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ON THE CHERN-TYPE PROBLEM IN AN INDEFINITE KÄHLER GEOMETRY

By

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§ 1. Introduction

The theory of indefinite complex submanifolds of an indefinite complex space form is one of interesting topics in differential geometry and it is investigated by many geometers from the various different points of view, see [1], [5], [8], [9], [12], [19] and [20] for examples. Romero [18] gave a nice survey in this direction. Their method in [1] and [3] seems to be interesting because they apply the Liouville-type inequality

$$\Delta f \geqq kf$$

for a non-negative function f, where k is positive constant.

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ of constant holomorphic sectional curvature c. Chern pointed out that it is interesting to study the distribution of the values of the squared norm h_2 of the second fundamental form α of M. The first value is of course 0 in the case where M is totally geodesic. The purpose of this paper is to investigate the Chern-type problem in the space-like Kähler geometry. The Chern-type problem in the space-like Kähler geometry can be written as follows

PROBLEM. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. Then does there exists a constant d in such a way that if it satisfies $h_2 > d$, then M is totally geodesic?*

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In this paper, we prove the following

THEOREM. *Let M be an $n(\geqq 3)$-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. If M is not totally geodesic and $p \leqq (1/2)n(n+1)$, then the squared norm h_2 of the second fundamental form α of M satisfies*

$$h_2 \geqq \frac{cnp(n+2)}{2(n+2p)},$$

where the equality holds if and only if M is a complex projective space $CP^n(c/2)$, α is parallel and $p=(1/2)n(n+1)$.

§ 2. Semi-definite Kähler Manifolds

We begin with recalling basic formulas on semi-definite Kähler manifolds. Let M be an $n(\geqq 2)$-dimensional connected semi-definite Kähler manifold equipped with a semi-definite Kähler metric tensor g and almost complex structure J. For the semi-definite Kähler structure $\{g,J\}$, it follows that J is integrable and the index of g is even, say $2s$ $(0 \leqq s \leqq n)$. In the case where the index $2s$ is contained in the range $0<s<n$, the structure $\{g,J\}$ is said to be *indefinite Kähler structure* and, in particular, in the case where $s=0$ or n, it is said to be *Kähler structure*.

In this section, we shall consider M an $n(\geqq 2)$-dimensional connected semi-definite Kähler manifold of index $2s$, $0 \leqq s \leqq n$. Then a local unitary frame field $\{E_j\} = \{E_1, \dots, E_n\}$ on a neighborhood of M can be chosen. This is a complex linear frame which is orthonormal with respect to the semi-definite Kähler metric g of M, that is, $g(E_j, E_k) = \varepsilon_j \delta_{jk}$, where

$$\varepsilon_j = -1 \text{ or } 1 \text{ according as } 0 \leqq j \leqq s \text{ or } s+1 \leqq j \leqq n.$$

Its dual frame field $\{\omega_j\} = \{\omega_1, \dots, \omega_n\}$ consists of complex valued 1-forms of $(1,0)$ on M such that $\omega_j(E_k) = \varepsilon_j \delta_{jk}$ and $\omega_1, \dots, \omega_n$, $\bar{\omega}_1, \dots, \bar{\omega}_n$ are linearly independent. Thus the natural extension g^c of the semi-definite Kähler metric g of M can be expressed as $g^c = 2\sum_j \varepsilon_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{E_j\}$, there exist complex valued forms ω_{ik}, where the indices i and k run over the range $1, \dots, n$. They are usually called *connection forms* on M such that they satisfy the structure equations of M:

$$d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{2.1}$$

$$d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.2}$$

$$\Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{2.3}$$

where $\Omega = (\Omega_{ij})$ (*resp.* $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (*resp.* the components of the semi-definite Riemannian curvature tensor R) of M. The second formula of (2.1) means the skew-Hermitian symmetricity of Ω_{ij}, which is equivalent to the symmetric condition

$$R_{\bar{i}jk\bar{l}} = \bar{R}_{\bar{j}il\bar{k}}.$$

Moreover, the first Bianchi identity implies the further symmetric relations

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}kj\bar{i}} = R_{\bar{l}jk\bar{i}}. \tag{2.4}$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$S = \sum_{i,j} \varepsilon_i \varepsilon_j (S_{i\bar{j}} \omega_i \otimes \bar{\omega}_j + S_{\bar{i}j} \bar{\omega}_i \otimes \omega_j), \tag{2.5}$$

where $S_{i\bar{j}} = \sum_k \varepsilon_k R_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j}$. The scalar curvature K of M is also given by

$$K = 2 \sum_j \varepsilon_j S_{j\bar{j}}. \tag{2.6}$$

The semi-definite Kähler manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{i\bar{j}} = \frac{K}{2n} \varepsilon_i \delta_{ij}.$$

Now, the components $R_{\bar{i}jk\bar{l}m}$ and $R_{\bar{i}jk\bar{l}\bar{m}}$ (*resp.* $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (*resp.* the Ricci tensor S) are obtained by

$$\begin{aligned} \sum_m \varepsilon_m (R_{\bar{i}jk\bar{l}m} \omega_m + R_{\bar{i}jk\bar{l}\bar{m}} \bar{\omega}_m) &= dR_{\bar{i}jk\bar{l}} \\ &\quad - \sum_m \varepsilon_m (R_{\bar{m}jk\bar{l}} \bar{\omega}_{mi} + R_{\bar{i}mk\bar{l}} \omega_{mj} + R_{\bar{i}jm\bar{l}} \omega_{mk} + R_{\bar{i}jk\bar{m}} \bar{\omega}_{ml}), \end{aligned}$$

$$\sum_k \varepsilon_k (S_{i\bar{j}k} \omega_k + S_{i\bar{j}\bar{k}} \bar{\omega}_k) = dS_{i\bar{j}} - \sum_k \varepsilon_k (S_{k\bar{j}} \omega_{ki} + S_{i\bar{k}} \bar{\omega}_{kj}).$$

The second Bianchi formula is given by

$$R_{\bar{i}jk\bar{l}m} = R_{\bar{i}jm\bar{l}k},$$

and hence we have

$$S_{\bar{i}jk} = S_{k\bar{j}i} = \sum_l \varepsilon_l R_{\bar{j}ik\bar{l}l}, \quad K_i = 2\sum_j \varepsilon_j S_{i\bar{j}j},$$

where $dK = \sum_j \varepsilon_j(K_j\omega_j + \bar{K}_j\bar{\omega}_j)$. The components $S_{\bar{i}jkl}$ and $S_{\bar{i}jk\bar{l}}$ of the covariant derivative of $S_{\bar{i}jk}$ are expressed by

$$(2.7) \qquad \sum_l \varepsilon_l(S_{\bar{i}jkl}\omega_l + S_{\bar{i}jk\bar{l}}\bar{\omega}_l) = dS_{\bar{i}jk} - \sum_l \varepsilon_l(S_{\bar{l}jk}\omega_{li} + S_{\bar{i}lk}\bar{\omega}_{lj} + S_{\bar{i}jl}\omega_{lk}).$$

By the exterior differentiation of the definition of $S_{\bar{i}jk}$ and taking account of (2.7), the Ricci formula for the Ricci tensor S is given by

$$S_{\bar{i}jk\bar{l}} - S_{\bar{i}j\bar{l}k} = \sum_m \varepsilon_m(R_{\bar{l}ki\bar{m}}S_{m\bar{j}} - R_{\bar{l}km\bar{j}}S_{i\bar{m}}).$$

A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate*, provided that the restriction $g_x|_P$ of g_x to P is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{u, v\}$ such that $g(u,u)g(v,v) - g(u,v)^2 \neq 0$, and a holomorphic plane spanned by u and Ju is non-degenerate if and only if it contains a vector v such that $g(v,v) \neq 0$. The sectional curvature of the non-degenerate holomorphic plane P spanned by u and Ju is called the *holomorphic sectional curvature*, which is denoted by $H(P) = H(u)$. The semi-definite Kähler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvatures $H(P)$ are constant for all non-degenerate holomorphic planes P and for all points of M. Then M is called a *semi-definite complex space form*, which is denoted by $M_s^n(c')$ provided that it is of constant holomorphic sectional curvature c', of complex dimension n and of index $2s$. The standard models of semi-definite complex space forms are the following three kinds which are given by Barros and Romero [4] and Wolf [21]: the semi-definite complex Euclidean space C_s^n, the semi-definite complex projective space $CP_s^n(c')$ or the semi-definite complex hyperbolic space $CH_s^n(c')$, according as $c' = 0$, $c' > 0$ or $c' < 0$. For any integer s $(0 \leqq s \leqq n)$, it is seen by [4] and [21] that they are only complete, simply connected and connected semi-definite complex space forms of dimension n and of index $2s$.

The Riemannian curvature tensor $R_{\bar{i}jk\bar{l}}$ of $M_s^n(c')$ is given by

$$(2.8) \qquad R_{\bar{i}jk\bar{l}} = \frac{c'}{2}\varepsilon_j\varepsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}).$$

§ 3. Semi-definite Complex Submanifolds

This section is concerned with semi-definite complex submanifolds of a semi-definite Kähler manifold. First of all, some basic formulas for the theory of semi-definite complex submanifolds are prepared.

Let (M', g') be an $(n+p)$-dimensional connected semi-definite Kähler manifold of index $2(s+t)$ $(0 \leqq s \leqq n, 0 \leqq t \leqq p)$ and let M be an n-dimensional connected semi-definite complex submanifold of index $2s$ of M'. Then M is the semi-definite Kähler manifold endowed with the induced metric tensor g. We choose a local unitary frame field $\{E_A\} = \{E_1, \ldots, E_{n+p}\}$ on a neighborhood of M' in such a way that restricted to $M, E_1, \ldots, E_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated:

$$A, B, \ldots = 1, \ldots, n, n+1, \ldots, n+p,$$

$$i, j, \ldots = 1, \ldots, n,$$

$$x, y, \ldots = n+1, \ldots, n+p.$$

With respect to the unitary frame field $\{E_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the Kähler metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_A$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space satisfy the structure equations

$$d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0,$$

$$d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \tag{3.1}$$

$$\Omega'_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D,$$

where $\Omega' = (\Omega'_{AB})$ (*resp.* $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form with respect to the unitary frame field $\{E_A\}$ (*resp.* components of the semi-definite Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced semi-definite Kähler metric tensor g of index $2s$ of M is given by $g = 2\sum_j \varepsilon_j \omega_j \otimes \bar{\omega}_j$. Then $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{E_j\}$, which consists of complex valued 1-forms of type $(1,0)$ on M. Moreover, $\omega_1, \dots, \omega_n$, $\bar{\omega}_1, \dots, \bar{\omega}_n$ are linearly independent, and they are said to be *canonical 1-forms* on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j \varepsilon_j h_{ij}^x \omega_j, \quad h_{ij}^x = h_{ji}^x. \tag{3.3}$$

The quadratic form $\sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x h_{ij}^x \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold M. From the structure equations of M', it follows that the structure equations for M are similarly given by

$$\begin{aligned} &d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{3.4}$$

where $\Omega = (\Omega_{ij})$ (*resp.* $R_{\bar{i}jk\bar{l}}$) denotes the curvature form with respect to the unitary frame field $\{E_i\}$ (*resp.* components of the semi-definite Riemannian curvature tensor R) of M.

Moreover, the following relationships are obtained:

$$\begin{aligned} &d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \\ &\Omega_{xy} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{3.5}$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that we have the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \varepsilon_x h_{jk}^x \bar{h}_{il}^x, \tag{3.6}$$

and by means of (3.1), (3.3) and (3.5), we have

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_j \varepsilon_j h_{kj}^x \bar{h}_{jl}^y.$$

Using (2.5), (2.6) and (3.6), components of the Ricci tensor S and the scalar curvature K of M are given by

$$S_{i\bar{j}} = \sum_k \varepsilon_k R'_{\bar{k}ki\bar{j}} - h^2_{i\bar{j}},$$

(3.7)
$$K = 2\left(\sum_{j,k} \varepsilon_j \varepsilon_k R'_{\bar{k}kj\bar{j}} - h_2\right),$$

where $h^2_{i\bar{j}} = h^2_{\bar{j}i} = \sum_{k,x} \varepsilon_k \varepsilon_x h^x_{ik} \bar{h}^x_{kj}$ and $h_2 = \sum_j \varepsilon_j h^2_{j\bar{j}}$.

Now, components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form of M are given by

$$\sum_k \varepsilon_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) = dh^x_{ij} - \sum_k \varepsilon_k (h^x_{kj}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y \varepsilon_y h^y_{ij}\omega_{xy}.$$

Then, substituting dh^x_{ij} into the exterior derivative of (3.3), we have

$$h^x_{ijk} = h^x_{jik} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -R'_{\bar{x}ij\bar{k}}.$$

Similarly components h^x_{ijkl} and $h^x_{ijk\bar{l}}$ of the covariant derivative of h^x_{ijk} can be defined by

$$\sum_l \varepsilon_l (h^x_{ijkl}\omega_l + h^x_{ijk\bar{l}}\bar{\omega}_l) = dh^x_{ijk} - \sum_l \varepsilon_l (h^x_{ljk}\omega_{li} + h^x_{ilk}\omega_{lj} + h^x_{ijl}\omega_{lk}) + \sum_y \varepsilon_y h^y_{ijk}\omega_{xy}$$

and by the simple calculation the Ricci formula for the second fundamental form are given by

$$h^x_{ijkl} = h^x_{ijlk},$$

$$h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} = \sum_r \varepsilon_r (R_{\bar{l}ki\bar{r}} h^x_{rj} + R_{\bar{l}kj\bar{r}} h^x_{ir}) - \sum_y \varepsilon_y R_{\bar{x}yk\bar{l}} h^y_{ij}.$$

In particular, let the ambient space be an $(n+p)$-dimensional semi-definite complex space form $M^{n+p}_{s+t}(c)$ of constant holomorphic sectional curvature c and of index $2(s+t)$ $(0 \leqq s \leqq n, 0 \leqq t \leqq p)$. Then, from (2.8), (3.6) and (3.7), we get

(3.8)
$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}\varepsilon_j \varepsilon_k (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \varepsilon_x h^x_{jk} \bar{h}^x_{il},$$

(3.9)
$$S_{i\bar{j}} = \frac{(n+1)c}{2}\varepsilon_i \delta_{ij} - h^2_{i\bar{j}}, \quad h^x_{ij\bar{k}} = 0.$$

And hence from (3.8) we obtain

(3.10)
$$h^x_{ijk\bar{l}} = \frac{c}{2}(\varepsilon_k h^x_{ij}\delta_{kl} + \varepsilon_i h^x_{jk}\delta_{il} + \varepsilon_j h^x_{ki}\delta_{jl}) - \sum_{r,y} \varepsilon_r \varepsilon_y (h^x_{ri} h^y_{jk} + h^x_{rj} h^y_{ki} + h^x_{rk} h^y_{ij}) \bar{h}^y_{rl}.$$

Here, we calculate the Laplacian of the squared norm $h_2 = |\alpha|_2$ of the second fundamental form α on M. The Laplacian Δh_2 of the function h_2 is by definition given as

$$\begin{aligned}\Delta h_2 &= 2\sum_{i,j,k,x} \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_x(h^x_{ij}\bar{h}^x_{ij})_{k\bar{k}} \\ &= 2\sum_{i,j,k,x} \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_x(h^x_{ijk\bar{k}}\bar{h}^x_{ij} + h^x_{ijk}\bar{h}^x_{ijk} + h^x_{ij\bar{k}}\bar{h}^x_{ij\bar{k}} + h^x_{ij}\bar{h}^x_{ij\bar{k}k}).\end{aligned}$$

Hence we have by the second equation of (3.9) and (3.10)

$$\Delta h_2 = c(n+2)h_2 - 4h_4 - 2TrA^2 + 2\sum_{i,j,k,x} \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_x h^x_{ijk}\bar{h}^x_{ijk}, \tag{3.11}$$

where $h_4 = \sum_{i,j} \varepsilon_i\varepsilon_j h^2_{i\bar{j}} h^2_{j\bar{i}}$, TrA^2 is the trace of the matrix A^2 and $A = (A^x_y) = \sum_{i,j} \varepsilon_i\varepsilon_j h^x_{ij}\bar{h}^y_{ij}$.

§4. Space-like Complex Submanifolds

Let $M' = CH^{n+p}_p(c)$ be an $(n+p)$-dimensional indefinite complex hyperbolic space of index $2p(>0)$ and let M be an $n(\geqq 2)$-dimensional space-like complex submanifold of M'. First of all, we will estimate the Laplacian of the squared norm h_2 of the second fundamental form. By (3.8), we have

$$R_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_x \varepsilon_x h^x_{jk}\bar{h}^x_{jk} \geqq \frac{c}{2}, \quad j \neq k, \tag{4.1}$$

where we have used the fact that $\varepsilon_x = -1$.

Since M is space-like, the normal space of M is time-like. So, the matrix $H = (h^2_{j\bar{k}})$ is a negative semi-definite Hermitian one and hence all eigenvalues μ_j of H are non-positive real valued functions on M. The matrix $A = (A^x_y)$ is a positive semi-definite Hermitian one and hence all eigenvalues μ_x of A are non-negative real valued functions on M. Thus it is easily seen that

$$\sum_j \mu_j = TrH = h_2, \qquad \sum_x \mu_x = TrA = -h_2,$$

$$h_2^2 \geqq h_4 = \sum_j \mu_j^2 \geqq \frac{1}{n}h_2^2, \tag{4.2}$$

$$h_2^2 \geqq TrA^2 = \sum_x \mu_x^2 \geqq \frac{1}{p}h_2^2.$$

Also from the estimating of the squared norm of

$$\sum_x \left\{ \varepsilon_x h_{jk}^x \bar{h}_{il}^x - \frac{h_2}{n(n+1)} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \right\},$$

it follows that

$$TrA^2 \geqq \frac{2}{n(n+1)} h_2^2, \tag{4.3}$$

where the equality holds if and only if M is a complex space form. By (3.11), (4.2) and (4.3), we have

$$\begin{aligned} \Delta h_2 &\leqq c(n+2)h_2 - 4h_4 - 2TrA^2 \\ &\leqq c(n+2)h_2 - \frac{4}{n(n+1)}(n+2)h_2^2, \end{aligned}$$

where the equality holds if and only if M is a complex space form and the second fundamental form of M is parallel. Let f be a non-negative function defined by $-h_2$. Then the above inequality is reduced to

$$\Delta f \geqq c(n+2)f + \frac{4}{n(n+1)}(n+2)f^2, \tag{4.4}$$

where the equality folds if and only if M is a complex space form and the second fundamental form of M is parallel.

On the other hand, the Laplacian Δh_2 of h_2 is also estimated in the different type by (3.11) and (4.2). That is, we have

$$\Delta h_2 \leqq c(n+2)h_2 - \frac{2}{np}(n+2p)h_2^2,$$

where the equality holds if and only if M is Einstein and the second fundamental form of M is parallel. So, the function f defined by $-h_2$ satisfies

$$\Delta f \geqq c(n+2)f + \frac{2}{np}(n+2p)f^2, \tag{4.5}$$

where the equality holds if and only if M is Einstein and the second fundamental form of M is parallel.

Now, applying the generalized maximum principle due to Omori [16] and Yau [22], Choi, Kwon and Suh [6] proved recently the following theorem.

THEOREM 4.1. *Let M be a complete Riemannian manifold whose Ricci tensor is bounded from below and let F be any polynomial of one variable x with constant* *coefficients* $c_0, \dots, c_{k+1}$ *such that*

$$F(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_k x^{n-k} + c_{k+1},$$

where $n \geqq 2$, $n - k > 0$ *and* $c_0 > c_{k+1}$. *If a* C^2*-function* f *satisfies* $\Delta f \geqq F(f)$, *then we have* $F(\sup f) \leqq 0$.

Owing to the above theorem, we estimate the squared norm h_2 of the second fundamental form α of M.

THEOREM 4.2. *Let* M *be an* $n(\geqq 2)$*-dimensional complete space-like complex submanifold of an* $(n+p)$*-dimensional indefinite complex hyperbolic space* $CH_p^{n+p}(c)$ *of constant holomorphic sectional curvature* c *and of index* $2p(>0)$. *Then the squared norm* h_2 *of the second fundamental form* α *of* M *satisfies*

$$h_2 \geqq \frac{c}{4} n(n+1) \quad \text{if } p \geqq \frac{1}{2} n(n+1),$$

or

$$h_2 \geqq \frac{c}{2(n+2p)} np(n+2) \quad \text{if } p \leqq \frac{1}{2} n(n+1),$$

where both equalities hold if and only if M *is a complex space form* $M^n(c/2)$, α *is parallel and* $p = (1/2)n(n+1)$.

PROOF. We can choose a suitable unitary frame field $\{E_1, \dots, E_n\}$ so that the negative semi-definite Hermitian matrix $H = (h^2_{j\bar{k}})$ can be diagonalized. Then the Ricci curvature $S_{j\bar{j}}$ of M is given by

$$S_{j\bar{j}} = \frac{c}{2}(n+1) - \mu_j,$$

where we have used (3.9) and μ_j is an eigenvalue of the negative semi-definite Hermitian matrix H. Thus the Ricci tensor is bounded from below. Moreover, the non-negative function $f = -h_2$ satisfies the Liouville type inequalities (4.4) and (4.5). If we define a polynomial $F(x)$ by

$$F(x) = \frac{1}{n(n+1)}(n+2)x\{cn(n+1) + 4x\}$$

$$(\textit{resp. } F(x) = \frac{1}{np} x\{cnp(n+2) + 2(n+2p)x\}),$$

then F satisfies conditions of Theorem 4.1. So, we can apply Theorem 4.1 to the function f and hence we obtain

$$F(\sup f) \leqq 0, \quad \text{i.e.,} \ \sup f\{cn(n+1) + 4 \sup f\} \leqq 0$$
$$(\textit{resp.}\ \sup f\{cnp(n+2) + 2(n+2p)\sup f\} \leqq 0).$$

This means that if M is not totally geodesic, then

$$cn(n+1) + 4 \sup f \leqq 0, \quad \text{i.e.,} \ 4h_2 \geqq cn(n+1)$$
$$(\textit{resp.}\ cnp(n+2) + 2(n+2p)\ \sup f \leqq 0, \ \text{i.e.,} \ 2(n+2p)h_2 \geqq cnp(n+2)),$$

where the equality holds if and only if M is a complex space form $M^n(c')$ (*resp.* Einstein) and α is parallel, then, since the scalar curvature K of M is given by

$$K = cn(n+1) - 2h_2. \tag{4.6}$$

Comparing this with (3.9), we see that the first equality holds if and only if $c' = c/2$. On the other hand, the second equality holds if and only if h_2 is a constant $(c/(2(n+2p)))np(n+2)$ and α is parallel. It implies that

$$h_2 = \frac{c}{2(n+2p)} np(n+2) = \frac{c}{4} n(n+1),$$

from which it follows that

$$p = \tfrac{1}{2} n(n+1).$$

It completes the proof. □

REMARK 4.1. Under the same assumption as stated in Theorem 4.2, we get

$$h_2 \geqq \frac{c}{4} n(n+1) \quad \text{and} \quad h_2 \geqq \frac{c}{2(n+2p)} np(n+2).$$

Here, in order to prove our main theorem, we will consider the totally real bisectional curvature of M. A plane section P in the tangent space T_xM of M at any point x in M is said to be *totally real* if P is orthogonal to JP. For the non-degenerate totally real plane P spanned by orthonormal vectors u and v, the *totally real bisectional curvature* $B(u, v)$ is defined by

$$B(u,v) = \frac{g(R(u,Ju)Jv, v)}{g(u,u)g(v,v)}. \tag{4.7}$$

For a space-like complex submanifold, using the first Bianchi identity to (4.7) and fundamental properties of the Riemannian curvature tensor of a space-like complex submanifold, we get

$$B(u,v)=g(R(u,v)v,u)+g(R(u,Jv)Jv,u)=K(u,v)+K(u,Jv), \tag{4.8}$$

where $K(u,v)$ means the sectional curvature of the plane spanned by u and v.

From now on, we suppose that u and v are space-like orthonormal vectors in the non-degenerate totally real plane P. If we put $u'=(1/\sqrt{2})(u+v)$ and $v'=(1/\sqrt{2})(u-v)$, then it is easily seen that

$$g(u',u')=1, \quad g(v',v')=1, \quad g(u',v')=0.$$

Thus we get

$$B(u',v')=g(R(u',Ju')Jv',v')=\tfrac{1}{4}\{H(u)+H(v)+2B(u,v)-4K(u,Jv)\},$$

where $H(u)=K(u,Ju)$ means the holomorphic sectional curvature of the holomorphic plane spanned by u and Ju. Hence we have

$$4B(u',v')-2B(u,v)=H(u)+H(v)-4K(u,Jv). \tag{4.9}$$

If we put $u''=(1/\sqrt{2})(u+Jv)$ and $v''=(1/\sqrt{2})(Ju+v)$, then we get

$$g(u'',u'')=1, \quad g(v'',v'')=1, \quad g(u'',v'')=0.$$

Using the similar method as in (4.9), we have

$$4B(u'',v'')-2B(u,v)=H(u)+H(v)-4K(u,v). \tag{4.10}$$

Summing up (4.9) and (4.10) and taking account of (4.8), we obtain

$$2B(u',v')+2B(u'',v'')=H(u)+H(v). \tag{4.11}$$

In the sequel, let $b(M)$ or $a(M)$ be the supremum or the infimum of the set B of totally real bisectional curvatures on M. Suppose that the totally real bisectional curvature is bounded from above (*resp.* below) by a constant b (*resp.* a). From (4.11), it follows that

$$H(u)+H(v)\leqq 4b \quad (\textit{resp.}\ \geqq 4a). \tag{4.12}$$

We can choose a unitary frame field $\{E_1,E_2,\dots,E_n\}$ on a neighborhood of M. With respect to this unitary frame field, let $\{\omega_1,\omega_2,\dots,\omega_n\}$ be a dual frame field. The holomorphic sectional curvature $H(E_j)$ of the holomorphic plane defined by E_j is given by

$$H(E_j)=g(R(E_j,\bar{E}_j)\bar{E}_j,E_j)=R_{\bar{j}j\bar{j}j}.$$

On the other hand, it is easily seen that the plane spanned by E_j and E_k $(j\neq k)$

is totally real and the totally real bisectional curvature $B(E_j, E_k)$ is given by

$$B(E_j, E_k) = g(R(E_j, \bar{E}_j)\bar{E}_k, E_k) = R_{\bar{j}jk\bar{k}}, \quad j \neq k. \tag{4.13}$$

From the inequality (4.12) for $u = E_j$ and $v = E_k$, we have

$$R_{\bar{j}jj\bar{j}} + R_{\bar{k}kk\bar{k}} \leqq 4b \ (resp. \ \geqq 4a), \quad j \neq k. \tag{4.14}$$

Thus we have

$$\sum_{j<k}(R_{\bar{j}jj\bar{j}} + R_{\bar{k}kk\bar{k}}) \leqq 2bn(n-1) \ (resp. \ \geqq 2an(n-1)), \tag{4.15}$$

which implies that

$$\sum_j R_{\bar{j}jj\bar{j}} \leqq 2bn \ (resp. \ \geqq 2an), \tag{4.16}$$

where the equality holds if and only if $R_{\bar{j}jj\bar{j}} = 2b$ (*resp.* $= 2a$) for any index j.

Since the scalar curvature K is given by

$$K = 2\sum_{j,k} R_{\bar{j}jk\bar{k}} = 2\left(\sum_j R_{\bar{j}jj\bar{j}} + \sum_{j\neq k} R_{\bar{j}jk\bar{k}}\right),$$

we have by (4.15)

$$K \leqq 2\sum_j R_{\bar{j}jj\bar{j}} + 2bn(n-1) \ \left(resp. \ \geqq 2\sum_j R_{\bar{j}jj\bar{j}} + 2an(n-1)\right),$$

from which we have

$$\sum_j R_{\bar{j}jj\bar{j}} \geqq \frac{K}{2} - bn(n-1) \ \left(resp. \ \leqq \frac{K}{2} - an(n-1)\right), \tag{4.17}$$

where the equality holds if and only if $R_{\bar{j}jk\bar{k}} = b$ (*resp.* $= a$) for any distinct indices j and k. In this case, M is locally congruent to $M^n(b)$(*resp.* $M^n(a)$) due to Houh [7]. Also (4.14) gives us $\sum_{j\neq k}(R_{\bar{j}jj\bar{j}} + R_{\bar{k}kk\bar{k}}) \leqq 4b(n-1)$ (*resp.* $\geqq 4a(n-1)$), so that

$$(n-2)R_{\bar{j}jj\bar{j}} + \sum_k R_{\bar{k}kk\bar{k}} \leqq 4b(n-1) \ (resp. \ \geqq 4a(n-1)).$$

From this together with (4.17), it follows that we have

$$(4.18)\qquad \begin{gathered}(n-2)R_{\bar{j}j\bar{j}j} \leqq b(n-1)(n+4) - \frac{K}{2} \\ \left(resp.\ \geqq a(n-1)(n+4) - \frac{K}{2}\right),\end{gathered}$$

for any index j, so that the holomorphic sectional curvature $R_{\bar{j}j\bar{j}j}$ is bounded from above (*resp.* below) for $n \geqq 3$. Moreover, the equality holds for some index j if and only if M is locally congruent to $M^n(2b)(resp.\ M^n(2a))$.

Since the Ricci curvature $S_{j\bar{j}}$ is given by

$$S_{j\bar{j}} = R_{\bar{j}j\bar{j}j} + \sum_{k \neq j} R_{\bar{j}jk\bar{k}},$$

we have by (4.13)

$$S_{j\bar{j}} \leqq R_{\bar{j}j\bar{j}j} + b(n-1) \quad (resp.\ \geqq R_{\bar{j}j\bar{j}j} + a(n-1))$$

and hence, from (4.18), we get

$$(4.19)\qquad \begin{gathered}S_{j\bar{j}} \leqq \frac{1}{2(n-2)}\{4b(n-1)(n+1) - K\} \\ \left(resp.\ \geqq \frac{1}{2(n-2)}\{4a(n-1)(n+1) - K\}\right).\end{gathered}$$

On the other hand, using (4.19), we get

$$\begin{aligned}K &= 2S_{j\bar{j}} + 2\sum_{k \neq j} S_{k\bar{k}} \\ &\leqq 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4b(n-1)(n+1) - K\} \\ &\left(resp.\ \geqq 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4a(n-1)(n+1) - K\}\right),\end{aligned}$$

and hence we have

$$(4.20)\qquad \begin{gathered}S_{j\bar{j}} \geqq \frac{1}{2(n-2)}\{(2n-3)K - 4b(n-1)^2(n+1)\} \\ \left(resp.\ \leqq \frac{1}{2(n-2)}\{(2n-3)K - 4a(n-1)^2(n+1)\}\right).\end{gathered}$$

Combining this with (4.18) and (4.20), we get

$$\varepsilon_k R_{\bar{j}jk\bar{k}} \geqq \frac{1}{n-2}\{(n-1)K - (2n^3 - 3n + 2)b\} \tag{4.21}$$
$$\left(resp. \leqq \frac{1}{n-2}\{(n-1)K - (2n^3 - 3n + 2)a\}\right)$$

for any distinct indices j and k.

First of all, before we estimate the supremum of B, we treat here the infimum $a(M)$.

THEOREM 4.3. *Let M be an $n(\geqq 3)$-dimensional complete space-like complex submanifold of $CH_p^{n+p}(c)$, $p > 0$. Then we have*

$$(1) \quad a(M) \leqq \frac{c}{4},$$

$$(2) \quad a(M) \leqq \frac{c}{2(n+1)(n+2p)} n(n+p+1).$$

PROOF. Since the totally real bisectional curvatures are bounded from below by (4.1), there exists a constant a such that

$$R_{\bar{j}jk\bar{k}} \geqq a \quad \text{for any } j, k (j \neq k).$$

Hence, by (4.16), (4.17) and (4.6), we have

$$2an \leqq \sum_j R_{\bar{j}jj\bar{j}} \leqq \frac{c}{2} n(n+1) - h_2 - an(n-1).$$

Thus we get

$$2h_2 \leqq (c - 2a)n(n+1). \tag{4.22}$$

From the estimate of h_2 in Theorem 4.2 together with (4.22), it follows that we get

$$(4a - c)n(n+1) \leqq 0.$$

It completes the proof of the first assertion.

Also, from Theorem 4.2 and (4.22), we can easily prove the second assertion.

□

REMARK 4.2. (1) The above first assertion is essentially proved by Ki and Suh [10]. But their one is unfortunately incomplete in order to apply another Liouville-type theorem, so the gap is here recovered.

(2) Theorem 4.3 can be restated by the following

$$a(M) \leqq \frac{c}{4} \quad \text{if } p \geqq \frac{1}{2}n(n+1),$$

$$a(M) \leqq \frac{c}{2(n+1)(n+2p)}n(n+p+1) \quad \text{if } p \leqq \frac{1}{2}n(n+1).$$

Next, we estimate the supremum $b(M)$ of the totally real bisectional curvatures of the space-like complex submanifold M.

THEOREM 4.4. *Let M be an $n(\geqq 3)$-dimensional complete space-like submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. Then the supremum $b(M)$ of the totally real bisectional curvatures of M satisfies*

$$b(M) < -\frac{c}{2(n-2)}(n^3 - 2n + 2).$$

PROOF. By Remark 4.1, it is seen that the squared norm h_2 of the second fundamental form of M is restricted by

$$0 \geqq h_2 \geqq \frac{c}{4}n(n+1), \tag{4.23}$$

where the second equality holds if and only if M is a complex space form $M^n(c/2)$ and the second fundamental form of M is parallel. By (4.21), we see that any totally real bisectional curvature $R_{\bar{j}jk\bar{k}}(j \neq k)$ satisfies

$$R_{\bar{j}jk\bar{k}} \leqq \frac{1}{n-2}\{(n-1)K - (2n^3 - 3n + 2)a(M)\}, \tag{4.24}$$

where the equality holds if and only if $a(M) = c/4$. By the definition of $b(M)$, we have

$$b(M) \leqq \frac{1}{n-2}\{(n-1)K - (2n^3 - 3n + 2)a(M)\}.$$

Together with (4.6) and the result $a(M) \geqq c/2$ by (4.1), we obtain

$$b(M) \leqq \frac{c}{2} - \frac{2}{n-2}(n-1)h_2. \tag{4.25}$$

where the equality holds if and only if $a(M) = c/2$. From (4.23) and (4.25), it turns out to be

$$b(M) \leqq -\frac{c}{2(n-2)}(n^3 - 2n + 2).$$

By conditions for the equalities of (4.24) and (4.25), we have the conclusion. □

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