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UNITARY-SYMMETRIC KÄHLERIAN MANIFOLDS AND POINTED BLASCHKE MANIFOLDS

By

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Introduction.

A unitary-symmetric Kählerian manifold is a Kählerian version of a rotationally symmetric (Riemannian) manifold (cf. Choi [3], Greene-Wu [5]). Precisely, a Kählerian manifold (M, g, J) of complex dimension n is unitary-symmetric at a point p of M if the linear isotropy group at p of the automorphism group of (M, g, J) is the unitary group $U(n)$. Of course, the complex space form is unitary-symmetric at every point.

The first purpose of this paper is to give one characterization of a connected, simply-connected, complete, unitary-symmetric Kählerian manifold. If M is compact, then the tangential cut locus C_p of p is spherical. Hence (M, g, J) is a Blaschke manifold at p and has a SL^p-structure (cf. Besse [1]). Then the second purpose is to give a sufficient condition in order that a connected, compact, unitary-symmetric Kählerian manifold has a SC^p-structure (Theorem D) (see Besse [1, p. 181]).

On the other hand, Greene-Wu [5, p. 85] introduced the notion of a Hermitian rotationally symmetric manifold of complex dimension 1 and Shiga [12] studied a Kählerian model, which is by defintion a Kählerian manifold with a pole p such that the linear isotropy group at p of the isometry group is $U(n)$. Note that their manifolds are unitary-symmetric Kählerian manifolds. The unitary-symmetric condition is a fairly strong one, because the result of Kaup [8, Folgerung 1.10] implies that a connected, unitary-symmetric Kählerian manifold is biholomorphic to one of the complex space forms. But there exist many complete unitary-symmetric Kählerian metrics, which are not isometric to them (see Mori-Watanabe [10]).

Throughout this paper, (M, g, J) is assumed to be a connected, complete Kählerian manifold of complex dimension $n \geqq 1$. To state our results, we prepapre the following. By Ω we denote the Kählerian form of (M, g, J). We frequently identify the tangent space $T_p(M)$ at a point p of M with the complex number n-space $\boldsymbol{C}^n$. Let $\exp_p$ be the exponential map of $T_p(M)$

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to M and δ be the distance from the origin O of $T_p(M)$ to the first conjugate locus Q_p in $T_p(M)$ of p. If M is simply-connected and $\delta=\infty$, i.e., p has no conjugate points, then M is diffeomorphic to $\boldsymbol{R}^{2n}$ (cf. Kobayashi-Nomizu [9, II, p. 105]). We put $S_\delta^{2n-1}=\{X\in T_p(M)\,;\,|X|=\delta\}$ $\tilde{B}_\delta=\{X\in T_p(M)\,;\,|X|<\delta\}$, where $|X|$ is the norm $\sqrt{g_p(X, X)}$ of X. On the other hand, it is well known (cf. Sasaki-Hatakeyama [11]) that there exists a Sasakian structure $(d\Theta^2, \phi, \xi, \eta)$ on the sphere S_1^{2n-1} in $\boldsymbol{C}^n$, called the standard one, where $d\Theta^2$ denotes the canonical metric of constant curvature 1. We set $\Psi(\ ,\ )=d\Theta^2(\phi\ ,\ )$.

THEOREM A. *Let (M, g, J) be a connected, complete Kählerian manifold of complex dimension n. If (M, g, J) is unitary-symmetric at a point p, then the Kählerian metric $\tilde{g}$ and the Kählerian from $\tilde{\Omega}$, pulled back under the exponential map* $\exp_p$, *are given by*

$$(*)\qquad \begin{aligned}\tilde{g}&=\exp_p^* g=dr^2+f(r)^2 d\Theta^2+f(r)^2(f'(r)^2-1)\eta\otimes\eta\\ \tilde{\Omega}&=\exp_p^*\Omega=2f(r)f'(r)\eta\wedge dr+f(r)^2\Psi\end{aligned}$$

on $\tilde{B}_\delta-\{O\}$ for some function $f(r)$ such that $f(r)>0$, $f'=dr/dr>0$ on $(0, \delta)$, where (r, Θ) is the usual polar coordinate system of $\boldsymbol{R}^{2n}$ and $(d\Theta^2, \phi, \xi, \eta)$ is the standard Sasakian structure on S_1^{2n-1}.

THEOREM B. *Let (M, g, J) be a connected, simply-connected, complete Kählerian manifold of complex dimension $n\geqq 2$. If there exists a point p in M such that* $\exp_p^* g$ *and* $\exp_p^*\Omega$ *satisfy* $(*)$, *then (M, g, J) is unitary-symmetric at p.*

COROLLARY C. *Under the assumption of Theorem B, if M is compact, then (M, g, J) is a Blaschke manifold at p and the cut locus $C(p)$ of p in M is a totally geodesic, complex hypersurface of M.*

REMARK. Let us consider S_1^{2n-1} as a principal circle bundle over the complex projective space $\boldsymbol{C}P^{n-1}$ with the canonical Kählerian metric $d\sigma^2$ of constant holomorphic curvature 4. Then, since $d\Theta^2=\pi^* d\sigma^2+\eta\otimes\eta$, $\tilde{g}$ may be represented by

$$(*)'\qquad \tilde{g}=dr^2+f(r)^2 f'(r)^2\eta\otimes\eta+f(r)^2\pi^* d\sigma^2,$$

where π denotes the canonical projection: $S_1^{2n-1}\to\boldsymbol{C}P^{n-1}$. Note that when $n=1$, $\tilde{g}=dr^2+f(r)^2 f'(r)^2 d\Theta^2$.

THEOREM D. *Let (M, g, J) be a connected, simply-connected, compact Kählerian manifold. Suppose that there exists a point p in M such that* $\exp_p^* g$ *and* $\exp_p^*\Omega$, *pulled back under* $\exp_p$, *satisfy the condition* $(*)$. *If its function $f(r)$ satisfies*

$f(\delta)f''(\delta)=-1$, *then any geodesic issuing from the point* p *is always closed.*

In § 1, we introduce some basic facts about Kählerian manifolds, complex hypersurfaces, almost contact metric manifolds and Sasakian manifolds. In § 2, by using the results of Ziller [16] and Kato-Motomiya [8] we study $U(n)$-invariant Kählerian structures on the open ball $\tilde{B}_\delta$, centered at the origin in $\boldsymbol{C}^n$ and then prove Theorem A in § 3. In § 4, we investigate the conjugate locus $Q(p)=\exp_p Q_p$ of a point p of a Kählerian manifold satisfying the conditions of Theorem B, and give a proof of Corollary C. § 5 is devoted to construct an automorphism F_A of M for each A of $U(n)$ and complete the proof of Theorem B. In the last section, we prove Theorem D, concerning with the closedness of geodesics issuing from one point.

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1. Preliminaries.

Let M be a complex manifold of complex dimension n. Then M admits an almost complex structure J on M, i.e., a tensor field J on M of type $(1, 1)$ such that $J^2X=-X$ for any vector field X on M. A Riemannian metric g on M is a Hermitian metric if

$$g(JX, JY)=g(X, Y) \tag{1.1}$$

holds for any vector fields X and Y on M. Here we define a 2-form Ω on M, called the fundamental 2-form; $\Omega(X, Y)=g(JX, Y)$. If in addition, J is parallel with respect to the Riemannian connection ∇ of g, then g(resp. Ω) is called a Kählerian metric (resp. a Kählerian form); (M, g, J) (resp. (g, J)) is then called a Kählerian manifold (resp. a Kählerian structure).

Let (M, g, J) be a connected Kählerian manifold of complex dimension n and let $\hat{M}$ be a connected complex hypersurface of M, i.e., there exists a complex analytic mapping $e:\hat{M}\to M$, whose differential e_* is 1-1 at each point of $\hat{M}$. All metric properties on $\hat{M}$ refer to the Hermitian metric $\hat{g}$ induced on $\hat{M}$ by the immersion e. In order to simplify the representation, we identify for each $\hat{x}\in\hat{M}$, the tangent space $T_{\hat{x}}(\hat{M})$ with $e_*(T_{\hat{x}}(\hat{M}))(\subset T_{e(\hat{x})}(M))$ by means of e_*. Since $e^*g=\hat{g}$ and $J\circ e_*=e_*\circ\hat{J}$, where $\hat{J}$ is the almost complex structure of $\hat{M}$, the structures $\hat{g}$ and $\hat{J}$ on $T_{\hat{x}}(\hat{M})$ are identified with restrictions of the structures g and J to the subspace $e_*(T_{\hat{x}}(\hat{M}))$ respectively. Then it follows that there exists a coordinate neighborhood $\hat{U}(\hat{x})$ of $\hat{x}$ in $\hat{M}$ on which there is a field ζ of unit vectors normal to $\hat{M}$. Now, if X and Y are vector fields on $\hat{U}(\hat{x})$, we may write

$$\nabla_X Y=\hat{\nabla}_X Y+h(X, Y)\zeta+k(X, Y)J\zeta,$$

where $\hat{\nabla}_X Y$ denotes the components of $\nabla_X Y$ tangent to $\hat{M}$. Then we have the Weingarten's formula (for example, cf. Smyth [13])

$$\nabla_X\zeta=-HX+s(X)J\zeta, \tag{1.2}$$

where HX is tangent to $\hat{M}$. Then H and s are tensor fields on $\hat{U}(\hat{x})$ of type (1, 1) and (0, 1), respectively. Further, H satisfies

$$h(X, Y)=\hat{g}(HX, Y), \qquad k(X, Y)=\hat{g}(\hat{J}HX, Y) \tag{1.3}$$

for any vectors X and Y tangent to $\hat{M}$ at a point of $\hat{U}(\hat{x})$.

On the other hand, an almost contact structure on an odd-dimensional manifold N is by definition a triple (ϕ, ξ, η), where ϕ is a tensor field of type (1, 1) on N, ξ is a vector field on N and η is a 1-form on N satisfying

$$\phi\xi=0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1, \quad \phi^2X=-X+\eta(X)\xi \tag{1.4}$$

for any vector field X on N. An almost contact structure is said to be normal if the torsion tensor N_{jk}^i (see [11, p. 255]) vanishes. If N has an associated Riemannian metric g such that

$$g(\xi, X)=\eta(X), \qquad g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y) \tag{1.5}$$

for any vector fields X and Y on N, then (N, g, ϕ, ξ, η) is called an almost contact Riemannian manifold: (g, ϕ, ξ, η) is then called an almost contact metric structure. If they satisfy

$$d\eta(X, Y)=2g(\phi X, Y), \qquad (\nabla_X\phi)Y=\eta(Y)X-g(X, Y)\xi \tag{1.6}$$

for any vector fields X and Y on N, (N, g, ϕ, ξ, η) is called a Sasakian manifold: (g, ϕ, ξ, η) is then called a Sasakian structure.

2. A $U(n)$-invariant Kählerian structure on an open ball in C^n.

In this section, we consider a $U(n)$-invariant Kählerian structure $(\tilde{g}, \tilde{J})$ on an open ball $\tilde{B}_l$ of radius l in C^n, centered at the origin O. Then, by the result of Kaup stated in the Introduction we may regard $\tilde{J}$ as the complex structure induced from the canonical one J_0 of C^n. Identifying C^n with R^{2n} naturally, we introduce the usual polar coordinate system (r, Θ) on $\tilde{B}_l-\{O\}$, centered at O. Then $\tilde{g}$ can be expressed in the form

$$\tilde{g}=dr^2+\bar{h}_{jk}(r, \Theta)d\theta^j\otimes d\theta^k \tag{2.1}$$

where (θ^i) denotes a local coordinate system of S_1^{2n-1} and small Latin indices run on the range $1, \cdots, 2n-1$. Note that for each fixed r $\bar{h}=\bar{h}_{jk}d\theta^j\otimes d\theta^k$ defines a Riemannian metric on S_r^{2n-1}.

On the other hand, if we set

$$\bar{\phi}_j^i=d\theta^i\Big(\tilde{J}\Big(\frac{\partial}{\partial\theta^j}\Big)\Big),\quad \bar{\xi}^i=d\theta^i\Big(\tilde{J}\Big(\frac{\partial}{\partial r}\Big)\Big)\quad\text{and}\quad \bar{\eta}_j=dr\Big(\tilde{J}\Big(\frac{\partial}{\partial\theta^j}\Big)\Big), \tag{2.2}$$

then $\tilde{J}$ is represented by

$$\tilde{J}=\begin{pmatrix}\bar{\phi}_j^i & -\bar{\eta}_j\\ \bar{\xi}^i & O\end{pmatrix} \tag{2.3}$$

with respect to the coordinate system. Since $(\tilde{g}, \tilde{J})$ is Hermitian, by (1.1) we have

$$\bar{\phi}_j^k\bar{\phi}_k^i=-\delta_j^i+\bar{\eta}_j\bar{\xi}^i,\quad \bar{\phi}_j^i\bar{\xi}^j=\bar{\phi}_j^i\bar{\eta}_i=0,\quad \bar{\eta}_i\bar{\xi}^i=1,$$
$$\bar{h}_{kh}\bar{\phi}_j^k\bar{\phi}_i^h=\bar{h}_{ji}-\bar{\eta}_j\bar{\eta}_i,\quad \bar{\eta}_i=\bar{h}_{ji}\bar{\xi}^j,\quad \bar{h}_{ji}\bar{\xi}^j\bar{\xi}^i=1.$$

Therefore, this implies that $\bar{\xi}=\bar{\xi}^i(\partial/\partial\theta^i)$, $\bar{\eta}=\bar{\eta}_i d\theta^i$ and $\bar{\phi}=\bar{\phi}_j^i(\partial/\partial\theta^i)\otimes d\theta^j$ define an almost contact metric structure on S_r^{2n-1}. Therefore, from the assumption that $(\tilde{g}, \tilde{J})$ is $U(n)$-invariant we see that $U(n)$ acts transitively on S_r^{2n-1} as a group of diffeomorphisms which leave the structure $(\bar{h}, \bar{\phi}, \bar{\xi}, \bar{\eta})$ invariant and from a result of Tanno [14, p. 25] that $(S_r^{2n-1}, \bar{h}, \bar{\phi}, \bar{\xi}, \bar{\eta})$ is normal and homogeneous. Thus, for each $r\in(0, l)$ we can regard $S_r^{2n-1}\cong U(n)/U(n-1)$ as a manifold having a normal almost contact metric structure $(\bar{h}, \bar{\phi}, \bar{\xi}, \bar{\eta})$ where $U(n-1)$ is the isotropy subgroup at the point $q_r=(r, 0, \cdots, 0)$ of S_r^{2n-1}.

We now are going to show that a splitting of the Lie algebra $\mathfrak{g}$ of $U(n)$ induces another $U(n)$-invariant almost contact metric structure on the homogeneous space $U(n)/U(n-1)$ and $(\bar{h}, \bar{\phi}, \bar{\xi}, \bar{\eta})$ is described by means of it. Let $\mathfrak{g}_0$ be the Lie algebra of $U(n-1)$. Then the splitting

$$\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{m} \tag{2.4}$$

is an $\mathrm{ad}\,\mathfrak{g}_0$-invariant, i.e., $[\mathfrak{g}_0, \mathfrak{m}]\subset\mathfrak{m}$. Then $\mathfrak{m}$ can be identified with the tangent space of $U(n)/U(n-1)$ at the coset $(U(n-1))$. The isotropy subgroup $U(n-1)$ acts on $\mathfrak{m}$ by the adjoint map and induces a splitting $\mathfrak{m}=\mathfrak{g}_1\oplus\mathfrak{g}_2$:

$$\mathfrak{g}_1=\left\{\begin{pmatrix}0 & -{}^t\bar{\boldsymbol{b}}\\ \boldsymbol{b} & O\end{pmatrix};\boldsymbol{b}\in\boldsymbol{C}^{n-1}\right\},\qquad \mathfrak{g}_2=\left\{\rho\begin{pmatrix}\sqrt{-1} & 0\\ 0 & O\end{pmatrix},\ \rho\in\boldsymbol{R}\right\} \tag{2.5}$$

where $\bar{\boldsymbol{b}}$ means the complex conjugate of $\boldsymbol{b}$. Let $\mathfrak{B}$ be a bi-invariant metric on $U(n)$. The $U(n)$-invariant metric $\bar{h}$ on $U(n)/U(n-1)$ can be uniquely described by giving its value on $\mathfrak{m}$, and is of the form

$$\langle\,,\rangle=\alpha\mathfrak{B}|_{\mathfrak{g}_1}+\mathfrak{k}|_{\mathfrak{g}_2}, \tag{2.6}$$

where $\alpha>0$ and $\mathfrak{k}$ is an arbitrary metric on $\mathfrak{g}_2$ (cf. Ziller [16]). The inclusion of

$1\times U(n-1)$ in $U(n)$ is the standard one. The metric (2.6) is identical with the one on the homogeneous space $SU(n)/SU(n-1)$, since $U(n)$ clearly also acts by isometries on the metrics in $SU(n)/SU(n-1)$ (cf. Ziller [16, p. 352]). But since $SU(n)$ is simple and $\mathfrak{B}|_{\mathfrak{g}_1}$ and the inner product

$$-\frac{1}{2n}\operatorname{trace} XY=\frac{1}{2n}\operatorname{trace} X^t\bar{Y} \qquad (X, Y\in\mathfrak{su}(n))$$

are $Ad(SU(n))$-invariant, where $\mathfrak{su}(n)$ is the Lie algebra of $SU(n)$, we have

$$\mathfrak{B}|_{\mathfrak{g}_1}(Z, W)=-\frac{1}{2n}\operatorname{trace} ZW=\operatorname{Re}(\boldsymbol{b}, \boldsymbol{c})\left(Z=\begin{pmatrix}0 & -{}^t\bar{\boldsymbol{b}}\\ \boldsymbol{b} & O\end{pmatrix}, W=\begin{pmatrix}0 & -{}^t\bar{\boldsymbol{c}}\\ \boldsymbol{c} & O\end{pmatrix}\right),$$

where $\operatorname{Re}(\,,\,)$ denotes the real part of the natural Hermitian inner product on $\boldsymbol{C}^{n-1}$. Therefore, from (2.6) we have

$$\langle\,,\,\rangle=\alpha\operatorname{Re}(\,,\,)+\lambda^*\mathfrak{u}\otimes{}^*\mathfrak{u} \tag{2.7}$$

for a positive constant λ, where $\mathfrak{u}=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & O\end{pmatrix}$ and ${}^*\mathfrak{u}$ is a 1-form on $\mathfrak{g}_2$ defined by ${}^*\mathfrak{u}(\mathfrak{u})=1$, ${}^*\mathfrak{u}(X)=0$ for all $X\in\mathfrak{g}_1$.

After some long calculations, we can confirm that $\mathfrak{g}_0$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ satisfy all conditions of Theorem 1 of Kato-Motomiya [7]. This implies that on the homogeneous space $U(n)/U(n-1)$ there is a unique $U(u)$-invariant normal almost contact structure (ϕ, ξ, η) with the initial condition $(-ad_{\mathfrak{m}}\mathfrak{u}, \mathfrak{u}, {}^*\mathfrak{u})$, where $ad_{\mathfrak{m}}\mathfrak{u}$ denotes the restriction of $ad\mathfrak{u}$ on $\mathfrak{m}$. In fact, let q be an arbitrary point of S_r^{2n-1}. Choose $A\in U(n)$ such that $A(q_r)=q$. We define $\xi_q=(\tau_A)_*\mathfrak{u}$ where τ_A denotes the left translation on $U(n)/U(n-1)$ given by $\tau_A(B\cdot U(n-1))=AB\cdot U(n-1)$, $B\in U(n)$. Hence we have a $U(n)$-invariant vector field ξ on S_r^{2n-1} such that $\xi_{q_r}=\mathfrak{u}$, where $T_{q_r}(S_r^{2n-1})$ is canonically identified with $\mathfrak{m}$. Similarly we can define a $U(n)$-invariant tensor field ϕ of type (1, 1) and a $U(n)$-invariant 1-form η on S_r^{2n-1} satisfying the initial conditions $\phi_{q_r}=-ad_{\mathfrak{m}}\mathfrak{u}$ and $\eta_{q_r}={}^*\mathfrak{u}$ respectively. Since $(\exp t\mathfrak{u})q_r=(re^{\sqrt{-1}t}, 0, \cdots, 0)$, we have $\mathfrak{u}=\xi_{q_r}=\sqrt{-1}q_r=J_oq_r$. Moreover, since

$$(-ad_{\mathfrak{m}}\mathfrak{u})(X)=\sqrt{-1}\begin{pmatrix}0 & {}^t\bar{\boldsymbol{b}}\\ \boldsymbol{b} & O\end{pmatrix} \qquad \left(X=\begin{pmatrix}0 & -{}^t\bar{\boldsymbol{b}}\\ \boldsymbol{b} & O\end{pmatrix}\in\mathfrak{g}_1\right)$$

holds, we see that ϕ is nothing but the standard tensor field of type (1, 1) on S_r^{2n-1}, introduced from J_o by Sasaki-Hatakeyama [11]. Therefore, between the two $U(n)$-invariant normal almost contact structures $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ and (ϕ, ξ, η) we obtain the following relations

$$\bar{\phi}=\phi, \quad \bar{\xi}=\frac{1}{\mu}\xi, \quad \bar{\eta}=\mu\eta \tag{2.8}$$

where $\mu=\sqrt{\tilde{g}_{q_r}(q_r, q_r)}$, by consequence of their initial conditions at the point

$q_r=(r, 0, \cdots, 0)\in S_r^{2n-1}$. Assigning ϕ, ξ and η to each sphere S_r^{2n-1} of radius r, we can naturally define a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η on $\tilde{B}_l-\{O\}$ respectively though they are written in the same letters. Then (2.8) implies that

$$\bar{\phi}=\phi, \quad \bar{\xi}=\frac{1}{\mu(r)}\xi, \quad \bar{\eta}=\mu(r)\eta, \tag{2.8$'$}$$

where $\mu(r)=|q_r|=\sqrt{\tilde{g}(q_r, q_r)}$ is a function on $(0, l)$, because of (2.1).

Let us turn to $\bar{h}$ in (2.1) again. Give an inner product

$$(\ ,\ )=\mathrm{Re}(\ ,\ )+{}^*\mathfrak{u}\otimes{}^*\mathfrak{u} \tag{2.9}$$

on $\mathfrak{m}$. Then by (2.7) and (2.9) we may put

$$\langle\ ,\ \rangle=\alpha(\ ,\ )+\beta{}^*\mathfrak{u}\otimes{}^*\mathfrak{u}, \tag{2.10}$$

where $\alpha+\beta>0$, because $\langle\ ,\ \rangle$ is positive definite. By $d\Theta^2$ we denote the $U(n)$-invariant Riemannian metric of constant curvature 1 on S_r^{2n-1}, induced from $(\ ,\ )$. Then from (2.10) we may write

$$\bar{h}=\alpha(r, \Theta)d\Theta^2+\beta(r, \Theta)\eta\otimes\eta, \tag{2.11}$$

where $d\Theta^2$ and $\eta\otimes\eta$ are usually regarded as tensor fields of type (0, 2) on $\tilde{B}_l-\{O\}$. Especially, we see from (2.9) and the statements of Example 10.5 in Kobayashi-Nomizu [9, II] that $(d\Theta^2, \phi, \xi, \eta)$ is nothing but the standard Sasakian structure on S_1^{2n-1}. As each field on $\tilde{B}_l-\{O\}$, induced from $(-ad_{\mathfrak{m}}\mathfrak{u}, \mathfrak{u}, {}^*\mathfrak{u})$, is defined independently of r, we may think that $(d\Theta^2, \phi, \xi, \eta)$ assigns the standard Sasakian structure to each sphere S_r^{2n-1} of radius r. Since $(\tilde{g}, \tilde{J})$ is Hermitian, the above facts imply that

$$\mu(r)=\sqrt{\alpha(r, \Theta)+\beta(r, \Theta)}, \tag{2.12}$$

taking account of (1.4)-(1.6), (2.3) and (2.11). From (2.6), $\alpha(r, \Theta)$ is a function of r only. Hence we have $\alpha=\alpha(r)$, $\beta=\beta(r)$ and further,

$$\mu(r)=\sqrt{\alpha(r)+\beta(r)}, \tag{2.12$'$}$$

form which $\tilde{g}$ and $\tilde{\Omega}$ are given by

$$\begin{aligned}\tilde{g}&=dr^2+\alpha(r)d\Theta^2+\beta(r)\eta\otimes\eta\\ \tilde{\Omega}&=\alpha(r)\Psi+2\sqrt{\alpha(r)+\beta(r)}\eta\wedge dr\end{aligned} \tag{2.13}$$

on $\tilde{B}_l-\{O\}$, where Ψ denotes $d\Theta^2\circ\phi$. A direct computation of $\tilde{\nabla}\tilde{\Omega}$, using (1.4), (1.5) and (1.6), implies that $d\alpha/dr=\sqrt{\alpha+\beta}$, where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{g}$, because of the Kählerian condition $\tilde{\nabla}\tilde{\Omega}=0$. Putting $\alpha=f(r)^2$ we have that $f'=df/dr$ is also positive on $(0, l)$. This implies that

$$\beta(r)=f(r)^2(f'(r)^2-1). \tag{2.14}$$

From (2.12)′, (2.13) and (2.14), we see that $\tilde{g}$ and $\tilde{\Omega}$ are given by

$$\begin{aligned}\tilde{g}&=dr^2+f(r)^2d\Theta^2+f(r)^2(f'(r)^2-1)\eta\otimes\eta\\ \tilde{\Omega}&=f(r)^2\Psi+2f(r)f'(r)\eta\wedge dr\end{aligned} \tag{2.15}$$

on $\tilde{B}_l-\{O\}$ respectively, where $f(r)$ is a positive function on $(0, l)$ such that $df/dr>0$, (r, Θ) is the usual polar coordinate system of $\boldsymbol{R}^{2n}$ and $(d\Theta^2, \phi, \xi, \eta)$ is the standard Sasakian structure on S_1^{2n-1}. Thus our purpose has been established.

3. Proof of Theorem A.

We regard $T_p(M)$ as a unitary space with the Hermitian inner product g_p and fix an orthonormal basis of $T_p(M)$ with respect to g_p. By exp we denote the exponential map of $T_p(M)$ to M. We define δ to be the distance from the origin to the first conjugate locus Q_p in $T_p(M)$. If $\delta=\infty$, then M is diffeomorphic to $\boldsymbol{C}^n$. At first, we shall show that for $\delta<\infty$ Q_p is the sphere $S_\delta^{2n-1}=\{X\in T_p(M)\,;\,|X|=\delta\}$. Let $\tilde{q}=X$ be a point of Q_p, $|X|=\delta$, and Y an arbitrary point of S_δ^{2n-1}. Then since $U(n)$ acts transitively on S_δ^{2n-1}, there exists $A\in U(n)$ such that $Y=AX$. From the assumption that (M, g, J) is unitary-symmetric at p it follows that there exists an automorphism Φ such that $\Phi(p)=p$ and $(\Phi_*)_p=A$. On the other hand, since $\tilde{q}$ is a conjugate point, there is a non-zero vector $v\in T_{\tilde{q}}(T_p(M))$ such that $(\exp_*)_{\tilde{q}}v=0$. Then, from the fact that the isometry Φ commutes with the exponential map (cf. Kobayashi-Nomizu [9, I, p. 225]) it follows that at $\tilde{q}'=A\tilde{q}$

$$(\exp_*)_{\tilde{q}}A_*v=(\exp_*)_{\tilde{q}}(\Phi_*)_pv=(\Phi_*)_{\exp\tilde{q}}(\exp_*)_{\tilde{q}}v=0.$$

Hence Q_p is the sphere S_δ^{2n-1} which consists of conjugate points of constant order. By the proof of Theorem 4.4 in [15] the tangential cut locus C_p of p coincides with Q_p and $\exp|_{\tilde{B}_\delta}$ is a diffeomorphism of $\tilde{B}_\delta=\{X\in T_p(M)\,;\,|X|<\delta\}$ onto $B_\delta=\exp\tilde{B}_\delta$. Then $\tilde{g}=\exp^*g$ and $\tilde{\Omega}=\exp^*\Omega$, pulled back under $\exp|_{\tilde{B}_\delta}:\tilde{B}_\delta\to B_\delta$, give a Kählerian structure on $\tilde{B}_\delta$. We now going to show that $\tilde{g}$ and $\tilde{\Omega}$ are $U(n)$-invariant on $\tilde{B}_\delta$. Let $\tilde{q}\in\tilde{B}_\delta$, $q=\exp\tilde{q}$ and $A\in U(n)$. Let $\tilde{X}$ and $\tilde{Y}$ be any tangent vectors at $\tilde{q}$. Then, using the fact that $\exp\circ A=\Phi\circ\exp$, we have

$$\begin{aligned}(A^*\tilde{g})_{\tilde{q}}(\tilde{X}, \tilde{Y})&=\tilde{g}_{A(\tilde{q})}(A_{*\tilde{q}}\tilde{X}, A_{*\tilde{q}}\tilde{Y})\\ &=g_{\exp A(\tilde{q})}((\exp_*)_{\tilde{q}'}(A_*)_{\tilde{q}}\tilde{X}, (\exp_*)_{\tilde{q}'}(A_*)_{\tilde{q}}\tilde{Y})\\ &=g_{\Phi(q)}((\Phi_*)_q(\exp_*)_{\tilde{q}}\tilde{X}, (\Phi_*)_q(\exp_*)_{\tilde{q}}Y)\\ &=\tilde{g}_{\tilde{q}}(\tilde{X}, \tilde{Y}),\end{aligned}$$

putting $\tilde{q}'=A(\tilde{q})$ and $q'=\exp\tilde{q}'$ and identifying $T_p(M)$ with $T_{\tilde{q}}(T_p(M))$. Similarly, we have

$$(A^*\tilde{\Omega})_{\tilde{q}'}(\tilde{X}, \tilde{Y})=\tilde{\Omega}_{\tilde{q}}(\tilde{X}, \tilde{Y})$$

for any vectors $\tilde{X}$, $\tilde{Y}$ at $\tilde{q}\in\tilde{B}_\delta$. Then we see that $(\tilde{g}, \tilde{J})$ is a $U(n)$-invariant Kählerian structure on $\tilde{B}_\delta$, where $\tilde{J}$ denotes the almost complex structure given by $\tilde{g}$ and $\tilde{\Omega}$. Therefore, (2.15) implies that $\tilde{g}$ and $\tilde{\Omega}$ are in the form

$$\tilde{g}=dr^2+f(r)^2d\Theta^2+f(r)^2f'(r)^2-1)\eta\otimes\eta\,, \tag{3.1}$$
$$\tilde{\Omega}=f(r)^2\Psi+2f(r)f'(r)\eta\wedge dr$$

on $\tilde{B}_\delta-\{O\}$ for some function f on $(0, \delta)$ with positive derivative $f'=df/dr$, where (r, Θ) is the usual polar coordinate system of $\boldsymbol{R}^{2n}$ and $(d\Theta^2, \phi, \xi, \eta)$ is the standard Sasakian structure on S_1^{2n-1}.

Finally, we shall show that f in (3.1) is extendible to a function $\tilde{f}$ defined on $(-\infty, \infty)$. For a unit tangent vector X at p, γ_X denotes the geodesic with $\gamma_X(0)=p$ and $\gamma'_X(0)=X$. Let E_0 be a unit vector at p, which is perpendicular to X and J_pX. By a direct computation from (3.1), using (1.4)-(1.6), we obtain

$$R(\gamma'_X, Y)\gamma'_X=-\frac{f''}{f}Y\,, \qquad R(\gamma'_X, J\gamma'_X)\gamma'_X=-\Big(\frac{3f''}{f}+\frac{f'''}{f'}\Big)J\gamma'_X \tag{3.2}$$

for any vector field Y along $\gamma_X|_{(0,\delta)}$ such that $g(\gamma'_X, Y)=g(J\gamma'_X, Y)=0$ where R denotes the curvature tensor of g (cf. Ejiri [4]). This implies that the Jacobi field V along γ_X with the initial conditions $V(0)=0$ and $(\nabla_{\gamma'_X}V)(0)=E_0$ satisfies

$$V(t)=f(t)E(t)$$

on $(0, \delta)$, where $E=E(t)$ is a parallel vector field along γ_X with the initial condition $E(0)=E_0$ (see §4 for detail). Now, from the assumption that M is connected and complete, we may define $\tilde{f}_X$ by

$$\tilde{f}_X(t)=g(V(t), E(t))$$

on $(-\infty, \infty)$. Then since $\tilde{f}_X=f$ on $(0, \delta)$, we see that $\tilde{f}_X$ is an extension of f. We now are going to show that the definition of $\tilde{f}_X$ is independent of the choice of a unit vector X at p. For an arbitrary vector $Y\in S_1^{2n-1}$ in $T_p(M)$, there exists $A\in U(n)$ such that $Y=AX$. From the assumption that M is unitary-symmetric at p, there exists an automorphism Φ of M onto itself such that $\Phi(p)=p$, $(\Phi_*)_p=A$. Let γ_{AX} be the geodesic such that $\gamma_{AX}(0)=p$, $\dot{\gamma}_{AX}(0)=AX$, where $(\dot{})$ denotes the derivative with respect to t. Then since AE_0 is perpendicular to both AX and $J_pAX=AJ_pX$ and $\Phi_*E(t)$ is parallel vector field along the geodesic $\Phi(\gamma_X(t))=\gamma_{AX}(t)$, the Jacobi field W along γ_{AX} with the initial conditions $W(0)=0$, $\nabla_{r)\dot{\gamma}_{AX}}W)(0)=AE_0$ satisfies $W(t)=f(t)\Phi_*E(t)$ on $(0, \delta)$. Summing up the above facts, it follows that

$$\tilde{f}_Y(t)=\tilde{f}_{AX}(t)=g(W(t),\ \Phi_*E(t))=g(\Phi_*V(t),\ \Phi_*E(t))=\tilde{f}_X(t).$$

Therefore, we may write $\tilde{f}$ instead of $\tilde{f}_X$ and adopt f instead of $\tilde{f}$. Thus the proof of Theorem A is complete.

4. Compact Kählerian manifolds satisfying the condition (*).

Let (M, g, J) be a complex $n(\geqq 2)$-dimensional, connected, simply-connected, compact Kählerian manifold satisfying the condition (*). Let p be the fixed point and exp be the exponential map of $T_p(M)$ onto M. By $\delta(>0)$ we denote the distance from the origin O of $T_p(M)$ to the first tangential conjugate locus Q_p in $T_p(M)$. We define $\tilde{B}_\delta=\{X\in T_p(M);\ |X|<\delta\}$ and $B_\delta=\exp\tilde{B}_\delta$, where $|X|$ is the norm $\sqrt{g_p(X, X)}$ of X. Then B_δ may possibly contain a cut point of p, but $\exp:\tilde{B}_\delta\to M$ is an immersion. So we calculate the geometric objects in B_δ in terms of the metric $\exp^*g|_{B_\delta}$. Let $\gamma=\exp rX$ be a geodesic issued from p such that $X\in S_1^{2n-1}$ and $\gamma'=\gamma'(r)$ be the tangent vector field along γ. Then $J\gamma'$ is a parallel unit vector field such that $g_{\gamma(r)}(\gamma', J\gamma')=0$, since J is parallel and satisfies (1.1). Recall the assumption

$$f(r)>0 \quad\text{and}\quad f'(r)>0 \tag{4.1}$$

on $(0, \delta)$. Then we have the following lemma.

LEMMA 4.1. *$f(r)$ satisfies*

$$\lim_{r\downarrow 0}f(r)=0,\qquad \lim_{r\downarrow 0}f'(r)=1. \tag{4.2}$$

PROOF. Let (x^A) be a normal coordinate system, centered at p with respect to g and let (r, Θ) be the geodesic polar coordinate system induced from (x^A). By (θ^i) we denote a local coordinate system of S_1^{2n-1}. Then we know that

$$x^A=ra^A$$

where $a^A=a^A(\theta^i)$ satisfies $\sum_{A=1}^{2n}a^Aa^A=1$. Choose a vector field Y along a geodesic γ issuing from p such that $g(Y, \gamma')=g(Y, J\gamma')=0$ and $d\Theta^2(Y, Y)=1$. Then we have

$$f(r)=r\Big(\frac{\partial a^A}{\partial\theta^i}\frac{\partial a^B}{\partial\theta^j}\tilde{g}_{AB}Y^iY^j\Big)^{1/2}$$

where $\tilde{g}_{AB}$ are the components of g with respect to (x^A) and Y^j are components of Y with respect to (θ^i). This implies (4.2).

Let γ be a geodesic issuing from p. Let $E=E(r)$ be a parallel vector field along γ such that $E(0)$ is perpendicular to the holomorphic section $\{\gamma'(0), J\gamma'(0)\}$. By (3.2) we have the following two kind of Jacobi fields Ξ and V along γ,

$$\Xi(r)=f(r)f'(r)J\gamma'(r), \qquad V(r)=f(r)E(r) \tag{4.3}$$

with the initial conditions

$$\Xi(0)=0 \quad (\nabla_{\gamma'}\Xi)(0)=J\gamma'(0), \qquad V(0)=0 \quad (\nabla_{\gamma'}V)(0)=E(0), \tag{4.4}$$

respectively.

From the assumption on δ, it follows that there exists a point $\tilde{q}=\delta X\in Q_p$, $X\in S_1^{2n-1}$. Since any Jacobi field along the geodesic $\gamma=\exp rX$ with the initial condition (4.4) is given by (4.3), Lemma 4.1 together with (4.3) implies that

$$f'(\delta)=0. \tag{4.5}$$

Hence it follows that the first conjugate locus Q_p in $T_p(M)$ of p is the sphere S_δ^{2n-1} and that the order of each point of it as a conjugate point must be constantly equal to 1.

Since $T_p(M)$ is a unitary space with the Hermitian inner product g_p, it can be naturally identified with $\boldsymbol{C}^n$. Further, identifying $T_p(M)$ with the tangent space $T_{\tilde{q}}(T_p(M))$ at each point $\tilde{q}$ of $T_p(M)$, we regard $T_p(M)$ as a flat Kählerian manifold with the canonical structure (ds_o^2, J_o). Since Q_p is S_δ^{2n-1} in $T_p(M)$, we can define a global unit vector field $\tilde{\xi}$ on Q_p by

$$\tilde{\xi}:\tilde{q}\longrightarrow\tilde{\xi}_{\tilde{q}}=J_oX$$

for $\tilde{q}=\delta X\in Q_p$, where X is regarded as a tangent vector to the ray rX at $\tilde{q}$. Then we see that $\tilde{\xi}$ is regular and that its maximal connected integral curve through $\delta X\in S_\delta^{2n-1}$ is a great circle in S_δ^{2n-1}, which is given by

$$X(\theta)=\delta(\cos\theta X+\sin\theta J_oX) \tag{4.6}$$

for $0\leqq\theta\leqq 2\pi$. Let $\hat{C}$ be the quotient space of Q_p obtained by identifying maximal connected integral curves of $\tilde{\xi}$ to points. Since Q_p is the sphere S_δ^{2n-1} in $\boldsymbol{C}^n$ and has the canonical differentiable structure induced from $\boldsymbol{C}^n$, from the regularity of $\tilde{\xi}$ we see that $\hat{C}$ has a natural manifold structure for which the projection $\pi: Q_p\to\hat{C}$ is a Riemannian submmersion. Thus $\hat{C}$ becomes a Kählerian manifold of positive constant holomorphic curvature (cf. Kobayashi-Nomizu [9, II, p. 134]).

First, we describe the relation of Jacobi fields to the exponential map in the following lemma.

LEMMA 4.2. (cf. Chavel [2]). *Let $p\in M$, $u\in T_p(M)$ and $v\in T_p(M)$ and $Y(t)$ be the Jacobi field along the geodesic $\gamma(t)=\exp_p tu$, determined by the initial conci-*

tions $Y(0)=0$, $(\nabla_u Y)(0)=v$. *Then we have*

$$(\exp_*)_{tu}v=\frac{1}{t}Y(t)$$

for $t\neq 0$, *where* v *is canonically identified with an element of the tangent space* $T_{tu}(T_p(M))$.

LEMMA 4.3. *Let* $\tilde{q}=\delta X$ *be a point of* Q_p, *and let* $\tilde{Y}_{\tilde{q}}$ *be a tangent vector of* $T_{\tilde{q}}(T_p(M))$ *such that* $\tilde{Y}_{\tilde{q}}$ *is perpendicular to* X *and* ξ_q. *Then we have*

(1) $$(\exp_*)_{\tilde{q}}\bar{\xi}_{\tilde{q}}=0$$

(2) $$(\exp_*)_{\tilde{q}}J_o\tilde{Y}_{\tilde{q}}=J_q(\exp_*)_{\tilde{q}}\tilde{Y}_{\tilde{q}},$$

where $q=\exp\tilde{q}$.

PROOF. Let $\gamma=\exp rX$ be the geodesic issuing from p such that $\gamma(0)=p$, $\gamma'(0)=X$. Recall that a Jacobi field Z along γ is uniquely determined by the initial values $Z(0)$ and $(\nabla_{\gamma'}Z)(0)$. Then using Lemma 4.2 together with (4.2)-(4.5), we have

$$(\exp_*)_{\tilde{q}}\bar{\xi}_{\tilde{q}}=\lim_{r\uparrow\delta}\frac{1}{r}\Xi(r)=\frac{1}{\delta}f(\delta)f'(\delta)J\gamma'(\delta)=0.$$

Next, let Y be a parallel vector field along γ such that $Y(0)=\tilde{Y}_{\tilde{q}}$. Since J is parallel, it follows from (3.2), (4.2) and (4.3) that the vector fields $V(r)$ and $W(r)$ defined by

$$V(r)=f(r)Y(r),\qquad W(r)=f(r)JY(r)$$

are both Jacobi fields along γ with the initial condition

$$V(0)=0\qquad (\nabla_{\gamma'}V)(0)=Y(0)=\tilde{Y}_{\tilde{q}},$$
$$W(0)=0\qquad (\nabla_{\gamma'}W)(0)=JY(0)=J_o\tilde{Y}_{\tilde{q}}$$

respectively. By using these and Lemma 4.2, we have

$$(\exp_*)_{\tilde{q}}J_o\tilde{Y}_{\tilde{q}}=\frac{1}{\delta}W(\delta)=\frac{f(\delta)}{\delta}J_qY(\delta)$$

and

$$J_q(\exp_*)_{\tilde{q}}\tilde{Y}_{\tilde{q}}=J_q\Big(\frac{1}{\delta}V(\delta)\Big)=\frac{f(\delta)}{\delta}J_qY(\delta).$$

This proves the assertion (2).

Here we define a mapping $e:\hat{C}\to M$,

(4.7) $$e(\pi(\tilde{q}))=\exp\tilde{q}$$

for any point $\tilde{q}$ of Q_p. This definition is well defined. In fact, if we set $X(\theta)=\delta(\cos\theta X+\sin\theta J_oX)$ for each $\tilde{q}=\delta X$ in Q_p, we have

$$\frac{d}{d\theta}\exp X(\theta)=(\exp_*)_{X(\theta)}J_oX(\theta)=0,$$

taking account of Lemma 4.3 (1).

We now are going to prove that the image of e is the first conjugate locus $Q(p)$ of p and that e is an immersion. For any point q of $Q(p)$, there exists a vector $X\in S_1^{2n-1}$ such that $q=\exp\delta X$. From this fact and (4.7) it follows that $q=e(\pi(\delta X))$, proving $e(\hat{C})=Q(p)$. Since $Q_p=S_\delta^{2n-1}$ is a principal circle bundle over $\hat{C}$, for each point $\hat{q}\in\hat{C}$ there exists an open neighborhood $\hat{U}$ of $\hat{q}$ in $\hat{C}$ and a diffeomorphism $\psi:\hat{U}\times S^1$ onto $\pi^{-1}(\hat{U})$. Using this diffeomorphism ψ, we have that for any $\hat{q}'\in\hat{U}$

$$e(\hat{q}')=e(\pi(\psi(\hat{q}',\ \theta_0)))=\exp\psi(\hat{q}',\ \theta_0),$$

from which the differentiablity of e follows. Then by using Lemmas 4.2 and 4.3 we can show that e is a C^∞-mapping of maximal rank. The following lemma implies that $(\hat{C},\ e)$ is a regular submanifold of M such that e is an imbedding and $e(\hat{C})=Q(p)$.

LEMMA 4.4 (cf. Warner [15, Lemma 3.3]). *Let (M, g, J) be a connected, simply-connected, compact Kählerian manifold of complex dimension $n\geqq 2$. If there exists a point p in M for which each point of the first conjugate locus Q_p in $T_p(M)$ has the constant order* 1, *then for any posnt q of $Q(p)=\exp Q_p$, $\exp^{-1}(q)\cap Q_p$ consists of a single, maximal, connected, integral curve of ξ.*

LEMMA 4.5. *Let $\hat{J}$ be the canonical complex structure on $\hat{C}$, induced from S_δ^{2n-1} in C^n. Give the canonical Kählerian metric $d\sigma^2$ of constant holomorphic curvature* 4 *on it, which is compatible with $\hat{J}$. Then we have*

(1) $$e_*\circ\hat{J}=J\circ e_*,$$

(2) $$e^*g=f(\delta)^2d\sigma^2$$

PROOF. Let d be any point of $\hat{C}$ and $\hat{Y}_d$, $\hat{Z}_d$ any tangent vectors of $T_d(\hat{C})$. Then there is a point $\tilde{q}\in S_\delta^{2n-1}$ such that $d=\pi(\tilde{q})$ and there are tangent vectors $\tilde{Y}_{\tilde{q}}$, $\tilde{Z}_{\tilde{q}}$ of $T_{\tilde{q}}(S_\delta^{2n-1})$ such that $(\pi_*)_{\tilde{q}}\tilde{Y}_{\tilde{q}}$, $(\pi_*)_{\tilde{q}}\tilde{Z}_{\tilde{q}}=\hat{Z}_d$. Then we have

$$(e_*)_d\hat{J}_d\hat{Y}_d=(e_*)_d((\pi_*)_d(J_o\tilde{Y}_{\tilde{q}}))=(\exp_*)_{\tilde{q}}(J_o\tilde{Y}_{\tilde{q}})=J_q(\exp_*)_{\tilde{q}}\tilde{Y}_{\tilde{q}},$$

taking account of (4.7) and Lemma 4.3 (2). Similarly we have

$$\begin{aligned}(e^*g)_d(\hat{Y}_d,\ \hat{Z}_d)&=g_q((e_*)_d\hat{Y}_d,\ (e_*)_d\hat{Z}_d)\\&=g_q((e_*)_d(\pi_*)_{\tilde{q}}\tilde{Y}_{\tilde{q}},\ (e_*)_d(\pi_*)_{\tilde{q}}\tilde{Z}_{\tilde{q}})\\&=g_q((\exp_*)_{\tilde{q}}\tilde{Y}_{\tilde{q}},\ (\exp_*)_{\tilde{q}}\tilde{Z}_{\tilde{q}})\\&=f(\delta)^2(d\sigma^2)_d(\hat{Y}_d,\ \hat{Z}_d).\end{aligned}$$

This shows the assertion (2).

PROPOSITION 4.6. *Let (M, g, J) be a connected, simply-connected, compact Kählerian manifold of complex dimension $n \geqq 2$. Suppose that there is a point p of M such that* $\exp^* g$ *and* $\exp^* \Omega$ *pulled back under* exp, *satisfy the condition* $(*)$. *Then the first conjugate locus $Q(p)$ of p is a totally geodesic, complex hypersurface of M.*

PROOF. Since we have already proved that $(\hat{C}, e)$ is a complex hypersurface of M in Lemma 4.5, we show only that $Q(p)$ is totally geodesic in M. Let $q = \exp \delta X$ be a point of $Q(p)$ and $\gamma = \exp rX$ a geodesic issuing from p. For any vector $v \in T_q(Q(p))$ there exists a unique Jacobi field $V(r) = f(r)E(r)$ along γ such that $V(\delta) = v$, because of (4.3) and (4.4), where $E(r)$ is a parallel vector field along γ and is perpendicular to γ' and $J\gamma'$. We put $w = E(0)$ and define a curve in S_1^{2n-1}

$$Z(t) = \cos(|w|t)X + \sin(|w|t)\frac{w}{|w|}. \tag{4.8}$$

Then we have

$$g_p(\delta Z(t), \delta \dot{Z}(t)) = g_p(J_o(\delta Z(t)), \delta \dot{Z}(t)) = 0, \tag{4.9}$$

where $\dot{Z}(t) = dZ/dt$ is a tangent vector to the curve $Z(t)$. Therefore, $c(t) = \exp \delta Z(t)$ is a curve in $Q(p)$. Moreover, we define a geodesic variation of γ by

$$\alpha(r, t) = \exp rZ(t). \tag{4.10}$$

Then it is easily seen from (4.9) and the Gauss's lemma that $\zeta = (\partial\alpha/\partial r)(\delta, t)$ is a normal vector field to $Q(p)$ along the curve $c(t)$. Especially, we see that

$$\zeta_0 = \frac{\partial \alpha}{\partial r}(\delta, 0) = \gamma'(\delta)$$

and from Lemma 4.3 that ζ_0 and $J_q\zeta_0$ span the normal space at q to $Q(p)$. Since $\alpha(r, t)$ is a geodesic variation of γ, it follows that the induced vector field $(\partial\alpha/\partial t)(r, 0)$ is a Jacobi field along γ and so that

$$\left(\frac{\partial \alpha}{\partial t}\right)(r, 0) = (\exp_*)_{rX} rw.$$

Then by consequence of their initial conditions we can show that $(\partial\alpha/\partial t)(r, 0)$ coincides with the Jacobi field $V = f(r)E(r)$. Recall the Weingarten's formula (1.2) on a complex hypersurface of a Kählerian manifold. Then by an elementary property of variation we have

$$-h_q(v, v)=g_q(\nabla_v\zeta, v)=g\Big(\nabla_{\partial\alpha/\partial t}\frac{\partial\alpha}{\partial r}, \frac{\partial\alpha}{\partial t}\Big)\Big|_{\substack{t=0\\ r=\delta}}=g\Big(\nabla_{\partial\alpha/\partial r}\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\Big)\Big|_{\substack{t=0\\ r=\delta}}$$
$$=g(\nabla_{\gamma'}V(r), V(r))_{r=\delta}=g_q(f'(\delta)E(\delta), f(\delta)E(\delta))=0,$$

taking account of (4.5). Similarly we have

$$-k_q(v, v)=g_q(\nabla_v J\zeta, v)=-g_q(\nabla_v\zeta, Jv)=-g\Big(\nabla_{\partial\alpha/\partial t}\frac{\partial\alpha}{\partial r}, J\frac{\partial\alpha}{\partial t}\Big)\Big|_{\substack{t=0\\ r=\delta}}$$
$$=-g(\nabla_{\gamma'}V(r), JV(r))|_{r=\delta}=-g_q(f'(\delta)E(\delta), f(\delta)J_qE(\delta))=0$$

by means of $\nabla J=0$. Hence both h_q and k_q vanish for all tangent vectors of $T_q(Q(p))$ at any point q of $Q(p)$. Thus we conclude that $Q(p)$ is totally geodesic. By Lemma 4.5 $(\hat{C}, e)$ is a totally geodisic, complex hypersurface of M. This completes the proof and also gives Corollary C.

5. Proof of Theorem B.

Our purpose in this section is to construct an automorphism F_A of M for each $A\in U(n)$ and to complete the proof of Theorem B. Let (M, g, J) be a connected, simply-connected, complete Kählerian manifold of complex dimension $n\geqq 2$. Suppose that there is a point $p\in M$ such that $\exp^*g$ and $\exp^*\Omega$, pulled back under exp, satisfy the condition $(*)$. If M is non compact, $\delta=\infty$, then exp is a diffeomorphism of $T_p(M)$ onto M as is described in Introduction. Then the reader will see that the discussions on B_δ in the case $\delta<\infty$ are just applicable to the case $\delta=\infty$. So in the following, M is assumed to be compact.

Since the first tangential conjugate locus Q_p of p in $T_p(M)$ is the sphere S_δ^{2n-1} and the order of each point of Q_p as a conjugate point is constantly equal to 1 as is seen in §4, by means of the proof of Theorem 4.4 in [15] Q_p coincides with the tangential cut locus C_p of p in $T_p(M)$. In the following, we write C_p for Q_p, and use the fact that M is a disjoint union of $B_\delta=\exp\tilde{B}_\delta$ and $C(p)=\exp C_p$ (cf. Kobayashi-Nomizu [9, II, p. 100]).

Since M is complete, we know from the theorem of Hopf-Rinow (cf. Helgason [6]) that any point q of M is written by $q=\exp rX$ for some $r\in\boldsymbol{R}$ and some unit vector X. Then for each $A\in U(n)$ we define a transformation $F_A: M\to M$,

$$F_A(q)=\exp rAX. \tag{5.1}$$

We show that the definition of F_A is well defined. Since $A(\tilde{B}_\delta)=\tilde{B}_\delta$ and $\exp|_{\tilde{B}_\delta}$ is a diffeomorphism of $\tilde{B}_\delta$ onto B_δ, it is obvious that $F_A|_{B_\delta}$ is a diffeomorphism of B_δ onto itself with the only fixed point p. Next, let $q=\exp\delta X$ be a point of $C(p)$. Then, it follows from (5.1) that $F_A(q)\in C(p)$. Suppose that q has another representation $q=\exp\delta Y$ such that $Y\in S_1^{2n-1}$. Then Lemma 4.4 implies that there is a number $t\in\boldsymbol{R}$ such that $Y=\cos tX+\sin tJ_oX$. Therefore we have

$$\begin{aligned}F_A(\exp\delta Y)&=\exp\delta A(\cos tX+\sin tJ_oX)\\&=\exp\delta(\cos tAX+\sin tJ_oAX)\\&=F_A(\exp\delta X),\end{aligned}$$

taking account of the properties $AX\in S_1^{2n-1}$ and $A\circ J_o=J_o\circ A$. This implies that F_A is well defined. Moreover, let $q=\exp\delta X$ be a point of $C(p)$ such that $X\in S_1^{2n-1}$. Since A is non singular, if we put $q'=\exp\delta A^{-1}X$, where A^{-1} denotes the inverse matrix of A, then

$$F_A(q')=\exp\delta AA^{-1}X=\exp\delta X=q.$$

This implies that F_A maps M onto M.

Let $q=\exp\delta X$ and $q'=\exp\delta Y$ be two points of $C(p)$ such that $F_A(q)=F_A(q')$, that is, $\exp\delta AX=\exp\delta AY$. Then by using Lemma 4.4 we see that there is a number $t\in\boldsymbol{R}$ such that $AY=\cos tAX+\sin tJ_oAX$. By the fact $J_o\circ A=A\circ J_o$, we have

$$Y=\cos tX+\sin tJ_oX,$$

from which it follows that

$$q'=\exp\delta Y=\exp\delta(\cos tX+\sin tJ_oX)=q.$$

This means that F_A is 1-1 on M.

First, we show that $F_A|_{B_\delta}$ and $F_A|_{C(p)}$ are differentiable and leave the Kählerian structure invariant on B_δ and $C(p)$ respectively. By these facts, it will be shown that F_A is an automorphism of (M, g, J).

We now consider about $F_A|_{B_\delta}$: Since $\exp|_{\tilde{B}_\delta}$ is a diffeomorphism of $\tilde{B}_\delta$ onto B_δ, we may write

$$(F_A|_{B_\delta})_*=(\exp)_*(A)_*(\exp|_{\tilde{B}_\delta})_*^{-1}. \tag{5.2}$$

In order to show that F_A leaves (g, J) invariant on B_δ, it is sufficient to prove that $\tilde{g}=\exp^*g$ and $\tilde{\Omega}=\exp^*\Omega$ are A-invariant on $\tilde{B}_\delta$. In fact, if $\tilde{g}$ and $\tilde{\Omega}$ are A-invariant, then

$$\begin{aligned}(F_A^*g)_q(X_q, Y_q)&=(\exp^*g)_{A(\tilde{q})}((A_*)_{\tilde{q}}(\exp_*^{-1})_qX_q, (A_*)_{\tilde{q}}(\exp_*^{-1})_qY_q)\\&=(\exp^*g)_{\tilde{q}}((\exp_*^{-1})_qX_q, (\exp_*^{-1})_qY_q)\\&=g_q(X_q, Y_q)\end{aligned}$$

for any tangent vectors X_q, Y_q of $T_q(B_\delta)$, where $q=\exp\tilde{q}$. Similarly we obtain

$$(F_A^*\Omega)=\Omega$$

on B_δ. We show that $\tilde{g}$ and $\tilde{\Omega}$ are A-invariant on $\tilde{B}_\delta$. Let $\tilde{q}=rX$ be a point of $\tilde{B}_\delta$ such that $X=(b^\alpha)\in S_1^{2n-1}$ and $\sum_{\alpha=1}^n b^\alpha\bar{b}^\alpha=1$. As is seen from the right hand side of (*), it is sufficient to show that $d\Theta^2$, η and Ψ are A-invariant. It is known (cf. Sasaki-Hatakeyama [11]) that they are represented by

$$d\Theta^2=\sum_{\alpha=1}^n db^\alpha d\bar{b}^\alpha, \quad \eta=\sqrt{-1}\sum_{\alpha=1}^n \bar{b}^\alpha db^\alpha, \quad \Psi=\sqrt{-1}\sum_{\alpha=1}^n db^\alpha\wedge d\bar{b}^\alpha,$$

from which by the property $\sum_{\beta=1}^n a_{\alpha\beta}\bar{a}_{\gamma\beta}=\delta_{\alpha\gamma}$ of $A=(a_{\alpha\beta})\in U(n)$, we have

$$A^*d\Theta^2=\sum_{\alpha,\beta,\gamma=1}^n d(a_{\alpha\beta}b^\beta)d(\bar{a}_{\gamma\alpha}\bar{b}^\gamma)=\sum_{\alpha,\beta,\gamma=1}^n a_{\alpha\beta}\bar{a}_{\gamma\alpha}db^\beta d\bar{b}^\gamma=\sum_{\alpha=1}^n db^\alpha d\bar{b}^\alpha$$

and similarly $A^*\eta=\eta$ and $A^*\Psi=\Psi$. Thus it follows that F_A leaves g and Ω invariant on B_δ.

We shall consider about the mapping $F_A|_{C(p)}$ in the following. Since $e:\hat{C}\to C(p)\subset M$ is diffeomorphic, the differentiability of $F_A|_{C(p)}$ follows from (4.7) and the fact that for $q=\exp\delta X\in C(p)$

$$F_A(q)=\exp\delta AX=e(\pi(A(\delta X)))=e\circ\hat{A}\circ\pi(\delta X)=e\circ\hat{A}\circ e^{-1}(q), \tag{5.3}$$

where $\hat{A}$ denotes a $U(n)$-action on $\hat{C}=CP^{n-1}$. Recall that the canonical Kählerian structure $(d\sigma^2, \hat{J})$ on $\hat{C}$ is $U(n)$-invariant (cf. Kobayashi-Nomizu [9, II, p. 273]). Then by (5.3) and Lemma 4.5, (2) we have

$$\begin{aligned}(F_A^*g)_q(Y_q, Z_q)&=g_{F_A(q)}((e_*)_{\hat{A}(d)}(\hat{A}_*)_d(e_*^{-1})_qY_q, (e^*)_{\hat{A}(d)}(\hat{A}_*)_d(e_*^{-1})_qZ_q)\\&=(e^*g)_{\hat{A}(d)}((\hat{A}_*)_d(e_*^{-1})_qY_q, (\hat{A}_*)_d(e_*^{-1})_qZ_q)\\&=f(\delta)^2(d\sigma^2)_{\hat{A}(d)}((\hat{A}_*)_d(e_*^{-1})_dY_q, (\hat{A}_*)_d(e_*^{-1})_qZ_q)\\&=g_q(Y_q, Z_q)\end{aligned}$$

for any tangent vectors Y_q, $Z_q\in T_q(C(p))$, where $d=e^{-1}(q)$ and $\hat{A}(d)=\hat{A}(\pi(\delta X))=\pi(\delta AX)$. Similarly we obtain

$$F_A^*\Omega=\Omega$$

on $C(p)$.

Though $F_A|_{B_\delta}$ and $F_A|_{C(p)}$ are differentiable, it remains to be shown that F_A is differentiable on M. Then by the following lemma (cf. Helgason [6, p. 61], Kobayashi-Nomizu [9, I, p, 169]) we now are going to show that F_A is an isometry of (M, g).

LEMMA 5.1 (Myers-Steenrod). *Let (N, g) be a connected Riemannian manifold and F a distance-preserving mapping of N onto itself, that is $d(F(p), F(q))=d(p, q)$ for $p, q\in N$. Then F is an isometry.*

First of all, we show that F_A is continuous on M. Since $F_A|_{B_\delta}$ is differntiable and B_δ is an open set of M, it remains to show that F_A is continuous at the point $q=\exp\delta X\in C(p)$. Let $q'=\exp Y$, $0<|Y|<\delta$, be a point sufficiently near q. Putting $q''=\exp\delta(Y/|Y|)$ and using the triangle inequality, we have $d(q', q'')\leqq d(q, q')$, from which $d(q, q'')\leqq 2d(q, q')$. Then we have

$$\begin{aligned} d(F_A(q), F_A(q'))&\leqq d(F_A(q), F_A(q''))+d(F_A(q''), F_A(q')) \\ &=d(q, q'')+d(q'', q')\leqq 3d(q, q'), \end{aligned}$$

taking account of the properties of $F_A|_{B_\delta}$ and $F_A|_{C(p)}$, since $C(p)$ is totally geodesic. This implies that F_A is continuous at q.

Next, we show that F_A is a distance-preserving mapping on (M, g). Let q and q' be two points of M. The set of all continuous piecewise C^1-curves from q to q' in M will be denoted by $\Gamma(q, q')$. Then for any curve c of $\Gamma(q, q')$, $F_A\circ c$ belongs to $\Gamma(F_A(q), F_A(q'))$ by virtue of continuity of F_A. Conversely, if $c\in\Gamma(F_A(q), F_A(q'))$, then $F_{A^{-1}}\circ c\in\Gamma(q, q')$. Then F_A induces a mapping of $\Gamma(q, q')$ onto $\Gamma(F_A(q), F_A(q'))$. Since $C(p)$ is a totally geodesic submanifold of M and since $F_A|_{C(p)}$ (resp. $F_A|_{B_\delta}$) is an isometry of $(C(p), g|_{C(p)})$ (resp. $(B_\delta, g|_{B_\delta})$) onto itself, we have to consider only the curves $c\in\Gamma(q, q')$ such that $c(a)=q\in B_\delta$, $c(b)=q'\in C(p)$ and $C([a, b))\subset B_\delta$. But for such curves c it can be easily shown that length of c=length of $F_A\circ c$. Thus F_A is a distance-preserving mapping of M onto itself. Thanks to Lemma 5.1, we establish that F_A is an isometry of (M, g) onto itself.

Finally, it remains to be shown that F_A is holomorphic on M, though $F_A|_{B_\delta}$ and $F_A|_{C(p)}$ are already so. But as is seen in (5.7), it is sufficient to show that $(F_{A*})_q J_q\gamma'_X(\delta)=J_{F_A(q)}(\gamma'_{AX}(\delta))$ at $q\in C(p)$, where γ_X denotes a geodesic issuing from p satisfying $\gamma'_X(0)=X$. Since F_A is differentiable, by (5.2) we have

$$\begin{aligned} (F_{A*})_q J_q\gamma'_X(\delta)&=\lim_{r\uparrow\delta}(\exp_*)_{A(rX)}(A_*)_{(rX)}\left(\frac{J_oX}{f(r)f'(r)}\right) \\ &=\lim_{r\uparrow\delta} J\gamma'_{AX}(r)=J\gamma'_{AX}(\delta), \end{aligned}$$

taking account of (4.3), (4.4) and $A\circ J_o=J_oA$. Therefore, F_A defined by (5.1) is an automorphism of M onto itself such that $F_A(p)=p$ and $(F_{A*})_p=A$. Thus (M, g, J) is unitary-symmetric at p and the proof of Theorem B is complete.

6. Proof of Theorem D.

Let X be a unit tangent vector in $T_p(M)$ and $\gamma_X=\gamma_X(r)$ $(0\leqq r\leqq\delta)$ be the geodesic issuing from p such that $\gamma'_X(0)=X$. For simplicity we put $X(\theta)=\cos\theta X+\sin\theta J_oX$ $(0\leqq\theta\leqq 2\pi)$ and define

(6.1) $$\omega(t, \theta)=\exp((\delta+t)X(\theta))$$

for $-\delta\leqq t\leqq 0$, $0\leqq\theta\leqq\pi$. Then by Lemma 4.2 we have

$$
\begin{aligned}
(6.2)\qquad \nabla_\theta\partial_t\omega|_{t=0} &= \nabla_t\partial_\theta\omega|_{t=0} \\
&= \nabla_t[(\exp_*)_{(\delta+t)X(\theta)}(\delta+t)J_oX(\theta)]|_{t=0} \\
&= \nabla_t[f(\delta+t)f'(\delta+t)J\gamma'_{X(\theta)}(\delta+t)]|_{t=0} \\
&= f(\delta)f''(\delta)J\gamma'_{X(\theta)}(\delta).
\end{aligned}
$$

Recall that as in the definition of the mapping $e:\hat{C}\to Q(p)\subset M$ we have $\omega(0, \theta)=q$ for each θ $(0\leqq\theta\leqq 2\pi)$. Therefore it follows that for each θ, $\rho(\theta)=(\partial_t\omega)(0, \theta)$ is a tangent vector in $T_q(M)$, from which $\nabla_\theta\rho(\theta)$ is always in $T_q(M)$. From this observation and Lemma 4.2, the assumption $f(\delta)f''(\delta)=-1$ together with (6.2) implies that $\rho=\rho(\theta)$ is a unit circle in $N_q=T_q(Q(p))^\perp$, whose tangent vectors are always of length 1, where $T_q(Q(p))^\perp$ is the 2-dimensional plane in $T_q(M)$ orthogonal to the tangent space $T_q(Q(p))$. Since $\{\gamma'_X(\delta), J\gamma'_X(\delta)\}$ is an orthonormal basis of N_q, $\rho(\theta)$ may be represented up to an orientation by

$$\rho(\theta)=\cos(\theta+\alpha)\gamma'_X(\delta)+\sin(\theta+\alpha)J\gamma'_X(\delta),$$

where α is a constant. This implies that $\rho(\pi)=-\rho(0)$, that is,

$$(6.3)\qquad \gamma'_{-X}(\delta)=-\gamma'_X(\delta),$$

Since geodesics in M are determined uniquely by their initial conditions at one point in M, by (6.3) we have

$$\exp(\delta-t)(-X)=\exp(\delta+t)X$$

for $0\leqq t\leqq\delta$, form which

$$\gamma_X(t)=\exp tX=\exp(2\delta-t)(-X)\qquad(0\leqq t\leqq 2\delta)$$

follows. Thus we see that any geodesic issuing from p is closed.

REMARK. Using Theorem D, Mori-Watanabe [10] has shown that there exist non-canonical SC^p-Kählerian structures on CP^n.

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