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DIAGONALIZATION OF AN ELEMENT P OF $\mathfrak{P}^C$ BY THE COMPACT LIE GROUP E_7

By

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1. Introduction

The following diagonalization problems are well-known. A Hermitian matrix $X \in M(n, K)$ over $K = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ can be transformed to a diagonal form by some element A of the groups $SO(n)$, $SU(n)$, $Sp(n)$, respectively. Moreover we know that a Hermitian matrix $X \in M(3, K)$ over $K = \mathfrak{C}$, $\mathfrak{C}^C$ can be transformed to a diagonal form by some element α of the compact Lie groups F_4, E_6, respectively (Freudenthal [1], Yokota [9]). Now, in this paper, we show that an element P of $\mathfrak{P}^C$ (which is a minimal dimensional representation space of the compact Lie group E_7) can be transformed to a diagonal form by some element α of E_7 (Theorem 5). In the last section, we give a canonical diagonal form of P of $\mathfrak{P}^C$ (Theorem 12). We would like to thank T. Miyashita for his calculation of Lemma 3.

2. Notations and Preliminaries

Although, throughout this paper, we use the same notations and definitions as in [10], we sketch briefly the definitions of the vector space $\mathfrak{P}^C$, the compact Lie group E_7 and its Lie algebra $\mathfrak{e}_7$. Let $\mathfrak{C}$ be the Cayley algebra and let $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\}$ be the exceptional Jordan algebra with the Jordan multiplication $X \circ Y = \frac{1}{2}(XY + YX)$, the inner product $(X, Y) = \mathrm{tr}(X \circ Y)$ and the Freudenthal multiplication $X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\,\mathrm{tr}(Y) - (X, Y))E)$. Let $\mathfrak{J}^C$ be the complexification of $\mathfrak{J}$ and define the Hermitian inner product $\langle X, Y \rangle$ by $\langle X, Y \rangle = (\tau X, Y)$ where τ is the complex conjugation in $\mathfrak{J}^C$. The simply connected compact Lie groups F_4, E_6 are defined as

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$$F_4 = \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

$$E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \mid \tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\},$$

respectively. The group E_6 contains F_4 as subgroup by

$$F_4 = \{\alpha \in E_6 \mid \alpha E = E\} = \{\alpha \in E_6 \mid \tau\alpha\tau = \alpha\}.$$

Moreover F_4, E_6 have subgroups $Spin(9)$, $Spin(10)$ as

$$Spin(9) = \{\alpha \in F_4 \mid \alpha E_1 = E_1\}, \quad Spin(10) = \{\alpha \in E_6 \mid \alpha E_1 = E_1\},$$

respectively, where $E_1 = \mathrm{diag}(1, 0, 0) \in \mathfrak{J} \subset \mathfrak{J}^C$. The C-vector space $\mathfrak{P}^C$ is defined by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C.$$

Hereafter, an element P of $\mathfrak{P}^C$,

$$P = \left(\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \eta_2 & y_1 \\ y_2 & \overline{y_1} & \eta_3 \end{pmatrix}, \xi, \eta \right), \quad \xi_k, \eta_k, \xi, \eta \in C, x_k, y_k \in \mathfrak{C}^C$$

is briefly denoted by

$$P = ((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta).$$

For $\phi \in \mathfrak{e}_6^C$, $A, B \in \mathfrak{J}^C$, $\nu \in C$, we define a C-linear mapping $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \nu) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi X - \frac{1}{3}\nu X + 2B \times Y + \eta A \\ 2A \times X - {}^t\phi Y + \frac{1}{3}\nu Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{pmatrix}.$$

For $P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define a C-linear mapping $P \times Q$ of $\mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \begin{cases} \phi = -\frac{1}{2}(X \vee W + Z \vee Y), \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X), \\ B = \frac{1}{4}(2X \times Z - \eta W - \omega Y), \\ \nu = \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)) \end{cases}$$

where $X \vee W \in \mathfrak{e}_6^C$ is defined by $(X \vee W)Z = \frac{1}{2}(W, Z)X + \frac{1}{6}(X, W)Z - 2W \times (X \times Z)$, $Z \in \mathfrak{J}^C$. Finally we define a Hermitian inner product $\langle P, Q \rangle$ in

$\mathfrak{P}^C$ by

$$\langle P, Q\rangle = \langle X, Z\rangle + \langle Y, W\rangle + (\tau\xi)\zeta + (\tau\eta)\omega,$$

where τ is the complex conjugation in $C = \boldsymbol{R}^C$. Now the simply connected compact Lie group E_7 is obtained ([5]) as

$$E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\},$$

and its Lie algebra $\mathfrak{e}_7$ is given by

$$\mathfrak{e}_7 = \{\Phi(\phi, A, -\tau A, \nu) \in \mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \phi \in \mathfrak{e}_6, A \in \mathfrak{J}^C, \nu \in i\boldsymbol{R}\}.$$

For $\alpha \in E_6$, if the mapping $\tilde{\alpha} : \mathfrak{P}^C \to \mathfrak{P}^C$ is defined by $\tilde{\alpha}(X, Y, \xi, \eta) = (\alpha X, \tau\alpha\tau Y, \xi, \eta)$ then $\tilde{\alpha} \in E_7$, so α and $\tilde{\alpha}$ will be identified. Hence the group E_7 contains E_6 as a subgroup by the identification:

$$E_6 = \{\tilde{\alpha} \in E_7 \mid \alpha \in E_6\}.$$

We define a C-linear mapping $\sigma(= \sigma_1) : \mathfrak{P}^C \to \mathfrak{P}^C$ by

$$\begin{aligned}&\sigma((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta)\\ &\quad = ((\xi_1, \xi_2, \xi_3; x_1, -x_2, -x_3), (\eta_1, \eta_2, \eta_3; y_1, -y_2, -y_3), \xi, \eta).\end{aligned}$$

$\sigma \in E_7$ and $\sigma^2 = 1$. And we know that the subgroup $(E_7)^\sigma = \{\alpha \in E_7 \mid \sigma\alpha = \alpha\sigma\}$ of E_7 is isomorphic to the group $(SU(2) \times Spin(12)/\boldsymbol{Z}_2)$ ([10], [8]) where $Spin(12)(= Spin_1(12))$ is a subgroup of E_7 given by

$$Spin(12) = \{\alpha \in E_7 \mid \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\}.$$

Here two C-linear mappings $\kappa(= \kappa_1)$, $\mu(= \mu_1) : \mathfrak{P}^C \to \mathfrak{P}^C$ are defined by

$$\begin{aligned}&\kappa((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta)\\ &\quad = ((-\xi_1, \xi_2, \xi_3; x_1, 0, 0), (\eta_1, -\eta_2, -\eta_3; -y_1, 0, 0), -\xi, \eta),\\ &\mu((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta)\\ &\quad = ((\eta, \eta_3, \eta_2; -y_1, 0, 0), (\xi, \xi_3, \xi_2; -x_1, 0, 0), \eta_1, \xi_1),\end{aligned}$$

respectively. If $\alpha \in E_7$ satisfies $\kappa\alpha = \alpha\kappa$ then α automatically satisfies $\sigma\alpha = \alpha\sigma$, because $\sigma = \exp(\pi i\kappa)$, so $Spin(12)$ is a subgroup of $(E_7)^\sigma$. Hence $Spin(12)$ leaves invariant the spaces $(\mathfrak{P}^C)_1 = \{P \in \mathfrak{P}^C \mid \sigma P = P\}$, $(\mathfrak{P}^C)_{-1} = \{P \in \mathfrak{P}^C \mid \sigma P = -P\}$ and we have $\mathfrak{P}^C = (\mathfrak{P}^C)_1 \oplus (\mathfrak{P}^C)_{-1}$. Hence, if we define $D_1(P)$ for $P \in \mathfrak{P}^C$ by

$$D_1(P) = \langle P_1, P_1\rangle, \quad P = P_1 + P_{-1}, P_1 \in (\mathfrak{P}^C)_1,\ P_{-1} \in (\mathfrak{P}^C)_{-1}.$$

LEMMA 1. *For* $P \in \mathfrak{P}^C$ *and* $\alpha \in Spin(12)$, *we have* $D_1(\alpha P) = D_1(P)$.

Similar to κ, μ we define C-linear mappings κ_2, κ_3 and μ_2, μ_3 by

$$\kappa_2 P = ((\xi_1, -\xi_2, \xi_3; 0, x_2, 0), (-\eta_1, \eta_2, -\eta_3; 0, -y_2, 0), -\xi, \eta),$$

$$\kappa_3 P = ((\xi_1, \xi_2, -\xi_3; 0, 0, x_3), (-\eta_1, -\eta_2, \eta_3; 0, 0, -y_3), -\xi, \eta)$$

$$\mu_2 P = ((\eta_3, \eta, \eta_1; 0, -y_2, 0), (\xi_3, \xi, \xi_1; 0, -x_2, 0), \eta_2, \xi_2)$$

$$\mu_3 P = ((\eta_2, \eta_1, \eta; 0, 0, -y_3), (\xi_2, \xi_1, \xi; 0, 0, -x_3), \eta_3, \xi_3),$$

where $P = ((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta) \in \mathfrak{P}^C$ and we define $Spin_2(12)$, $Spin_3(12)$ by $Spin_k(12) = \{\alpha \in E_7 \mid \kappa_k\alpha = \alpha\kappa_k, \mu_k\alpha = \alpha\mu_k\}$, $k = 2, 3$, respectively.

3. Some elements of $Spin(12) (\subset E_7)$

LEMMA 2. ([9]). (1) *For* $a \in \mathfrak{C}$, $a \neq 0$, *the mapping* $\alpha(a) : \mathfrak{J} \to \mathfrak{J}$ *defined by* $\alpha(a)X = X'$,

$$\begin{cases} \xi_1' = \xi_1 \\ \xi_2' = \dfrac{\xi_2 + \xi_3}{2} + \dfrac{\xi_2 - \xi_3}{2}\cos 2|a| + \dfrac{(a, x_1)}{|a|}\sin 2|a| \\ \xi_3' = \dfrac{\xi_2 + \xi_3}{2} - \dfrac{\xi_2 - \xi_3}{2}\cos 2|a| - \dfrac{(a, x_1)}{|a|}\sin 2|a| \end{cases}$$

$$\begin{cases} x_1' = x_1 - \dfrac{(\xi_2 - \xi_3)a}{2|a|}\sin 2|a| - \dfrac{2(a, x_1)a}{|a|^2}\sin^2|a| \\ x_2' = x_2\cos|a| - \dfrac{\overline{x_3 a}}{|a|}\sin|a| \\ x_3' = x_3\cos|a| + \dfrac{\overline{a x_2}}{|a|}\sin|a| \end{cases}$$

belongs to $Spin(9) \subset F_4 \subset E_7$.

(2) *For* $a \in \mathfrak{C}$, $a \neq 0$, *the mapping* $\beta(a) : \mathfrak{J}^C \to \mathfrak{J}^C$ *defined by* $\beta(a)X = X'$,

$$\begin{cases} \xi_1' = \xi_1 \\ \xi_2' = \dfrac{\xi_2 - \xi_3}{2} + \dfrac{\xi_2 + \xi_3}{2}\cos 2|a| + i\dfrac{(a, x_1)}{|a|}\sin 2|a| \\ \xi_3' = -\dfrac{\xi_2 - \xi_3}{2} + \dfrac{\xi_2 + \xi_3}{2}\cos 2|a| + i\dfrac{(a, x_1)}{|a|}\sin 2|a| \end{cases}$$

$$\begin{cases} x_1' = x_1 + i\dfrac{(\xi_2+\xi_3)a}{2|a|}\sin 2|a| - \dfrac{2(a,x_1)a}{|a|^2}\sin^2|a| \\ x_2' = x_2\cos|a| + i\dfrac{\overline{x_3 a}}{|a|}\sin|a| \\ x_3' = x_3\cos|a| + i\dfrac{\overline{a x_2}}{|a|}\sin|a| \end{cases}$$

belongs to $Spin(10) \subset E_6 \subset E_7$.

LEMMA 3. (1) *For* $a \in \mathfrak{C}$, $a \neq 0$, *the mapping* $\gamma(a) : \mathfrak{P}^C \to \mathfrak{P}^C$ *defined by* $\gamma(a)(X, Y, \xi, \eta) = (X', Y', \xi', \eta')$,

$$\begin{cases} \xi_1' = \dfrac{\xi_1-\xi}{2} + \dfrac{\xi_1+\xi}{2}\cos 2|a| + \dfrac{(a,y_1)}{|a|}\sin 2|a| \\ \xi_2' = \xi_2 \\ \xi_3' = \xi_3 \\ x_1' = x_1 + \dfrac{(\eta_1+\eta)a}{2|a|}\sin 2|a| - \dfrac{2(a,x_1)a}{|a|^2}\sin^2|a| \\ x_2' = x_2\cos|a| - \dfrac{\overline{y_3 a}}{|a|}\sin|a| \\ x_3' = x_3\cos|a| - \dfrac{\overline{a y_2}}{|a|}\sin|a| \end{cases}$$

$$\begin{cases} \eta_1' = \dfrac{\eta_1-\eta}{2} + \dfrac{\eta_1+\eta}{2}\cos 2|a| - \dfrac{(a,x_1)}{|a|}\sin 2|a| \\ \eta_2' = \eta_2 \\ \eta_3' = \eta_3 \\ y_1' = y_1 - \dfrac{(\xi_1+\xi)a}{2|a|}\sin 2|a| - \dfrac{2(a,y_1)a}{|a|^2}\sin^2|a| \\ y_2' = y_2\cos|a| + \dfrac{\overline{x_3 a}}{|a|}\sin|a| \\ y_3' = y_3\cos|a| + \dfrac{\overline{a x_2}}{|a|}\sin|a| \end{cases}$$

$$\begin{cases} \xi' = -\dfrac{\xi_1-\xi}{2} + \dfrac{\xi_1+\xi}{2}\cos 2|a| + \dfrac{(a,y_1)}{|a|}\sin 2|a| \\ \eta' = -\dfrac{\eta_1-\eta}{2} + \dfrac{\eta_1+\eta}{2}\cos 2|a| - \dfrac{(a,x_1)}{|a|}\sin 2|a| \end{cases}$$

belongs to $Spin(12) \subset E_7$.

(2) *For* $a \in \mathfrak{C}$, $a \neq 0$, *the mapping* $\delta(a) : \mathfrak{P}^C \to \mathfrak{P}^C$ *defined by* $\delta(a)(X, Y, \xi, \eta) = (X', Y', \xi', \eta')$,

$$\begin{cases} \xi_1' = \dfrac{\xi_1 + \xi}{2} + \dfrac{\xi_1 - \xi}{2}\cos 2|a| - i\dfrac{(a, y_1)}{|a|}\sin 2|a| \\ \xi_2' = \xi_2 \\ \xi_3' = \xi_3 \\ x_1' = x_1 - i\dfrac{(\eta_1 - \eta)a}{2|a|}\sin 2|a| - \dfrac{2(a, x_1)a}{|a|^2}\sin^2 |a| \\ x_2' = x_2 \cos |a| + i\dfrac{\overline{y_3 a}}{|a|}\sin |a| \\ x_3' = x_3 \cos |a| + i\dfrac{\overline{a y_2}}{|a|}\sin |a| \end{cases}$$

$$\begin{cases} \eta_1' = \dfrac{\eta_1 + \eta}{2} + \dfrac{\eta_1 - \eta}{2}\cos 2|a| - i\dfrac{(a, x_1)}{|a|}\sin 2|a| \\ \eta_2' = \eta_2 \\ \eta_3' = \eta_3 \\ y_1' = y_1 - i\dfrac{(\xi_1 - \xi)a}{2|a|}\sin 2|a| - \dfrac{2(a, y_1)a}{|a|^2}\sin^2 |a| \\ y_2' = y_2 \cos |a| + i\dfrac{\overline{x_3 a}}{|a|}\sin |a| \\ y_3' = y_3 \cos |a| + i\dfrac{\overline{a x_2}}{|a|}\sin |a| \end{cases}$$

$$\begin{cases} \xi' = \dfrac{\xi_1 + \xi}{2} - \dfrac{\xi_1 - \xi}{2}\cos 2|a| + i\dfrac{(a, y_1)}{|a|}\sin 2|a| \\ \eta' = \dfrac{\eta_1 + \eta}{2} - \dfrac{\eta_1 - \eta}{2}\cos 2|a| + i\dfrac{(a, x_1)}{|a|}\sin 2|a| \end{cases}$$

belongs to $Spin(12) \subset E_7$.

PROOF. (1) For $\Phi = \Phi(0, F_1(a), -F_1(a), 0) \in \mathfrak{spin}(12) \subset \mathfrak{e}_7$ (where $F_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \bar{a} & 0 \end{pmatrix} \in \mathfrak{J}^C$), we have $\gamma(a) = \exp \Phi$. Hence $\gamma(a) \in Spin(12) \subset E_7$.

(2) For $\Phi = \Phi(0, iF_1(a), iF_1(a), 0) \in \mathfrak{spin}(12) \subset \mathfrak{e}_7$, we have $\delta(a) = \exp \Phi$. Hence $\delta(a) \in Spin(12) \subset E_7$.

4. Diagonalization of $P \in \mathfrak{P}^C$ by E_7

PROPOSITION 4. (1) *Any element $P \in \mathfrak{P}^C$ can be transformed to the following form by some $\alpha \in Spin(12)$:*

$$\alpha P = (\begin{pmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & x_1 \\ x_2 & \overline{x_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_3 & \overline{y_2} \\ \overline{y_3} & 0 & y_1 \\ y_2 & \overline{y_1} & 0 \end{pmatrix}, 0, 0), \quad x_k, y_k \in \mathfrak{C}^C.$$

(2) *Any element $P \in \mathfrak{P}^C$ can be transformed to the following form by some $\alpha \in Spin(12)$:*

$$\alpha P = (\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \overline{y_2} \\ \overline{y_3} & \eta_2 & 0 \\ y_2 & 0 & \eta_3 \end{pmatrix}, \xi, \eta), \quad \xi_1, \eta_k, \xi, \eta \in C, x_k, y_k, \in \mathfrak{C}^C.$$

PROOF. (1) For a given element

$$P = ((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta)$$

of $\mathfrak{P}^C$, choose $a_1 \in \mathfrak{C}$ such that $(a_1, x_1) = (a_1, y_1) = 0, |a_1| = \pi/4$ and act $\gamma(a_1)$ of Lemma 3.(1) on P, then

$$\gamma(a_1)P = ((\xi_1', \xi_2', \xi_3'; x_1', *, *), (\eta_1', \eta_2', \eta_3'; y_1', *, *), -\xi_1', -\eta_1') = P_1, \quad \begin{array}{l} \xi_2' = \xi_2, \ \xi_3' = \xi_3, \\ \eta_2' = \eta_2, \ \eta_3' = \eta_3. \end{array}$$

Choose $a_2 \in \mathfrak{C}$ such that $(a_2, x_1') = (a_2, y_1') = 0$, $|a_2| = \pi/4$ and act $\delta(a_2)$ of Lemma 3.(2) on P_1, then

$$\delta(a_2)P_1 = ((0, \xi_2'', \xi_3''; x_1'', *, *), (0, \eta_2'', \eta_3''; y_1'', *, *), 0, 0) = P_2, \quad \begin{array}{l} \xi_2'' = \xi_2', \ \xi_3'' = \xi_3', \\ \eta_2'' = \eta_2', \ \eta_3'' = \eta_3'. \end{array}$$

Choose $a_3 \in \mathfrak{C}$ such that $(a_3, x_1'') = (a_3, y_1'') = 0$, $|a_3| = \pi/4$ and act $\beta(a_3)$ of Lemma 2.(2) on P_2, then

$$\beta(a_3)P_2 = ((0, \xi_2^{(3)}, -\xi_2^{(3)}; x_1^{(3)}, *, *), ((0, \eta_2^{(3)}, -\eta_2^{(3)}; y_1^{(3)}, *, *), 0, 0) = P_3.$$

Choose $a_4 \in \mathfrak{C}$ such that $(a_4, x_1^{(3)}) = (a_4, y_1^{(3)}) = 0$, $|a_4| = \pi/4$ and act $\alpha(a_4)$ of Lemma 2.(1) on P_3, then

$$\alpha(a_4)P_3 = ((0, 0, 0; x_1^{(4)}, *, *), (0, 0, 0; y_1^{(4)}, *, *), 0, 0) = P_4.$$

This P_4 is the required form of (1).

(2) In the form of (1), let $x_1 = P + iq, p, q \in \mathfrak{C}$ and assume $q \neq 0$. Put $a_5 = \pi q/4|q|$ and act $\delta(a_5)$ of Lemma 3.(2) on P_4, then

$$\delta(a_5)P_4 = ((\xi_1^{(5)}, 0, 0; p', *, *), ((\eta_1^{(5)}, 0, 0; y_1^{(5)}, *, *), -\xi_1^{(5)}, -\eta_1^{(5)}) = P_5$$

where $P' \in \mathfrak{C}$. Assume $p' \neq 0$. Let $a_6 = \pi p'/4|p'|$ and act $\gamma(a_6)$ of Lemma 3.(1) on P_5, then

$$\gamma(a_6)P_5 = ((\xi_1^{(6)}, 0, 0; 0, *, *), ((\eta_1^{(6)}, 0, 0; y_1^{(6)}, *, *), \xi^{(6)}, \eta^{(6)}) = P_6.$$

Let $y_1^{(6)} = u + iv, u, v \in \mathfrak{C}$ and assume $u \neq 0$. Let $a_7 = \pi u/4|u|$ and act $\beta(a_7)$ of Lemma 2.(2) on P_6, then

$$\beta(a_7)P_6 = ((\xi_1^{(7)}, 0, 0; 0, *, *), ((\eta_1^{(7)}, \eta_2^{(7)}, \eta_3^{(7)}; iv', *, *), \xi^{(7)}, \eta^{(7)}) = P_7, \quad \eta_2^{(7)} = \eta_3^{(7)}$$

where $v' \in \mathfrak{C}$. Assume $v' \neq 0$. Put $a_8 = \pi v'/4|v'|$ and act $\alpha(a_8)$ of Lemma 2.(1) on P_7, then

$$\alpha(a_8)P_7 = ((\xi_1^{(8)}, 0, 0; 0, *, *), ((\eta_1^{(8)}, \eta_2^{(8)}, \eta_3^{(8)}; 0, *, *), \xi^{(8)}, \eta^{(8)}) = P_8.$$

This P_8 is the required form of (2).

THEOREM 5. *Any element $P \in \mathfrak{P}^C$ can be transformed to a diagonal form by some $\alpha \in E_7$:*

$$\alpha P = (\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix}, \lambda, \nu), \quad \lambda_k, \nu_k, \lambda, \nu \in C.$$

PROOF. For a general element $P = (X, Y, \xi, \eta) \in \mathfrak{P}^C$, we denote by $D(P)$ the square sum of the absolute values of the diagonal elements of X, Y, ξ and η:

$$D(P) = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + |\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + |\xi|^2 + |\eta|^2.$$

Then $D_1(P)$ of Section 2 is

$$D_1(P) = D(P) + 2|x_1|^2 + 2|y_1|^2, \quad P \in \mathfrak{P}^C.$$

Now, for a given $P \in \mathfrak{P}^C$, consider a space $\mathfrak{X} = \{\alpha P \mid \alpha \in E_7\}$. Since E_7 is compact, $\mathfrak{X}$ is also compact. Let $D(\tilde{P})$ be the maximal values of $\{D(P') \mid P' \in \mathfrak{X}\}$. Then we show that $\tilde{P} = (\tilde{X}, \tilde{Y}, \tilde{\xi}, \tilde{\eta})$ is diagonal. Suppose that $\tilde{x}_1 \neq 0$ of $\tilde{X}$ or $\tilde{y}_1 \neq 0$ of $\tilde{Y}$. From Proposition 4.(2), $\tilde{P}$ can be transformed to the form

$$\alpha\tilde{P} = ((\tilde{\xi}_1', 0, 0; 0, *, *), (\tilde{\eta}_1', \tilde{\eta}_2', \tilde{\eta}_3'; 0, *, *), \tilde{\xi}', \tilde{\eta}') \tag{i}$$

by some $\alpha \in Spin(12)$. Then

$$\begin{array}{ccccccc} D(\tilde{P}) & < & D_1(\tilde{P}) & = & D_1(\alpha\tilde{P}) & = & D(\alpha\tilde{P}) \\ & \begin{pmatrix} \tilde{x}_1 \neq 0 \\ \text{or } \tilde{y}_1 \neq 0 \end{pmatrix} & & \begin{pmatrix} \alpha \in Spin(12) \\ \text{and Lemma 1} \end{pmatrix} & & \text{(i)} & \end{array}$$

This contradicts the maximality of $D(\tilde{P})$. Hence we get $\tilde{x}_1 = \tilde{y}_1 = 0$. Similarly we can prove that the other entries of $\tilde{X}$ and $\tilde{Y}$ except the diagonals are zero by means of $Spin_k(12)$, $k = 2, 3$. Thus the proof of Theorem 5 is completed.

5. The group E_8 and subgroups $SU(2) \subset E_7$, $SU(2) \subset E_8$

As is mentioned in Introduction, any element $X \in \mathfrak{J}^C$ can be transformed to a diagonal form by some $\alpha \in E_6$:

$$\alpha X = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_k \in C.$$

However, $\lambda_1, \lambda_2, \lambda_3$ are not eigenvalues of X. (These depend on the choice of $\alpha \in E_6$). In fact, we can choose $\alpha \in E_6$ so that two of $\lambda_1, \lambda_2, \lambda_3$ are non-negative real numbers. Moreover some αX has the following canonical form:

$$\alpha X = \begin{pmatrix} r_1 e^{i\theta} & 0 & 0 \\ 0 & r_2 e^{i\theta} & 0 \\ 0 & 0 & r_3 e^{i\theta} \end{pmatrix}, \quad \theta, r_k \in \boldsymbol{R},\ 0 \leq r_1 \leq r_2 \leq r_3.$$

Now, we shall find a canonical form of an element P of $\mathfrak{P}^C$. For this purpose, we prepare subgroups isomorphic to $SU(2)$ in E_7 and E_8.

Let $\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ be the complex Lie algbera of type E_8 with the Lie bracket

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t),$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s = \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1 \end{cases}$$

where $\{P_1, P_2\}$ is defined by $\{P_1, P_2\} = (X_1, Y_2) - (X_2, Y_1) + \xi_1 \eta_2 - \xi_2 \eta_1$ for $P_k = (X_k, Y_k, \xi_k, \eta_k) \in \mathfrak{P}^C$, $k = 1, 2$. The Killing form B_8 of $\mathfrak{e}_8^C$ is given by

$$B_8(R_1, R_2) = \tfrac{5}{3}B_7(\Phi_1, \Phi_2) + 15\{Q_1, P_2\} - 15\{P_1, Q_2\} + 120r_1r_2 + 60t_1s_2 + 60s_1t_2$$

$(R_k = (\Phi_k, P_k, Q_k, r_k, s_k, t_k) \in \mathfrak{e}_8^C, k = 1, 2)$, where B_7 is the Killing form of the Lie algebra $\mathfrak{e}_7^C$. We define C-linear transformations $\lambda : \mathfrak{P}^C \to \mathfrak{P}^C$, $\tilde{\lambda} : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ and $\boldsymbol{R}$-linear transformations $\tau : \mathfrak{P}^C \to \mathfrak{P}^C$, $\tilde{\tau} : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ by

$$\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi),$$

$$\tilde{\lambda}(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s),$$

$$\tau(X, Y, \xi, \eta) = (\tau X, \tau Y, \tau\xi, \tau\eta),$$

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t),$$

respectively, and a Hermitian inner product $\langle R_1, R_2\rangle$ of $\mathfrak{e}_8^C$ by $\langle R_1, R_2\rangle = -\frac{1}{15}B_8(\tilde{\tau}\tilde{\lambda}R_1, R_2)$. Then the group

$$E_8 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle\alpha R_1, \alpha R_2\rangle = \langle R_1, R_2\rangle\}$$

is a simply connected compact Lie group of type E_8. The Lie algebra $\mathfrak{e}_8$ of E_8 has the form

$$\mathfrak{e}_8 = \{(\Phi, P, -\tau\lambda P, r, s, -\tau s) \mid \Phi \in \mathfrak{e}_7,\ P \in \mathfrak{P}^C,\ r \in i\boldsymbol{R},\ s \in C\}.$$

For $\alpha \in E_7$, if the mapping $\alpha : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ is defined by $\tilde{\alpha}(\Phi, P, Q, r, s, t) = (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, s, t)$, then $\tilde{\alpha} \in E_8$ so α and $\tilde{\alpha}$ will be identified. Hence the group E_8 contains E_7 as a subgroup by the identification:

$$E_7 = \{\tilde{\alpha} \in E_8 \mid \alpha \in E_7\}.$$

LEMMA 6. (1) ([8]). *The group E_7 has a subgroup $\phi(SU(2)) = \{\phi(A) \in E_7 \mid A \in SU(2)\}$ which is isomorphic to the group $SU(2) = \{A \in M(2, C) \mid (\tau{}^tA)A = E$, $\det A = 1\}$, where $\phi(A) : \mathfrak{P}^C \to \mathfrak{P}^C$ is defined by*

$$\phi(A)((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta)$$
$$= ((\xi_1', \xi_2', \xi_3'; x_1', x_2', x_3'), (\eta_1', \eta_2', \eta_3'; y_1', y_2', y_3'), \xi', \eta'),$$

$$\begin{pmatrix}\xi_1'\\ \eta'\end{pmatrix} = A\begin{pmatrix}\xi_1\\ \eta\end{pmatrix}, \quad \begin{pmatrix}\xi'\\ \eta_1'\end{pmatrix} = A\begin{pmatrix}\xi\\ \eta_1\end{pmatrix}, \quad \begin{pmatrix}\eta_2'\\ \xi_3'\end{pmatrix} = A\begin{pmatrix}\eta_2\\ \xi_3\end{pmatrix}, \quad \begin{pmatrix}\eta_3'\\ \xi_2'\end{pmatrix} = A\begin{pmatrix}\eta_3\\ \xi_2\end{pmatrix},$$

$$\begin{pmatrix}x_1'\\ y_1'\end{pmatrix} = (\tau A)\begin{pmatrix}x_1\\ y_1\end{pmatrix}, \quad \begin{pmatrix}x_2'\\ y_2'\end{pmatrix} = \begin{pmatrix}x_2\\ y_2\end{pmatrix}, \quad \begin{pmatrix}x_3'\\ y_3'\end{pmatrix} = \begin{pmatrix}x_3\\ y_3\end{pmatrix}.$$

(2) ([6]). *The group E_8 has a subgroup $\varphi(SU(2)) = \{\varphi(A) \in E_8 \mid A \in SU(2)\}$ which is isomorphic to the group $SU(2)$, where $\varphi(A) : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ is defined by*

$$\varphi(\begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & -\tau b1 & 0 & 0 & 0 \\ 0 & b1 & \tau a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\tau a)a-(\tau b)b & -(\tau a)b & a(\tau b) \\ 0 & 0 & 0 & 2a(\tau b) & a^2 & -(\tau b)^2 \\ 0 & 0 & 0 & 2(\tau a)b & -b^2 & (\tau b)^2 \end{pmatrix}.$$

We identify $P \in \mathfrak{P}^C$ with $\tilde{P} = (0, P, -\tau\lambda P, 0, 0, 0) \in \mathfrak{e}_8^C$ hereafter. Then the groups $\phi(SU(2))$ and $\varphi(SU(2))$ act on $\mathfrak{P}^C$ and we have

Lemma 7. *For $P \in \mathfrak{P}^C$ and $A, B \in SU(2)$, we have*

$$\phi(A)\varphi(B)P = \varphi(B)\phi(A)P, \quad D_1(\varphi(B)\phi(A)P) = D_1(P).$$

6. A canonical diagonal form of $P \in \mathfrak{P}^C$ by E_7

Proposition 8. *Any element $P \in \mathfrak{P}^C$ can be transformed to the following form by some $\alpha \in E_7$:*

$$\alpha P = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & \overline{y} & 0 \end{pmatrix}, 0, 0), \quad x, y \in \mathfrak{C}^C.$$

Proof. We start from a diagonal form of P of Theorem 5 and repeat the same process in the proof of Proposition 4.(1). Then we have the required form.

Lemma 9. *In the form of αP of Proposition 8, let $x = p + iq$, $y = u + iv$, $p, q, u, v \in \mathfrak{C}$. Then we may assume that $(p,u) = (p,v) = (q,u) = (q,v) = 0$, under the action of $\varphi(B)\phi(A), A, B \in SU(2)$.*

Proof. We denote $\varphi(DC)\phi(BA)$, $A = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$, $B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$, $C = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$, $D = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}$ by $\varphi(\delta, \gamma, \beta, \alpha)$. For $P = ((0,0,0; p+iq, 0,0), (0,0,0; u+iv, 0,0), 0, 0)$, let

$$l_1 = (p,v) + (q,u), \quad l_2 = (p,u) - (q,v), \quad l_3 = (p,v) - (q,u),$$

$$l_4 = (p,u) + (q,v), \quad l_5 = (p,q) - (u,v), \quad l_6 = (p,q) + (u,v).$$

$$k_1 = \tfrac{1}{2}(|p|^2 + |q|^2 - |u|^2 - |v|^2), \quad k_2 = \tfrac{1}{2}(|p|^2 - |q|^2 - |u|^2 + |v|^2),$$

$$k_3 = \tfrac{1}{2}(|p|^2 - |q|^2 + |u|^2 - |v|^2).$$

Let $P' = \varphi(\delta, \gamma, \beta, \alpha)P$, then P' has also the form

$$P' = ((0,0,0;p' + iq',0,0),(0,0,0;u' + iv',0,0),0,0).$$

We shall show that $l_1' = l_2' = l_3' = l_4' = 0$ under some actions of type $\varphi(\delta, \gamma, \beta, \alpha)$.

Step 1. We can deform to $k_2'k_3' + l_5'l_6' = 0$, $k_2'l_1' - l_2'l_5' = 0$. In fact, let $\beta = 0$, $\delta = \pi/4$ and choose γ such that

$$-(k_2l_3 + l_4l_6)\sin 2\gamma + (k_3l_4 - l_3l_5)\cos 2\gamma = 0,$$

then we have $k_2'k_3' + l_5'l_6' = 0$. Next choose α satisfying

$$(l_3l_2 \sin 2\gamma + l_3l_1 \cos 2\gamma - k_1l_6 \sin 2\gamma + k_1k_3 \cos 2\gamma)\sin 2\alpha$$
$$- (l_4l_2 \sin 2\gamma + l_4l_1 \cos 2\gamma + k_1k_2 \sin 2\gamma + k_1l_5 \cos 2\gamma)\cos 2\alpha = 0.$$

then we have $k_2'l_1' - l_2'l_5' = 0$.

Step 2. We can deform to $k_1' = l_3' = l_4' = 0$. In fact, it can be assumed that $k_2k_3 + l_5l_6 = 0$, $k_2l_1 - l_2l_5 = 0$ by Step 1. Choose γ satisfying

$$k_3 \sin 2\gamma + l_6 \cos 2\gamma = 0, \quad l_5 \sin 2\gamma - k_2 \cos 2\gamma = 0 \quad \text{and} \quad l_1 \sin 2\gamma - l_2 \cos 2\gamma = 0.$$

Next let $\beta = 0$, $\delta = \pi/4$, then we have $k_1' = l_3' = l_4' = 0$.

Step 3. We can deform to $l_1' = l_2' = l_3' = l_4' = 0$. In fact, it can be assumed that $k_1 = l_3 = l_4 = 0$ by Step 2. Let $\delta = 0$, then $l_3' = l_4' = 0$. Choose α such that

$$l_1(l_6 \sin 2\alpha + k_2 \cos 2\alpha) + l_2(k_3 \sin 2\alpha - l_5 \cos 2\alpha) = 0.$$

Next choose γ satisfying

$$(l_6 \sin 2\alpha + k_6 \cos 2\alpha)\sin 2\gamma - (k_3 \sin 2\alpha - l_5 \cos 2\alpha)\cos 2\gamma = 0$$
$$\text{and} \qquad l_2 \sin 2\gamma + l_1 \cos 2\gamma = 0.$$

Then we have $l_1' = 0$. Finally choose β such that

$$((l_6 \cos 2\gamma + k_3 \sin 2\gamma)\sin 2\alpha - (l_5 \sin 2\gamma - k_2 \cos 2\gamma)\cos 2\alpha)\sin 2\beta$$
$$- (l_1 \sin 2\gamma - l_2 \cos 2\gamma)\cos 2\beta = 0.$$

Then we have $l_2' = 0$.

LEMMA 10. (1) ([9]). *For $t \in \boldsymbol{R}$, the mapping $\varepsilon(t) : \mathfrak{J}^C \to \mathfrak{J}^C$,*

$$\varepsilon(t)X = \begin{pmatrix} \xi_1 & e^{it/2}x_3 & e^{-it/2}\overline{x_2} \\ e^{it/2}\overline{x_3} & e^{it}\xi_2 & x_1 \\ e^{-it/2}x_2 & \overline{x_1} & e^{-it}\xi_3 \end{pmatrix}$$

belongs to $Spin(10) \subset E_6 \subset E_7$.

(2) ([5]) *For* $t \in \boldsymbol{R}$, *the mapping* $\epsilon(t) : \mathfrak{P}^C \to \mathfrak{P}^C$,

$$\epsilon(t)(X, Y, \xi, \eta) = (\begin{pmatrix} e^{2it}\xi_1 & e^{it}x_3 & e^{it}\overline{x_2} \\ e^{it}\overline{x_3} & \xi_2 & x_1 \\ e^{it}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2it}\eta_1 & e^{-it}y_3 & e^{-it}\overline{y_2} \\ e^{-it}\overline{y_3} & \eta_2 & y_1 \\ e^{-it}y_2 & \overline{y_1} & \eta_3 \end{pmatrix}, e^{-2it}\xi, e^{2it}\eta)$$

belongs to E_7.

PROPOSITION 11. *Any element* $P \in \mathfrak{P}^C$ *can be transformed to the following form by some* $\alpha \in \varphi(SU(2))E_7(\subset E_8)$,

$$\alpha P = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, 0, r), \quad r_k, r \in \boldsymbol{R},\ 0 \leq r_1 \leq r_2 \leq r_3 \leq r.$$

PROOF. We start from the form of P of Proposition 8. Let $x = p + iq$, $y = u + iv$, $p, q, u, v \in \mathfrak{C}$. We may assume $(p, u) = (p, v) = (q, u) = (q, v) = 0$ from Lemma 9. Repeat the same process in the proof of Proposition 4.(2), then we have the form

$$((0,0,0;0,0,0),(\eta_1,\eta_2,\eta_3;0,0,0),0,\eta) = P_1.$$

We give the polar expression of P_1:

$$P_1 = ((0,0,0;0,0,0),(r_1e^{i\theta_1}, r_2e^{i\theta_2}, r_3e^{i\theta_3};0,0,0),0,re^{i\theta}), \quad \theta, \theta_k, r, r_k \in \boldsymbol{R}, 0 \leq r_k, 0 \leq r.$$

Using ε, ϵ of Lemma 10 and ψ in the proof Lemma 9, we have

$$\psi(0,t_4,0,t_3)\epsilon(t_2)\varepsilon(t_1)P_1 = ((0,0,0;0,0,0),(r_1,r_2,r_3;0,0,0),0,r) = P_2,$$

where

$$t_1 = \tfrac{1}{2}(\theta_2 - \theta_3), \quad t_2 = \tfrac{1}{4}(\theta_1 - \theta),$$
$$t_3 = \tfrac{1}{4}(\theta_1 - \theta_2 - \theta_3 + \theta), \quad t_4 = -\tfrac{1}{4}(\theta_1 + \theta_2 + \theta_3 + \theta).$$

If necessary we can change the order of r, r_1, r_2, r_3 of P_2, using elements of $Spin_k(9)$, $Spin_k(12)$, $k = 1, 2, 3$. For example, $\alpha(\pi/2)$ of Lemma 2. (1) changes r_2 for r_3 and $\gamma(\pi/2)$ of Lemma 3.(1) changes r_1 for r. Thus we have required form of Proposition 11.

THEOREM 12. *Any element* $P \in \mathfrak{P}^C$ *can be transformed to the following form by some* $\alpha \in E_7$:

$$\alpha P = (\begin{pmatrix} ar_1 & 0 & 0 \\ 0 & ar_2 & 0 \\ 0 & 0 & ar_3 \end{pmatrix}, \begin{pmatrix} br_1 & 0 & 0 \\ 0 & br_2 & 0 \\ 0 & 0 & br_3 \end{pmatrix}, ar, br), \quad \begin{array}{l} |a|^2 + |b|^2 = 1, a, b \in C, \\ 0 \leq r_1 \leq r_2 \leq r_3 \leq r. \end{array}$$

PROOF. For a given element $P \in \mathfrak{P}^C$, there exist $A = \begin{pmatrix} b & -a \\ \tau a & \tau b \end{pmatrix} \in SU(2)$ and $\alpha \in E_7$ such that

$$\varphi(A^{-1})\alpha P = ((0,0,0;0,0,0),(r_1,r_2,r_3;0,0,0),0,r)$$

from Proposition 11. Therefore

$$\begin{aligned} \alpha P &= \varphi(A)((0,0,0;0,0,0),(r_1,r_2,r_3;0,0,0),0,r) \\ &= ((ar_1,ar_2,ar_3;0,0,0),(br_1,br_2,br_3;0,0,0),ar,br). \end{aligned}$$

Thus the proof is completed.

7. Appendix

In the above sections, Theorem 12 is proved as a generalization of Yokota [9]. Now, we show that Theorem 12 can be also obtained from the following universal result.

LEMMA 13 (cf. [2], [3], [4], [7]). *Let G be a real semisimple Lie group with the Lie algebra $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, that is, there exists an involutive automorphism ι of $\mathfrak{g}$ such that*

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \iota X = X\}, \quad \mathfrak{m} = \{X \in \mathfrak{g} \mid \iota X = -X\}$$

and the corresponding Lie subgroup K of G with the Lie algebra $\mathfrak{k}$ is compact. If $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{m}$, then $K \cdot \mathfrak{a} = \mathfrak{m}$ with respect to the adjoint action $\cdot$ of K on $\mathfrak{m}$.

Consider the real simple Lie group E_8 with the Lie algebra $\mathfrak{e}_8$ with the Cartan decomposition $\mathfrak{e}_8 = \mathfrak{k} \oplus \mathfrak{m}$ with respect to the involutive automorphism $v : \mathfrak{e}_8 \to \mathfrak{e}_8$ defined by

$$v(\Phi, P, -\tau\lambda P, r, s, -\tau s) = (\Phi, -P, \tau\lambda P, r, s, -\tau s).$$

Then

$$\mathfrak{k} = \{(\Phi, 0, 0, r, s, -\tau s) \mid \Phi \in \mathfrak{e}_7, r \in \boldsymbol{R}, s \in C\},$$

$$\mathfrak{m} = \tilde{\mathfrak{P}}^C = \{\tilde{P} = (0, P, -\tau\lambda P, 0, 0, 0) \in \mathfrak{e}_8 \mid P \in \mathfrak{P}^C\}.$$

Note that the corresponding Lie subgroup K of G with the Lie algebra $\mathfrak{k}$ is equal to the compact Lie subgroup $\tilde{E}_7\varphi(SU(2))$, where $\tilde{E}_7 = \{\tilde{\alpha} \in \mathrm{Iso}_C(\mathfrak{e}_8^C) \mid \alpha \in E_7\}$ (see Section 5 for the notation $\tilde{\alpha}$), and the adjoint action on $\mathfrak{m}$ is equivariant to the action on $\mathfrak{P}^C$ with respect to the correspondence:

$$\mathfrak{P}^C \to \mathfrak{m} = \tilde{\mathfrak{P}}^C, \quad P \mapsto \tilde{P} = (0, P, -\tau\lambda P, 0, 0, 0).$$

PROPOSITION 14. *Denote* $\tilde{\mathfrak{a}} = \{\tilde{P} \mid P \in \mathfrak{a}\}$ *such that*

$$\mathfrak{a} = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, 0, r \right) \middle| \, r, r_k \in \boldsymbol{R} \right\}.$$

Then $\tilde{\mathfrak{a}}$ *is a maximal abelian subspace of* $\mathfrak{m}$.

PROOF. From the straightforward computation of the Lie bracket of the realized $\mathfrak{e}_8$, it is observed that

$$\mathfrak{a} = \{P \in \mathfrak{P}^C \mid [\tilde{P}, \tilde{\mathfrak{a}}] = 0\}.$$

(The detail computation is concerned with the ones on $\mathfrak{J}$, so it is omitted).

Because of Lemma 13 combined with Proposition 14, we have $\mathfrak{m} = K \cdot \tilde{\mathfrak{a}} = (\tilde{E}_7\varphi(SU(2))) \cdot \tilde{\mathfrak{a}}$.

PROPOSITION 15. $(\tilde{E}_7\varphi(SU(2))) \cdot \tilde{\mathfrak{a}} = (\tilde{E}_7\varphi(SU(2))) \cdot \tilde{\mathfrak{a}}_o$, *where* $\tilde{\mathfrak{a}}_o = \{\tilde{P} \mid P \in \mathfrak{a}_o\}$ *such that*

$$\mathfrak{a}_o = \{((0,0,0;0,0,0), (r_1, r_2, r_3; 0,0,0), 0, r) \in \mathfrak{a} \mid 0 \leq r_1 \leq r_2 \leq r_3 \leq r\}.$$

PROOF. As similar to the last half of the Proof of Prop. 11, it can be proved that any element of $\tilde{\mathfrak{a}}$ can be transformed to an element of $\tilde{\mathfrak{a}}_o$ by the action of $\varepsilon(\boldsymbol{R})\epsilon(\boldsymbol{R})\phi(SU(2))\varphi(SU(2))(\subset \tilde{E}_7\varphi(SU(2)))$, as required.

Note that $\varphi(SU(2))) \cdot \tilde{\mathfrak{a}}_o = \tilde{\mathfrak{b}}$, where $\tilde{\mathfrak{b}} = \{\tilde{P} \mid P \in \mathfrak{b}\}$ such that

$$\mathfrak{b} = \left\{ P = \left(\begin{pmatrix} ar_1 & 0 & 0 \\ 0 & ar_2 & 0 \\ 0 & 0 & ar_3 \end{pmatrix}, \begin{pmatrix} br_1 & 0 & 0 \\ 0 & br_2 & 0 \\ 0 & 0 & br_3 \end{pmatrix}, ar, br \right) \middle| \begin{array}{l} |a|^2 + |b|^2 = 1, \\ r, r_k \in \boldsymbol{R}, a, b \in C \\ 0 \leq r_1 \leq r_2 \leq r_3 \leq r \end{array} \right\}.$$

Then $\tilde{E}_7 \cdot \tilde{\mathfrak{b}} = \tilde{E}_7 \cdot (\varphi(SU(2)) \cdot \tilde{\mathfrak{a}}_o) = \mathfrak{m} = \tilde{\mathfrak{P}}^C$. Hence $\mathfrak{P}^C = E_7 \cdot \mathfrak{b}$. Thus Theorem 12 is proved.

Now, we show that the above proof of Theorem 12 leads to some generalization. For $K = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ and the Cayley Algebra $\mathfrak{C}$, let $\mathfrak{J}_K = \{X \in M(3, K) \mid X^* = X\}$, $\mathfrak{P}_K^C = \mathfrak{J}_K^C \oplus \mathfrak{J}_K^C \oplus C \oplus C$ and

$$E_{7K} = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}_K^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}.$$

Then we have

Theorem 16. *Any element of* $\mathfrak{P}_K^C$ *can be transformed to an element of* $\mathfrak{b}$ *by* E_{7K}.

Proof. For $K = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ and the Cayley algebra $\mathfrak{C}$, let

$$\mathfrak{e}_{6K} = \{\phi \in \mathrm{Hom}_C(\mathfrak{J}_K^C, \mathfrak{J}_K^C) \mid \tau\phi\tau(X \times X) = 2\phi X \times X, \langle \phi X, Y \rangle = -\langle X, \phi Y \rangle\},$$

$$\mathfrak{e}_{7K} = \{\Phi(\phi, A, -\tau A, \nu) \in \mathrm{Hom}_C(\mathfrak{P}_K^C, \mathfrak{P}_K^C) \mid \phi \in \mathfrak{e}_{6K}, A \in \mathfrak{J}_K^C, \nu \in i\boldsymbol{R}\},$$

$$\mathfrak{e}_{8K}^C = \mathfrak{e}_{7K}^C \oplus \mathfrak{P}_K^C \oplus \mathfrak{P}_K^C \oplus C \oplus C,$$

$$\mathfrak{e}_{8K} = \mathfrak{e}_8 \cap \mathfrak{e}_{8K}^C, \mathfrak{k}_K = \mathfrak{k} \cap \mathfrak{e}_{8K}^C, \mathfrak{m}_K = \mathfrak{m} \cap \mathfrak{e}_{8K}^C = \tilde{\mathfrak{P}}_K^C.$$

Then $\mathfrak{e}_{8K} = \mathfrak{k}_K \oplus \mathfrak{m}_K$ is a Cartan decomposition, and that $\tilde{\mathfrak{a}}$ is a maximal abelian subspace of $\mathfrak{m}_K$ by Proposition 14. Hence

$$(\tilde{E}_{7K}\varphi(SU(2))) \cdot \tilde{\mathfrak{a}} = \mathfrak{m}_K = \tilde{\mathfrak{P}}_K^C.$$

Because of the similar argument to the proof of Proposition 15, any element of $\tilde{\mathfrak{a}}$ can be transformed to an element of $\tilde{\mathfrak{a}}_o$ by the action of $(\varepsilon(\boldsymbol{R})\epsilon(\boldsymbol{R})\phi(SU(2))\varphi(SU(2))) \mid \tilde{\mathfrak{P}}_K^C \subset \tilde{E}_{7K}\varphi(SU(2)) \mid \tilde{\mathfrak{P}}_K^C$. Then

$$\tilde{E}_{7K} \cdot (\varphi(SU(2))) \cdot \tilde{\mathfrak{a}}_o) = (\tilde{E}_{7K}\varphi(SU(2))) \cdot \tilde{\mathfrak{a}}_o = \tilde{\mathfrak{P}}_K^C.$$

Because of $\tilde{\mathfrak{b}} = \{\tilde{P} \mid P \in \mathfrak{b}\} = \varphi(SU(2)) \cdot \tilde{\mathfrak{a}}_o$, one has that $\tilde{E}_{7K} \cdot \tilde{\mathfrak{b}} = \tilde{\mathfrak{P}}_K^C$, so that $E_{7K} \cdot \mathfrak{b} = \mathfrak{P}_K^C$, as required.

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