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GLOBAL SOLVABILITY FOR THE GENERALIZED DEGENERATE KIRCHHOFF EQUATION WITH REAL-ANALYTIC DATA IN R^n

By

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1. Introduction

Kirchhoff equation was proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string and it is expressed as follows

$$\partial_t^2 u(t,x) - \left(\varepsilon^2 + \frac{1}{2l}\int_0^l |\partial_x u(t,x)|^2 dx\right)\partial_x^2 u(t,x) = 0, \tag{1.1}$$

where $t > 0$, $l > 0$, $\varepsilon > 0$ and $x \in [0, l]$. In 1940 S. Bernstein [B] proved the global solvability for analytic initial data and local solvability for C^m-*class* initial data to the following initial boundary value problem:

$$\begin{cases} \partial_t^2 u(t,x) - \left(a + b\int_0^{2\pi} |\partial_x u(t,x)|^2 dx\right)\partial_x^2 u(t,x) = 0 \quad (t > 0, x \in [0, 2\pi]), \\ u(t,x) = 0 \quad (t \geq 0, x = 0, 2\pi), \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.2}$$

where $a > 0$ and $b > 0$. In 1971, T. Nishida [Nd] proved Bernstein's result in case of $a = 0$. Equation (1.2) can be regarded as the following more generalized equation:

$$\begin{cases} \partial_t^2 u(t,x) - M\left(\int_\Omega |\nabla_x u(t,x)|^2 dx\right)\Delta_x u(t,x) = 0 \quad (t > 0, x \in \Omega), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \quad x \in \Omega \subset R^n, \end{cases} \tag{1.3}$$

with boundary condition

$$u(t,x) = \varphi \quad \text{on } [0,\infty) \times \partial\Omega. \tag{1.4}$$

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In case of (1.2), $\Omega = [0, 2\pi]$, $\varphi = 0$ and $M(\eta) = a + b\eta$. In 1975, S. I. Pohožaev [P] proved the existence and uniqueness of time global real-analytic solution for the problem (1.3)–(1.4) under the assumption of $n \geq 1$ and $M(\eta) \in C^1([0, \infty))$ where Ω is bounded and $\varphi = 0$.* On the other hand, in case that $\Omega = \boldsymbol{R}^n$, Y. Yamada [Yd] proved the existence and uniqueness of global solution of (1.3) in 1980. In 1984, K. Nishihara [Nh] showed the global existence of the quasi-analytic solution in case that $M(\eta)$ is locally Lipschitz continuous and non-degenerate. In that year, A. Arosio and S. Spagnolo [AS] proved the existence of time global 2π-periodic solution for real-analytic data in case that $\Omega = [0, 2\pi]^n$ under some assumptions for $M(\eta) \in C^0$. In 1992, P. D'Ancona and S. Spagnolo [DS] relaxed the assumptions in [AS] to any $M(\eta) \in C^0$. Moreover, the equation (1.3)–(1.4) can be generalized as

$$\begin{cases} \partial_t^2 u(t,x) + M((Au(t,\cdot), u(t,\cdot))_\Omega)Au(t,x) = f(t,x) & (t > 0, x \in \Omega), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega \subset \boldsymbol{R}^n, \end{cases} \tag{1.5}$$

with boundary condition

$$u(t,x) = \varphi \quad \text{on } [0,\infty) \times \partial\Omega. \tag{1.6}$$

Here A is a degenerate elliptic operator of second order defined as $Au(t,x) = \sum_{i,j=1}^n D_{x_j}(a_{ij}(x) D_{x_i} u(t,x))$, $D_{x_i} = ((1/\sqrt{-1})(\partial/\partial x_j))$. Suppose that $[a_{ij}(x)]_{i,j=1,\ldots,n}$ is a real-analytic symmetric matrix which satisfies that

$$a(x,\xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0 \tag{1.7}$$

and there are $c_0 > 0$ and $\rho_0 > 0$ such that

$$|D_x^\alpha a_{ij}(x)| \leq c_0 \rho_0^{-|\alpha|} |\alpha|!, \quad i,j = 1, \ldots, n, \tag{1.8}$$

for $x \in \boldsymbol{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$, (Au, u) is an inner product of $Au(x)$ and $u(x)$ in $L_x^2(\Omega)$ and $M(\eta)$ satisfies

$$M(\eta) \in C^0([0,\infty)) \quad \text{and} \quad M(\eta) \geq 0. \tag{1.9}$$

If $a_{ij}(x) = \delta_{ij}$ and $f(t,x) \equiv 0$, then equation (1.5) coincides with equation (1.3), where δ_{ij} is Kronecker's delta. In 1994 K. Kajitani and K. Yamaguti [KY] proved the existence and uniqueness of time global real-analytic solution for (1.5) in case

*In fact he proved the existence and uniqueness of time global solution to more general problem on some suitable Hilbert space.

that $\Omega = \boldsymbol{R}^n$, $u_0(x)$, $u_1(x) \in L^2(\boldsymbol{R}^n) \cap C^\omega(\boldsymbol{R}^n)$, $M(\eta) \in C^1([0, \infty))$, $M(\eta) \geq 0$, and $a_{ij}(x) \geq 0$ are $C^\omega(\boldsymbol{R}^n)$ functions, respectively, where $C^\omega(\boldsymbol{R}^n)$ is the set of real analytic functions in $\boldsymbol{R}^n$. In 1995 K. Yamaguti [Yg] extended the result of [KY] for quasi-analytic data under the assumption of $M(\eta) > 0$.

Our main theorem in this paper is an extention of the result of [KY] in case of $M(\eta) \in C^0$. At first we introduce some definitions in order to state our main theorem.

Definition 1.1. For $s \in \boldsymbol{R}$ and $\rho > 0$, we define the function space H_ρ^s by

$$H_\rho^s = \{u(x) \in L_x^2(\boldsymbol{R}^n);\ \langle\xi\rangle^s e^{\rho\langle\xi\rangle}\hat{u}(\xi) \in L_\xi^2(\boldsymbol{R}^n)\}, \tag{1.10}$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\langle\xi\rangle = (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2}$, and $\hat{u}(\xi)$ stands for Fourier transform of u. If we introduce the inner product $(\cdot,\cdot)_{H_\rho^s}$ of H_ρ^s such that

$$(u, v)_{H_\rho^s} = (e^{\rho\langle\cdot\rangle}\hat{u}(\cdot), e^{\rho\langle\cdot\rangle}\hat{v}(\cdot))_s, \tag{1.11}$$

then H_ρ^s is a Hilbert space, where $(\cdot,\cdot)_s$ is an inner product of H^s which is the normal Sobolev space (See [Ku]). For $\rho < 0$ we define H_ρ^s as the dual space of $H_{-\rho}^{-s}$.

Definition 1.2. For $\rho \in \boldsymbol{R}$, define the operator $e^{\rho\langle D\rangle}$ from H_ρ^s into H^s as follows:

$$e^{\rho\langle D\rangle}u(x) = \int_{\boldsymbol{R}_\xi^n} e^{ix\cdot\xi + \rho\langle\xi\rangle}\hat{u}(\xi)\tilde{d}\xi, \tag{1.12}$$

for $u \in H_\rho^s$, where $x = (x_1, \dots, x_n)$, $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ and $\tilde{d}\xi = (2\pi)^{-n}d\xi$. Note that $(e^{\rho\langle D\rangle})^{-1} = e^{-\rho\langle D\rangle}$ is a mapping from H^s into H_ρ^s.

Hilbert space H_ρ^s and the operator $e^{\rho\langle D\rangle}$ were introduced in [Ka] and [KY]. In this paper we define the new space $H_{\rho,\delta,\kappa}^s$ as a weighted subspace of H_ρ^s.

Definition 1.3. For $s, \rho, \delta \in \boldsymbol{R}$ and $\kappa > 0$, we define $H_{\rho,\delta,\kappa}^s$ as

$$H_{\rho,\delta,\kappa}^s = \{u(x) \in \mathscr{S}';\ \langle D\rangle^s\{\langle x\rangle_\kappa^\delta e^{\rho\langle D\rangle}u(x)\} \in L_x^2(\boldsymbol{R}^n)\}, \tag{1.13}$$

where $\langle x\rangle_\kappa = (\kappa^2 + x_1^2 + \cdots + x_n^2)^{1/2}$ and $\mathscr{S}'$ is the dual space of the Schwartz space $\mathscr{S}$ of rapidly decreasing functions in $\boldsymbol{R}^n$. And we define the inner product $(\cdot,\cdot)_{H_{\rho,\delta,\kappa}^s}$ of $H_{\rho,\delta,\kappa}^s$ as follows:

$$(u, v)_{H_{\rho,\delta,\kappa}^s} = (\langle\cdot\rangle_\kappa^\delta e^{\rho\langle D\rangle}u(\cdot), \langle\cdot\rangle_\kappa^\delta e^{\rho\langle D\rangle}v(\cdot))_s. \tag{1.14}$$

The principal method of the proof of this theorem is based on [Ka] and [KY]. In this paper we introduce the new space $H^s_{\rho,\delta,\kappa}$ which is a weighted subspace of H^s_ρ for $\delta > 0$, and we consider the global solvablity for the equation in it. For positive real numbers ρ and κ and for non-negative real numbers s and δ, the function spaces H^s_ρ and $H^s_{\rho,\delta,\kappa}$ are included the intersection of $L^2(\boldsymbol{R}^n)$ and $C^\omega(\boldsymbol{R}^n)$. Our main theorem in this paper is the global existence of the real-analytic solution which has initial condition in $H^s_{\rho,\delta,\kappa}$.

MAIN THEOREM. *Assume that* (1.7), (1.8) *and* (1.9) *are valid. Let* $0 < \rho_1 < \rho_0/\sqrt{n}$, $\delta > 0$, $\kappa > 0$ *and put* $\rho(t) = \rho_1 e^{-\gamma t}$ *for* $\gamma > 0$. *Then there exists* $\gamma > 0$ *such that for any* $u_0 \in H^2_{\rho_1,\delta,\kappa}$, $u_1 \in H^1_{\rho_1,\delta,\kappa}$ *and for any* $f(t,x)$ *satisfying* $\langle x\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle} f(t,x) \in C^0([0,\infty); H^1)$, *the Cauchy problem* (1.5) *with* $\Omega = \boldsymbol{R}^n$ *has a solution* $u(t,x)$ *that satisfies* $\langle x\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle} u(t,x) \in \bigcap_{j=0}^2 C^{2-j}([0,\infty); H^j)$.

2. Preliminaries

In this section we introduce some propositions and lemmas to prove the following lemmas and our main theorem.

PROPOSITION 2.1. *Assume that* $a(x,\xi) \in S^2$ *is non-negative. Then there are positive constants* C_1 *and* C_2 *such that*

$$\Re(Op(a)u,u)_s \geq -C_1\|u\|_s \tag{2.1}$$

and

$$\sum_{|\alpha|=1}\{\|Op(a_{(\alpha)})u\|^2_{s-1} + \|Op(a^{(\alpha)})u\|^2_s\} \leq C_2\{2C_1\|u\|^2_s + \Re(Op(a)u,u)_s\} \tag{2.2}$$

for $u \in H^{s+2}$, *where* S^m *is the symbol-class of pseudo-differential operator of order* m *(See* [Ku]*),* $Op(a)$ *is the pseudo-differential operator defined as*

$$Op(a)u = \int_{\boldsymbol{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\bar{d}\xi$$

for $u(x) \in \mathscr{S}$, *where* $\|\cdot\|_s$ *is a norm of* H^2.

For a proof of this proposition, refer to [FP].

PROPOSITION 2.2. (i) *Let* $a(x,\xi) \in C^\infty(\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi)$ *be a 'double order' symbol in the 'double order symbol space'* $SG_1^{(m_1,m_2)}$:

$$SG_1^{(m_1,m_2)} = \{a(x,\xi) \in C^\infty(\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi);\ a^{(\alpha)}_{(\beta)}(x,\xi) = O(\langle\xi\rangle^{m_1-|\alpha|}\langle x\rangle^{m_2-|\beta|})\} \tag{2.3}$$

for $(m_1, m_2) \in \boldsymbol{R} \times \boldsymbol{R}$ *where* $a_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta a(x,\xi)$, *and if we* $a(x,\xi)$ *define the operator* $Op(a)$ *by*

$$(Op(a))(x,D)f(x) = \int_{\boldsymbol{R}_\xi^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\tilde{d}\xi, \quad f \in \mathscr{S}, \tag{2.4}$$

then $Op(a)$ *is the bounded linear operator from* $H_{\rho,s_2,\kappa}^{s_1}$ *into* $H_{\rho,s_2-m_2,\kappa}^{s_1-m_1}$ *for each* $s_1, s_2 \in \boldsymbol{R}$.

(ii) *If* $s > s'$ *and* $\delta > \delta'$, *then the embedding* $H_{\rho,\delta,\kappa}^{s} \hookrightarrow H_{\rho,\delta',\kappa}^{s'}$ *is compact.*

(iii) *Let* $c(x,\xi)$ *be the symbol of the product* $Op(a)Op(b)$ *of* $a \in SG_1^{(l_1,l_2)}$ *and* $b \in SG_1^{(m_1,m_2)}$, *then* $c(x,\xi)$ *has the asymptotic expansion*:

$$c(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} a^{(\alpha)}(x,\xi) b_{(\alpha)}(x,\xi). \tag{2.5}$$

This proposition is introduced in [S].

LEMMA 2.3. (i) *Let* $u \in H_{\rho,0}^s = H_\rho^s$, *then for* $\rho > 0$,

$$\|D^\alpha u\|_s \leq \|u\|_{H_\rho^s} \rho^{-|\alpha|} |\alpha|! \tag{2.6}$$

and

$$|D_x^\alpha u(x)| \leq C_n \|u\|_{H_\rho^s} \rho^{-(|\alpha|+n+|s|)} (|\alpha| + n + |s|)! \tag{2.7}$$

for $x \in \boldsymbol{R}^n$ *and* $\alpha \in \boldsymbol{N}^n$.

(ii) *Let* $u(x)$ *be a function in* H^∞ *and* $s \in \boldsymbol{R}$. *If* $u(x)$ *satisfies*

$$\|D^\alpha u\|_{H^s} \leq c_0 \rho_1^{-|\alpha|} |\alpha|! \tag{2.8}$$

for every multi-index $\alpha \in \boldsymbol{N}^n$, *then* $u(x) \in H_\rho^s$ *for* $\rho < \rho_1/\sqrt{n}$.

For a proof of this lemma, refer to [KY].

LEMMA 2.4. *Let* $\delta \geq 0$, $c > 0$ *and* $\varepsilon \in (0,1]$, *then* $\langle x \rangle_c^{-\delta}$ *is a real-analytic function satisfying*

$$|D_x^\alpha \langle x \rangle_c^{-\delta}| \leq (8\varepsilon^{-1})^{|\alpha|} (1+\varepsilon)^\delta |\alpha|! \langle x \rangle_c^{-\delta-|\alpha|}, \tag{2.9}$$

for $x \in \boldsymbol{R}^n$. *Moreover if* $0 \leq \delta \leq 1$, *then*

$$|D_x^\alpha \langle x \rangle_c^{-\delta}| \leq 4^{|\alpha|} |\alpha|! \langle x \rangle_c^{-\delta-|\alpha|} \tag{2.10}$$

for $x \in \boldsymbol{R}^n$.

For a proof, refer to [Ka].

Let $a(x)$ be a real-analytic function in $\boldsymbol{R}^n$ satisfies that there are $c_0 > 0$ and $\rho_0 > 0$ such that

$$|D_x^\alpha a(x)| \le c_0 \rho_0^{-|\alpha|} |\alpha|! \tag{2.11}$$

for any $x \in \boldsymbol{R}^n$ and any multi-index $\alpha \in \boldsymbol{N}^n$. Define the multiplier $a\cdot$ as $(a \cdot u)(x) = a(x)u(x)$. Let us define $a(\rho; x, D)u(x) = e^{\rho\langle D\rangle} a \cdot e^{-\rho\langle D\rangle} u(x)$ for $u(x) \in L^2(\boldsymbol{R}^n)$ and denote by $a(\rho; x, \xi)$ its symbol.

PROPOSITION 2.5. (i) *$a(\rho; x, D)$ is a pseudo-differential operator of order* 0 *and its symbol has the following expansion*:

$$a(\rho; x, \xi) = a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi), \tag{2.12}$$

where

$$a_1(x, \xi) = -\sum_{j=1}^{n} D_{x_j} a(x) \partial_{\xi_j} \langle \xi \rangle, \tag{2.13}$$

and a_2 and r respectively satisfy

$$|a_{2(\beta)}^{(\alpha)}(\rho; x, \xi)| \le C_{\alpha\beta\rho_0} \langle \xi \rangle^{-|\alpha|}, \tag{2.14}$$

$$|r_{(\beta)}^{(\alpha)}(\rho; x, \xi)| \le C_{\alpha\beta\rho_0} \langle \xi \rangle^{-1-|\alpha|} \tag{2.15}$$

for $x, \xi \in \boldsymbol{R}^n$, $|\rho| < \rho_0/\sqrt{n}$ and $a, \beta \in \boldsymbol{N}^n$.

(ii) *If $\rho = \rho(t) \in C^0([0, T])$ for $T > 0$, then $a(\rho(t); x, \xi) \in C^0([0, T]; S^0)$.*

For a proof of (i), refer to [KY] and for (ii) refer to [Ka].

COROLLARY 2.6. *Define the opeartior A_Λ by*

$$A_\Lambda u(x) = e^{\rho\langle D\rangle}(A e^{-\rho\langle D\rangle} u(x)) \tag{2.16}$$

for $A = \sum_{i,j=1}^n D_j(a_{ij}(x) D_i)$. Then A_Λ and $\langle x \rangle_\kappa^\delta A_\Lambda \langle x \rangle_\kappa^{-\delta}$ are pseudo-differential operators of order 2 *and their symbols have the following expansions respectively;*

$$\sigma(A_\Lambda)(x, \xi) = \sum_{i,j=1} (a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r_1(\rho; x, \xi)) \xi_j \xi_i, \tag{2.17}$$

$$\sigma(\langle x \rangle_\kappa^\delta A_\Lambda \langle x \rangle_\kappa^{-\delta})(x, \xi) = \sum_{i,j=1} (a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r_2(\rho; x, \xi)) \xi_j \xi_i, \tag{2.18}$$

where $\sigma(P)(x,\xi)$ *denotes the symbol of a pseudo-differential operator* $P(x,D)$, $a_1 = a_{1_{ij}}$ *and* $a_2 = a_{2_{ij}}$ *are defined in Proposition* 2.5, *and both* $r_1 = r_{1_{ij}}$ *and* $r_2 = r_{2_{ij}}$ *belong to* S^{-1}. *Moreover, for* $\rho(t) \in C^0([0,T])$, $\sigma(A_\Lambda)(t,x,\xi)$ *and* $\sigma(\langle x\rangle_\kappa^\delta A_\Lambda \langle x\rangle_\kappa^{-\delta})(t,x,\xi)$ *belong to* $C^0([0,T]; S^2)$.

PROOF. It is obvious by Proposition 2.2 and Proposition 2.5.

LEMMA 2.7. *If* $u(x) \in H^s_{\rho,\delta,\kappa}$ *for* $\delta > 0$, *then* $u(x)$ *is a real-analytic function whose radius of convergence is* ρ_1, *where* $\rho_1 \le \min\{\kappa/8, \rho_0\}$ *and* $0 < \rho_0 < \rho$.

PROOF. Note that $\langle x\rangle_\kappa^\delta u(x) \in H^s_\rho$ if $u(x) \in H^s_{\rho,\delta,\kappa}$.

$$\begin{aligned}
|D_x^\alpha u(x)| &= |D_x^\alpha(\langle x\rangle_\kappa^{-\delta}\langle x\rangle_\kappa^\delta u(x))| \\
&\le \sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|D_x^{\alpha-\alpha'}\langle x\rangle_\kappa^{-\delta}||D_x^{\alpha'}(\langle x\rangle_\kappa^\delta u(x))| \\
&\le C_1\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\alpha'|!|\alpha-\alpha'|!\left(\frac{\kappa}{8}\right)^{-|\alpha-\alpha'|}\rho_0^{-|\alpha'|} \\
&\le C_2\rho_1^{-|\alpha|}|\alpha|!, \qquad (2.19)
\end{aligned}$$

where $\rho_1 \le \min\{\kappa/8, \rho_0\}$, $0 < \rho_0 < \rho$ and we used Lemma 2.3, Lemma 2.4 and the estimate;

$$\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\alpha'|!|\alpha-\alpha'|!\eta_1^{-|\alpha'|}\eta_2^{-|\alpha-\alpha'|} \le \frac{\eta_1}{\eta_1-\eta_2}\eta_2^{-|\alpha|}|\alpha|!, \qquad (2.20)$$

for $0 < \eta_2 < \eta_1$. □

3. Existence of solutions for the linear problem

In this section, we consider the local existence for the following linear Cauchy problem:

$$\begin{cases} \partial_t^2 u(t,x) + m(t)Au(t,x) = f(t,x), \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \qquad (3.1)$$

where $m(t)$ is a non-negative continuous function in $[0,\infty)$.

At first we introduce a proposition to prove the existence of the linear problem (3.1).

Let $P(t) = [p_{ij}(t,x,D)]_{i,j=1,\dots,d}$ be a matrix consisting of pseudo-differential operators whose symbols $p_{ij}(t,x,\xi)$ all belong to the class $C^0([0,T];S^1)$. Let us consider the following linear Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}U(t) = P(t)U(t) + F(t), & t \in (0,T], \\ U(0) = U_0, \end{cases} \tag{3.2}$$

where $U(t) = {}^t(U_1(t),\dots,U_d(t))$ is an unknown vector valued function, $F(t) = {}^t(F_1(t),\dots,F_d(t))$ and $U_0 = {}^t(U_{01},\dots,U_{0d})$ are known vector valued functions. Then the following proposition is concluded.

PROPOSITION 3.1. *Suppose that* $\det(\lambda I - p(t,x,\xi)) \neq 0$ *for* $\lambda \in C^1(\boldsymbol{R}^n)$ *with* $\Re\lambda > -c_0\langle\xi\rangle$ *for some positive constant* c_0, $t \in [0,T]$ *and* $|\xi| \gg 1$. *Take an arbitrary real number* s. *Then for any* $U_0 \in (H^{s+1})^d$ *and for any* $F(t) \in C^0([0,T];(H^{s+1})^d)$, *there exists a unique solution* $U(t) \in C^1([0,T];(H^s)^d) \cap C^0([0,T];(H^{s+1})^d)$ *of* (3.2).

This proposition was introduced as Proposition 4.5 in [M]. For the proof of the proposition, refer to [M].

Let $v(t,x) = \langle x\rangle_\kappa^\delta e^{\Lambda(t)}u(t,x)$ and transform the equation (3.1) of $u(t,x)$ to the equation of $v(t,x)$ such that

$$\begin{cases} \langle x\rangle_\kappa^\delta(\partial_t - \Lambda)^2\langle x\rangle_\kappa^{-\delta}v(t,x) + m(t)\langle x\rangle_\kappa^\delta A_\Lambda\langle x\rangle_\kappa^{-\delta}v(t,x) = g(t,x), \\ v(0,x) = v_0(x), \qquad \partial_t v(0,x) = v_1(x), \end{cases} \tag{3.3}$$

where $\Lambda = \Lambda(t) = \rho(t)\langle D\rangle$, $\Lambda_t = \Lambda_t(t) = \rho_t(t)\langle D\rangle$, $\rho(t) = \rho_1 e^{-\gamma t}$ for $\rho_1 > 0$, $\gamma > 0$ and $g(t,x) = \langle x\rangle_\kappa^\delta e^{\Lambda(t)}f(t,x)$. Then the following lemma is concluded for the Cauchy problem (3.3).

LEMMA 3.2. *Assume that* $v_0 \in H^{s+2}$, $v_1 \in H^{s+1}$ *and* $g(t,x) \in C^0([0,T];H^{s+1})$, *then there is* $\gamma_0 > 0$ *and the Cauchy problem* (3.3) *has a unique solution* $v(t,x)$ *such that*

$$v(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^{s+j})$$

for all $\gamma \geq \gamma_0$.

PROOF. Now let us put $V(t) = {}^t(V_1(t),V_2(t))$, $V_0 = {}^t(V_{01},V_{02})$, $F(t) =$

${}^{t}(0, g(t))$ and

$$P(t) = \begin{pmatrix} \langle D\rangle\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1} & \langle D\rangle \\ -m(t)\langle x\rangle_\kappa^\delta A_\Lambda \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1} & \langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta} \end{pmatrix}. \tag{3.4}$$

Where A_Λ is defined by (2.16). Then we consider the following linear Cauchy problem:

$$\begin{cases} \dfrac{d}{dt} V(t) = P(t)V(t) + F(t), \quad t \in (0, T], \\ V(0) = V_0. \end{cases} \tag{3.5}$$

At first we show that the symbols of pseudo-differential operator $P(t)$ satisfies the conditions of Proposition 3.1. Clearly $\sigma(\langle D\rangle\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1})\cdot(t,x,\xi)$, $\sigma(\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta})(t,x,\xi)$ and $\sigma(\langle x\rangle_\kappa^\delta A_\Lambda \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1})(t,x,\xi)$ belong to $C^0([0,T];S^1)$ by Corollary 2.6.

$$\begin{aligned} &\det(\lambda I - \sigma(P)(t,x,\xi)) \\ &\quad = (\lambda - \sigma(\langle D\rangle\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1})(t,x,\xi))(\lambda - \sigma(\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta})(t,x,\xi)) \\ &\qquad + m(t)\sigma(\langle x\rangle_\kappa^\delta A_\Lambda \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1})(t,x,\xi)\langle\xi\rangle \\ &\quad = (\lambda - \rho'(t)\langle\xi\rangle - \rho'(t)p_1^0(x,\xi))(\lambda - \rho'(t)\langle\xi\rangle - \rho'(t)p_2^0(x,\xi)) \\ &\qquad + m(t)(\sigma(A_\Lambda)(t,x,\xi) + p_3^1(t,x,\xi)), \end{aligned} \tag{3.6}$$

where $\sigma(P) = [\sigma(P_{ij})]_{i,j=1,2}$, $p_j^0(x,\xi) \in S^0 (j = 1,2)$ and $p_3^1(t,x,\xi) \in ([0,T]; S^1)$, and they satisfy

$$\sigma(\langle D\rangle\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1})(t,x,\xi) = \rho'(t)\langle\xi\rangle + \rho'(t)p_1^0(x,\xi) \tag{3.7}$$

$$\sigma(\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta})(t,x,\xi) = \rho'(t)\langle\xi\rangle + \rho'(t)p_2^0(x,\xi) \tag{3.8}$$

$$\sigma(\langle x\rangle_\kappa^\delta A_\Lambda \langle x\rangle_\kappa^{-\delta}\langle D\rangle^{-1})(t,x,\xi)\langle\xi\rangle = \sigma(A_\Lambda)(t,x,\xi) + p_3^1(t,x,\xi). \tag{3.9}$$

Therefore we have

$$\begin{aligned} &\det(\lambda I - \sigma(P)(t,x,\xi)) \\ &\quad = \lambda^2 - \rho'(t)\lambda(2\langle\xi\rangle + p_1^0(x,\xi) + p_2^0(x,\xi)) \\ &\qquad + \rho'(t)^2(\langle\xi\rangle + p_1^0(x,\xi))(\langle\xi\rangle + p_2^0(x,\xi)) \\ &\qquad + m(t)(\sigma(A_\Lambda)(t,x,\xi) + p_3^1(t,x,\xi)). \end{aligned} \tag{3.10}$$

Let $\det(\lambda I - \sigma(P)(t,x,\xi)) = 0$ and solve it in λ, then we have

$$\begin{aligned}\lambda &= \rho'(t)(2\langle\xi\rangle + p_1^0(x,\xi) + p_2^0(x,\xi)) \\ &\quad \pm [\rho'(t)^2\{-2\langle\xi\rangle(p_1^0(x,\xi) + p_2^0(x,\xi)) - 3p_1^2(x,\xi)p_2^0(x,\xi) + p_1^0(x,\xi)^2 + p_2^0(x,\xi)^2\} \\ &\quad - 4m(t)\sum_{i,j=1}^{n}\{a(x) + \rho(t)a_1(x,\xi) + \rho(t)^2 a_2(\rho(t);x,\xi) + r_2(\rho(t);x,\xi)\}\xi_j\xi_i \\ &\quad + p_3^1(t,x,\xi)]^{1/2},\end{aligned} \tag{3.11}$$

where a, a_1, a_2 and r_2 are defined in (2.18). Then the order of $\Re\lambda$ is as follows:

$$\Re\lambda = -\gamma\rho_1 e^{-\gamma t}O(\langle\xi\rangle) \pm \{m(t)\rho_1 e^{-\gamma t}O(|\xi|) + O(|\xi|^{1/2})\}. \tag{3.12}$$

Hence, obviously there are $\gamma_0 > 0$ and $c_0 > 0$ such that $\det(\lambda I - \sigma(P)(t,x,\xi)) > 0$ for any γ satisfying $\gamma > \gamma_0$, $|\xi| \gg 1$ and $\Re\lambda > -c_0\langle\xi\rangle$. Therefore equation (3.5) has a unique solution $V(t) = (V_1(t), V_2(t))$ satisfying

$$V_1(t), V_2(t) \in C^1([0,T];H^s) \cap C^0([0,T];H^{s+1}) \tag{3.13}$$

for V_{01}, $V_{02} \in H^{s+1}$. Now, if we let $v(t) = \langle D\rangle^{-1}V_1(t)$, then $v(t)$ satisfies

$$v(t,x) \in C^1([0,T];H^{s+1}) \cap C^0([0,T];H^{s+2}) \tag{3.14}$$

for $v(0) = v_0 \in H^{s+2}$. Then we know that $v(t,x)$ satisfying

$$\partial_t\langle D\rangle v(t,x) = \langle D\rangle\langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta} v(t,x) + \langle D\rangle V_2(t), \tag{3.15}$$

and obviously $V_2(t)$ is represented by $v(t,x)$ such that

$$V_2(t) = \partial_t v(t,x) - \langle x\rangle_\kappa^\delta \Lambda_t \langle x\rangle_\kappa^{-\delta} v(t,x), \qquad V_2(0) = V_{02} \in H^{s+1}. \tag{3.16}$$

Then by (3.5), $v(t,x)$ satisfies

$$\langle x\rangle_\kappa^\delta(\partial_t - \Lambda_t)^2\langle x\rangle_\kappa^{-\delta} + m(t)\langle x\rangle_\kappa^\delta A_\Lambda \langle x\rangle_\kappa^{-\delta} v(t,x) = g(t,x). \tag{3.17}$$

It shows that $v(t,x)$ is a solution of (3.3) satisfying

$$v(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^{s+j}). \qquad \square \tag{3.18}$$

By Lemma 3.2, obviously we have the following lemma.

LEMMA 3.3. *For* $u_0 \in H^{s+2}_{\rho_1,\delta,\kappa}$, $u_1 \in H^{s+1}_{\rho_1,\delta,\kappa}$ *and* $\langle x\rangle_\kappa^\delta e^{\Lambda(t)} f(t,x) \in C^0([0,T];H^{s+1})$, *there exists a positive constant* γ_0 *and the Cauchy problem* (3.1) *has a unique solution* $u(t,x)$ *such that*

$$\langle x\rangle_\kappa^\delta e^{\Lambda(t)} u(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^{s+j}). \tag{3.19}$$

for all $\gamma \geq \gamma_0$.

4. A priori estimate of solution for the linear problem

Let $0 < T < \infty$, $m(t)$ be a non-negative function in $C^0([0,T])$, $\rho(t)$ a positive function in $C^1([0,T)) \cap C^0([0,T])$ such that $\rho_t(t) < 0$, $\varphi(t)$ a positive function in $C^1([0,T])$ satisfying $\varphi'(t) \leq 0$ for $t \geq 0$ and $m_\varepsilon(t) = \int_0^T \chi_{\tilde\varepsilon}(t-\tau)m(\tau)d\tau + \varepsilon$, where $\tilde\varepsilon(\varepsilon)$ satisfies $0 < \tilde\varepsilon < \varepsilon$ and $|\int_0^T \chi_{\tilde\varepsilon}(t-\tau)m(\tau)d\tau - m(t)| < \varepsilon$, and $\chi_\varepsilon(t) = \varepsilon^{-1}\chi(\varepsilon^{-1}t)$, $\chi(t) \in C_0^\infty((0,1))$ satisfying $\chi(t) \geq 0$ and $\int_0^1 \chi(t)dt = 1$ for $0 \leq t \leq T$. Then we define $E_s(t)$ as follows:

$$E_s(t)^2 = \frac{1}{2}\left\{\|\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta} v(t)\|_s^2 + \varphi(t)\|v(t)\|_{s+1}^2 + m_\varepsilon(t)(A\langle D\rangle^s v(t), \langle D\rangle^s v(t))\right\}. \tag{4.1}$$

for the solution $v(t,x)$ of (3.3).

LEMMA 4.1. *Assume that* $m(t)$ *is a non-negative function in* $C^0([0,T])$, $\varphi(t) = e^{-2\gamma t}$, $\rho(t) = \rho_1 e^{-\gamma t}$ *and* $v(t,x)$ *is a solution of* (3.3) *satisfying* $v(t,x) \in \bigcap_{j=0}^2 C^{2-j}([0,T);H^{s+j})$, *then there exist positive constants* ε, γ_0, c *and* c_0 *such that*

$$E_s(t) \leq E_s(0)e^{\int_0^t q(\tau)d\tau} + \frac{1}{2}\int_0^t e^{\int_\tau^t q(\mu)d\mu}\|g(\tau)\|_s d\tau, \tag{4.2}$$

for $t \in [0,T)$ *and for any* $\gamma \geq \gamma_0$, *where*

$$q(t) = \frac{c}{2}\left(|\rho_t(t)| + \frac{|m_\varepsilon'(t)|}{m_\varepsilon(t)} + \frac{m(t)^2}{\varphi(t)|\rho_t(t)|} + \frac{m(t)^2\rho(t)^2}{\varphi(t)|\rho_t(t)|} + \frac{m_\varepsilon(t)|\rho_t(t)|}{\varphi(t)} + c_0\right). \tag{4.3}$$

PROOF. Note that $m_\varepsilon(t) \to m(t)$ in $L^1([0,t])$ for arbitrary $t \in [0,T]$.

Differentiating both sides in (4.1), we have

$$2E_s'(t)E_s(t) = \frac{d}{dt}\left\{\frac{1}{2}\|\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t)\|_s^2\right\} \tag{4.4}$$

$$+\frac{d}{dt}\left(\frac{1}{2}\varphi(t)\|v(t)\|_{s+1}^2\right) \tag{4.5}$$

$$+\frac{d}{dt}\left(\frac{1}{2}m_\varepsilon(t)(A\langle D\rangle^s v(t), \langle D\rangle^s v(t))\right). \tag{4.6}$$

$$\begin{aligned}
(4.4) &= \Re(\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)^2\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\
&\quad + \Re(\langle\cdot\rangle_\kappa^\delta\Lambda_t(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\
&= \Re(g(t), \langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\
&\quad - m(t)\Re(\langle\cdot\rangle_\kappa^\delta A_\Lambda\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\
&\quad + \Re(\Lambda_t\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\
&\quad + \Re(\rho_t p_1^0\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s
\end{aligned}$$

$$\le \|g(t)\|_s E_s(t) \tag{4.7}$$

$$- m(t)\Re(|\Lambda_t|^{-1/2}\langle\cdot\rangle_\kappa^\delta A_\Lambda\langle\cdot\rangle_\kappa^{-\delta}v(t), |\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \tag{4.8}$$

$$- \||\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t)\|_s^2 \tag{4.9}$$

$$+ C_1|\rho_t|E_s(t)^2, \tag{4.10}$$

where $p_1^0(x, D) \in Op(S^0)$, and we used an equality; $\|Pu\|_s \le C_s\|u\|_{s+m}$ for some positive constant C_s provided $P \in Op(S^m)$ and $u \in H^s$ (See [Ku]).

$$\begin{aligned}
(4.5) &= \frac{1}{2}\varphi'(t)\|v(t)\|_{s+1}^2 \\
&\quad + \varphi(t)\Re(\langle D\rangle\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle D\rangle v(t))_s \\
&\quad + \varphi(t)\Re(\langle D\rangle\langle\cdot\rangle_\kappa^\delta\Lambda_t\langle\cdot\rangle_\kappa^{-\delta}v(t), \langle D\rangle v(t))_s
\end{aligned}$$

$$\leq \frac{\varphi'(t)}{\varphi(t)} E_s(t)^2 \tag{4.11}$$

$$+\frac{1}{2}\||\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t)\|_s^2 \tag{4.12}$$

$$+\frac{\varphi(t)^2}{2|\rho_t|}\|v(t)\|_{s+3/2}^2 \tag{4.13}$$

$$+\varphi(t)\rho_t\|v(t)\|_{s+3/2}^2 \tag{4.14}$$

$$+C_2 E_s(t)^2, \tag{4.15}$$

$$\begin{aligned}(4.6) = &\frac{1}{2}m_\varepsilon'(t)(A\langle D\rangle^s v(t), \langle D\rangle^s v(t))_s \\ &+ m_\varepsilon(t)\Re(|\Lambda_t|^{-1/2}\langle D\rangle^{-s}A\langle D\rangle^s v(t), |\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\ &+ m_\varepsilon(t)\Re(|\Lambda_t|^{-1/2}\langle D\rangle^{-s}A\langle D\rangle^s v(t), |\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta\Lambda_t\langle\cdot\rangle_\kappa^{-\delta}v(t))_s\end{aligned}$$

$$\leq \frac{|m_\varepsilon'(t)|}{m_\varepsilon(t)} E_s(t)^2 \tag{4.16}$$

$$+ m_\varepsilon(t)\Re(|\Lambda_t|^{-1/2}\langle D\rangle^{-s}A\langle D\rangle^s v(t), |\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \tag{4.17}$$

$$+ m_\varepsilon(t)\rho_t\Re(\langle D\rangle^{1/2}A\langle D\rangle^s v(t), \langle D\rangle^{s+(1/2)}v(t)) \tag{4.18}$$

$$+ m_\varepsilon(t)\rho_t\Re(\langle D\rangle^{-s}A\langle D\rangle^s v(t), p_2^0 v(t))_s, \tag{4.19}$$

where $p_2^0(x, D) \in Op(S^0)$ and we used

$$(\langle D\rangle^{-s}A\langle D\rangle^s u, v)_s = (u, \langle D\rangle^{-s}A\langle D\rangle^s v)_s$$

which is verified by the symmetry of $[a_{ij}]_{i,j=1,\dots,n}$.

$$(4.18) + (4.19) \leq m_\varepsilon(t)\rho_t\Re(A\langle D\rangle^{s+(1/2)}v(t), \langle D\rangle^{s+(1/2)}v(t)) \tag{4.20}$$

$$+\frac{C_3 m_\varepsilon(t)|\rho_t|}{\varepsilon(t)} E_s(t)^2, \tag{4.21}$$

$$\begin{aligned}(4.8) + (4.17) \leq\ &|(|\Lambda_t|^{-1/2}m_\varepsilon(t)\langle D\rangle^{-s}A\langle D\rangle^s v(t), |\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s \\ &- (|\Lambda_t|^{-1/2}m(t)\langle\cdot\rangle_\kappa^\delta A_\Lambda\langle\cdot\rangle_\kappa^{-\delta}v(t), |\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t - \Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t))_s|.\end{aligned} \tag{4.22}$$

Then, using the equality:

$$\begin{aligned} &m_\varepsilon(t)\langle D\rangle^{-s}A\langle D\rangle^{s} - m(t)\langle x\rangle_\kappa^\delta A_\Lambda\langle x\rangle_\kappa^{-\delta} \\ &\quad = m(t)(A_\Lambda - \langle x\rangle_\kappa^\delta A_\Lambda\langle x\rangle_\kappa^{-\delta}) + m(t)(A - A_\Lambda) \\ &\qquad + m(t)(\langle D\rangle^{-s}A\langle D\rangle^{s} - A) + \{m_\varepsilon(t) - m(t)\}\langle D\rangle^{-s}A\langle D\rangle^{s}, \end{aligned} \tag{4.23}$$

we obtain the estimate;

$$\begin{aligned} &\||\Lambda_t|^{-1/2}\{m_\varepsilon(t)\langle D\rangle^{-s}A\langle D\rangle^{s} - m(t)\langle\cdot\rangle_\kappa^\delta A_\Lambda\langle\cdot\rangle_\kappa^{-\delta}\}v(t)\|_s \\ &\quad \le |m_\varepsilon(t) - m(t)|\||\Lambda_t|^{-1/2}\langle D\rangle^{-s}A\langle D\rangle^{s}v(t)\|_s \\ &\qquad + m(t)\{\||\Lambda_t|^{-1/2}(\langle D\rangle^{-s}A\langle D\rangle^{s} - A)v(t)\|_s \\ &\qquad + \||\Lambda_t|^{-1/2}(A_\Lambda - \langle\cdot\rangle_\kappa^\delta A_\Lambda\langle\cdot\rangle_\kappa^{-\delta})v(t)\|_s \\ &\qquad + \||\Lambda_t|^{-1/2}(A - A_\Lambda)v(t)\|_s\} \\ &\quad \le C_2|m_\varepsilon(t) - m(t)||\rho_t|^{-1/2}\|v(t)\|_{s+3/2} \\ &\qquad + m(t)(\||\Lambda_t|^{-1/2}p^1(\cdot,D)v(t)\|_s \rho(t)\||\Lambda_t|^{-1/2}\tilde{a}_1(\cdot,D)v(t)\|_s \\ &\qquad + \rho(t)^2\||\Lambda_t|^{-1/2}\tilde{a}_2(\rho;\cdot,D)v(t)\|_s + \||\Lambda_t|^{-1/2}\tilde{r}(\rho;\cdot,D)v(t)\|_s) \\ &\quad \le (C_2|m_\varepsilon(t) - m(t)| + C_3 m(t)\rho(t)^2)|\rho_t|^{-1/2}\|v(t)\|_{s+3/2} && (4.24) \\ &\qquad + C_4 m(t)|\rho_t|^{-1/2}\|v(t)\|_{s+1} && (4.25) \\ &\qquad + m(t)\rho(t)|\rho_t|^{-1/2}\|\tilde{a}_1(\cdot,D)v(t)\|_{s-1/2}, && (4.26) \end{aligned}$$

where $p^1(x,\xi) \in S^1$, $\tilde{a}_1(x,\xi) = \sum_{i,j=1}^n a_1\xi_1\xi_j$, $\tilde{a}_2(\rho,x,\xi) = \sum_{i,j=1}^n a_2\xi_i\xi_j$ and $\tilde{r}(\rho;x,\xi) = \sum_{i,j=1}^n r_1\xi_i\xi_j$, a_1, a_2 and r_1 defined in (2.17). Besides, by Proposition 2.1, (4.26) is estimated in the following:

$$\begin{aligned} \|\tilde{a}_1(\cdot,D)v(t)\|_{s-1/2}^2 &= \Big\|\sum_{|\alpha|=1}\tilde{a}_{(\alpha)}(\cdot,D)D^\alpha\langle D\rangle^{-1}v(t)\Big\|_{s-1/2}^2 \\ &\le C_5\sum_{|\alpha|=1}\|\tilde{a}_{(\alpha)}(\cdot,D)v(t)\|_{s-1/2}^2 + C_6\|v(t)\|_{s+1/2}^2 \\ &\le C_7\Re(\tilde{a}(\cdot,D)v(t),v(t))_{s+1/2} + C_8\|v(t)\|_{s+1/2}^2 \\ &\le C_7\Re(A\langle D\rangle^{s+1/2}v(t),\langle D\rangle^{s+1/2}v(t)) && (4.27) \\ &\quad + C_9\varphi(t)^{-1}E_s(t)^2, && (4.28) \end{aligned}$$

where $\tilde{a}(x,\xi)=\sigma(A)(x,\xi)$. Therefore (4.8) + (4.17) is estimated as below

$$(4.8)+(4.17)\le 2\{C_2^2|m_\varepsilon(t)-m(t)|^2+C_3^2m(t)^2\rho^4\}|\rho_t|^{-1}\|v(t)\|_{s+3/2}^2 \tag{4.29}$$

$$+\{4C_4^2m(t)^2\varphi(t)^{-1}+C_{10}m(t)^2\rho^2\varphi^{-1}\}|\rho_t|^{-1}E_s(t)^2 \tag{4.30}$$

$$+C_7m(t)^2\rho^2|\rho_t|^{-1}\Re(A\langle D\rangle^{s+1/2}v(t),\langle D\rangle^{s+1/2}v(t)) \tag{4.31}$$

$$+\frac{1}{2}\||\Lambda_t|^{1/2}\langle\cdot\rangle_\kappa^\delta(\partial_t-\Lambda_t)\langle\cdot\rangle_\kappa^{-\delta}v(t)\|_s^2. \tag{4.32}$$

Note that C_j $(j=1,\dots,10)$ are positive constants independent of t and γ. Hence combing the preceding estimates, we have the following estimate for (4.1);

$$2E_s'(t)E_s(t)\le\|g(t)\|_sE_s(t) \tag{4.33}$$

$$+c\left(|\rho_t(t)|+\frac{|m_\varepsilon'(t)|}{m_\varepsilon(t)}+\frac{m(t)^2}{\varphi(t)|\rho_t(t)|}+\frac{m(t)^2\rho(t)^2}{\varphi(t)|\rho_t(t)|}+\frac{m_\varepsilon(t)|\rho_t(t)|}{\varphi(t)}+c_0\right)E_s(t)^2 \tag{4.34}$$

$$+c^2\left(\frac{\varphi(t)^2}{|\rho_t(t)|}+\varphi(t)\rho_t(t)+\frac{|m_\varepsilon(t)-m(t)|^2}{|\rho_t(t)|}+\frac{m(t)^2\rho(t)^4}{|\rho_t(t)|}\right)\|v(t)\|_{s+3/2}^2 \tag{4.35}$$

$$+c^2(m_\varepsilon(t)\rho_t(t)+m(t)^2\rho(t)^2|\rho_t(t)|^{-1})\Re(A\langle D\rangle^{s+1/2}v(t),\langle D\rangle^{s+1/2}v(t)). \tag{4.36}$$

Thus, if we let $\gamma>0$ and $\varepsilon>0$ satisfying

$$\varepsilon\le e^{-2\gamma T},\quad \gamma^2\ge\max\left\{\sup_{0\le t\le T}\left\{\frac{\rho_1M_0^2}{m_\varepsilon(t)}\right\},\frac{2}{\rho_1^2}+M_0^2\rho_1^2\right\}, \tag{4.37}$$

where $M_0=\max_{0\le t\le T}m(t)$, then the third and the fourth terms are non-positive. □

Lemma 4.2. *Assume that $m(t)$ is a non-negative function satisfying $m(t)\in C^0([0,T))\cap L^1([0,T])$ and $v(t,x)\in\bigcap_{j=0}^2C^{2-j}([0,T);H^{s+j})$. Then there are $\rho(t)$ and $\varphi(t)$ in $C^1([0,T))$ with $\rho_t(t)\in L^1([0,T])$, $\rho(0)=\rho_1$ and $\varepsilon>0$ such that the estimate* (4.2) *is established for* (4.3).

Proof. If we choose $\rho(t)$ and $\varepsilon>0$ suitably, we can prove that (4.35) and (4.35) are non-positive. Indeed, put $\varphi(t)$ and $\rho(t)$;

$$\varphi(t)=\rho_1^2e^{-2c\left\{t+\int_0^tm(\tau)(1+1/\sqrt{m_\varepsilon(\tau)})d\tau\right\}}, \tag{4.38}$$

$$\rho(t)=\left(\rho_1e^{-ct}-c\int_0^t\frac{\rho_1}{\varphi(T)}|m_\varepsilon(\tau)-m(\tau)|d\tau\right)e^{-c\int_0^tm(\tau)(1+1/\sqrt{m_\varepsilon(\tau)})d\tau}, \tag{4.39}$$

then $\varphi(t)$ and $\rho(t)$ belong to $C^1([0,T))$ with $\rho_t \in L^1([0,T])$ and $\rho(t) > 0$ for sufficientry small $\varepsilon > 0$, and they satisfy

$$\begin{cases} \rho(0) = \rho_1, \\ \rho_t(t) \le -c\left(\dfrac{|m_\varepsilon(t) - m(t)|}{\sqrt{\varphi(t)}} + \dfrac{m(t)\rho(t)^2}{\sqrt{\varphi(t)}} + \dfrac{m(t)\rho(t)}{\sqrt{m_\varepsilon(t)}} + \sqrt{\varphi(t)}\right) \end{cases} \tag{4.40}$$

for $t \in (0,T)$. Hence we obtain (4.2). □

LEMMA 4.3. *Assume that $m(t)$, $\varphi(t)$ and $\rho(t)$ satisfy the conditions of Lemma* 4.1 *and that $u(t,x)$ is a solution of the Cauchy problem* (3.1) *satisfying* (3.19), *then $u(t,x)$ has the inequality as*

$$\begin{aligned}
&(e^{-2\gamma t}\|\langle\cdot\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}u(t)\|_{s+1}^2 + \|\langle\cdot\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}\partial_t u(t)\|_s^2)^{1/2} \\
&\le ce^{\int_0^t q(\tau)d\tau}\Bigg(\|\langle\cdot\rangle_\kappa^\delta e^{\rho_1\langle D\rangle}u_0\|_{s+1} + \|\langle\cdot\rangle_\kappa^\delta e^{\rho_1\langle D\rangle}u_1\|_s \\
&\quad + \int_0^t \|\langle\cdot\rangle_\kappa^\delta e^{\rho(\tau)\langle D\rangle}f(\tau)\|_s d\tau\Bigg),
\end{aligned} \tag{4.41}$$

for $t \in [0,T]$, where $q(\tau)$, γ and ε are given by Proposition 4.1, *and the positive constant c is independent of γ.*

PROOF. It is obvious by Lemma 4.1.

5. Local existence of solutions for the nonlinear problem

Let $0 \le \tau < T_1$. For $T \in (\tau, T_1]$ we consider the Cauchy problem:

$$\begin{cases} \partial_t^2 u(t,x) + M((Au(t),u(t)))Au(t,x) = f(t,x), & \tau < t < T, \\ u(\tau,x) = u_0(x), \quad \partial_t u(\tau,x) = u_1(x). \end{cases} \tag{5.1}$$

THEOREM 5.1. *Assume that* (1.4), (1.5) *and* (1.6) *are valid. Let $0 < \rho_1 < \rho_0/\sqrt{n}$. Then for any $u_0(x) \in H^{s+2}_{\rho_1,\delta,\kappa}$, $u_1(x) \in H^{s+1}_{\rho_1,\delta,\kappa}$ and $\langle x\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}f(t,x) \in C^0([0,T_1];H^{s+1})$ with $\rho(t) = \rho_1 e^{-\gamma(t-\tau)}$, there exist $T \in (\tau, T_1]$ and $\gamma_0 > 0$ such that the Cauchy problem* (5.1) *has a solution satisfying*

$$\langle x\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}u(x,t) \in \bigcap_{j=0}^{2} C^{2-j}([\tau,T];H^{s+j}) \tag{5.2}$$

for any $\gamma \ge \gamma_0$.

PROOF. We may assume $\tau = 0$ without loss of generality. We shall prove the existence of the solution of (5.1) by Schauder's fixed point theorem. For $T > 0$ and $s \in \boldsymbol{R}$, we introduce a space of functions;

$$X^s_{T,\delta,\kappa} = \{w(t,x); \langle x\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle} w(t,x) \in C^0([0,T];H^{s+1}) \cap C^1([0,T];H^s)\} \tag{5.3}$$

equipped with its norm $\|\cdot\|_{X^s_{T,\delta,\kappa}}$ as

$$\|w\|_{X^s_{T,\delta,\kappa}} = \sup_{0\le t\le T}\left\{\frac{1}{2}(\|\langle\cdot\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle} w(t)\|^2_{s+1} + \|\langle\cdot\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle}\partial_t w(t)\|^2_s)\right\}^{1/2} \tag{5.4}$$

for every $w \in X^s_{T,\delta,\kappa}$. Let $B^s_{T,\delta,\kappa}(R)$ be a convex subspace of $X^{s+1}_{T,\delta,\kappa}$ such that

$$B^s_{T,\delta,\kappa}(R) = \left\{u \in X^{s+1}_{T,\delta,\kappa}; \langle x\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle} u(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^{s+j}),\ \|u\|_{X^{s+1}_{T,\delta,\kappa}} \le R\right\}, \tag{5.5}$$

for $R \gg 1$. We now define the two functions

$$m(t) = m(t;w) = M(\eta(t;w)), \quad \eta(t;w) = \sum_{i,j=1}^{n}(a_{ij}D_i w(t), D_j w(t)), \tag{5.6}$$

for each $w \in X^{s'+1}_{T,0,\kappa}$, where $s' < s$. Note that $m(t) = M(\eta(t;w)) \in C^0([0,T])$, and if $w \in B^s_{T,0,\kappa}(R)$ for $R > 0$, then for arbitrary fixed $\nu > 0$, there exists a positive constant ε independent of w such that

$$\int_0^T |m_\varepsilon(t;w) - m(t;w)|dt < \nu, \tag{5.7}$$

where $m_\varepsilon(t;w) = \int_0^T \chi_\varepsilon(t-\tau)m(\tau;w)d\tau + \varepsilon$ and $\chi_\varepsilon(t)$ is defined in section 4. Then we define the mapping Φ from $w \in X^{s+1}_{T,0,\kappa}$ into $u \in X^{s+1}_{T,0,\kappa}$ such that

$$\partial_t^2 u(t,x) + M(\eta(t;w))Au(t,x) = f(t,x). \tag{5.8}$$

We shall prove that Ψ is a compact mapping from $B^{s'}_{T,0,\kappa}(R)$ into itself for $s' < s$ and sufficiently small T. By Lemma 3.3, $u(t,x)$ in (5.8) satisfies

$$\langle x\rangle^\delta_\kappa e^{\rho(t)\langle D\rangle} u(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^{s+j}) \tag{5.9}$$

for $u_0 \in H^{s+2}_{\rho_1,\delta,\kappa}$, $u_1 \in H^{s+1}_{\rho_1,\delta,\kappa}$ and every fixed $w \in B^{s'}_{T,0,\kappa}(R)$. Then by Lemma 4.1, we have

$$\left\{\frac{1}{2}(\|\langle\cdot\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}u(t)\|_{s+1}^2 + \|\langle\cdot\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}\partial_t u(t)\|_s^2)\right\}^{1/2}$$

$$\leq e^{\gamma t}\left\{\frac{1}{2}(e^{-2\gamma t}\|\langle\cdot\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}u(t)\|_{s+1}^2 + \|\langle\cdot\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}\partial_t u(t)\|_s^2)\right\}^{1/2}$$

$$\leq e^{\gamma t}\left\{ce^{\int_0^t q(\tau)d\tau}\left(\|\langle\cdot\rangle_\kappa^\delta e^{\rho_1\langle D\rangle}u_0\|_{s+1} + \|\langle\cdot\rangle_\kappa^\delta e^{\rho_1\langle D\rangle}u_1\|_s + \int_0^t \|\langle\cdot\rangle_\kappa^\delta e^{\rho(\tau)\langle D\rangle}f(\tau)\|_s d\tau\right)\right\}$$

$$\leq c' e^{\int_0^T (q(\tau)+\gamma)d\tau}, \tag{5.10}$$

where c' is independent of T and R. Therefore for sufficiently large R, we can find $T(R) = T > 0$ such that

$$c' e^{\int_0^T (q(\tau)+\gamma)d\tau} = R. \tag{5.11}$$

On the other hand, by Proposition 2.2, we have obviously that the embedding $B^s_{T,\delta,\kappa}(R) \hookrightarrow B^{s'}_{T,0,\kappa}(R)$ is compact for $s' < s$ and $\delta > 0$. Hence the mapping Ψ defined (5.8) is a compact mapping from $B^{s'}_{T,0,\kappa}(R)$ into itself. Then by Schauder's fixed point theorem, Ψ has a fixed point $u(t,x)$ in $B^{s'}_{T,0,\kappa}$. Further by Lemma 3.3, the fixed point is a solution of (5.1) satisfying

$$\langle x\rangle_\kappa^\delta e^{\rho(t)\langle D\rangle}u(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T];H^{s+j}) \tag{5.12}$$

for $u_0 \in H^{s+2}_{\rho,\delta,\kappa}$ and $u_1 \in H^{s+1}_{\rho,\delta,\kappa}$. □

6. Global existence of solution for the non-linear problem

In this section we shall prove our main theorem. Now we introduce the following energy:

$$E(t)^2 = \frac{1}{2}(\|\partial_t u(t) + u(t)\|^2 + \|u(t)\|^2 + F(\eta(t))), \tag{6.1}$$

where $F(\eta) = \int_0^\eta M(\lambda)d\lambda$ and $\eta(t) = (Au(t), u(t))$. Then for the energy $E(t)$, according to [DS] and [KY], the following energy estiamte is concluded.

Proposition 6.1. *Assume that $M(\eta)$ is a non-negative continuous function in $[0,\infty)$ and $f(t,x) \in C^0([0,T];L^2)$. If $u(t,x)$ is a solution of the Cauchy problem*

(1.3) *in* $(0, T)$ *such that* $u(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0,T); H^j)$, *then we have the energy estimate*:

$$E(t)^2 + \int_0^t e^{3(t-\tau)} M(\eta(\tau))\eta(\tau)d\tau \le E(0)^2 e^{3t} + \frac{1}{2}\int_0^t e^{3(t-\tau)} \|f(\tau)\|^2 d\tau \tag{6.2}$$

for $t \in [0, T)$.

Proof. Differenting (6.1), from the equation (1.3) we get,

$$\begin{aligned} 2E'(t)E(t) &= \Re(f(t) + \partial_t u(t), \partial_t u(t) + u(t)) + \Re(\partial_t u(t), u(t)) - M(\eta(t))\eta(t) \\ &\le \frac{1}{2}\|f(t)\|^2 + 3E(t)^2 - M(\eta(t))\eta(t) \end{aligned} \tag{6.3}$$

for $t \in [0, T)$, which yields (6.2). □

Corollary 6.2. *If* (6.2) *holds and* $T < \infty$, *then* $M(\eta(t)) \in L^1([0,T])$.

Proof. From (6.2), it is evident that $M(\eta(t))\eta(t) \in L^1([0,T])$. On the other hand

$$\begin{aligned} \int_0^t M(\eta(\tau))d\tau &= \int_{[0,t]\cap\{\tau;\eta(\tau)>1\}} M(\eta(\tau))d\tau + \int_{[0,t]\cap\{\tau;\eta(\tau)\le 1\}} M(\eta(\tau))d\tau \\ &\le \int_0^t M(\eta(\tau))\eta(\tau)d\tau + \sup_{0\le\eta\le 1} M(\eta)t \end{aligned} \tag{6.4}$$

for all $t \in [0, T)$, which implies that $M(\eta(t)) \in L^1([0,T])$. □

Now we can prove our main theorem. Let $\Lambda(t,\gamma) = \rho_1 e^{-\gamma t}\langle D\rangle$ and T^* the real number defined by

$$T^* = \max\left\{ T > 0; \text{there exist } \gamma > 0 \text{ and a solution } u(t,x) \text{ satisfying (1.3)} \right.$$
$$\left. \text{in } (0,T) \text{ such that } \langle x\rangle_\kappa^\delta e^{\Lambda(t,\gamma)} u(t,x) \in \bigcap_{j=1}^{2} C^{2-j}([0,T); H^j) \right\}.$$

Theorem 5.1 ensures $T^* > 0$. We shall claim $T^* = \infty$. Suppose that $T^* < 0$. Then it follows from Proposition 6.2 that $m(t) = M(Au(t), u(t))$ belongs to $L^1([0, T^*])$. Hence, Proposition 3.2 and the fact that $m(t) \in C^0([0,T^*)) \cap L^1([0,T^*])$ yield that $v(t,x) = \langle x\rangle_\kappa^\delta e^{\Lambda(t)} u(t,x)$ which satisfies (3.19) with $s = 0, 1$ and $T = T^*$, where $\Lambda(t) = \rho(t)\langle D\rangle$ and $\rho(t)$ is introduced in (4.39). Let us take $\gamma > 0$ such that

$\rho_1 e^{-\gamma t} \leq \rho(t)$ for $t \in [0, T^*)$. Then the definition of T^* and (4.2) imply $\langle x \rangle_\kappa^\delta e^{\Lambda(t,\gamma)} u(t,x) \in \bigcap_{j=0}^2 C^{2-j}([0, T^*]; H^j)$, where $\Lambda(t,\gamma) = \rho_1 e^{-\gamma t} \langle D \rangle$. Hence we have the limits $u(T^* - 0) \in H^2$ and $\partial_t u(T^* - 0)$ which satisfy $\langle x \rangle_\kappa^\delta e^{\Lambda(T^*,\gamma)} u(T^* - 0) \in H^1$. Therefore, applying Theorem 5.1 with $\rho_2 = \rho_1 e^{\gamma T^*}$, we have a solution $\tilde{u}(t,x)$ of the Cauchy problem (5.1) in (T^*, T), $T > T^*$ with initial data $\tilde{u}(T^*) = u(T^* - 0)$ and $\partial_t \tilde{u}(T^*) = \partial_t u(T^* - 0)$, which satisfies

$$\langle x \rangle_\kappa^\delta \exp(\rho_2 e^{-\gamma(t-T^*)} \langle D \rangle) \tilde{u}(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([T^*, T]; H^j). \tag{6.5}$$

Then $\Lambda(t,\gamma) = \rho_2 e^{-\gamma(T-T^*)} \langle D \rangle$ implies that

$$\langle x \rangle_\kappa^\delta e^{\Lambda(t,\gamma)} \tilde{u}(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([T^*, T]; H^j). \tag{6.6}$$

Now let us define

$$w(t,x) = \begin{cases} u(t,x), & t \in (0, T^*) \\ \tilde{u}(t,x), & t \in [T^*, T). \end{cases} \tag{6.7}$$

Then $w(t,x)$ has to satisfy (1.3) in $(0, T)$ and

$$\langle x \rangle_\kappa^\delta e^{\Lambda(t,\gamma)} w(t,x) \in \bigcap_{j=0}^{2} C^{2-j}([0, T); H^j). \tag{6.8}$$

This result contradicts the definition of T^*. Thus, we have proved that $T^* = \infty$. □

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