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ON PRIMES IN ARITHMETIC PROGRESSIONS

In honorem Saburô Uchiyama annos LXX nati

By

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1. Introduction

The Dirichlet theorem says that, for any coprime integers q and a, there are infinitely many primes which are congruent to a modulo q. See [16, Kap. IV], for instance. Then, for $(q,a)=1$, let $P(q,a)$ be the least prime in an arithmetic progression $p \equiv a \pmod{q}$. The extended Riemann hypothesis gives that

$$P(q,a) \ll q^{2+\varepsilon} \tag{1}$$

for any $\varepsilon > 0$. However it is conjectured that this exponent 2 could be replaced by 1.

The Linnik theorem unconditionally shows that

$$P(q,a) \ll q^{L}$$

with some absolute constant L, vide [16, Kap. X]. Many works have been done to obtain an explicit value of this Linnik constant. The best known result is $L=5.5$ due to D. R. Heath-Brown [14].

The Bombieri-Vinogradov theorem, see [7, §28], has the same power as the extended Riemann hypothesis in some sense. Indeed, it yields (1) for any given $a \neq 0$ and almost all q. In 1980 E. Fouvry and H. Iwaniec [10, 11] made a significant step beyond the extended Riemann hypothesis. Their ideas have been surprisingly developed by E. Fouvry [8, 9] and E. Bombieri, J. B. Friedlander and H. Iwaniec [4, 5]. In particular, it follows from [5] that, for any fixed $a \neq 0$ and almost all q,

$$P(q,a) \ll q^{2-\delta} \tag{2}$$

where $0 < \delta = \delta(q) \to 0$ as $q \to \infty$.

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In 1986, for the first time, B. Rousselet [17] proved (2) with an absolute constant $\delta = 10^{-100}$. E. Bombieri, J. B. Friedlander and H. Iwaniec [6] also got the similar result without an explicit value of the exponent. Later R. C. Baker and G. Harman [2] showed that

$$P(q, a) \ll q^{25/13+\varepsilon}$$

for any fixed $a \neq 0$ and almost all q.

Our aim is to make a modest improvement upon it.

THEOREM. *Let $K > 32/17$ and $A, B > 0$ be given. Let a be an integer and Q be large with $0 < |a| < (\log Q)^B$. Then, except possibly for $O(Q(\log Q)^{-A})$ integers q with $(q, a) = 1$ and $Q < q \leq 2Q$, we have*

$$P(q, a) \ll q^K$$

where the implied constants depend only on A, B and K.

As well as [17, 6, 2], our argument is a combination of the mean value theorems, which are established by [8, 5], and the sieve identity methods, vide [3, 12, 13] for instance. We add no new result on the former. Our idea is, if exists, concerned with the latter. We introduce the incomplete sum

$$\sum_{\substack{d \mid n \\ d < \sqrt{n}}} \mu(d)$$

into the sieve of Eratosthenes, inspired by K. Alladi [1, Lemma 3] and I. M. Vinogradov [18, Chap. 2, ex. 25].

The sieve identity methods decompose "primes" into "products". We here count primes $p \leq x$, $p \equiv a \pmod{q}$, for $(q, a) = 1$, $Q < q \leq 2Q$. E. Fouvry's result, see Lemma A in section 5, says that one may manage a "product" if its "good" factor falls in the interval $(Q^2 x^{-1}, x^{5/6} Q^{-4/3})$. So this "permissible" interval has to be wide enough to pick a prime up. Actually our sieve procedure requires that it should be of the form (α, β) with $\alpha^2 \leq \beta$. This imposes the restriction $(Q^2 x^{-1})^2 \leq x^{5/6} Q^{-4/3}$ or $Q \leq x^{17/32}$, and then the smallest "permissible" interval becomes $(x^{1/16}, x^{1/8})$. Thus, as far as based upon E. Fouvry's fundamental result, the exponent $K > 32/17$ seems to be the limit of our argument below.

The reader may skip the whole of section 4, in which numerical integrations are estimated by hand computation, if one can accept the statement that a routine calculation shows $C > 0.01$.

We change a bit the usual notation in the sieve theory. This will be explained in the next section. Except these, we use the standard notation in Number Theory. Especially, the letter p is reserved for primes. $a \equiv b(q)$ is short for $a \equiv b \pmod{q}$. $n \sim N$ means $N_1 < n \leq N_2$ with some $N \leq N_1$, $N_2 \leq 2N$. For a set S, $|S|$ stands for its cardinality or measure. We use the abbreviation $\mathscr{L} = \log x$.

The present paper is a detailed and modified version of my talk delivered at the Oberwolfach Institute, March 1996, and the Kansai Seminar House, May 1996. I would like to thank the organizers for kind invitation and the participants for patience.

2. Sieve of Eratosthenes

To begin with, for $z \geq 2$, we introduce the arithmetical functions:

$$\Phi_z(n) = \begin{cases} 1, & \text{if } p|n \text{ implies } p \geq z \text{ or } n = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\Psi_z(n) = \begin{cases} 1, & \text{if } p|n \text{ implies } p < z \text{ or } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We notice that, in the usual notation in the sieve theory, $S(\mathscr{A}, z) = \sum_{n \in \mathscr{A}} \Phi_z(n)$, and the sieve of Eratosthenes reads $\Phi_z(n) = \sum_{d|n} \mu(d)\Psi_z(d)$. We then observe that both Φ and Ψ are completely multiplicative. Let $p(n)$ denote, as usual, the least prime factor of an integer $n > 1$.

LEMMA 1.

$$\Phi_z(n) = 1 - \sum_{\substack{p|n \\ p<z}} \Phi_p\left(\frac{n}{p}\right).$$

LEMMA 2. *For $D > 2$, we have*

$$\Phi_z(n) = \sum_{\substack{d|n \\ d<D}} \mu(d)\Psi_z(d) + \sum_{\substack{d|n \\ d/p(d)<D\leq d}} \mu(d)\Psi_z(d)\Phi_{p(d)}\left(\frac{n}{d}\right).$$

LEMMA 3. *Suppose that, as* $x \to \infty$, $z = z(x) \to \infty$ *and* $\log x/\log z > \log\log x$. *Then we have*

$$\sum_{n \le x} \Psi_z(n) \ll x \exp\left(-\frac{\log x}{\log z}\right).$$

Lemma 1 is Buchstab's identity. Lemma 2 may be produced by an iterative usage of Lemma 1. For an elegant simple proof, see [3]. Lemma 3 is [16, Kap. V, Lemma 5.2]. Lemmas 2 and 3 form a prototype of the fundamental lemma in the sieve theory, vide [16, Kap. VI, Satz 6.1]. Lemma 4 below is the core of our proof of Theorem and verified by a straightforward argument.

LEMMA 4. *For square-free n, we have*

$$\sum_{\substack{d|n \\ d<\sqrt{n}}} \mu(d) = \begin{cases} 0, & \text{if } \mu(n) = 1 \\ 1, & \text{if } n = p \\ 0, & \text{if } p|n \text{ with } \sqrt{n} < p < n \\ -2, & \text{if } n = p_1p_2p_3 \text{ with } p_3 < p_2 < p_1 < \sqrt{n} \\ -20, & \text{if } \nu(n) = 7 \text{ with } p(n) > n^{1/8}. \end{cases}$$

For n having five prime factors, the above weight takes various values depending on its prime factorization. To handle this, we define the sets

$$\mathscr{E}' = \{n \in N \mid n = p_1p_2p_3p_4p_5, p_5 < p_4 < p_3 < p_2 < p_1\}$$

and, with a parameter $t \ge 1$,

$$\mathscr{H}(t) = \{d \in N \mid \mu^2(d) = 1, \nu(d) = 3, \sqrt{t} < d < \sqrt{tp(d)}\}. \tag{3}$$

LEMMA 5. *For* $n \in \mathscr{E}'$, *we have*

$$0 \le \sum_{\substack{d|n \\ d \in \mathscr{H}(n)}} 2 - \sum_{\substack{d|n \\ d<\sqrt{n}}} \mu(d) \le \begin{cases} 2, & \text{if } p_2p_3 < p_1p_5 \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Let Σ denote the incomplete sum of Möbius function in question. Our starting point is $\sum_{d|n} \mu(d) = 0$. We divide the sum over $d|n, d > \sqrt{n}$, according as $p(n)|d$ or not. For $p(n) \nmid d|n$, put $d' = dp(n)$. Then $p(n)|d'|n$, $d' > \sqrt{n}p(n)$ and $\mu(d) = -\mu(d')$. We have that

$$\Sigma = -\sum_{\substack{p(n)|d|n \\ d>\sqrt{n}}} \mu(d) + \sum_{\substack{p(n)|d'|n \\ d'>\sqrt{n}p(n)}} \mu(d')$$

$$= -\sum_{\substack{p(n)|d|n \\ \sqrt{n}<d<\sqrt{n}p(n)}} \mu(d)$$

$$= -|\mathcal{Q}_2| + |\mathcal{Q}_3| - |\mathcal{Q}_4|,$$

where

$$\mathcal{Q}_k = \{d \mid \nu(d) = k, p(n)|d|n, \sqrt{n} < d < \sqrt{n}p(n)\}.$$

For $p(n)|d|n$, define $d^\dagger = (n/d)p(n)$. If $d \in \mathcal{Q}_k$ then $\nu(d^\dagger) = 6 - k$, $p(n)|d^\dagger|n$ and $\sqrt{n} < d^\dagger < \sqrt{n}p(n)$. Hence $\dagger$ is a bijection between $\mathcal{Q}_2$ and $\mathcal{Q}_4$, and a permutation of $\mathcal{Q}_3$. It follows that

$$|\mathcal{Q}_2| = |\mathcal{Q}_4|,$$

and that, in view of $dd^\dagger = np(n)$,

$$|\mathcal{Q}_3| = 2|\{d \mid \nu(d) = 3, p(n)|d|n, \sqrt{n} < d < \sqrt{np(n)}\}| = 2|\mathcal{Q}_3^\dagger|, \quad \text{say.}$$

Since $p(n)|d|n$ means $p(d) = p(n)$, $\mathcal{Q}_3^\dagger$ may be written as

$$\begin{aligned}
&\{d \mid \nu(d) = 3, p(n)|d|n, \sqrt{n} < d < \sqrt{np(d)}\} \\
&\quad = \{d \mid d \in \mathcal{H}(n), p(n)|d|n\} \\
&\quad = \{d \mid d \in \mathcal{H}(n), d|n\} \backslash \{d \mid d \in \mathcal{H}(n), p(n) \nmid d|n\} \\
&\quad = \mathcal{S} \backslash \mathcal{R}, \quad \text{say.}
\end{aligned}$$

We therefore have that

$$\Sigma = 2(|\mathcal{Q}_3^\dagger| - |\mathcal{Q}_2|) = 2(|\mathcal{S}| - |\mathcal{R}| - |\mathcal{Q}_2|),$$

or

$$2|\mathcal{S}| - \Sigma = 2(|\mathcal{R}| + |\mathcal{Q}_2|),$$

from which the first part of Lemma 5 follows.

To see the second part, for $d|n$, define $d^* = n/d$. Write $\mathcal{R}^* = \{d^* \mid d \in \mathcal{R}\}$.

Then $d \in \mathscr{R}$ means that $\nu(d^*) = 2$, $\sqrt{n/p(n/d^*)} < d^* < \sqrt{n}$ and $p(n)|d^*|n$. Hence we find that $*$ is an injection from $\mathscr{R}$ to $\mathscr{R}^*$, $\mathscr{R}^* \cap \mathscr{Q}_2 = \varnothing$ and

$$\mathscr{R}^* \cup \mathscr{Q}_2 \subset \left\{ d \mid \nu(d) = 2, p(n)|d|n, \sqrt{\frac{n}{p(n/d)}} < d < \sqrt{n}p(n) \right\} = \mathscr{Q}, \quad \text{say.}$$

Thus

$$|\mathscr{R}| + |\mathscr{Q}_2| \le |\mathscr{R}^*| + |\mathscr{Q}_2| = |\mathscr{R}^* \cup \mathscr{Q}_2| \le |\mathscr{Q}|.$$

Now, we plainly write $\mathscr{E}' \ni n = p_1p_2p_3p_4p_5$, $p_5 < p_4 < p_3 < p_2 < p_1$. Then $d \in \mathscr{Q}$ must be of the form p_ip_5, $i = 1,2,3,4$, because of $p(n)|d$. However, unless $i = 1$, $d^2 = (p_ip_5)^2 < p_1p_2p_4p_5 = n/p_3 \le n/p(n/d)$, which contradicts the condition of $d \in \mathscr{Q}$. Thus the possible member of $\mathscr{Q}$ is $d = p_1p_5$ only, in particular, $|\mathscr{Q}| \le 1$. $\mathscr{Q} \ni d = p_1p_5$ implies $(p_1p_5)^2 = d^2 > n/p(n/d) = p_1p_2p_3p_5$, whence $p_1p_5 > p_2p_3$. Namely, $|\mathscr{Q}| = 0$ unless $p_1p_5 > p_2p_3$, as claimed.

We then have to count the divisors $d|n$ with $d \in \mathscr{H}(n)$. Put

$$\mathscr{D} = \{n \mid n = p_1p_2p_3p_4, p_4 < p_3 < p_2 < p_1 < p_2p_3p_4, p_2p_3 < p_1\}.$$

LEMMA 6. *For square-free n, we have*

$$\sum_{\substack{d|n \\ d \in \mathscr{H}(n)}} 2 \le \begin{cases} 0, & \text{if } \nu(n) \le 4 \text{ and } n \notin \mathscr{D} \\ 2, & \text{if } n \in \mathscr{D} \\ 20, & \text{if } \nu(n) = 6 \\ 0, & \text{if } \nu(n) = 7 \text{ with } p(n) > n^{1/8}. \end{cases}$$

PROOF. Let $\mathscr{S}_n$ denote the set of divisors $d|n$ with $d \in \mathscr{H}(n)$ in question. If $\nu(n) \le 3$ then plainly $\mathscr{S}_n$ is empty. We consider the case of $\nu(n) = 4$; $n = p_1p_2p_3p_4$, $p_4 < p_3 < p_2 < p_1$. Let $d \in \mathscr{S}_n$. Obviously, $p_4 \nmid d/p(d)$. Suppose $p_1|d/p(d)$. Then $d/p(d) = p_1p_j$, $j = 2,3$. However $(d/p(d))^2 = (p_1p_j)^2 > p_1p_2p_3p_4 > n/p(d)$, which is impossible. Hence we have $p_1 \nmid d/p(d)$, so that $d/p(d) = p_2p_3$ and $p(d) = p_4$. Thus, $|\mathscr{S}_n| = 1$ or 0, according as $p_2p_3p_4 \in \mathscr{H}(n)$ or not.

If $\nu(n) = 6$, then the number of divisors $d|n$ with $\nu(d) = 3$ is $\binom{6}{3}$. In view of the correspondence $d \leftrightarrow n/d$, the condition $d > \sqrt{n}$ is fulfilled by the half of them. Hence we see that $|\mathscr{S}_n| \le \binom{6}{3}/2 = 10$.

Finally, if $\nu(n) = 7$ and $p(n) > n^{1/8}$ then, for any divisor $d|n$ with $\nu(d) = 3$, $d < np(n)^{-4} < \sqrt{n}$. Hence $\mathscr{S}_n$ is empty.

3. Proof of Theorem

Put $x = Q^K$, $y = x\mathscr{L}^{-4}$ and $\mathscr{M} = \{q \mid Q < q \le 2Q, (q,a) = 1\}$. We shall show that

$$(4) \qquad |\{p \mid x - y < p \le x, p \equiv a \pmod q\}| \ge \frac{y(1 + O(\mathscr{L}^{-1}))}{100\varphi(q)\mathscr{L}} + R(x, y; q, a)$$

with

$$(5) \qquad \sum_{q \in \mathscr{M}} |R(x, y; q, a)| \ll x\mathscr{L}^{-A-6}.$$

To derive Theorem from these, let $\mathscr{M}' = \{q \in \mathscr{M} \mid P(q,a) > q^K\}$. If $q \in \mathscr{M}'$ then, since $q^K > Q^K = x$, the left hand side of (4) is zero. Namely,

$$|R(x, y; q, a)| \gg yQ^{-1}\mathscr{L}^{-2} = xQ^{-1}\mathscr{L}^{-6}$$

uniformly for $q \in \mathscr{M}'$. Thus, (5) shows that

$$xQ^{-1}\mathscr{L}^{-6}|\mathscr{M}'| \ll x\mathscr{L}^{-A-6},$$

from which Theorem follows.

Obviously, we may assume that K is close to 32/17, so that $Q = x^{17/32-\delta}$ with some sufficiently small $\delta = \delta(K) > 0$. We call a function $F(x, y; q, a)$ "admissible" remainder term, if $\sum_{q \in \mathscr{M}} |F(x, y; q, a)| \ll x\mathscr{L}^{-E}$ for any fixed $E > 0$, and indicate such a function by "$A.R.$" in a formula. To prove (4) and (5), we define

$$(6) \qquad \Theta = \Theta(x, y; q, a) = \sum_{\substack{k \in \mathscr{I} \\ k \equiv a(q)}} \Bigg(\sum_{\substack{d|k \\ d<I}} \mu(d) \Bigg) \Phi_H(k)$$

where $H = x^{1/8}$, $I = x^{1/2}$ and $\mathscr{I} = (x - y, x]$. We begin by giving an asymptotic formula for Θ. We postpone our proof of the following evaluation until section 7.

Proposition 1.

$$\Theta = \frac{1}{\varphi(q)} \sum_{\substack{k \in \mathscr{I} \\ (k,q)=1}} \Bigg(\sum_{\substack{d|k \\ d<I}} \mu(d) \Bigg) \Phi_H(k) + O(yq^{-1}\mathscr{L}^{-3}) + A.R..$$

For the sum in the right hand side of the above formula, we drop the condition $(k,q)=1$, then restrict $k \in \mathscr{J}$ to square-free integers and replace $d < I = x^{1/2}$ by $d < \sqrt{k}$. The resulting cost is

$$\begin{aligned}
&\ll \sum_{k \in \mathscr{J}} \left(\sum_{\substack{e|q \\ e|k \\ e \geq H}} 1 + \sum_{\substack{p^2|k \\ p \geq H}} 1 + \sum_{\substack{d|k \\ d^2 \in \mathscr{J}}} 1 \right) \\
&\ll \sum_{H \leq e|q} \frac{y}{e} + \sum_{H \leq p \leq I} \left(\frac{y}{p^2} + 1 \right) + \sum_{d^2 \in \mathscr{J}} \frac{y}{d} \\
&\ll y\mathscr{L}^{-4}.
\end{aligned}$$

Then we put

$$\Xi(k) = \mu^2(k) \left(\sum_{\substack{d|k \\ d<\sqrt{k}}} \mu(d) \right) \Phi_H(k). \tag{7}$$

By Lemma 4, we find that

$$\begin{aligned}
\sum_{\substack{k \in \mathscr{J} \\ (k,q)=1}} \left(\sum_{\substack{d|k \\ d<I}} \mu(d) \right) \Phi_H(k) &= \sum_{k \in \mathscr{J}} \Xi(k) + O(y\mathscr{L}^{-4}) \\
&= \sum_{p \in \mathscr{J}} (+1) + \sum_{k \in \mathscr{C}} (-2) + \sum_{k \in \mathscr{E}} \Xi(k) \\
&\quad + \sum_{k \in \mathscr{G}} (-20) + O(y\mathscr{L}^{-4})
\end{aligned} \tag{8}$$

where

$$\mathscr{C} = \{p_1p_2p_3 \in \mathscr{J} \mid H \leq p_3 < p_2 < p_1 < p_2p_3\},$$

$$\mathscr{E} = \{p_1p_2p_3p_4p_5 \in \mathscr{J} \mid H \leq p_5 < p_4 < p_3 < p_2 < p_1\},$$

$$\mathscr{G} = \{p_1p_2p_3p_4p_5p_6p_7 \in \mathscr{J} \mid H \leq p_7 < p_6 < p_5 < p_4 < p_3 < p_2 < p_1\}.$$

As well as $|\{p \in \mathscr{J}\}|$, the cardinalities $|\mathscr{C}|$ and $|\mathscr{G}|$ can be evaluated by the prime number theorem and partial summation. We leave the sum over $k \in \mathscr{E}$ untouched. It therefore follows from Proposition 1 and (8) that

$$\Theta = (1 - 2C_3 - 20C_7 + O(\mathscr{L}^{-1})) \frac{y}{\varphi(q)\mathscr{L}} + \frac{1}{\varphi(q)} \sum_{k \in \mathscr{E}} \Xi(k) + A.R., \tag{9}$$

where

$$C_3 = \iint_{\substack{1/8<t_3<t_2 \\ 2t_2+t_3<1 \\ t_2+t_3>1/2}} \frac{dt_2 dt_3}{(1-t_2-t_3)t_2 t_3} \tag{10}$$

and

$$C_7 = \iiiint\!\!\iint_{\substack{1/8<t_7<t_6<t_5<t_4<t_3<t_2 \\ 2t_2+t_3+t_4+t_5+t_6+t_7<1}} \frac{dt_2 dt_3 dt_4 dt_5 dt_6 dt_7}{(1-t_2-t_3-t_4-t_5-t_6-t_7)t_2 t_3 t_4 t_5 t_6 t_7}. \tag{11}$$

Next we shall deduce an upper bound for Θ. In the definition (6) of Θ, we change the summand by Ξ, which is given by (7). The resulting error is

$$\ll \sum_{\substack{k \in \mathscr{I} \\ k \equiv a(q)}} \Bigg(\sum_{\substack{p^2 | k \\ p \geq H}} 1 + \sum_{\substack{d | k \\ d^2 \in \mathscr{I}}} 1 \Bigg).$$

The first term is admissible. In fact, the averaged sum over $q \in \mathscr{M}$ is

$$\ll \sum_{k \in \mathscr{I}} \sum_{p^2 | k,\, p \geq H} \tau(k-a) \ll x^{\varepsilon} \sum_{H \leq p \leq I} \left(\frac{y}{p^2} + 1 \right) \ll x^{7/8+\varepsilon}$$

for any $\varepsilon > 0$. To bound the second error term, we appeal to C. Hooley's argument [15, Chapter 1]. We first express it by means of the function $\psi(t) = t - [t] + 1/2$, then expand ψ as a Fourier series, and employ bounds for incomplete Kloosterman sums. Thus, for any $\varepsilon > 0$, the above second term is

$$\ll \sum_{d^2 \in \mathscr{I}} \frac{y}{qd} + q^{1/2+\varepsilon} + x^{1-\varepsilon} q^{-1} \ll y q^{-1} \mathscr{L}^{-4}. \tag{12}$$

By Lemma 4 again, we then see that

$$\begin{aligned} \Theta &= \sum_{\substack{k \in \mathscr{I} \\ k \equiv a(q)}} \Xi(k) + O(yq^{-1}\mathscr{L}^{-4}) + A.R. \\ &= \sum_{\substack{p \in \mathscr{I} \\ k \equiv a(q)}} (+1) + \sum_{\substack{k \in \mathscr{C} \\ k \equiv a(q)}} (-2) + \sum_{\substack{k \in \mathscr{E} \\ k \equiv a(q)}} \Xi(k) + \sum_{\substack{k \in \mathscr{G} \\ k \equiv a(q)}} (-20) \\ &\quad + O(yq^{-1}\mathscr{L}^{-4}) + A.R.. \end{aligned} \tag{13}$$

Trivially,

$$\sum_{\substack{k \in \mathscr{G} \\ k \equiv a(q)}} (-20) \leq 0. \tag{14}$$

We proceed to the sum over $k \in \mathscr{E}$. Put

$$\Upsilon(k) = \mu^2(k)\left(\sum_{\substack{d|k \\ d \in \mathscr{H}(k)}} 2\right)\Phi_H(k)$$

in the notation of section 2. Then, Lemma 5 shows that

$$\begin{aligned}(15) \qquad \sum_{\substack{k \in \mathscr{E} \\ k \equiv a(q)}} \Xi(k) &\le \sum_{\substack{k \in \mathscr{E} \\ k \equiv a(q)}} \Upsilon(k) \\ &\le \sum_{\substack{k \in \mathscr{J} \\ k \equiv a(q)}} \left(\sum_{\substack{d|k \\ d \in \mathscr{H}(x)}} 2\right)\Phi_H(k) + O\left(\sum_{\substack{k \in \mathscr{J} \\ k \equiv a(q)}} \sum_{\substack{d|k \\ d^2 \in \mathscr{J}}} 1\right).\end{aligned}$$

The above O-term becomes $O(yq^{-1}\mathscr{L}^{-4})$ as before in (12). We put our proof of the following formula on the above first term off until section 8.

PROPOSITION 2.

$$\sum_{\substack{k \in \mathscr{J} \\ k \equiv a(q)}} \left(\sum_{\substack{d|k \\ d \in \mathscr{H}(x)}} 2\right)\Phi_H(k) = \frac{1}{\varphi(q)} \sum_{\substack{k \in \mathscr{J} \\ (k,q)=1}} \left(\sum_{\substack{d|k \\ d \in \mathscr{H}(x)}} 2\right)\Phi_H(k) + O(yq^{-1}\mathscr{L}^{-3}) + A.R..$$

As for the sum in the right hand side, we take the same route as (15) in the opposite derection. Lemmas 5 and 6 show that

$$\begin{aligned}(16) \qquad & \sum_{\substack{k \in \mathscr{J} \\ (k,q)=1}} \left(\sum_{\substack{d|k \\ d \in \mathscr{H}(x)}} 2\right)\Phi_H(k) \\ &\le \sum_{\substack{k \in \mathscr{J} \\ (k,q)=1}} \left(\sum_{\substack{d|k \\ d \in \mathscr{H}(k)}} 2\right)\Phi_H(k) + O\left(\sum_{k \in \mathscr{J}} \sum \sum_{\substack{ef|k \\ e^2 f \in \mathscr{J} \\ ef \le x^{3/5}}} 1\right) \\ &= \sum_{k \in \mathscr{J}} \Upsilon(k) + O\left(\sum_{k \in \mathscr{J}} \left(\sum_{\substack{p^2|k \\ p \ge H}} 1 + \sum_{\substack{d|q \\ d|k \\ d \ge H}} 1\right)\right) + O\left(\sum\sum_{e^2 f \in \mathscr{J}} \frac{y}{ef}\right) \\ &\le \sum_{k \in \mathscr{D}} 2 + \sum_{k \in \mathscr{E}} \Xi(k) + \sum_{k \in \mathscr{E}^*} 2 + \sum_{k \in \mathscr{F}} 20 + O(y\mathscr{L}^{-3})\end{aligned}$$

where

$$\mathscr{D} = \{p_1p_2p_3p_4 \in \mathscr{J} \mid H \le p_4 < p_3 < p_2 < p_1 < p_2p_3p_4, p_2p_3 < p_1\},$$

$$\mathscr{E}^* = \{p_1p_2p_3p_4p_5 \in \mathscr{J} \mid H \le p_5 < p_4 < p_3 < p_2 < p_1, p_2p_3 < p_1p_5\},$$

$$\mathscr{F} = \{p_1p_2p_3p_4p_5p_6 \in \mathscr{J} \mid H \le p_6 < p_5 < p_4 < p_3 < p_2 < p_1\}.$$

As before, by the prime number theorem, (16) becomes

$$\sum_{k\in\mathscr{E}} \Xi(k) + (2C_4 + 2C_5^* + 20C_6 + O(\mathscr{L}^{-1}))y\mathscr{L}^{-1} \tag{17}$$

where

$$C_4 = \iiint_{\substack{1/8<t_4<t_3<t_2\\ 2t_2+2t_3+t_4<1\\ t_2+t_3+t_4>1/2}} \frac{dt_2dt_3dt_4}{(1-t_2-t_3-t_4)t_2t_3t_4}, \tag{18}$$

$$C_5^* = \iiiint_{\substack{1/8<t_5<t_4<t_3<t_2\\ 2t_2+2t_3+t_4<1}} \frac{dt_2dt_3dt_4dt_5}{(1-t_2-t_3-t_4-t_5)t_2t_3t_4t_5} \tag{19}$$

and C_6 is similar to C_7 given by (11). In conjunction with (15), Proposition 2, (16) and (17), we have that

$$\sum_{\substack{k\in\mathscr{E}\\ k\equiv a(q)}} \Xi(k) \le \frac{1}{\varphi(q)}\sum_{k\in\mathscr{E}} \Xi(k) + (2C_4+2C_5^*+20C_6+O(\mathscr{L}^{-1}))\frac{y}{\varphi(q)\mathscr{L}} + A.R.. \tag{20}$$

We turn to the sum over $k \in \mathscr{C}$ in (13). We define the subset $\mathscr{B} \subset \mathscr{C}$ by

$$\mathscr{B} = \{p_1p_2p_3 \in \mathscr{C} \mid p_2^3p_3^2 \le x^{\theta+1-2\delta}, p_2^3p_3^4 \le x^{2-\delta}, p_2p_3 \ge x^\theta\} \tag{21}$$

with $\theta = 17/32$. Here we remember that $Q = x^{\theta-\delta}$ and that $\delta = \delta(K) > 0$ is supposed to be sufficiently small. In section 9 we shall show the following formula on the sum over $\mathscr{B}$.

PROPOSITION 3.

$$\sum_{\substack{k\in\mathscr{B}\\ k\equiv a(q)}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{k\in\mathscr{B}\\ (k,q)=1}} 1 + A.R..$$

It therefore turns out that

$$\begin{aligned}\sum_{\substack{k\in\mathscr{C}\\ k\equiv a(q)}} (-2) &\le \frac{1}{\varphi(q)} \sum_{\substack{k\in\mathscr{B}\\ (k,q)=1}} (-2) + A.R. \\ &= (-2C_3' + O(\mathscr{L}^{-1}))\frac{y}{\varphi(q)\mathscr{L}} + O(y\varphi(q)^{-1}\mathscr{L}^{-4}) + A.R.\end{aligned} \tag{22}$$

where

$$C_3' = C_3'(\delta) = \iint_{\substack{1/8<t_3<t_2 \\ 3t_2+2t_3<\theta+1-2\delta \\ 3t_2+4t_3<2-\delta \\ t_2+t_3>\theta}} \frac{dt_2dt_3}{(1-t_2-t_3)t_2t_3}. \tag{23}$$

Substituting (14), (20) and (22) into (13), we conclude that

$$\begin{aligned}\Theta \le \sum_{\substack{p\in\mathcal{J} \\ p\equiv a(q)}} 1 + (-2C_3' + 2C_4 + 2C_5^* + 20C_6 + O(\mathcal{L}^{-1}))\frac{y}{\varphi(q)\mathcal{L}} \\ + \frac{1}{\varphi(q)}\sum_{k\in\mathcal{E}} \Xi(k) + A.R..\end{aligned} \tag{24}$$

We are now in the final step. Combining (24) with (9), we have that

$$\sum_{\substack{p\in\mathcal{J} \\ p\equiv a(q)}} 1 \ge (C + O(\mathcal{L}^{-1}))\frac{y}{\varphi(q)\mathcal{L}} + A.R.$$

where $C = C(\delta) = 1 - 2((C_3 - C_3'(\delta)) + C_4 + C_5^* + 10C_6 + 10C_7)$. In the next section, we shall check

$$C = C(\delta) \ge 0.01, \tag{25}$$

providing that $\delta = \delta(K) > 0$ is small enough. This shows (4) and (5).

We thus get Theorem, apart from the verification of Propositions 1, 2, 3 and (25).

4. Numerical Integrations

In this section we shall verify (25). We begin with $C_3 - C_3'(\delta)$. Recall the definitions (10) and (23). For simplicity, we rewrite $(t_3, t_2) = (u, v) \in \boldsymbol{R}^2$. Let $\mathcal{N}_0 = \{(u,v) \mid 1/8 < u < v, 2v + u < 1, 1/2 < v + u\}$ the integral region of C_3. Note that $v > 1/4$ and $u < 1/3$. Put $\mathcal{N} = \{(u,v) \in \mathcal{N}_0 \mid 3v + 2u > \theta + 1$ or $3v + 4u > 2$ or $v + u < \theta\}$. Then we find that, with some absolute constant $C_0 > 0$,

$$C_3 - C_3'(\delta) \le \iint_{\mathcal{N}} \frac{dvdu}{(1-v-u)vu} + C_0\delta = C_3^* + C_0\delta, \quad \text{say.} \tag{26}$$

We shall divide $\mathcal{N}$ into five subsets. Let

$$\mathcal{N} = \{(u,v) \in \mathcal{N}_0 \mid 3v + 2u > \theta + 1 \text{ or } 3v + 4u > 2\} \cup \{(u,v) \in \mathcal{N}_0 \mid v + u < \theta\}$$

$$= \mathcal{N}' \cup \mathcal{N}'', \quad \text{say.}$$

Since $v + u < \theta$ implies $3v + 2u < 3\theta - u < 3\theta - 1/8 < \theta + 1$ and $3v + 4u < 4\theta - v < 4\theta - 1/4 < 2$, we see that $\mathcal{N}' \cap \mathcal{N}'' = \varnothing$ and that the condition $1/2 < v + u$ of $\mathcal{N}_0$ is absorbed by the additional condition of $\mathcal{N}'$.

We first deal with $\mathcal{N}'$. If $u \le (1-\theta)/2$ then $3v + 4u > 2$ means $3v + 2u > 2 - 2u \ge 2 - (1-\theta) = \theta + 1$, and $3v + 2u > \theta + 1$ implies $3v > \theta + 1 - 2u \ge \theta + 1 - (1-\theta) = 2\theta > 1 > 3u$. If $(1-\theta)/2 < u \le 2/7$ then $3v + 2u > \theta + 1$ means $3v + 4u > \theta + 1 + 2u > \theta + 1 + (1-\theta) = 2$, and $3v + 4u > 2$ implies $3v > 2 - 4u \ge 3u$. If $2/7 < u$ then $u < v$ implies $3v + 4u > 7u > 2$. Hence $\mathcal{N}'$ is written as the mutually disjoint union $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ where

$$\mathcal{N}_1 = \{(u,v) \mid 1/8 < u \le (1-\theta)/2, 3v + 2u > \theta + 1, 2v + u < 1\},$$

$$\mathcal{N}_2 = \{(u,v) \mid (1-\theta)/2 < u \le 2/7, 3v + 4u > 2, 2v + u < 1\},$$

$$\mathcal{N}_3 = \{(u,v) \mid 2/7 < u < v, 2v + u < 1\}.$$

We turn to $\mathcal{N}''$. The additional condition $v + u < \theta$ of $\mathcal{N}''$ implies $2v + u < 2\theta - u < 2\theta - 1/8 < 1$. If $u \le 1/4$ then $1/2 < v + u$ implies $v > 1/2 - u \ge u$. If $u > 1/4$ then $u < v$ implies $v + u > 2u > 1/2$. Hence $\mathcal{N}''$ is the disjoint union $\mathcal{N}_4 \cup \mathcal{N}_5$ where

$$\mathcal{N}_4 = \{(u,v) \mid 1/8 < u \le 1/4, 1/2 < v + u < \theta\},$$

$$\mathcal{N}_5 = \{(u,v) \mid 1/4 < u < v, v + u < \theta\}.$$

Now we replace $\mathcal{N}$ the integral region of C_3^* by $\mathcal{N}_j$ $(1 \le j \le 5)$ and write the resulting integral by N_j. Then, $C_3^* = \sum_{j=1}^{5} N_j$. For $j = 1, 2, 5$, we use the simple inequality:

$$N_j \le |\mathcal{N}_j| \sup_{(u,v) \in \mathcal{N}_j} ((1 - v - u)vu)^{-1}.$$

We begin with N_1:

$$\int_{1/8}^{(1-\theta)/2} \int_{(\theta+1-2u)/3}^{(1-u)/2} \frac{dvdu}{(1 - v - u)vu}.$$

Then

$$|\mathcal{N}_1| = \int_{1/8}^{(1-\theta)/2} \frac{1}{6}(u - (2\theta - 1))\, du = \Big/_{1/8}^{15/64} \frac{1}{12}\left(u - \frac{1}{16}\right)^2 = \frac{1}{12}\frac{7}{64}\frac{15}{64}.$$

As for the intgrand, we see that

$$\begin{aligned}(1-v-u)vu &\geq (1-(\theta+1-2u)/3-u)((\theta+1-2u)/3)u\\ &= (1/9)(2-\theta-u)(\theta+1-2u)u\\ &\geq (1/9)\min\{(2-\theta-1/8)(\theta+1-1/4)/8, (3-\theta)\theta(1-\theta)/2\}\\ &= \frac{1}{9}\min\left(\frac{43}{32}\frac{41}{32}\frac{1}{8}, \frac{79}{32}\frac{17}{32}\frac{15}{32}\frac{1}{2}\right),\end{aligned}$$

in the intgral region $\mathcal{N}_1$. Hence, $N_1 \leq 7\cdot 15\cdot 3/(2\cdot 43\cdot 41) = 315/3526$. Next

$$N_2 = \int_{(1-\theta)/2}^{2/7}\int_{(2-4u)/3}^{(1-u)/2}\frac{dvdu}{(1-v-u)vu}.$$

As above,

$$|\mathcal{N}_2| = \int_{(1-\theta)/2}^{2/7}\frac{1}{6}(5u-1)\,du = \Big/_{15/64}^{2/7}\frac{1}{60}(5u-1)^2 = \frac{1}{60}\frac{115}{7\cdot 64}\frac{269}{7\cdot 64},$$

and, in $\mathcal{N}_2$,

$$\begin{aligned}(1-v-u)vu &\geq (1-(2-4u)/3-u)((2-4u)/3)u\\ &= (2/9)(1+u)u(1-2u)\\ &\geq \frac{2}{9}\min\left(\frac{79}{64}\frac{15}{64}\frac{17}{32}, \frac{9}{7}\frac{2}{7}\frac{3}{7}\right).\end{aligned}$$

Hence, $\mathcal{N}_2 \leq 23\cdot 269\cdot 4/(7\cdot 7\cdot 79\cdot 5\cdot 17) = 24748/329035$. We turn to N_3.

$$\begin{aligned}N_3 &= \int_{2/7}^{1/3}\int_{u}^{(1-u)/2}\frac{dvdu}{(1-v-u)vu}\\ &\leq \int_{2/7}^{1/3}\frac{\frac{1}{2}(1-u)-u}{(1-2u)u^2}\,du\\ &= \frac{1}{2}\int_{2/7}^{1/3}\left(\frac{-2}{1-2u}-\frac{1}{u}+\frac{1}{u^2}\right)du\\ &= \frac{1}{2}\Big/_{2/7}^{1/3}\log\left(\frac{1}{u}-2\right)-\frac{1}{u}\\ &= \frac{1}{2}\left(\frac{1}{2}-\log\frac{3}{2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}
N_4 &= \int_{1/8}^{1/4}\int_{1/2-u}^{\theta-u} \frac{dvdu}{(1-v-u)vu} \\
&\le \int_{1/8}^{1/4} \frac{\theta-1/2}{(1-(1/2-u)-u)(1/2-u)u}\,du \\
&= 4\left(\theta-\frac{1}{2}\right)\int_{1/8}^{1/4}\left(\frac{1}{u}+\frac{1}{1/2-u}\right)du \\
&= \frac{1}{8}\Big/_{1/8}^{1/4} \log\left(\frac{2u}{1-2u}\right) \\
&= \frac{1}{8}\log 3.
\end{aligned}$$

Finally,

$$N_5 = \int_{1/4}^{\theta/2}\int_{u}^{\theta-u} \frac{dvdu}{(1-v-u)vu}.$$

Then

$$|\mathscr{N}_5| = \int_{1/4}^{\theta/2}(\theta-2u)\,du = \Big/_{1/4}^{\theta/2} -\frac{1}{4}(\theta-2u)^2 = \frac{1}{4}\left(\frac{1}{32}\right)^2.$$

Since, in $\mathscr{N}_5$, $(1-v-u)vu \ge (1-2u)u^2 \ge 1/2(1/4)^2$, we see $N_5 \le 1/128$. On summing up, we have that

$$C_3^* \le \frac{315}{3526}+\frac{24748}{329035}+\frac{1}{2}\left(\frac{1}{2}-\log\frac{3}{2}\right)+\frac{1}{8}\log 3+\frac{1}{128} < 0.35696. \tag{27}$$

Next we estimate C_4 given by (18):

$$\iiint_{\mathscr{T}} \frac{dwdvdu}{(1-w-v-u)wvu}.$$

where $\mathscr{T} = \{(u,v,w) \in \boldsymbol{R}^3 \mid 1/8 < u < v < w,\ \ 2w+2v+u < 1, 1/2 < w+v+u\}$. We divide $\mathscr{T}$ into three mutually disjoint subsets $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$, by adding

$$u > 1/6,\ 4v+2u > 1; \quad u \le 1/6, 4v+2u > 1; \quad 4v+2u \le 1.$$

For $\mathscr{T}_1$, $1/6 < u < v < w$ implies $w+v+u > 3u > 1/2$ and $4v+2u > 6u > 1$. For $\mathscr{T}_2$, $u \le 1/6$ and $4v+2u > 1$ imply $4v > 1-2u \ge 4u$. And $v < w$ and

$4v+2u>1$ imply $w+v+u>2v+u>1/2$. For $\mathscr{T}_3$, $4v+2u\le 1$ and $1/2<w+v+u$ imply $2v\le 1-2u-2v<2w$. Hence we see that

$$\mathscr{T}_1=\{(u,v,w)\,|\,1/6<u<v<w, 2w+2v+u<1\},$$

$$\mathscr{T}_2=\{(u,v,w)\,|\,1/8<u\le 1/6, 4v+2u>1, v<w, 2w+2v+u<1,\},$$

$$\mathscr{T}_3=\{(u,v,w)\,|\,1/8<u<v, 4v+2u\le 1, 1/2<w+v+u, 2w+2v+u<1\}.$$

Let $\mathscr{T}$ the integral domain of C_4 replace by $\mathscr{T}_j$ $(j=1,2,3)$ and denote the resulting integral by T_j. Then $C_4=\sum_{j=1}^{3}T_j$. We plainly bound each of T_j by the supremum of the integrand times the volume of the integral domain, as before.

To start with,

$$T_1=\int_{1/6}^{1/5}\int_{u}^{(1-u)/4}\int_{v}^{(1-2v-u)/2}\frac{dwdvdu}{(1-w-v-u)wvu}.$$

The volume of $\mathscr{T}_1$ is

$$\int_{1/6}^{1/5}\int_{u}^{(1-u)/4}\frac{1}{2}((1-u)-4v)\,dvdu=\int_{1/6}^{1/5}\frac{1}{16}(1-5u)^2\,du=\frac{1}{16}\frac{1}{15}\left(1-\frac{5}{6}\right)^3.$$

Since, in $\mathscr{T}_1$, $(1-w-v-u)wvu\ge(1-2v-u)v^2u\ge(1-3u)u^3\ge(1-3/6)(1/6)^3$, we find that $T_1\le 1/120$. Next,

$$T_2=\int_{1/8}^{1/6}\int_{(1-2u)/4}^{(1-u)/4}\int_{v}^{(1-2v-u)/2}\frac{dwdvdu}{(1-w-v-u)wvu}.$$

The volume of $\mathscr{T}_2$ is

$$\int_{1/8}^{1/6}\int_{(1-2u)/4}^{(1-u)/4}2\left(\frac{1}{4}(1-u)-v\right)dvdu=\int_{1/8}^{1/6}\left(\frac{u}{4}\right)^2du=\frac{1}{16}\frac{1}{3}\left(\left(\frac{1}{6}\right)^3-\left(\frac{1}{8}\right)^3\right).$$

Since $(1-w-v-u)wvu\ge(1-2v-u)v^2u\ge(1/2)(1/16)(1-2u)^2u\ge(1/2)\cdot(1/16)(1-1/4)^2(1/8)$ in $\mathscr{T}_2$, we have that $T_2\le 37/1458$. Finally,

$$T_3=\int_{1/8}^{1/6}\int_{u}^{(1-2u)/4}\int_{1/2-v-u}^{(1-2v-u)/2}\frac{dwdvdu}{(1-w-v-u)wvu}.$$

The volume of $\mathscr{T}_3$ is equal to

$$\int_{1/8}^{1/6}\frac{u}{2}\frac{1-6u}{4}\,du=\Big/_{1/8}^{1/6}\frac{1}{16}u^2(1-4u)=\frac{1}{16}\left(\left(\frac{1}{6}\right)^2\frac{1}{3}-\left(\frac{1}{8}\right)^2\frac{1}{2}\right).$$

Since $(1-w-v-u)wvu \geq (1/2)(1/2-v-u)vu \geq (1/2)(1/2-2u)u^2 \geq (1/8)^3$ in $\mathscr{T}_3$, it follows that $T_3 \leq 5/108$. On summing up the above estimates, we obtain that

$$C_4 \leq \frac{1}{120} + \frac{37}{1458} + \frac{5}{108} < 0.08001. \tag{28}$$

We proceed to C_5^* given by (19):

$$\iiiint_{\substack{1/8<t_5<t_4<t_3<t_2 \\ 2t_2+2t_3+t_4<1}} \frac{dt_2dt_3dt_4dt_5}{(1-t_2-t_3-t_4-t_5)t_2t_3t_4t_5}.$$

The volume of the integral domain is equal to

$$\begin{aligned}
&\int_{1/8}^{1/5}\int_{t_5}^{1/5}\int_{t_4}^{(1-t_4)/4}\int_{t_3}^{(1-2t_3-t_4)/2} dt_2dt_3dt_4dt_5 \\
&= \int_{1/8}^{1/5}\int_{t_5}^{1/5}\int_{t_4}^{(1-t_4)/4} \frac{1}{2}((1-t_4)-4t_3)\, dt_3dt_4dt_5 \\
&= \int_{1/8}^{1/5}\int_{t_5}^{1/5} \frac{1}{16}(1-5t_4)^2\, dt_4dt_5 \\
&= \int_{1/8}^{1/5} \frac{1}{16}\frac{1}{15}(1-5t_5)^3\, dt_5 \\
&= \frac{1}{16\cdot 15\cdot 20}\left(1-\frac{5}{8}\right)^4.
\end{aligned}$$

Since, in the domain in question,

$$\begin{aligned}
(1-t_2-t_3-t_4-t_5)t_2t_3t_4t_5 &\geq (1-2t_3-t_4-t_5)t_3^2t_4t_5 \\
&\geq (1-3t_4-t_5)t_4^3t_5 \\
&\geq (1-4t_5)t_5^4 \\
&\geq (1-\tfrac{4}{8})(\tfrac{1}{8})^4,
\end{aligned}$$

we find that

$$C_5^* \leq \frac{27}{800} = 0.03375. \tag{29}$$

We then turn to

$$C_6 = \iiiiint_{\substack{1/8<t_6<t_5<t_4<t_3<t_2 \\ 2t_2+t_3+t_4+t_5+t_6<1}} \frac{dt_2dt_3dt_4dt_5dt_6}{(1-t_2-t_3-t_4-t_5-t_6)t_2t_3t_4t_5t_6}.$$

The volume of the integral domain equals

$$\int_{1/8}^{1/6}\int_{t_6}^{(1-t_6)/5}\int_{t_5}^{(1-t_5-t_6)/4}\int_{t_4}^{(1-t_4-t_5-t_6)/3}\int_{t_3}^{(1-t_3-t_4-t_5-t_6)/2} dt_2dt_3dt_4dt_5dt_6$$

$$= \int_{1/8}^{1/6}\int_{t_6}^{(1-t_6)/5}\int_{t_5}^{(1-t_5-t_6)/4}\int_{t_4}^{(1-t_4-t_5-t_6)/3} \frac{1}{2}((1-t_4-t_5-t_6)-3t_3)\, dt_3dt_4dt_5dt_6$$

$$= \int_{1/8}^{1/6}\int_{t_6}^{(1-t_6)/5}\int_{t_5}^{(1-t_5-t_6)/4} \frac{1}{2!3!}((1-t_5-t_6)-4t_4)^2\, dt_4dt_5dt_6$$

$$= \int_{1/8}^{1/6}\int_{t_6}^{(1-t_6)/5} \frac{1}{3!4!}((1-t_6)-5t_5)^3\, dt_5dt_6$$

$$= \int_{1/8}^{1/6} \frac{1}{4!5!}(1-6t_6)^4\, dt_6$$

$$= \frac{1}{5!6!}\left(1-\frac{6}{8}\right)^5.$$

Since $(1-t_2-t_3-t_4-t_5-t_6)t_2t_3t_4t_5t_6 \geq (1-5t_6)t_6^5 \geq (1-5/8)(1/8)^5$ in the domain under consideration, we see that

$$C_6 \leq \frac{2^9}{(6!)^2} < 0.000988. \tag{30}$$

Similarly,

$$C_7 \leq \frac{1}{6!7!}\left(1-\frac{7}{8}\right)^6\left(1-\frac{6}{8}\right)^{-1}8^6 = \frac{4}{6!7!} < 0.000002. \tag{31}$$

In conjunction with (27), (28), (29), (30) and (31), we conclude that

$$\begin{aligned} &C_3^* + C_4 + C_5^* + 10C_6 + 10C_7 \\ &\qquad < 0.35696 + 0.08001 + 0.03375 + 0.00988 + 0.00002 = 0.48062. \end{aligned}$$

It follows from this and (26) that

$$\begin{aligned} C = C(\delta) &= 1 - 2((C_3 - C_3'(\delta)) + C_4 + C_5^* + 10C_6 + 10C_7) \\ &> 1 - 2 \times 0.481 - C_0\delta \\ &= 0.038 - C_0\delta \geq 0.01, \end{aligned}$$

providing that $\delta = \delta(K) > 0$ is small enough. This shows (25), as required.

5. Mean Value Estimates

In this section we quote three mean value theorems from [5, 8] so as to provide for our proof of Propositions. The following Lemma A is [8, Théorème 1]. Lemmas B and C are [5, Theorem 3] and [5, Theorem 5*], respectively.

For a sequence $\alpha = (\alpha_k)$, $x \ll k \ll x$, put

$$S((\alpha); Q) = \sum_{\substack{Q < q \le 2Q \\ (q,a)=1}} \left| \sum_{k \equiv a(q)} \alpha_k - \frac{1}{\varphi(q)} \sum_{(k,q)=1} \alpha_k \right|.$$

Here $0 < |a| < \mathscr{L}^B$ with a constant $B > 0$. Let $\kappa \ge 2$ be an integer. If $\alpha_k \ll \tau_\kappa(k)$ we call α of order κ. All sequences in this section are supposed to be of order κ. Let $\varepsilon > 0$ be a fixed small number. Our goal is to get that

$$S((\alpha); Q) \ll x\mathscr{L}^{-A} \tag{32}$$

for any $A > 0$, under some assumption on α. Here the implied $\ll$-constant may depend on B, κ, ε and A.

Lemma A. *Let* $x \ll LM \ll x$; $L, M \gg x^\varepsilon$;

$$\zeta = (\zeta_l), \quad l \sim L; \quad \xi = (\xi_m), \quad m \sim M; \quad \alpha = \zeta * \xi.$$

Suppose that, for any $d \ge 1$, $b \ne 0$, $r \ge 1$, $(r, b) = 1$ *and* $E > 0$,

$$\sum_{\substack{l \equiv b(r) \\ (l,d)=1}} \zeta_l = \frac{1}{\varphi(r)} \sum_{\substack{(l,r)=1 \\ (l,d)=1}} \zeta_l + (\tau_\kappa(d) L \mathscr{L}^{-E}). \tag{33}$$

Then we have (32), *provided that*

$$Q^2 x^{\varepsilon - 1} \ll L \ll x^{5/6 - \varepsilon} Q^{-4/3}.$$

Lemma B. *Let* $x \ll LMN \ll x$; $L, M, N \gg x^\varepsilon$;

$$\zeta = (\zeta_l), \quad l \sim L; \quad \xi = (\xi_m), \quad m \sim M; \quad \eta = (\eta_n), \quad n \sim N; \quad \alpha = \zeta * \xi * \eta.$$

Suppose that if $p | lmn$ *with* $p < \exp(\mathscr{L}(\log \mathscr{L})^{-2})$ *then* $\zeta_l \xi_m \eta_n = 0$. *Let* ζ *fulfill* (33). *Then we have* (32), *provided that*

$$LM \gg Qx^\varepsilon,$$

$$L^3 M^2 \ll Qx^{1-\varepsilon},$$

$$(L + M) L^2 M^4 \ll x^{2-\varepsilon}.$$

Lemma C. *Let* $x \ll LMN \ll x$; $L, M, N \gg x^\varepsilon$; $2 \le z \ll \exp(\mathscr{L}(\log \mathscr{L})^{-2})$. *Define*

$$\zeta_l = \begin{cases} \Phi_z(l), & \text{if } L \le l \le L_1 < 2L \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi = (\xi_m), m \sim M$; $\eta = (\eta_n), n \sim N$; $\alpha = \zeta * \xi * \eta$. *Then we have* (32), *provided that*

$$M \ll x^{1-\varepsilon} Q^{-1},$$

$$MN^4 \ll x^{2-\varepsilon} Q^{-1},$$

$$MN^2 \ll x^{2-\varepsilon} Q^{-2}.$$

6. Proof of Proposition 1

First of all, we remember the definition (6) of Θ. We shall decompose the summand of Θ into some suitable form for Lemmas A and C. To this end, for $k \in \mathscr{J}$, put

$$\Xi_0(k) = \Bigg(\sum_{\substack{d|k \\ d<I}} \mu(d)\Bigg) \Phi_H(k).$$

Since Φ is completely multiplicative, we see

$$\Xi_0(k) = \sum_{\substack{dl=k \\ d<I}}\sum \mu(d)\Phi_H(d)\Phi_H(l). \tag{34}$$

We first handle $\Phi_H(l)$. Let $\Delta = 1 + \mathscr{L}^{-8}$ and $\mathscr{U} = (H\Delta^{-i})_{1 \le i \le I_0}$ where I_0 is determined by the inequality $H\Delta^{-(I_0-1)} > \exp(\mathscr{L}(\log \mathscr{L})^{-2}) \ge H\Delta^{-I_0} = Z$, say. Then we see that $[Z, H) = \bigcup_{U \in \mathscr{U}} [U, U\Delta)$ and $|\mathscr{U}| = I_0 \ll \mathscr{L}^9$.

Now, Lemma 1 shows that

$$\Phi_H(l) = \Phi_Z(l) - \sum_{\substack{pn=l \\ Z \le p < H}}\sum \Phi_p(n).$$

For $p \in [U, U\Delta)$, $U \in \mathscr{U}$, we change $\Phi_p(n)$ by $\Phi_U(n)$ with the cost of

$$\Phi_p(n) - \Phi_U(n) = \sum_{\substack{p'm=n \\ U \le p' < p}}\sum \Phi_{p'}(m)$$

by Lemma 1. Thus we have that

$$\Phi_H(l) = \Phi_Z(l) - \sum_{U \in \mathscr{U}} \sum_{\substack{pn=l \\ U \le p < U\Delta}}\sum \Phi_U(n) + O\Bigg(\sum_{U \in \mathscr{U}} \sum_{\substack{p'pm=l \\ U \le p' < p < U\Delta}}\sum\sum 1\Bigg). \tag{35}$$

When $Z \le U < G := x^{1/16}$, we decompose $\Phi_U(n)$ one more. Lemma 2 shows that

$$(36)\qquad \Phi_U(n) = \Phi_U(n)\Phi_Z(n) = \sum_{\substack{el'=n \\ e<4G/U}}\sum \mu\Psi_U\Phi_Z(e)\Phi_Z(l') + \sum_{\substack{el'=n \\ 4G/U\le e \\ e/p(e)<4G/U}}\sum \mu\Psi_U\Phi_Z(e)\Phi_{p(e)}(l').$$

In the above second term, we see that $Z \le p(e) < U$, because of $\Psi_U\Phi_Z(e)$. Then, for $p(e) \in [V, V\Delta)$ with $V \in \mathscr{U}$ and $V < U$, we replace $\Phi_{p(e)}(l')$ by $\Phi_V(l')$. By Lemma 1, the resulting error in (36) is

$$(37)\qquad \ll \sum_{el'=n}\sum \Psi_U(e) \sum_{\substack{V\in\mathscr{U} \\ V<U}} \sum_{\substack{p''m=l' \\ V\le p''<p(e)<V\Delta}}\sum \Phi_{p''}(m)$$

$$\ll \sum_{\substack{V\in\mathscr{U} \\ V<U}} \sum_{\substack{p'fp''m=n \\ V\le p''<p'<V\Delta}}\sum\sum\sum \Psi_U(f),$$

which contributes to (35)

$$(38)\qquad \ll \sum_{V\in\mathscr{U}} \sum_{\substack{p'p''mg=l \\ V\le p''<p'<V\Delta}}\sum\sum\sum\sum \left(\sum_{U\in\mathscr{U}} \sum_{\substack{pf=g \\ U\le p<U\Delta}}\sum \Psi_U(f)\right).$$

Since the above inner sum is

$$(39)\qquad \le \sum_{U\in\mathscr{U}} \sum_{\substack{pf=g \\ U\le p<U\Delta}}\sum \Psi_p(f) = \sum_{\substack{pf=g \\ Z\le p<H}}\sum \Psi_p(f) \le \Psi_H(g),$$

the O-term in (35) is absorbed by (38), with $g = 1$. It therefore follows from (35), (36), (38) and (39) that

$$(40)\qquad \Phi_H(l) = \Phi_Z(l) - \sum_{\substack{U\in\mathscr{U} \\ U<G}} \sum_{\substack{pel'=l \\ U\le p<U\Delta \\ e<4G/U}}\sum\sum \mu\Psi_U\Phi_Z(e)\Phi_Z(l')$$

$$- \sum_{\substack{U,V\in\mathscr{U} \\ V<U<G}}\sum \sum_{\substack{pel'=l \\ U\le p<U\Delta \\ 4G/U\le e<4G/Up(e) \\ V\le p(e)<V\Delta}}\sum\sum \mu\Psi_U\Phi_Z(e)\Phi_V(l')$$

$$- \sum_{\substack{U\in\mathscr{U} \\ U\ge G}} \sum_{\substack{pn=l \\ U\le p<U\Delta}}\sum \Phi_U(n) + O\left(\sum_{U\in\mathscr{U}} \sum_{\substack{p'pmg=l \\ U\le p'<p<U\Delta}}\sum\sum\sum\sum \Psi_H(g)\right)$$

$$= \sum_{j=1}^{5} \rho_j(l), \quad \text{say.}$$

Substituting this into (34), we have that $\Xi_0(k) = \sum_{j=1}^{5} \Xi_j(k)$ where

$$\Xi_j(k) = \sum\sum_{\substack{dl=k\\ d<I}} \mu\Phi_H(d)\rho_j(l).$$

First we deal with Ξ_2. Put $pe = g$. Then we observe that $g < U\Delta \cdot 4G/U = 4\Delta G < 8G$, and that the coefficient of g has the same property as (39). Thus Ξ_2 is written as

$$\sum_{U\in\mathscr{U}} \sum\sum\sum_{\substack{dgl=k\\ d<I\\ g<8G}} \mu\Phi_H(d)\alpha_U(g)\Phi_Z(l) \tag{41}$$

where $\sum_{U\in\mathscr{U}} |\alpha_U(g)| \ll \Psi_H\Phi_Z(g)$, in particular, $\alpha_U(g) \ll \Psi_H\Phi_Z(g)$. Also Ξ_1 has the same expression as above, with $g = 1$.

We turn to Ξ_3. Put $dl' = m$ and $pe = l$. The coefficient of m is $O(\tau(m))$ simply. For given $V < U < G$, we see that $4G = U \cdot 4G/U \leq l = pe < U\Delta \cdot 4G/U \cdot V\Delta \leq 4U\Delta G \leq 4G^2 = 4H$. Let $\varpi_U(n)$ denote the characteristic function of primes $\in [U, U\Delta)$. Then the coefficient of l is written as the convolution

$$(\varpi_U * \alpha_{U,V})(l) \tag{42}$$

with some $\alpha_{U,V}(e) \ll \Psi_U(e)$. Hence Ξ_3 is put into the form

$$\sum\sum_{U,V\in\mathscr{U}} \sum\sum_{\substack{lm=k\\ G\leq l<4H}} \zeta_{U,V}(l)\xi_V(m) \tag{43}$$

where $\zeta_{U,V}(l) \ll 1$ has the expression (42) and $\xi_V(m) \ll \tau(m)$.

Ξ_4 has also the same form as (43), with $\zeta = \varpi$. Next, $\Xi_5(k)$ is bounded by

$$\ll \sum_{U\in\mathscr{U}} \sum\sum\sum\sum\sum_{\substack{p'pmgd=k\\ U\leq p'<p<U\Delta}} \Psi_H(g)\Phi_H(d) \ll \sum_{U\in\mathscr{U}} \sum\sum\sum_{\substack{p'pn=k\\ U\leq p'<p<U\Delta}} \tau(n). \tag{44}$$

Thus $\Xi_0(k)$ is written as the sum of (41), (43) and (44).

We have to rearrange (41) furthermore. We shall split up $\mu(d)\Phi_H(d)$ in (41). Since if $d > 1$ then $x^{1/8} = H \leq d < I = x^{1/2}$, d has at most three prime factors. Thus, for $d > 1$, we have that

$$\mu(d)\Phi_H(d) \doteq -\Phi_H(d) + \sum\sum_{\substack{p'p=d\\ H\leq p', p}} 4 + O\left(\sum_{\substack{p^2|d\\ H\leq p<x^{1/4}=H^2}} 1\right). \tag{45}$$

Lemma 2 shows, as in (36) and (37), that

$$\Phi_H(d) = \sum_{\substack{ef=d \\ e<H}}\sum \mu\Psi_H\Phi_Z(e)\Phi_Z(f) + \sum_{V\in\mathscr{U}} \sum_{\substack{ef=d \\ H\le e<Hp(e) \\ V\le p(e)<V\Delta}}\sum \mu\Psi_H\Phi_Z(e)\Phi_V(f) \tag{46}$$

$$+ O\left(\sum_{V\in\mathscr{U}} \sum_{\substack{p'pn=d \\ V\le p'<p<V\Delta}}\sum\sum \tau(n)\right).$$

In the above second sum, we see that $H \le e < Hp(e) < H^2$, because of $\Psi_H(e)$. Combining (46) with (45), we have that

$$\mu(d)\Phi_H(d) = \sum_{\substack{ef=d \\ e<H}}\sum \beta(e)\Phi_Z(f) + \sum_{V\in\mathscr{U}} \sum_{\substack{ef=d \\ H\le e<H^2}}\sum \beta_V(e)\gamma_V(f) + r(d) \tag{47}$$

where $\beta, \beta_V, \gamma_V \ll \Phi_Z$ and r consists of O-terms in (45) and (46). Here we notice that the second term in the right hand side of (45) is put into that in (47), since $\min(p', p)^2 \le p'p < I = H^4$. By substituting (47), (41) becomes

$$\sum_{U\in\mathscr{U}} \sum_{\substack{efgl=k \\ e<H \\ ef<I,\, g<8G}}\sum\sum\sum \beta(e)\Phi_Z(f)\alpha_U(g)\Phi_Z(l) \tag{48}$$

$$+ \sum_{U,V\in\mathscr{U}}\sum \sum_{\substack{efgl=k \\ H\le e<H^2 \\ ef<I,\, g<8G}}\sum\sum\sum \beta_V(e)\gamma_V(f)\alpha_U(g)\Phi_Z(l)$$

$$+ O\left(\sum_{U\in\mathscr{U}} \sum_{dgl=k}\sum\sum |r(d)|\,|\alpha_U(g)|\Phi_Z(l)\right).$$

The above O-term is

$$\ll \sum_{dm=k}\sum |r(d)|\tau(m) \tag{49}$$

$$\ll \sum_{\substack{p^2n=k \\ H\le p<H^2}}\sum \tau_3(n) + \sum_{V\in\mathscr{U}} \sum_{\substack{p'pn=k \\ V\le p'<p<V\Delta}}\sum\sum \tau_4(n),$$

which is larger than (44). As for the second term of (48), we write $fg = m$ and $e = n$. Then $Hm < em = efg < 8IG = 8IHG^{-1}$ or $m < 8IG^{-1}$, and $H \le n < H^2$. We treat the condition $nf = ef < I$ by using

$$\int_{-1/2}^{1/2} e^{-2\pi itn}\left(\sum_{mf<I} e^{2\pi itm}\right) dt = \int_{-1/2}^{1/2} e^{-2\pi itn}T(t; f)\, dt, \quad \text{say.}$$

Since $T(t;f) \ll \min(I,|t|^{-1})$, the second term in (48) is written as

$$(50)\qquad \sum\sum_{U,V\in\mathscr{U}}\int_{-1/2}^{1/2}\left(\sum\sum\sum_{\substack{lmn=k\\ m<8IG^{-1}\\ H\le n<H^2}}\Phi_Z(l)\xi_{t,U,V}(m)\eta_{t,V}(n)\right)\min(I,|t|^{-1})\,dt$$

where $\xi_{t,U,V}(m)=\sum\sum_{fg=m}\gamma_V(f)\alpha_U(g)T(t;f)/\min(I,|t|^{-1}) \ll \tau(m)$ and $\eta_{t,V}(n) = \beta_V(n)e^{-2\pi itn} \ll 1$.

We proceed to the first term in (48). Lemma 2 shows that

$$\Phi_Z(f)=\sum\sum_{\substack{dm=f\\ d<F}}\mu(d)\Psi_Z(d)+O\left(\sum\sum_{\substack{dm=f\\ F\le d<FZ}}\Psi_Z(d)\right)$$

where $F=x^\eta$ with a fixed small number $\eta>0$. We also have the similar decomposition of $\Phi_Z(l)$. Then we replace $\Phi_Z(f)\Phi_Z(l)$ by

$$\sum\sum_{\substack{dm=f\\ d<F}}\mu(d)\Psi_Z(d)\sum\sum_{\substack{rn=l\\ r<F}}\mu(r)\Psi_Z(r).$$

The resulting error term contributes to (48)

$$(51)\qquad \ll \sum\sum_{\substack{dm=k\\ F\le d<FZ}}\Psi_Z(d)\tau_5(m).$$

Write $ed=h$. Then $h<HF$ and $hm=edm=ef<I$. Therefore the first term in (48) becomes

$$(52)\qquad \sum_{U\in\mathscr{U}}\sum\sum\sum\sum_{\substack{ghmn=k\\ g<8GF\\ h<HF,\,hm<I}}\xi_U(g)\eta(h)$$

where

$$\sum_{U\in\mathscr{U}}|\xi_U(g)| \ll \sum_{U\in\mathscr{U}}\sum\sum_{g'r=g}|\alpha_U(g')|\Psi_Z(r) \ll \sum\sum_{g'r=g}\Phi_Z(g')\Psi_Z(r)\ll 1,$$

$$\eta(h)\ll\sum\sum_{ed=h}|\beta(e)|\Psi_Z(d)\ll\sum\sum_{ed=h}\Phi_Z(e)\Psi_Z(d)\ll 1.$$

On summing up the above decomposition, Ξ_0 is written as the sum of "type I" (52), (50), "type II" (43) and "error" (51), (49). Consequently,

$$(53)\qquad D_0=D_0(x,y;q,a):=\sum_{\substack{k\in\mathscr{J}\\ k\equiv a(q)}}\Xi_0(k)-\frac{1}{\varphi(q)}\sum_{\substack{h\in\mathscr{J}\\ (k,q)=1}}\Xi_0(k)$$

$$=D_{I/1}+D_{I/2}+D_{II}+D_{III}.$$

Here $D_{I/1}$ comes from "type I" (52):

$$D_{I/1} \ll \sum_{g<8GH} \sum_{h<HF} \left| \sum_{\substack{ghmn \in \mathcal{J} \\ ghmn \equiv a(q) \\ hm<I}} \sum 1 - \frac{1}{\varphi(q)} \sum_{\substack{ghmn \in \mathcal{J} \\ (ghmn,q)=1 \\ hm<I}} \sum 1 \right|. \tag{54}$$

The contribution of "type I" (50) is $D_{I/2}$, which will be found admissible.

$$\begin{aligned} S_I &:= \sum_{q \in \mathcal{M}} |D_{I/2}| \\ &\ll \mathcal{L}^{19} \sup \sum_{\substack{Q<q\le 2Q \\ (q,a)=1}} \left| \sum_{\substack{lmn \in \mathcal{J} \\ lmn \equiv a(q) \\ m<8IG^{-1} \\ H \le n < H^2}} \sum\sum \Phi_Z(l)\xi_m\eta_n - \frac{1}{\varphi(q)} \sum_{\substack{lmn \in \mathcal{J} \\ (lmn,q)=1 \\ m<8IG^{-1} \\ H \le n < H^2}} \sum\sum \Phi_Z(l)\xi_m\eta_n \right| \\ &= \mathcal{L}^{19} \sup S_I', \quad \text{say,} \end{aligned} \tag{55}$$

where the supremum is over all ξ and η satisfying $\xi_m \ll \tau(m)$ and $\eta_n \ll 1$. Next D_{II} arises from "type II" (43).

$$\begin{aligned} S_{II} := \sum_{q \in \mathcal{M}} |D_{II}| &\ll \mathcal{L}^{18} \sup \sum_{\substack{Q<q\le 2Q \\ (q,a)=1}} \left| \sum_{\substack{lm \in \mathcal{J} \\ lm \equiv a(q) \\ G \le l < 4H}} \sum \zeta_l \xi_m - \frac{1}{\varphi(q)} \sum_{\substack{lm \in \mathcal{J} \\ (lm,q)=1 \\ G \le l < 4H}} \sum \zeta_l \xi_m \right| \\ &= \mathcal{L}^{18} \sup S_{II}', \quad \text{say,} \end{aligned} \tag{56}$$

where the supremum is taken over all ζ having the expression (42) and all ξ with $\xi_m \ll \tau(m)$. Finally D_{III}, which corresponds to "error" (51) and (49), is divided into three parts.

$$D_{III} = D_{III/1} + D_{III/2} + D_{III/3}$$

where

$$D_{III/1} \ll \sum_{\substack{dn \in \mathcal{J} \\ dn \equiv a(q) \\ F \le d < FZ}} \sum \Psi_Z(d)\tau_5(n) + \varphi(q)^{-1} \sum_{\substack{dn \in \mathcal{J} \\ F \le d < FZ}} \sum \Psi_Z(d)\tau_5(n), \tag{57}$$

$$D_{III/2} \ll \sum_{U \in \mathcal{U}} \sum_{\substack{p'pn \in \mathcal{J} \\ p'pn \equiv a(q) \\ U \le p' < p < U\Delta}} \sum\sum \tau_4(n) + \varphi(q)^{-1} \sum_{U \in \mathcal{U}} \sum_{\substack{p'pn \in \mathcal{J} \\ U \le p' < p < U\Delta}} \sum\sum \tau_4(n), \tag{58}$$

$$S_{III} := \sum_{q \in \mathcal{M}} |D_{III/3}| \ll \sum_{\substack{p^2 n \in \mathcal{J} \\ H \le p < H^2}} \sum \tau_3(n)(\tau(p^2 n - a) + \mathcal{L}). \tag{59}$$

We shall estimate these sums in the next section.

7. Proof of Proposition 1, Continued

In remains to show that $D_i \ll yq^{-1}\mathscr{L}^{-4}$, $(i = I/1, III/1, III/2)$, and that, for any $A > 0$, $S_i \ll x\mathscr{L}^{-A}$, $(i = I, II, III)$. We begin with S_{III} given by (59). We have that, for any $\varepsilon > 0$,

$$S_{III} \ll x^{\varepsilon} \sum_{H \le p < H^2} xp^{-2} \ll x^{1+\varepsilon}H^{-1} \ll x^{7/8+\varepsilon},$$

which is negligible. The first term of $D_{III/2}$ given by (58) is

$$\begin{aligned}
&\ll \sum_{U \in \mathscr{U}} \sum_{U \le p' < p < U\Delta} \frac{y\mathscr{L}^3}{qp'p} \\
&\ll yq^{-1}\mathscr{L}^3 \sum_{U \in \mathscr{U}} \sum_{U \le p' < U\Delta} \frac{1}{p' \log p'} \sum_{U \le p < U\Delta} \frac{\log p}{p} \\
&\ll yq^{-1}\mathscr{L}^3 \sum_{Z \le p < H} \frac{1}{p \log p} \log \Delta \\
&\ll yq^{-1}\mathscr{L}^{-5},
\end{aligned}$$

which is also acceptable. The second term of $D_{III/2}$ is easier to treat and bounded similarly. The first part of $D_{III/1}$ given by (57) is

$$\begin{aligned}
&\ll \sum_{F \le d < FZ} \Psi_Z(d) \frac{y\mathscr{L}^4}{qd} \\
&\ll yq^{-1}\mathscr{L}^4 \exp(-\log F/\log Z) \\
&\ll yq^{-1}\mathscr{L}^4 \exp(-\eta\mathscr{L}^2)
\end{aligned}$$

by Lemma 3. This is satisfactory as well. The second part is similar. Hence

$$D_{III} = O(yq^{-1}\mathscr{L}^{-4}) + A.R.. \tag{60}$$

Next we handle $D_{I/1}$ given by (54). Since $|\{n \in \mathscr{I} \mid (n,q) = 1\}| = \varphi(q)/q|\mathscr{I}| + O(\tau(q))$ for any interval $\mathscr{I}$, we have that

$$D_{I/1} \ll \sum_{\substack{g < 8GF \\ (gh,q)=1}} \sum_{h < HF} \left| \sum_{\substack{hm < I \\ (m,q)=1}} \left(\sum_{\substack{(x-y)/ghm < n \le x/ghm \\ n \equiv \overline{ghm}a(q)}} 1 - \frac{y}{qghm} + O\left(\frac{\tau(q)}{\varphi(q)}\right) \right) \right|.$$

It follows from the method deriving (12) that the sum inside the absolute value symbol is bounded by

$$\ll \left(q^{1/2+\varepsilon} + \frac{x^{1-\varepsilon}}{qgh}\right) + \sum_m q^{\varepsilon-1}$$

for any $\varepsilon > 0$. Since $q \ll x^{17/32}$, we have that

$$\begin{aligned} D_{I/1} &\ll q^{1/2} GHx^{\varepsilon+2\eta} + x^{1-\varepsilon/2}q^{-1} + GIx^{\eta+\varepsilon}q^{-1} \\ &\ll q^{1/2}x^{3/16+\varepsilon+2\eta} + x^{1-\varepsilon/2}q^{-1} \\ &\ll x^{1-\varepsilon/2}q^{-1}, \end{aligned} \tag{61}$$

providing that both $\eta > 0$ and $\varepsilon > 0$ are small enough.

We proceed to S_I given by (55). Lemmas 2 and 3 yield that, for any interval $\mathscr{I}$ and $(b, q) = 1$,

$$\begin{aligned} \sum_{\substack{l \in \mathscr{I} \\ l \equiv b(q)}} \Phi_Z(l) - \varphi(q)^{-1} \sum_{\substack{l \in \mathscr{I} \\ (l,q)=1}} \Phi_Z(l) &\ll \sum_{d<F} \Psi_Z(d) \left| \sum_{\substack{l \in \mathscr{I}, d|l \\ l \equiv b(q)}} 1 - \varphi(q)^{-1} \sum_{\substack{l \in \mathscr{I}, d|l \\ (l,q)=1}} 1 \right| \\ &\quad + \sum_{F \le d < FZ} \Psi_Z(d) \left(\sum_{\substack{l \in \mathscr{I}, d|l \\ l \equiv b(q)}} 1 + \varphi(q)^{-1} \sum_{\substack{l \in \mathscr{I}, d|l \\ (l,q)=1}} 1 \right) \\ &\ll \sum_{d<F} 1 + \sum_{F \le d < FZ} \Psi_Z(d) \left(\frac{|\mathscr{I}|}{qd} + 1 \right) \\ &\ll FZ + |\mathscr{I}| q^{-1} \exp(-\log F / \log Z) \\ &\ll x^{2\eta} + |\mathscr{I}| q^{-1} \exp(-\eta (\log \mathscr{L})^2). \end{aligned}$$

Hence, the part of S_I with $m < H$ is

$$\begin{aligned} &\ll \mathscr{L}^{19} \sum_{\substack{Q<q\le 2Q \\ (q,a)=1}} \sum_{\substack{m<H \\ H \le n < H^2}} \sum \tau(m) \left(x^{2\eta} + \frac{x}{qmn} \exp(-\eta(\log \mathscr{L})^2) \right) \\ &\ll \mathscr{L}^{19} (QH^3 \mathscr{L} x^{2\eta} + x \mathscr{L}^3 \exp(-\eta (\log \mathscr{L})^2)) \\ &\ll x^{29/32+3\eta} + x\mathscr{L}^{21} \exp(-\eta(\log \mathscr{L})^2), \end{aligned} \tag{62}$$

which is satisfactory.

When $m \ge H$, we appeal to Lemma C, on taking $0 < \varepsilon < \delta$. We have to remove the condition $lmn \in \mathscr{J}$ from S_I'. To this end, we divide the summation ranges into short intervals, as a standard way. Write $k \approx X$ for $k \in [X, X(1+\mathscr{L}^{-W}))$, where $W \geq 1$ is a constant. This splitting argument shows that

$$S_I \ll \mathscr{L}^{19}((\mathscr{L}^{1+W})^3 \sup S_I'' + x\mathscr{L}^4\mathscr{L}^{-W})$$

where S_I'' is the same expression as S_I', except for the new condition $l \approx L$, $m \approx M$, $n \approx N$ in place of $lmn \in \mathscr{J}$, and the supremum is additionally taken over L, M, N such that

$$x \ll LMN \ll x, \quad H \ll M \ll IG^{-1}, \quad H \ll N \ll H^2.$$

For given $E > 0$, we choose $W = E + 23$. Then S_I'' satisfies all assumptions of Lemma C, since

$$x^{1/8} \ll M \ll x^{7/16} < x^{15/32},$$

$$MN^4 \ll IG^{-1}H^8 \ll x^{1+7/16} < x^{1+15/32},$$

$$MN^2 \ll IG^{-1}H^4 = x^{15/16}$$

and $N \gg x^{1/8}$. It therefore follows that

$$S_I \ll x\mathscr{L}^{-E}, \tag{63}$$

together with (62).

To estimate S_{II} given by (56), we use Lemma A. Since (ζ_l) contains the characteristic function of primes in the interval $[U, U(1+\mathscr{L}^{-8}))$ with $U \gg \exp(\mathscr{L}(\log \mathscr{L})^{-2})$ as its convolution factor, the Siegel-Walfisz theorem [16, Kap. IV, Satz 8.3] ensures that (ζ_l) fulfilles the assumption (33) of Lemma A. After removing the condition $lm \in \mathscr{J}$ as above, we have that

$$S_{II} \ll \mathscr{L}^{18}((\mathscr{L}^{W+1})^2 \sup S_{II}'' + x\mathscr{L}^3\mathscr{L}^{-W})$$

where $''$ in S_{II}'' indicates the condition $lm \in \mathscr{J}$ of S_{II}' is replaced by $l \approx L$, $m \approx M$, and the supremum is taken, in addition, over L, M such that

$$x \ll LM \ll x, \quad G \ll L \ll H.$$

For given $E > 0$, we choose $W = E + 21$. And we take $0 < \varepsilon < \delta$ in Lemma A. Since

$$Q^2x^{\varepsilon-1} \ll x^{1/16} = G \quad \text{and} \quad x^{5/6-\varepsilon}Q^{-4/3} \gg x^{1/8} = H,$$

Lemma A shows

$$S_{II} \ll x\mathscr{L}^{-E}. \tag{64}$$

Therefore Proposition 1 follows from (60), (61), (63) and (64).

8. Proof of Proposition 2

We remember the definition (3) of $\mathscr{H}(x)$. Put, for $k \in \mathscr{J}$,

$$\Upsilon_0(k) = \left(\sum_{\substack{d|k \\ d \in \mathscr{H}(x)}} 1\right)\Phi_H(k) = \sum\sum\sum_{p_1 p_2 p_3 l = k}^{*}\sum \Phi_H(l)$$

where $*$ indicates the condition:

$$H \le p_3 < p_2 < p_1, \quad I = x^{1/2} < p_1 p_2 p_3, \quad (p_1 p_2)^2 p_3 < x. \tag{65}$$

We decompose $\Phi_H(l)$ into the same form as (40) or the combination of (41), (43) and (44) to obtain that

$$\Upsilon_0(k) = \sum_{U \in \mathscr{U}}\sum\sum\sum^{*}_{\substack{p_1 p_2 p_3 g l = k \\ g < 8G}}\sum\sum \alpha_U(g)\Phi_Z(l) + \Upsilon_2(k) + \Upsilon_3(k) \tag{66}$$

where Υ_2 and Υ_3 are written as the form (43) and (44), respectively. Then, as before, Υ_2 and Υ_3 are handled in the same way as S_{II} and $D_{III/2}$, respectively.

Let Υ_1 denote the first term in (66). Put $p_1 p_2 = m$ and $p_3 g = n$. Then, since (65) implies $p_3^5 < (p_1 p_2)^2 p_3 < x$ and $p_3 \ge H$, we see that

$$H^2 < m = p_1 p_2 \le (x/p_3)^{1/2} \le (x/H)^{1/2} = IG^{-1},$$

$$mn = p_1 p_2 p_3 g = (p_1 p_2 p_3^{1/2}) p_3^{1/2} g < x^{1/2}(x^{1/5})^{1/2} 8G = 8x^{3/5}G,$$

$$H < n = p_3 g < 8x^{1/5}G.$$

We treat the condition (65) on p_3 by the integration

$$\int_{-1/2}^{1/2}\left(\sum_{\substack{H \le h < p_2 \\ I < p_1 p_2 h \\ (p_1 p_2)^2 h \le x}} e^{2\pi i t h}\right) e^{-2\pi i t p_3}\,dt = \int_{-1/2}^{1/2} T(t; p_1, p_2) e^{-2\pi i t p_3}\,dt, \quad \text{say.}$$

Here note that $T(t; p_1, p_2) \ll \min(I, |t|^{-1})$. To deal with the boundary condition $k \in \mathscr{J}$, we split up the summation ranges into short intervals. Let $n \approx N$ denote the condition $n \in [N, N(1 + \mathscr{L}^{-11}))$. Therefore, $\Upsilon_1(k)$, $k \in \mathscr{J}$, is written as

$$\Upsilon_1(k) = \sum_{U \in \mathscr{U}}\sum_{L}\sum_{M}\sum_{N}\int_{-1/2}^{1/2}\left(\sum\sum\sum_{\substack{lmn = k \\ l \approx L m \approx M n \approx N}} \Phi_Z(l)\xi_t(m)\eta_{t,U}(n)\right)\min(I, |t|^{-1})\,dt$$

where L, M, N run through the powers of $\Delta = (1 + \mathscr{L}^{-11})$ satisfying

(67) $\quad x \ll LMN \ll x, \quad H^2 \ll M \ll IG^{-1}, \quad H \ll N \ll x^{1/5}G, \quad MN \ll x^{3/5}G,$

and

$$\xi_t(m) = \sum_{\substack{p_1 p_2 = m \\ p_2 < p_1}}\sum T(t; p_1, p_2)/\min(I, |t|^{-1}) \ll 1, \quad \eta_{t,U}(n) = \sum_{\substack{p_3 g = n \\ H \le p_3}}\sum \alpha_U(g) e^{-2\pi i t p_3}.$$

Here we notice that, by (41),

$$\sum_{U \in \mathscr{U}} |\eta_{t,U}(n)| \ll \sum_{pg=n} \sum_{H \le p} \sum_{U \in \mathscr{U}} |\alpha_U(g)| \ll \sum_{pg=n} \sum_{H \le p} \Psi_H(g) \le 1.$$

Let Υ' be the part of Υ_1 restricted by the condition $x - y < LMN$ and $LMN\Delta^3 \le x$, that implies $lmn \in \mathscr{I}$. Let Υ'' be the part of Υ_1 with the condition $LMN \le x - y$ or $x < LMN\Delta^3$, that means $lmn \le (x - y)\Delta^3$ or $lmn > x\Delta^{-3}$. Then we have that

$$\sum_{\substack{k \in \mathscr{I} \\ k \equiv a(q)}}\sum \Upsilon''(k) \ll \mathscr{L} \sum_{\substack{x - y < k \le (x-y)\Delta^3 \\ \text{or } x\Delta^{-3} < k \le x \\ k \equiv a(q)}} \tau_3(k) \ll \mathscr{L} x \mathscr{L}^{-11} q^{-1} \mathscr{L}^2 \ll x q^{-1} \mathscr{L}^{-8} = y q^{-1} \mathscr{L}^{-4},$$

which is acceptable. Similarly,

$$\varphi(q)^{-1} \sum_{\substack{k \in \mathscr{I} \\ (k,q)=1}} \Upsilon''(k) \ll y \varphi(q)^{-1} \mathscr{L}^{-4},$$

which is also acceptable. Moreover we have that

$$\sum_{\substack{Q < q \le 2Q \\ (q,a)=1}} \left| \sum_{\substack{k \in \mathscr{I} \\ k \equiv a(q)}} \Upsilon'(k) - \frac{1}{\varphi(q)} \sum_{\substack{k \in \mathscr{I} \\ (k,q)=1}} \Upsilon'(k) \right|$$

$$\ll \mathscr{L}^9 (\mathscr{L}^{12})^3 \mathscr{L}$$

$$\cdot \sup \sum_{\substack{Q < q \le 2Q \\ (q,a)=1}} \left| \sum_{\substack{l \approx L m \approx M n \approx N \\ lmn \equiv a(q)}}\sum\sum \Phi_Z(l) \xi_m \eta_n - \frac{1}{\varphi(q)} \sum_{\substack{l \approx L m \approx M n \approx N \\ (lmn,q)=1}}\sum\sum \Phi_Z(l) \xi_m \eta_n \right|$$

where the supremum is taken over (ξ_m) with $\xi_m \ll \tau(m)$, (η_n) with $\eta_n \ll 1$ and L, M, N satisfying (67). We take $0 < \varepsilon < \delta$ in Lemma C. Then all assumptions of Lemma C are satisfied. Indeed, by (67),

$$M \ll IG^{-1} = x^{7/16} < x^{15/32}$$

$$MN^4 \ll x^{3/5}G(x^{1/5}G)^3 = x^{6/5}G^4 = x^{1+9/20} < x^{1+15/32}$$

$$MN^2 \ll x^{3/5}Gx^{1/5}G = x^{4/5}G^2 = x^{37/40} < x^{15/16}.$$

Thus Proposition 2 follows.

9. Proof of Proposition 3

We recall the definition (21) of $\mathscr{B}$. To make the variables p_j $(j = 1, 2, 3)$ separated, we define the sequence

$$\mathscr{P} = (H\Delta^m)_{0 \le m \le M_0}; \quad \Delta = 1 + \mathscr{L}^{-W}, \quad M_0 = (1/2)\mathscr{L}^{W+1}$$

where $W \ge 1$ is a constant. For $P \in \mathscr{P}$, we write $n \approx P$ for $P \le n < P\Delta$. We divide $\mathscr{B}$ by the restriction $p_j \approx P_j \in \mathscr{P}$ $(j = 1, 2, 3)$. Let $\mathscr{R}$ denote the set of $(P_1, P_2, P_3) \in \mathscr{P}^3$ such that

$$(x - y) < P_1P_2P_3, \quad P_1P_2P_3\Delta^3 \le x, \quad P_3 < P_2 < P_1 < P_2P_3,$$

$$(P_2\Delta)^3(P_3\Delta)^2 \le x^{\theta+1-2\delta}, \quad (P_2\Delta)^3(P_3\Delta)^4 \le x^{2-\delta}, \quad P_2P_3 > x^{\theta}.$$

If $(P_1, P_2, P_3) \in \mathscr{R}$ then $p_j \approx P_j$ $(j = 1, 2, 3)$ implies all conditions of $p_1p_1p_3 \in \mathscr{B}$. Hence

$$(68) \quad \sum_{q \in \mathscr{M}} \left| \sum_{\substack{k \in \mathscr{B} \\ k \equiv a(q)}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{k \in \mathscr{B} \\ (k,q)=1}} 1 \right|$$

$$\ll \sum\sum\sum_{\substack{P_j \ (j=1,2,3) \\ (P_1,P_2,P_3) \in \mathscr{R}}} \sum_{q \in \mathscr{M}} \left| \sum\sum\sum_{\substack{p_j \approx P_j \ (j=1,2,3) \\ p_1p_2p_3 \equiv a(q)}} 1 - \frac{1}{\varphi(q)} \sum\sum\sum_{\substack{p_j \approx P_j \ (j=1,2,3) \\ (p_1p_2p_3, q)=1}} 1 \right|$$

$$+ \sum\sum\sum_{\substack{P_j \ (j=1,2,3) \\ (P_1,P_2,P_3) \notin \mathscr{R}}} \left(\sum\sum\sum_{\substack{p_1p_2p_3 \in \mathscr{B} \\ p_j \approx P_j \ (j=1,2,3)}} \tau(p_1p_2p_3 - a) + \mathscr{L} \sum\sum\sum_{\substack{p_1p_2p_3 \in \mathscr{B} \\ p_j \approx P_j \ (j=1,2,3)}} 1 \right)$$

$$= S_1 + S_2, \quad \text{say.}$$

If $(P_1, P_2, P_3) \notin \mathscr{R}$, $p_j \approx P_j$ $(j = 1, 2, 3)$ and $p_1 p_2 p_3 \in \mathscr{B}$ then at least one of the following eight conditions holds:

$$
\begin{gathered}
x - y < p_1 p_2 p_3 \leq (x - y)\Delta^3; \quad x\Delta^{-3} < p_1 p_2 p_3 \leq x; \\
p_3 < p_2 < p_3\Delta; \quad p_2 < p_1 < p_2\Delta; \quad p_1 < p_2 p_3 < p_1 \Delta^2; \qquad (69) \\
x^{\theta+1-2\delta}\Delta^{-5} < p_2^3 p_3^2 \leq x^{\theta+1-2\delta}; \quad x^{2-\delta}\Delta^{-7} < p_2^3 p_3^4 \leq x^{2-\delta}; \quad x^{\theta} < p_2 p_3 < x^{\theta}\Delta^2.
\end{gathered}
$$

Thus it is not hard to see that

$$\sum\sum\sum_{\substack{P_j\ (j=1,2,3) \\ (P_1, P_2, P_3) \notin \mathscr{R}}} \sum\sum\sum_{\substack{p_1 p_2 p_3 \in \mathscr{B} \\ p_j \approx P_j\ (j=1,2,3)}} 1 \ll \sum\sum\sum_{\substack{x - y < p_1 p_2 p_3 \leq x \\ \text{one of (69)}}} 1 \ll x \mathscr{L}^{-W}.$$

Hence Cauchy's inequality shows that

$$S_2 \ll x \mathscr{L}^{2-W/2}. \tag{70}$$

As for S_1, we appeal to Lemma B, by taking $0 < \varepsilon < \delta$, $L = P_2$, $M = P_3$ and $N = P_1$. Then all conditions of Lemma B are fulfilled. Actually the set $\mathscr{B} \in \mathscr{C}$ is determined in this way. Hence we have that

$$S_1 \ll (\mathscr{L}^{W+1})^3 x \mathscr{L}^{-D} \tag{71}$$

for any $D > 0$.

For given $E > 0$, we choose $W = 2E + 2$ and $D = 7E + 9$. Then Proposition 3 follows from (68), (70) and (71).

This completes our proof of Theorem.

References

[1] K. Alladi, Moments of additive functions and the sequence of shifted primes, Pacific J. Math., **118** (1985), 261–275.

[2] R. C. Baker and G. Harman, The Brun-Titchmarsh theorem on average, Analytic Number Theory, vol. 1, Progress in Math., **138** (1996), 39–103.

[3] A. Balog, On sums over primes, Banach Center Publ., **17** (1985), 9–19.

[4] E. Bombieri, J. B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli, Acta Math., **156** (1986), 203–251.

[5] E. Bombieri, J. B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli II, Math. Ann., **277** (1987), 361–393.

[6] E. Bombieri, J. B. Friedlander and H. Iwaniec, Primes in arithmetic progressions to large moduli III, J. American Math. Soc., **2** (1989), 215–224.

[7] H. Davenport, Multiplicative Number Theory, (2nd ed. rev. by H. L. Montgomery), Springer-Verlag, 1980.

[8] E. Fouvry, Autor du théorème de Bombieri-Vinogradov, Acta Math., **152** (1984), 219–244.

[9] E. Fouvry, Autor du théorème de Bombieri-Vinogradov II, Ann. sciet. Éc. Norm. Sup., **20** (1987), 617–640.

[10] E. Fouvry and H. Iwaniec, On a theorem of Bombieri-Vinogradov type, Mathematika **27** (1980), 135–152.
[11] E. Fouvry and H. Iwaniec, Primes in arithmetic progressions, Acta Arith., **42** (1983), 197–218.
[12] G. Harman, Eratosthenes, Legendre, Vinogradov and beyond. The hidden power of the simplest sieve, Sieve Methods, Exponential Sums, and their Applications in Number Theory, London Math. Soc. Lecture Note Series, **237** (1997), 161–173.
[13] D. R. Heath-Brown, Sieve identities and gaps between primes, Astérisque, **94** (1982), 61–65.
[14] D. R. Heath-Brown, Zero-free regions for Dirichlet L-functions and the least prime in an arithmetic progression, Proc. London Math. Soc., **64** (1992), 265–338.
[15] C. Hooley, Applications of Sieve Methods to the Theory of Numbers, Cambridge Univ. Press, 1976.
[16] K. Prachar, Primzahlverteilung, Springer-Verlag, 1957.
[17] B. Rousselet, Inegalités de type Brun-Titchmarsh en moyenne, Groupe de tavaile en théorie analytique et elementaire des nombres, Publ. Math. d'Orsay, **88-01** (1988), 91–123.
[18] I. M. Vinogradov, Basic Number Theory, (6-th ed.), Moscow-Leninglad, 1952.

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