

## ON EXPONENTIAL SUMS OVER PRIMES IN ARITHMETIC PROGRESSIONS

By

Hiroshi MIKAWA

### 1. Introduction

I. M. Vinogradov's proof of the ternary Goldbach problem is based upon bounds for the exponential sum

$$(1) \quad \sum_{n \leq x} \Lambda(n) e(\alpha n)$$

with a wide uniformity in real  $\alpha$ , where  $\Lambda$  is the von Mangoldt function and, for real  $\theta$ ,  $e(\theta) = \exp(2\pi i\theta)$ . By using a combinatorial identity, R. C. Vaughan presented an elegant simple argument on it, see [2], for instance.

J.-r. Chen's theorem on the binary Goldbach problem is built upon the linear sieve and the mean prime number theorem, vide [5]. According to H. Iwaniec [6], the Rosser's weight of the linear sieve has the well-factorable property. An arithmetic function  $\lambda$  is called "well-factorable of level  $D$ ", if for any  $D_1, D_2 \geq 1$ ,  $D = D_1 D_2$ , there exist two functions  $\lambda_1$  and  $\lambda_2$  supporting in  $(0, D_1]$  and  $(0, D_2]$  respectively such that  $|\lambda_1| \leq 1$ ,  $|\lambda_2| \leq 1$  and  $\lambda = \lambda_1 * \lambda_2$ . Also the mean prime number theorem has been surprisingly developed by E. Fouvry and H. Iwaniec [4], E. Fouvry [3] and E. Bombieri, J.-B. Friedlander and H. Iwaniec [1]. In [1] they established a non-trivial bound of the averaging sum

$$(2) \quad \sum_{(d,c)=1} \lambda(d) \left( \sum_{\substack{n \leq x \\ n \equiv c \pmod{d}}} \Lambda(n) - \frac{x}{\varphi(d)} \right)$$

for any fixed integer  $c \neq 0$  and for any well-factorable function  $\lambda$  of level  $D = x^{4/7-\varepsilon}$ ,  $\varepsilon > 0$ .

Recently D. I. Tolev mixed the ternary problem with the binary problem, and was led to a blend of (1) and (2):

$$(3) \quad \sum_{\substack{d \leq D \\ (d,c)=1}} \gamma(d) \sum_{\substack{n \leq x \\ n \equiv c \pmod{d}}} \Lambda(n)e(\alpha n).$$

In [8] he successfully estimated (3) with a wide uniformity in  $\alpha$ , providing that  $\gamma \ll 1$  and  $D = x^{1/3}(\log x)^{-B}$  where  $B > 0$  is some constant. As the sequence  $\gamma$  is regarded as sieving weights, it is of some interest to extend the level of distribution  $D$  in (3). Thus the purpose of this paper is to show that, if  $\gamma$  is well-factorable, then the above exponent  $1/3$  may be replaced by  $4/9$ .

**THEOREM.** *Suppose that  $|\alpha - a/q| \leq q^{-2}$  with  $(a, q) = 1$ . Let  $c \neq 0$  be an integer. Let  $B > 0$  be given. Then, for any well-factorable function  $\lambda$  of level  $D = x^{4/9}(\log x)^{-B}$ , we have that*

$$\sum_{(d,c)=1} \lambda(d) \sum_{\substack{n \leq x \\ n \equiv c \pmod{d}}} \Lambda(n)e(\alpha n) \ll x^{7/8}(xq^{-1} + x(\log x)^{-4B} + q)^{1/8}(\log x)^{13}$$

where the implied constant depends only on  $B$ .

This assertion would be applicable to the problems of [7, 8, 9] and capable to make a modest improvement upon these results. As well as [8], our argument is elementary.

The notation of this paper is standard in Number Theory. Although the symbol  $\|\cdot\|$  is used in two different meanings, there would be no confusion. For real  $\theta$ ,  $\|\theta\|$  is the distance from  $\theta$  to the nearest integer. For sequence  $a = (a(n))$ ,  $\|a\|$  stands for the  $l^2$ -norm.  $n \sim N$  means that  $N < n \leq cN$  with some constant  $0 < c \leq 2$ . We use the abbreviation  $L = \log x$ .

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### 2. Proof of Theorem

We may assume that  $q \leq x$ , for otherwise our assertion is trivial. We choose the parameters of the well-factorable property as  $D_1 = x^{1/3}L^{-B}$  and  $D_2 = x^{1/9}$ , so that  $D = D_1D_2 = x^{4/9}L^{-B}$ . By a dyadic decomposition of summation ranges, it is sufficient to show that

$$(4) \quad P := \sum_{\substack{m \sim M \\ (m,c)=1}} \sum_{\substack{n \sim N \\ (n,c)=1}} f(m)g(n) \sum_{\substack{k \leq x \\ k \equiv c \pmod{mn}}} \Lambda(k)e(\alpha k) \ll x^{7/8}(xq^{-1} + xL^{-4B} + q)^{1/8}L^{11}$$

uniformly for

$$(5) \quad 1 \ll M \ll x^{1/3}L^{-B}, \quad 1 \ll N \ll x^{1/9}; \quad f \ll 1, \quad g \ll 1.$$

We next decompose  $\Lambda$  by means of the combinatorial identity of R. C. Vaughan. We take the parameters in [2, §24] as  $U = V = x^{1/3}$ . Then  $\Lambda$  is written as the sum of  $\Lambda_0$  and  $\Lambda_{ij}$ 's,  $1 \leq i, j \leq 2$ , where

$$\Lambda_0(k) = \begin{cases} \Lambda(k) & k \leq x^{1/3} \\ 0 & \text{otherwise;} \end{cases}$$

$$\Lambda_{1j}(k) = \sum_{\substack{th=k \\ t \leq x^{1/3}}} a_j(t)l_j(h); \quad \Lambda_{2j}(k) = \sum_{\substack{th=k \\ x^{1/3} < t, h \leq x^{2/3}}} b_j(t)d_j(h)$$

with  $a_1(n) = b_1(n) \ll \log n$ ,  $a_2(n) \ll 1$ ,  $l_1(n) = d_1(n) = 1$ ,  $l_2(n) = \log n$ ,  $b_2(n) \ll \Lambda(n)$  and  $d_2(n) \ll \tau(n)$ .

The contribution of  $\Lambda_0$  to  $P$  is at most

$$\sum_{m \sim M} \sum_{n \sim N} L \left( \frac{x^{1/3}}{mn} + 1 \right) \ll (x^{1/3} + MN)L \ll x^{1/2}.$$

Let  $Q_{ij}$  be the partial sum of  $P$  corresponding to  $\Lambda_{ij}$ ,  $1 \leq i, j \leq 2$ . Then

$$(6) \quad P \ll x^{1/2} + \sum_{i=1,2} \sum_{j=1,2} |Q_{ij}|.$$

We first consider the ‘‘type I’’ sum  $Q_{1j}$ ,  $j = 1, 2$ . Since  $l_1(h) = 1$ , we see that

$$Q_{11} \ll \sum_{\substack{m \sim M \\ (m, c)=1}} \sum_{\substack{n \sim N \\ (n, c)=1}} \sum_{t \leq x^{1/3}} |a_1(t)| \left| \sum_{\substack{th \leq x \\ th \equiv c \pmod{mn}}} e(\alpha th) \right|.$$

The above congruence is soluble if and only if  $(t, mn) = 1$ , and equivalent to  $h \equiv r \pmod{mn}$  with some  $r$ . Writing  $h = r + mnk$ , we change the variable  $h$  for  $k$ . Then  $k$  runs through some interval of length  $\leq x(tmn)^{-1}$ . Here we note that  $tmn \ll x^{1/3}MN \ll x$  or  $x(tmn)^{-1} \gg 1$ . Hence we have that

$$(7) \quad Q_{11} \ll L \sum_m \sum_n \sum_t \left| \sum_k e(\alpha tmnk) \right|$$

$$\ll L \sum_{\substack{m \sim M \\ (m, ct)=1}} \sum_{\substack{n \sim N \\ (n, ct)=1}} \sum_{t \leq x^{1/3}} \min \left( \frac{x}{tmn}, \frac{1}{\|\alpha tmn\|} \right)$$

$$\ll L \sum_{k \ll MNx^{1/3}} \tau_3(k) \min \left( \frac{x}{k}, \frac{1}{\|\alpha k\|} \right)$$

$$\begin{aligned} &\ll L \left( \sum_k \tau_3(k)^2 \frac{x}{k} \right)^{1/2} \left( \sum_k \min \left( \frac{x}{k}, \frac{1}{\|\alpha k\|} \right) \right)^{1/2} \\ &\ll L(xL^9)^{1/2} ((xq^{-1} + MNx^{1/3} + q)L)^{1/2} \\ &\ll x^{1/2} (xq^{-1} + MNx^{1/3} + q)^{1/2} L^6 \end{aligned}$$

by Cauchy’s inequality and [2, §25, (3)]. The estimation of  $Q_{12}$  is similar.

We proceed to the “type II” sum  $Q_{2j}$ ,  $j = 1, 2$ . Put

$$\begin{aligned} R &= R(M, N, U, V; f, g, r, s) \\ &= \sum_{\substack{m \sim M \\ (m, c)=1}} \sum_{\substack{n \sim N \\ (n, c)=1}} f(m)g(n) \sum_{u \sim U} \sum_{\substack{v \sim V \\ uv \leq x \\ uv \equiv c \pmod{mn}}} r(u)s(v)e(\alpha uv). \end{aligned}$$

By a dyadic decomposition of summation ranges, we find that

$$(8) \quad |Q_{21}| + |Q_{22}| \ll L^2 \sup |R|$$

where the supremum is taken over all parameters  $M, N, U, V$  and all sequences  $f, g, r, s$  satisfying (5) and

$$(9) \quad x^{1/3} \ll U, \quad V \ll x^{2/3}; \quad r(k) \ll \log k, \quad s(k) \ll \tau(k).$$

In the next section we shall show that

$$(10) \quad |R|^2 \ll \|r\|^2 \|s\|^2 x^{3/4} (xq^{-1} + xL^{-4B} + q)^{1/4} L^{13}$$

uniformly. We here note that, by symmetry, we may assume

$$(11) \quad V \ll U.$$

Therefore, since  $\|r\|^2 \|s\|^2 \ll xL^5$ , (4) follows from (6), (7), (8) and (10). Our proof of Theorem is thus reduced to the estimation (10) for  $R$  under the conditions (5), (9) and (11).

### 3. Type II Sum

In order to show (10), we first arrange  $R$  in the following three ways:

$$\sum_u \left| \sum_m \sum_n \sum_v \right|; \quad \sum_u \sum_m \left| \sum_n \sum_v \right|; \quad \sum_u \sum_m \sum_n \left| \sum_v \right|.$$

We then examine each of these, and compare the three resulting bounds for  $R$ .

We begin by taking the second way. It follows from Cauchy's inequality that

$$(12) \quad |R|^2 \leq \|r\|^2 M \sum_{u \sim U} \sum_{\substack{m \sim M \\ (m,c)=1}} \left| \sum_{\substack{n \sim N \\ (n,c)=1}} g(n) \sum_{\substack{v \sim V \\ uv \leq x \\ uv \equiv c \pmod{mm}}} s(v) e(\alpha uv) \right|^2$$

$$= \|r\|^2 MS, \text{ say.}$$

We expand the square is  $S$  and bring the sum over  $u$  inside to obtain

$$S = \sum_{\substack{m \sim M \\ (m,c)=1}} \sum_{\substack{n_1 \sim N \\ (n_1,c)=1}} \sum_{\substack{n_2 \sim N \\ (n_2,c)=1}} g(n_1) \overline{g(n_2)} \sum_{v_1 \sim V} \sum_{v_2 \sim V} s(v_1) \overline{s(v_2)} \sum_{\substack{u \sim U \\ uv_1, uv_2 \leq x \\ uv_1 \equiv c \pmod{mn_1} \\ uv_2 \equiv c \pmod{mn_2}}} e(\alpha u(v_1 - v_2)).$$

The above simultaneous congruences are soluble if and only if  $(v_1, mn_1) = (v_2, mn_2) = 1$  and  $v_1 \equiv v_2 \pmod{m(n_1, n_2)}$ , and reduce to the single equation  $u \equiv b \pmod{m[n_1, n_2]}$  with some  $b$ . Writing  $v_1 = v_2 + m(n_1, n_2)k$  and  $u = b + m[n_1, n_2]l$ , we change the variables  $(v_1, v_2, u)$  for  $(k, v, l)$ . Then we see that

$$(13) \quad |m(n_1, n_2)k| = |v_1 - v_2| \leq V,$$

and that  $l$  runs through some interval of length  $\leq U(m[n_1, n_2])^{-1}$ . Also

$$\begin{aligned} u(v_1 - v_2) &= (b + m[n_1, n_2]l)m(n_1, n_2)k \\ &= bm(n_1, n_2)k + m^2n_1n_2kl. \end{aligned}$$

Hence we have that

$$S \ll \sum_m \sum_{n_1} \sum_{n_2} \sum_k \sum_v |s(v + m(n_1, n_2)k)| |s(v)| \left| \sum_l e(\alpha m^2 n_1 n_2 kl) \right|.$$

The terms with  $k = 0$  contribute

$$(14) \quad \sum_m \sum_{n_1} \sum_{n_2} \|s\|^2 \sum_l 1 \ll \|s\|^2 \sum_{m \sim M} \sum_{n_1 \sim N} \sum_{n_2 \sim N} \left( \frac{U}{m[n_1, n_2]} + 1 \right)$$

$$\ll \|s\|^2 (UL^3 + MN^2).$$

As for the terms with  $k \neq 0$ , we may assume  $k > 0$ . Put  $n_1 n_2 k = j$ . Then, by (13), the condition on  $j$  becomes

$$0 < mj = mn_1 n_2 k = [n_1, n_2] m(n_1, n_2) k \ll N^2 V.$$

Also the trivial bound for the sum over  $l$  is

$$\ll \frac{U}{m[n_1, n_2]} + 1 = \frac{Um(n_1, n_2)k}{m^2 n_1 n_2 k} + 1 \ll \frac{UV}{m^2 j} + 1 \ll \frac{x}{m^2 j} + 1.$$

Moreover the sum over  $v$  is  $O(\|s\|^2)$  because of  $ab \ll a^2 + b^2$ . Hence the sum under consideration is bounded by

$$(15) \quad \sum_m \sum_j \tau_3(j) \|s\|^2 \left| \sum_l e(\alpha m^2 j l) \right| \ll \|s\|^2 \sum_{m \sim M} \sum_{mj \ll N^2 V} \tau_3(j) \min\left(\frac{x}{m^2 j} + 1, \frac{1}{\|\alpha m^2 j\|}\right).$$

Here we note that  $\min(a + 1, b) \leq \min(a, b) + 1$ . Thus, substituting (14) and (15) into (12), we have that

$$(16) \quad |R|^2 \ll \|r\|^2 \|s\|^2 \cdot \left\{ M \sum_{m \sim M} \sum_{mj \ll N^2 V} \tau_3(j) \min\left(\frac{x}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right) + MUL^3 + M^2 N^2 + MN^2 VL^3 \right\}.$$

Now, in the above double sum, we split up the summation range for  $j$ . We then appeal to

LEMMA. For  $0 < M, J \leq x$ , we have that

$$G := M \sum_{m \sim M} \sum_{j \sim J} \tau_3(j) \min\left(\frac{x}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right) \ll M^2 J L^3 + x^{3/4} (xq^{-1} + xM^{-1} + q)^{1/4} L^8.$$

We put our proof of this lemma off until the next section. Therefore, through the second way, we reach the following estimation.

$$(17) \quad |R|^2 \ll \|r\|^2 \|s\|^2 L^9 \{x^{3/4} (xq^{-1} + xM^{-1} + q)^{1/4} + MU + M^2 N^2 + MN^2 V\} = \|r\|^2 \|s\|^2 L^9 Y, \text{ say.}$$

Next, turning back to the begining, we take the third way. In place of the form  $\sum_u \sum_m |\sum_n \sum_v|$ , our starting point is now  $\sum_u \sum_m \sum_n |\sum_v|$ . Then, by the similar argument as above, we get the similar bound to (16), in which the pair of parameters  $(M, N)$  is replaced by  $(MN, 1)$ . We thus have that

(18)

$$\begin{aligned}
 |R|^2 &\ll \|r\|^2 \|s\|^2 L^3 \left\{ MN \sum_{d \sim MN} \sum_{dj < V} \min\left(\frac{x}{d^2 j}, \frac{1}{\|\alpha d^2 j\|}\right) + MNU + M^2 N^2 + MNV \right\} \\
 &\ll \|r\|^2 \|s\|^2 L^{12} \{x^{3/4}(xq^{-1} + x(MN)^{-1} + q)^{1/4} + MNU + M^2 N^2 + MNV\} \\
 &= \|r\|^2 \|s\|^2 L^{12} Z, \text{ say,}
 \end{aligned}$$

by Lemma again.

Finally we take the first way. Restarting from  $\sum_u |\sum_m \sum_n \sum_v|$ , we argue as before. We then have the similar estimation to (16), replacing  $(M, N)$  by  $(1, MN)$ . Hence we see that

$$|R|^2 \ll \|r\|^2 \|s\|^2 \left\{ \sum_{h \ll M^2 N^2 V} \tau_5(h) \min\left(\frac{x}{h}, \frac{1}{\|\alpha h\|}\right) + UL^8 + M^2 N^2 + M^2 N^2 V \right\}.$$

The square of the above sum over  $h$  is at most

$$\sum_{k \ll M^2 N^2 V} \tau_5(k)^2 \frac{x}{k} \sum_{h \ll M^2 N^2 V} \min\left(\frac{x}{h}, \frac{1}{\|\alpha h\|}\right) \ll x(xq^{-1} + M^2 N^2 V + q)L^{26},$$

by Cauchy's inequality and [2, §25, (3)]. Hence, going through the first way, we get that

$$\begin{aligned}
 (19) \quad |R|^2 &\ll \|r\|^2 \|s\|^2 L^{13} \{x^{1/2}(xq^{-1} + M^2 N^2 V + q)^{1/2} + U + M^2 N^2 + M^2 N^2 V\} \\
 &= \|r\|^2 \|s\|^2 L^{13} X, \text{ say.}
 \end{aligned}$$

In conjunction with (17), (18) and (19), we conclude that

$$(20) \quad |R|^2 \ll \|r\|^2 \|s\|^2 L^{13} \min(X, Y, Z).$$

Now we recall the conditions (5), (9) and (11). It follows from (17) and (18) that

$$\min(Y, Z) \ll x^{3/4}(xq^{-1} + xM^{-1} + q)^{1/4} + M^2 N^2 + \min(MU + MN^2 V, MNU)$$

since  $\min(a + b, a + c) = a + \min(b, c)$ ,  $N \gg 1$  and  $V \ll U$ . The above last term is

$$\begin{aligned}
 &\leq MU + \min(MN^2 V, MNU) \\
 &\leq MU + (MN^2 V)^{1/2} (MNU)^{1/2} \\
 &\ll Mx^{2/3} + MN^{3/2} x^{1/2} \\
 &\ll xL^{-B}.
 \end{aligned}$$

Here we used the inequality that  $\min(a, b) \leq a^s b^t$ ,  $s + t = 1$ ,  $s, t \geq 0$ . Hence

$$\begin{aligned} \min(Y, Z) &\ll x^{3/4}(xq^{-1} + xL^{-4B} + q)^{1/4} + xM^{-1/4} \\ &= W + xM^{-1/4}, \text{ say.} \end{aligned}$$

Also, from (19), we see that

$$X \ll W + x^{1/2}(M^2 N^2 V)^{1/2} + M^2 N^2 V$$

because of  $1 \leq q \leq x$ . In consequence, it turns out that

$$\begin{aligned} \min(X, Y, Z) &= \min(X, \min(Y, Z)) \\ &\ll W + \min(x^{1/2}(M^2 N^2 V)^{1/2} + M^2 N^2 V, xM^{-1/4}) \\ &\ll W + (x^{1/2}(M^2 N^2 V)^{1/2})^{1/5} (xM^{-1/4})^{4/5} + (M^2 N^2 V)^{1/9} (xM^{-1/4})^{8/9} \\ &\ll W + x^{9/10} (N^2 V)^{1/10} + x^{8/9} (N^2 V)^{1/9} \\ &\ll W \end{aligned}$$

since  $N^2 V \ll x^{8/9}$ .

Substituting this into (20), we get the required bound (5) for  $R$ . Therefore we have Theorem, except for the verification of Lemma.

#### 4. Proof of Lemma

It remains to estimate  $G$ . To this end, we employ a well-known Fourier series: For  $H > 2$ ,

$$\min(H, \|\theta\|^{-1}) = \sum_{h \in \mathbb{Z}} w_h e(\theta h)$$

where

$$w_h = w_h(H) \ll \min\left(\log H, \frac{H}{|h|}, \frac{H^2}{h^2}\right).$$

Put  $H = x(M^2 J)^{-1}$ . Unless  $H > 2$ , we trivially have that

$$G \ll M \sum_{m \sim M} \sum_{j \sim J} \tau_3(j) \ll M^2 J L^2.$$

So we may use the above expansion to obtain

$$\min\left(H, \frac{1}{\|\alpha m^2 j\|}\right) = O(L) + \sum_{0 < |h| \leq H^2} w_h e(\alpha m^2 jh).$$

Substituting this into  $G$ , we see that

$$(21) \quad G \ll M^2 J L^3 + M \sum_{0 < |h| \leq H^2} |w_h| \sum_{j \sim J} \tau_3(j) \left| \sum_{m \sim M} e(\alpha m^2 jh) \right| \\ = M^2 J L^3 + F, \text{ say.}$$

Here we consider

$$\left| \sum_{m \sim M} e(\alpha m^2 jh) \right|^2 = \sum_{m_1 \sim M} \sum_{m_2 \sim M} e(\alpha(m_1^2 - m_2^2)jh).$$

We write  $m_1 - m_2 = g$ , so that  $|g| \leq M$  and  $m_1^2 - m_2^2 = 2m_2g + g^2$ . The above sum is then bounded by

$$(22) \quad \ll \sum_{m \sim M} 1 + \sum_{g \leq M} \left| \sum_{m \sim M} e(\alpha 2mgjh) \right| \\ \ll M + \sum_{g \leq 2M} \min\left(M, \frac{1}{\|\alpha gjh\|}\right).$$

Hence it follows from (21), Cauchy's inequality and (22) that

$$(23) \quad F^2 \ll M^2 \sum_{k \leq H^2} |w_k| \sum_{l \sim J} \tau_3(l)^2 \sum_{h \leq H^2} |w_h| \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m^2 jh) \right|^2 \\ \ll M^2 H J L^9 \left\{ H J M L + \sum_{h \leq H^2} \min\left(\log H, \frac{H}{h}\right) \sum_{j \sim J} \sum_{g \leq 2M} \min\left(M, \frac{1}{\|\alpha gjh\|}\right) \right\} \\ = x L^9 \left\{ \frac{x}{M} L + E \right\}, \text{ say.}$$

We proceed to  $E$ . Dividing the interval  $(0, H^2]$  into the subintervals  $(0, H]$  and  $(H2^{k-1}, H2^k]$ ,  $1 \leq k \ll L$ , we find that

$$(24) \quad E \ll L \max_{1 \leq T < H} \frac{1}{T} \sum_{h \leq 2HT} \sum_{j \sim J} \sum_{g \leq 2M} \min\left(M, \frac{1}{\|\alpha gjh\|}\right).$$

Put  $l = gjh$ . Then  $l \ll MJHT \ll (x/M)T$  or  $M \ll xT/l$ . Hence, by Cauchy's

inequality and [2, §25, (3)], the triple sum in (24) is at most

$$\begin{aligned} & \sum_{l \ll (x/M)T} \tau_3(l) \min\left(M, \frac{1}{\|\alpha l\|}\right) \\ & \ll \left( \sum_{l \ll (x/M)T} \tau_3(l)^2 M \right)^{1/2} \left( \sum_{l \ll (x/M)T} \min\left(\frac{xT}{l}, \frac{1}{\|\alpha l\|}\right) \right)^{1/2} \\ & \ll Tx^{1/2} \left( \frac{x}{q} + \frac{x}{M} + \frac{q}{T} \right)^{1/2} L^5. \end{aligned}$$

We therefore have that

$$E \ll x^{1/2} \left( \frac{x}{q} + \frac{x}{M} + q \right)^{1/2} L^6.$$

Combining this with (23) and (21), we get the required bound for  $G$ .

This completes our proof of Theorem.

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Institute of Mathematics  
University of Tsukuba  
Tsukuba 305-8571  
Japan  
mikawa@math.tsukuba.ac.jp