \subseteq D'$ we can consider the commutative diagram

$$\begin{array}{ccc} M^C & \xrightarrow{(-)_\varphi} & M^D \\ {\scriptstyle (-)_i}\uparrow & & \uparrow{\scriptstyle (-)_j} \\ M^{C'} & \xrightarrow{(-)_{\varphi'}} & M^{D'} \end{array}$$

where $j : D' \to D$ is the inclusion map. If $(-)_\varphi$ is separable, then $(-)_\varphi \circ (-)_i = (-)_j \circ (-)_{\varphi'}$ is separable. By [5, Lemma 1.1.(3)], $(-)_{\varphi'}$ is a separable functor.

(3) Assume that C is co-semi-simple. Then $C = \bigoplus C_i$, where the C_i's are simple subcoalgebras. If we put $D_i = \varphi(C_i)$, and we denote by $\varphi_i : C_i \to D_i$ the induced coalgebra morphism, we have by hypothesis that $(-)_{\varphi_i}$ is a separable functor. For every i, there is a D_i-bicomodule morphism $\psi_i : C_i \square_{D_i} C_i \to C_i$ such that $\psi_i \circ \overline{\Delta_{C_i}} = 1_{C_i}$. We have that $C_i \square_{D_i} C_i = C_i \square_D C_i$ and

$$C \square_D C = \bigoplus_i (C_i \square_D C_i) \oplus \bigoplus_{i \neq j} (C_i \square_D C_j)$$

Then the maps $\{\psi_i\}$ give a bicomodule map $\psi : C \square_D C \to C$ if we put $\psi = \bigoplus \psi_i$ on $\bigoplus_i (C_i \square_D C_i)$ and zero on $\bigoplus_{i \neq j} (C_i \square_D C_j)$. Clearly, $\psi \circ \overline{\Delta_C} = 1_C$. By Theorem 2.4, $(-)_\varphi$ is a separable functor. $\square$

A k-coalgebra C is *coseparable* (see [2]) if there exists k-linear map $\tau : C \otimes C \to k$ such that $(I \otimes \tau)(\Delta \otimes I) = (\tau \otimes I)(I \otimes \Delta)$ and $\tau\Delta = \varepsilon$. As it was observed in [2, p. 41], C is coseparable if and only if there exists a C-bicomodule map $\pi : C \otimes C \to C$ such that $\pi\Delta = I$. Now, the counit $\varepsilon : C \to K$ is a morphism of coalgebras and, in this case, $C \square_k C = C \otimes C$. It follows from Theorem 2.4 that C is coseparable if and only if the corestriction functor $(-)_\varepsilon : M^C \to k - \mathrm{Mod}$ is separable. From Corollary 2.6 every coseparable coalgebra is co-semi-simple. This result is also given in [2]. Moreover, from Proposition 3.2.3, if C is co-semi-simple then C is coseparable if and only if any finite-dimensional subcoalgebra of C is coseparable.

If C is a k-coalgebra and $k \subseteq K$ is any field extension, then we can define the K-coalgebra $C \otimes K = C \otimes_k K$, with comultiplication given by

$$\Delta_{C \otimes K} = \Delta_C \otimes 1_K : C \otimes K \to (C \otimes C) \otimes K \cong (C \otimes K) \otimes_K (C \otimes K)$$

and counit given by

$$\varepsilon_{C\otimes K} = \varepsilon \otimes 1_K : C \otimes K \to k \otimes K \cong K$$

With this notation, we can prove the following result.

PROPOSITION 3.3. *Let $\varphi : C \to D$ be a morphism of k-coalgebras. The following statements are equivalent.*

(i) *The functor $(-)_\varphi : M^C \to M^D$ is separable.*
(ii) *For any field extension $k \subseteq K$ the functor $(-)_{\varphi\otimes 1_K} : M^{C\otimes K} \to M^{D\otimes K}$ is separable.*

PROOF. (i) $\Rightarrow$ (ii) By Theorem 2.4, $\overline{\Delta_C}$ is a splitting monomorphism of C-bicomodules. It is not difficult to see that $(C \otimes K)\square_{D\otimes K}(C \otimes K) \cong (C\square_D C) \otimes K$. Therefore, $\overline{\Delta_{C\otimes K}} = \overline{\Delta_C} \otimes 1_K$ is a splitting monomorphism of $C \otimes K$-bicomodules.
(ii) $\Rightarrow$ (i) This is clear. □

The following theorem gives a "classical" interpretation of the notion of coseparable coalgebra.

THEOREM 3.4. *A k-coalgebra C is coseparable if and only if $C \otimes K$ is cosemi-simple for every field extension $k \subseteq K$.*

PROOF. Assume that C is coseparable. By Proposition 3.3, $C \otimes K$ is a K-coalgebra coseparable for every field extension $k \subseteq K$. By Corollary 2.6, $C \otimes K$ is a co-semi-simple coalgebra.

Conversely, assume that $C \otimes K$ is co-semi-simple for every field extension $k \subseteq K$. By Proposition 3.2 we have only to prove that every finite-dimensional subcoalgebra of C is coseparable. Indeed, if $E \leq C$ is a finite-dimensional subcoalgebra then $E \otimes K$ is co-semi-simple by Corollary 2.6. Therefore,

$$\mathrm{Hom}_K(E \otimes K, K) = (E \otimes K)^* \cong E^* \otimes K$$

is a semisimple K-algebra for every field extension $k \subseteq K$. This entails that E^* is a separable k-algebra. By Proposition 3.1, E is a coseparable k-coalgebra. □

REMARK 3.5. If H is a Hopf k-algebra then H is coseparable as k-coalgebra if and only if H is co-semi-simple. This fact follows from Theorem 3.4 and [1, Theorem 3.3.2].

References

[1] Abe, E., "Hopf Algebras", Cambridge University Press, 1977.
[2] Doi, Y., Homological coalgebra, *J. Math. Soc. Japan* **33** (1981), 31–50.
[3] Gabriel, P., Des catégories abeliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
[4] Năstăsescu, C. and Torrecillas, B., Torsion Theories for Coalgebras, *J. Pure Appl. Algebra* **97** (1994), 203–220.
[5] Năstăsescu, C. Van Oystaeyen, F. and Van den Bergh, M., Separable functors applied to graded rings, *J. Algebra* **123** (1989), 397–413.
[6] Rafael, M. D., Separable functors revisited, *Comm. Algebra* **18** (1990), 1445–1459.
[7] Sullivan, J. B., The uniqueness of integrals for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras, *J. Algebra* **19** (1971), 426–440.
[8] Sweedler, M. E., "Hopf Algebras", Benjamin, New York, 1969.
[9] Takeuchi, M., Morita theorems for categories of comodules, *J. Fac. Sci. Univ. Tokyo* **24** (1977), 629–644.

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