Simple exactly solvable models of non-Fermi-liquids

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We generalize the model of Hatsugai and Kohmoto and find ground states which do not show the properties of Fermi liquids. We work in two space dimensions, but it is straightforward to generalize to higher dimensions. The ground state is highly degenerate and there is no discontinuity in the momentum distribution; i.e., there is no Fermi surface. The Green's function generically has a branch cut. [S0163-1829(98)11303-6]

Since the discovery of high-T_c superconductivity, strongly correlated electron systems have been intensely studied, both experimentally and theoretically. Much attention has been given to two-dimensional models, because the high-T_c cuprates have a layered structure. One of the most studied models is the Hubbard model, but many fundamental questions have not yet been answered. A special version of the model, in the limit that the space dimensionality goes to infinity, has recently been studied.[1] There are attempts to compare the results with experimental data, in spite of the fact that real materials have at most three dimensions. Similarly, a simple exactly solvable model which describes a metal-insulator transition, and has an altered ground-state momentum distribution, was proposed.[2] Like the Sherrington-Kirkpatrick model of a spin glass,[3] the interparticle interaction is independent of distance. The Hamiltonian is

$$H = \sum_k H_k,$$

$$H_k = \varepsilon(k)(n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow} n_{k\downarrow}, \tag{1}$$

where $n_{k\sigma} = c_{k\sigma}^\dagger c_{k\sigma}$. This gives non-Fermi-liquid behavior in any dimension. Since the Hamiltonian is diagonal in k space, we refer to this model as the HK model. A similar model was discussed by Baskaran[4] as part of an effort to understand high-T_c superconductors. The metal-insulator transition of the HK model which was found in Ref. 2 has been discussed from a scaling point of view by Continentino and Coutinho-Filho,[5] who also formulated a boson version of the model. Nogueira and Anda[6] established the equivalence of the HK model with infinite range hopping to the Hubbard model with infinite range hopping.

In this paper, we extend the HK model to include coupling between k modes with the same absolute values of k. Namely, an electron with momentum k interacts with many other electrons with the same magnitude of momentum. For simplicity, we will work with spinless fermions in two dimensions. Generalization to higher dimensions and to include spin are straightforward.

We study

$$H = \sum_k \varepsilon(\mathbf{k}) n_\mathbf{k} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}'} f_{\mathbf{k},\mathbf{k}'} n_\mathbf{k} n_{\mathbf{k}'}, \tag{2}$$

where V is the volume of the system. The summations are taken over $0 < |\mathbf{k}| < k_c$, where k_c represents the short-range cutoff. It is interesting to note that if $n_\mathbf{k}$ is replaced by the difference between the interacting and the noninteracting distributions, the interaction energy of Eq. (2) has a resemblance to that in the Fermi-liquid theory. In fact, Khodel and Shaginyan,[7] starting from the Fermi-liquid equation, proposed the existence of "fermion condensation" which gives large degeneracy at the Fermi surface. Within the Hartree-Fock treatment, Nozieres[8] criticized such phenomenon. (See also Ref. 9.) The highly degenerate ground state of our model looks similar to the "fermion condensation." However, in our case there is no definite Fermi surface and the model is exactly solvable, so Nozieres' critique does not apply.

In the thermodynamic limit $V \to \infty$, the interaction term in Eq. (2) can be written

$$f(\mathbf{k},\mathbf{k}') = (2\pi)^2 U(k) \delta(k - k') \frac{g(\phi)}{k}, \tag{3}$$

where $k = |\mathbf{k}|$, $k' = |\mathbf{k}'|$, and ϕ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. The factor $1/k$ comes from the measure in two-dimensional polar coordinates. The function $g(\phi)$ is periodic $g(\phi) = g(\phi + 2\pi)$ and satisfies $g(\phi) = g(-\phi)$, assuming parity invariance. Throughout this article we will work with the condition that $g(\phi)$ takes the *maximum* for $\phi = \pi$ ($\mathbf{k}' = -\mathbf{k}$), and monotonically decreases toward $\phi = 0$. For this class of models, we can obtain the exact ground states with high degeneracy. The momentum distribution is shown to be continuous and does not have a jump in contrast to the Fermi liquids. It will be shown that the Green's functions generically have a branch cut instead of a pole.

Let us obtain the ground state and its momentum distribution. Due to the angular dependence of the interaction $g(\phi)$ the electrons will bunch up in an arc of angle ϕ_k. So, the momentum distribution is
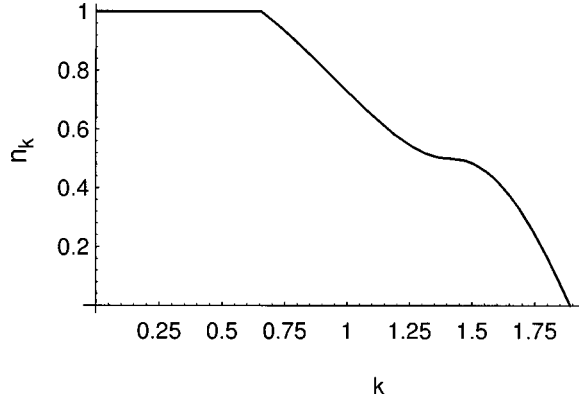
FIG. 1. Ground-state momentum distribution for $g(\phi)=(\pi^2-\phi^2)^{-1/2}$ and $U=1/2$.

$$n_k=\frac{\phi_k}{2\pi}. \tag{4}$$

The ground-state energy including a chemical potential μ is

$$\langle H-\mu N\rangle_G=\frac{V}{(2\pi)^2}\int_0^{k_c}dk\,k\left[(\varepsilon(k)-\mu)\phi_k\right.$$

$$\left.+U(k)\int_0^{\phi_k}d\phi\int_0^{\phi}d\phi'g(\phi')\right], \tag{5}$$

where $\langle\rangle_G$ is the ground-state expectation value, and $\varepsilon(\mathbf{k})$ is assumed to be independent of ϕ. Since ϕ_k must minimize Eq. (5), we obtain, by taking the derivative of $\langle H-\mu N\rangle_G$ with respect to ϕ_k,

$$\varepsilon(k)+U(k)\int_0^{\phi_k}g(\phi)d\phi=\mu. \tag{6}$$

The ground state is specified by ϕ_k, but it is highly degenerate due to the freedom of choice of an arc of angle ϕ_k at each k. The momentum distribution (4) is obtained by solving Eq. (6). Let k_1 be the largest value of k which gives $\phi_k=2\pi$ and let k_0 be the smallest value of k which gives $\phi_k=0$. Then $n_k=1$ for $k<k_1$, $0<n_k<1$ for $k_1<k<k_0$, and $n_k=0$ for $k_0<k$. Thus n_k has cusps at k_1 and k_0. We also
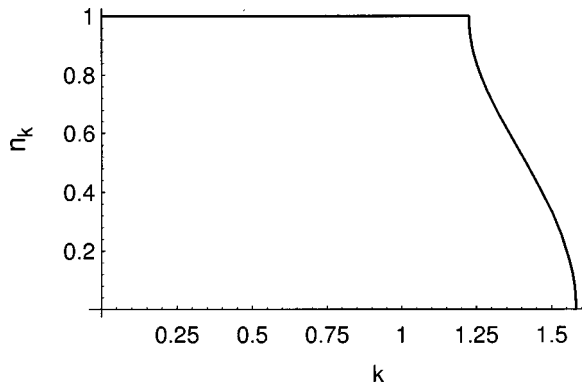

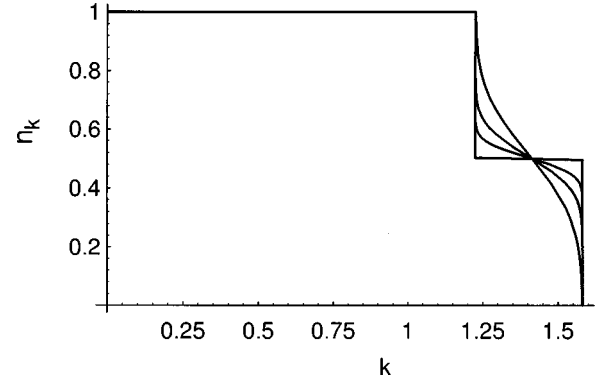FIG. 2. Ground-state momentum distribution for $g(\phi)=\sin(\phi/2)$ and $U=1/8$.

FIG. 3. Ground-state momentum distribution for $g(\phi)=(1/\sigma\pi^{1/2})\exp[(\pi-\phi)^2/\sigma^2]$ and $U=1/2$, $\sigma=\pi/512$, $\pi/8$, $\pi/4$, $\pi/2$.

define $k_{1/2}$ by the condition $\phi_{k_{1/2}}=\pi$. It is possible to have $n_k<1$ for all k, i.e., the absence of k_1, if the interaction is strong enough.

Suppose that $U(k)$ is not a singular function and $g(\phi)$ is a function whose singularity is weaker than the integrable ones. Then ϕ_k, the solution of Eq. (6), cannot have a discontinuity. Namely, $n_k=\phi_k/2\pi$ is continuous and we do not have a Fermi surface associated with discontinuity of momentum distributions.

Let us observe whether the Mott metal-insulator transition may occur or not. Note that due to the cutoff k_c, there is a bandwidth $W=\varepsilon(k_c)-\varepsilon(0)$. Let us examine the following two cases:

(i) If the interaction satisfies the conditions $f_{\mathbf{k},\mathbf{k}'}\sim O(1)$ for $\mathbf{k}\neq-\mathbf{k}'$ and $f_{\mathbf{k},-\mathbf{k}}>VW$, the angle ϕ_k is exactly equal to π for any k at the half filling. A finite amount of energy is needed to add one particle to this ground state. We should note that the condition $f_{\mathbf{k},-\mathbf{k}}>VW$, namely that f has a divergence of order V at $\mathbf{k}=-\mathbf{k}'$, means that the function $g(\phi)$ contains a δ-function-like singularity at $\phi=\pi$.

(ii) If the condition $f_{\mathbf{k},-\mathbf{k}}>VW$ is not satisfied for some k, it costs almost no energy to add a particle to the ground state. Namely, there is no gap between the ground state and the low-energy states. Thus, the Mott transition will not happen.

To show the smooth falling off of the momentum distribution of the ground state, we will give some examples. For simplicity, $U(k)$ is taken to be constant U.
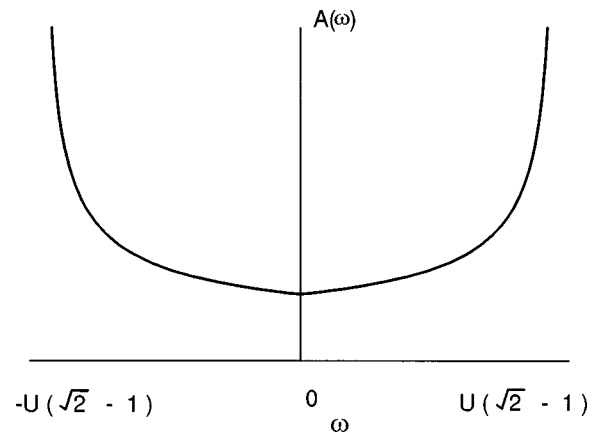

FIG. 4. Spectral function for $g(\phi)=\sin(\phi/2)$, at $k_{1/2}$.

(I) $g(\phi) = \dfrac{1}{(\pi^2 - \phi^2)^{1/2}}$ for $0 < \phi < \pi$,   (7)

$$= \dfrac{1}{(\pi^2 - (2\pi - \phi)^2)^{1/2}}$$

for $\pi < \phi < 2\pi$.   (8)

This interaction has an integrable singularity at $\phi = \pi$, causing the tangent to n_k to be horizontal at $n_k = 1/2$. The derivative of n_k is discontinuous at k_1 and k_0. See Fig. 1.

$$n_k = \begin{cases} 1 - \dfrac{1}{2}\sin\dfrac{\mu - \varepsilon(k)}{U} & \text{for } k_1 < k < k_{1/2}, \\[2ex] \dfrac{1}{2}\sin\dfrac{\mu - \varepsilon(k)}{U} & \text{for } k_{1/2} < k < k_0. \end{cases} \quad (9)$$

The derivative of n_k is discontinuous at k_1 and k_0. We have $\lim_{k \to k_1^+} dn/dk \propto -(k - k_1)^{-1/2}$ and $\lim_{k \to k_0^-} dn/dk \propto -(k_0$

(II) $g(\phi) = \sin\dfrac{\phi}{2}$.   (10)

$-k)^{-1/2}$. At $k = k_{1/2}$ the second derivative changes sign. See Fig. 2.

$$n_k = \begin{cases} 1 - \dfrac{1}{\pi}\arccos\left(\dfrac{\mu - \varepsilon(k)}{2U} - 1\right) & \text{for } k_1 < k < k_{1/2}, \\[2ex] \dfrac{1}{\pi}\arccos\left(1 - \dfrac{\mu - \varepsilon(k)}{2U}\right) & \text{for } k_{1/2} < k < k_0. \end{cases} \quad (11)$$

(III) $g(\phi) = \dfrac{1}{\sigma \pi^{1/2}}\exp\left[-\dfrac{(\pi - \phi)^2}{\sigma^2}\right]$.   (12)

In this example, we set $\varepsilon(k) = k^2/2m$. As $\sigma \to 0$, the interaction approaches a δ function and the Hamiltonian approaches the original HK model. This can be seen in Fig. 3. For the smallest σ, the momentum distribution appears to have two pseudo-Fermi surfaces, as in the HK model. Note that, if σ is finite, there is no Mott transition at half filling, as mentioned earlier. We may obtain ϕ_k by inverting

$$k = \begin{cases} \sqrt{2m\left[\mu - \dfrac{U}{2}\mathrm{erf}\left(\dfrac{\pi - \phi_k}{\sigma}, \dfrac{\pi}{\sigma}\right)\right]} & \text{for } \phi_k < \pi, \\[3ex] \sqrt{2m\left[\mu - \dfrac{U}{2}\mathrm{erf}\left(0, \dfrac{\pi}{\sigma}\right) - \dfrac{U}{2}\mathrm{erf}\left(0, \dfrac{\phi_k - \pi}{\sigma}\right)\right]} & \text{for } \phi_k > \pi, \end{cases} \quad (13)$$

where $\mathrm{erf}(x, y) = 2\pi^{-1/2}\int_x^y e^{-t^2}dt$.

Next, let us study the Green's function. The ground states of our models are highly degenerate. For each k in the partially occupied zone, we may rotate the electrons in k space by an arbitrary angle and obtain another ground state. We will show that after averaging over all such ground states, the single-particle Green's function develops a branch cut, a signature of a non-Fermi-liquid.

The Green's function is

$$G(\mathbf{k}, t) = \theta(t)G^>(\mathbf{k}, t) - \theta(-t)G^<(\mathbf{k}, t), \quad (14)$$

where $G^{\gtrless}(\mathbf{k}, t)$ are the correlation functions

$$G^>(\mathbf{k}, t) = \int_0^{2\pi}\dfrac{d\varphi}{2\pi}\langle\varphi|e^{i(H - \mu N)t}c_\mathbf{k}e^{-i(H - \mu N)t}c_\mathbf{k}^\dagger|\varphi\rangle,$$

$$G^<(\mathbf{k}, t) = \int_0^{2\pi}\dfrac{d\varphi}{2\pi}\langle\varphi|c_\mathbf{k}^\dagger e^{i(H - \mu N)t}c_\mathbf{k}e^{-i(H - \mu N)t}|\varphi\rangle. \quad (15)$$

Suppose that the electrons are on an arc between φ and $\varphi + \phi_k$ of a circle with radius k. Using circular symmetry, the correlation functions are obtained and their Fourier transforms $G^{\gtrless}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty}dt\, e^{i\omega t}G^{\gtrless}(\mathbf{k}, t)$ are

$$G^>(\mathbf{k}, \omega) = -2\,\mathrm{Im}\int_0^{2\pi - \phi_k}\dfrac{d\varphi}{2\pi}\dfrac{1}{\omega - \varepsilon(k) + \mu - UP_k(\varphi) + i\delta},$$

$$G^<(\mathbf{k}, \omega) = -2\,\mathrm{Im}\int_{2\pi - \phi_k}^{2\pi}\dfrac{d\varphi}{2\pi}\dfrac{1}{\omega - \varepsilon(k) + \mu - UP_k(\varphi) + i\delta}, \quad (16)$$

where

$$P_k(\varphi) = \dfrac{1}{2}\int_\varphi^{\varphi + \phi_k}d\varphi'\, g(\varphi'). \quad (17)$$

The integrands in Eq. (16) have a pole at $\omega = \varepsilon(k) - \mu + UP_k(\varphi)$. Thus, after integration over φ we have a branch-cut in the Green's function instead of a pole which corresponds to a quasiparticle.

The spectral function, relevant to photoemission, is

$$A(\mathbf{k},\omega) = G^>(\mathbf{k},\omega) + G^<(\mathbf{k},\omega)$$

$$= -2 \, \mathrm{Im} \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{1}{\omega - \varepsilon(k) + \mu - UP_k(\varphi) + i\delta}. \tag{18}$$

For example, if one takes the interaction $g(\theta) = \sin(\theta/2)$,

$$A(\mathbf{k},\omega)|_{\mathbf{k}=\mathbf{k}_{1/2}}$$

$$= \begin{cases} \dfrac{\sqrt{2}}{U \sqrt{1 - \dfrac{1}{2}(1 + |\omega|/U)^2}} & \text{if } |\omega| < U(\sqrt{2} - 1), \\[20pt] 0 & \text{otherwise.} \end{cases} \tag{19}$$

This is shown in Fig. 4.

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$$H = \sum_k (\varepsilon(k) - \mu) c_{k\sigma}^\dagger c_{k\sigma} + J\left(\sum_k S_k\right)^2 - J\sum_k S_k^2$$
$$S_k \equiv c_{k\alpha}^\dagger \sigma_{\alpha\beta} c_{k\beta},$$

would be the same as the model of Hatsugai and Kohmoto were it not for the second term which partially lifts the degeneracy.

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