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COVERING PROPERTIES IN COUNTABLE PRODUCTS

By

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1. Introduction.

A space X is said to be *subparacompact* if every open cover of X has a σ-discrete closed refinement, and *metacompact* (*countably metacompact*) if every open cover (countable open cover) of X has a point finite open refinement. A space X is said to be *metalindelöf* if every open cover of X has a point countable open refinement. A collection $\mathcal{U}$ of subsets of a space X is said to be *interior-preserving* if $\mathrm{int}(\cap\mathcal{V})=\cap\{\mathrm{int}\, V : V\in\mathcal{V}\}$ for every $\mathcal{V}\subset\mathcal{U}$. Clearly, an open collection $\mathcal{U}$ is interior-preserving if and only if $\cap\mathcal{V}$ is open for every $\mathcal{V}\subset\mathcal{U}$. A space X is said to be *orthocompact* if every open cover of X has an interior-preserving open refinement. Every paracompact Hausdorff space is subparacompact and metacompact, and every metacompact space is countably metacompact, metalindelöf and orthocompact. The reader is refered to D. K. Burke [4] for a complete treatment of these covering properties and some informations of their role in general topology.

Let $\mathcal{DC}$ be the class of all spaces which have a discrete cover by compact sets. The topological game $G(\mathcal{DC}, X)$ was introduced and studied by R. Telgársky [19]. The games are played by two persons called Players I and II. Players I and II choose closed subsets of II's previous play (or of X, if $n=0$): Player I's choice must be in the class $\mathcal{DC}$ and II's choice must be disjoint from I's. We say that Player I *wins* if the intersection of II's choices is empty. Recall from [19] that a space X is said to be $\mathcal{DC}$*-like* if Player I has a winning strategy in $G(\mathcal{DC}, X)$. The class of $\mathcal{DC}$-like spaces includes all spaces which admit a σ-closure-preserving closed cover by compact sets, and regular subparacompact, σ-C-scattered spaces.

Paracompactness and Lindelöf property of countable products have been studied by several authors. In particular, if X is a separable metric space or X is a regular Čech-complete Lindelöf space or X is a regular C-scattered Lindelöf space, then $X^{\omega}\times Y$ is Lindelöf for every regular hereditarily Lindelöf space Y. The first result is due to E. Michael (cf. [14]) and the second one

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is due to Z. Frolik [9] and the third one is due to K. Alster [1]. K. Alster [2] also proved that if Y is a perfect paracompact Hausdorff space and X_n is a scattered paracompact Hausdorff space for each $n\in\omega$, then $Y\times\prod_{n\in\omega}Y_n$ is paracompact. Furthermore, the author [17] proved that if Y is a perfect paracompact Hausdorff (regular hereditarily Lindelöf) space and X_n is a paracompact Hausdorff (regular Lindelöf) $\mathcal{DC}$-like space for each $n\in\omega$, then $Y\times\prod_{n\in\omega}X_n$ is paracompact (Lindelöf).

The aim of this paper is to consider subparacompactness, metacompactness, metalindelöf property and orthocompactness of countable products. We show that if Y is a perfect subparacompact space and X_n is a regular subparacompact $\mathcal{DC}$-like space for each $n\in\omega$, then $Y\times\prod_{n\in\omega}X_n$ is subparacompact. We also prove that if X_n is a regular metacompact $\mathcal{DC}$-like (C-scattered) space for each $n\in\omega$, then $\prod_{n\in\omega}X_n$ is metacompact. Furthermore, let Y be a hereditarily metacompact space and X_n be a regular metacompact $\mathcal{DC}$-like (C-scattered) space for each $n\in\omega$. Then the following statements are equivalent: (a) $Y\times\prod_{n\in\omega}X_n$ is metacompact; (b) $Y\times\prod_{n\in\omega}X_n$ is countably metacompact and (c) $Y\times\prod_{n\in\omega}X_n$ is orthocompact. For metalindelöf property, it will be shown that if Y is a hereditarily metalindelöf space and X_n is a regular metalindelöf $\mathcal{DC}$-like (C-scattered) space for each $n\in\omega$, then $Y\times\prod_{n\in\omega}X_n$ is metalindelöf.

In this paper, we deal with infinite spaces. No separation axioms are assumed. However, regular spaces are assumed to be T_1. Let $|A|$ denote the cardinality of a set A. The letter ω denotes the set of natural numbers.

Given a cover $\mathcal{U}$ of a space X, and $Y\subset X$, let $\mathcal{U}|Y=\{U\cap Y: U\in\mathcal{U}\}$. For each $x\in X$, let $\mathcal{U}_x=\{U\in\mathcal{U}: x\in U\}$ and let $\mathrm{ord}(x,\mathcal{U})=|\mathcal{U}_x|$. Let $\mathcal{U}^F$ be the collection of all finite unions of elements of $\mathcal{U}$.

We use the finite sequences in the proofs. So we adopt the following notations for them: Let A be a set, and let $\mathcal{P}(A)$ be the set of all nonempty subsets of A. Let $A^0=\{\emptyset\}$. For each $n\geqq 1$, A^n denotes the set of all n-sequences of elements of A and $A^{<\omega}=\bigcup_{n\in\omega}A^n$. If $\tau=(a_0,\cdots,a_n)\in A^{<\omega}$ and $a\in A$, then $\tau\oplus a$ denotes the sequence $(a_0,\cdots,a_n,a)$ and $\tau_-=(a_0,\cdots,a_{n-1})$ if $n\geqq 1$ and $\tau_-=\emptyset$ if $n=0$.

2. Topological games.

For the class $\mathcal{DC}$ and a space X, the *topological game* $G(\mathcal{DC}, X)$ is defined as follows: There are two players I and II (the pursuer and evader). They alternatively choose consecutive terms of a sequence $\langle E_0, F_0, E_1, F_1, \cdots, E_n, F_n,$

$\cdots\rangle$ of subsets in X. When each player chooses his term, he knows $\mathcal{DC}$, X and their previous choices.

For a space X, let 2^X denote the set of all closed subsets of X. A sequence $\langle E_0, F_0, E_1, F_1, \cdots, E_n, F_n, \cdots\rangle$ of subsets in X is a *play* of $G(\mathcal{DC}, X)$ if it satisfies the following conditions: For each $n\in\omega$,

(1) E_n is the choice of Player I,

(2) F_n is the choice of Player II,

(3) $E_n\in 2^X\cap\mathcal{DC}$,

(4) $F_n\in 2^X$,

(5) $E_n\cup F_n\subset F_{n-1}$, where $F_{-1}=X$,

(6) $E_n\cap F_n=\emptyset$.

Player I wins if $\bigcap_{n\in\omega} F_n=\emptyset$ (Player II has no place to run away). Otherwise Player II wins.

A finite sequence $\langle E_0, F_0, E_1, F_1, \cdots, E_m, F_m\rangle$ is said to be *admissible* if it satisfies the above conditions (1)-(6) for each $n\leqq m$.

Let s' be a function from $\bigcup_{n\in\omega}(2^X)^{n+1}$ into $2^X\cap\mathcal{DC}$. Let

$$\mathcal{S}_0=\{F : \langle s'(X), F\rangle \text{ is admissible for } G(\mathcal{DC}, X)\}\,.$$

Moreover, we can inductively define

$$\begin{aligned}\mathcal{S}_n=\{(F_0, F_1, \cdots, F_n) : {} & \langle E_0, F_0, E_1, F_1, \cdots, E_n, F_n\rangle \\ & \text{is admissible for } G(\mathcal{DC}, X), \text{ where } F_{-1}=X \text{ and} \\ & E_i=s'(F_0, F_1, \cdots, F_{i-1}) \text{ for each } i\leqq n\}\,.\end{aligned}$$

Then the restriction s of s' to $\bigcup_{n\in\omega}\mathcal{S}_n$ is said to be a *strategy* for Player I in $G(\mathcal{DC}, X)$. We say that the strategy s is a *winning* one if Player I wins every play $\langle E_0, F_0, E_1, F_1, \cdots, E_n, F_n, \cdots\rangle$ such that $E_n=s(F_0, F_1, \cdots, F_{n-1})$ for $n\in\omega$.

Next, we define another (winning) strategy for Player I in $G(\mathcal{DC}, X)$, which depends only on the preceding choice of Player II.

A function s from 2^X into $2^X\cap\mathcal{DC}$ is said to be a *stationary strategy* for Player I in $G(\mathcal{DC}, X)$ if $s(F)\subset F$ for each $F\in 2^X$. We say that the s is *winning* if he wins every play $\langle s(X), F_0, s(F_0), F_1, s(F_1), \cdots\rangle$. That is, a function s from 2^X into $2^X\cap\mathcal{DC}$ is a stationary winning strategy if and only if it satisfies

(i) $s(F)\subset F$ for each $F\in 2^X$,

(ii) if $\{F_n : n\in\omega\}$ is a decreasing sequence of closed subsets of X such that $s(F_n)\cap F_{n+1}=\emptyset$ for each $n\in\omega$, then $\bigcap_{n\in\omega} F_n=\emptyset$.

The following lemma shows that there is no essential difference between the winning strategy and the stationary winning strategy.

LEMMA 2.1 (F. Galvin and R. Telgársky [10]). *Player I has a winning strategy in $G(\mathcal{DC}, X)$ if and only if he has a stationary winning strategy in it.*

As described in the introduction, a space X is $\mathcal{DC}$-like if Player I has a winning strategy in $G(\mathcal{DC}, X)$.

LEMMA 2.2 (R. Telgársky [19]). *If a space X has a countable closed cover by $\mathcal{DC}$-like sets, then X is a $\mathcal{DC}$-like space.*

Recall that a space X is *scattered* if every non-empty subset A of X has an isolated point of A, and *C-scattered* if for every non-empty closed subset A of X, there is a point of A which has a compact neighborhood in A. Then scattered spaces and locally compact Hausdorff spaces are C-scattered. Let X be a space. For each $F \in 2^X$, let

$$F^{(1)} = \{x \in F : x \text{ has no compact neigborhood in } F\}.$$

Let $X^{(0)} = X$. For each successor ordinal α, let $X^{(\alpha)} = (X^{(\alpha-1)})^{(1)}$. If α is a limit ordinal, let $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$. Notice that a space X is C-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. If X is C-scattered, let $\varepsilon(X) = \inf\{\alpha : X^{(\alpha)} = \emptyset\}$. We say that $\varepsilon(X)$ is the *C-scattered height* of X. For each $x \in X$, we denote by $\alpha_X(x)$ the ordinal such that $x \in X^{(\alpha_X(x))} - X^{(\alpha_X(x)+1)}$. Let X be a regular C-scattered space. If A is either open or closed in X, then A is C-scattered. More precisely, if A is an open subset of X, then $A^{(\alpha)} = X^{(\alpha)} \cap A$ for each $\alpha < \varepsilon(X)$ and if A is a closed subset of X, then $A^{(\alpha)} \subset A \cap X^{(\alpha)}$ for each $\alpha < \varepsilon(X)$. Therefore, if $x \in A$, then $\alpha_A(x) \leqq \alpha_X(x)$ and hence, $\varepsilon(A) \leqq \varepsilon(X)$. A space X is said to be *σ-scattered* (*σ-C-scattered*) if X is the union of countably many closed scattered (C-scattered) subspaces.

LEMMA 2.3 (R. Telgársky [19]). (a) *If a space X has a σ-closure-preserving closed cover by compact sets, then X is a $\mathcal{DC}$-like space.*

(b) *If X is a regular subparacompact, σ-C-scattered space, then X is $\mathcal{DC}$-like space.*

LEMMA 2.4 (G. Gruenhage and Y. Yajima [11], Y. Yajima [21]). (a) *If X is a regular subparacompact (metacompact) $\mathcal{DC}$-like space, then $X \times Y$ is subparacompact (metacompact) for every subparacompact (metacompact) space Y.*

(b) *If X is a regular C-scattered metacompact space, then $X \times Y$ is metacompact for every metacompact space Y.*

For topological games, the reder is refered to R. Telgársky [18], [19] and Y. Yajima [21].

3. Preliminaries.

Let Z be a space and $\{Y_i : i\in\omega\}$ be a countable collection of spaces. For $Z\times\prod_{i\in\omega} Y_i$, we denote by $\mathcal{B}$ the collection of all basic open subsets of $Z\times\prod_{i\in\omega} Y_i$. Let us denote by $\mathcal{R}$ the collection of closed subsets of $Z\times\prod_{i\in\omega} Y_i$ consisting of sets of the form $R=E_R\times\prod_{i\in\omega} R_i$, where E_R is a closed subset of Z and there is an $n\in\omega$ such that for each $i\leqq n$, R_i is a closed subset of Y_i and for each $i>n$, $R_i=Y_i$. For each $B=U_B\times\prod_{i\in\omega} B_i\in\mathcal{B}$ and $R=E_R\times\prod_{i\in\omega} R_i\in\mathcal{R}$, we define $n(B)=\inf\{i\in\omega : B_j=Y_j \text{ for } j\geqq i\}$ and $n(R)=\inf\{i\in\omega : R_j=Y_j \text{ for } j\geqq i\}$. We call $n(B)$ and $n(R)$ the *length* of B and R respectively. Let $\mathcal{K}=\{\prod_{i\in\omega} K_i : K_i$ is a compact subset of Y_i for each $i\in\omega\}$. For each $z\in Z$ and $K\in\mathcal{K}$, let $K_{(z,K)}=\{z\}\times K$.

LEMMA 3.1 (D. K. Burke [3], [4]). *The following are equivalent for a space X.*

(a) *X is subparacompact,*

(b) *Every open cover of X has a σ-locally finite closed refinement,*

(c) *For every open cover $\mathcal{U}$ of X, there is a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of open refinements of $\mathcal{U}$ such that for each $x\in X$, there is an $n\in\omega$ with* $\operatorname{ord}(x, \mathcal{V}_n)=1$.

It is well known that a space X is metacompact (metalindelöf) if and only if for every open cover $\mathcal{U}$ of X, $\mathcal{U}^F$ has a point finite (point countable) open refinement. In order to study subparacompactness of $Z\times\prod_{i\in\omega} Y_i$, we need the following lemma.

LEMMA 3.2. *Let Z be a space and $\{Y_i : i\in\omega\}$ be a countable collection of spaces. Assume that all finite subproducts of $Z\times\prod_{i\in\omega} Y_i$ are subparacompact. If, for every open cover $\mathcal{O}$ of $Z\times\prod_{i\in\omega} Y_i$, $\mathcal{O}^F$ has a σ-locally finite refinement consisting of elements of $\mathcal{R}$, then $Z\times\prod_{i\in\omega} Y_i$ is subparacompact.*

PROOF. Let $\mathcal{O}$ be an open cover of $Z\times\prod_{i\in\omega} Y_i$. We may assume that $\mathcal{O}\subset\mathcal{B}$. By the assumption, there is a σ-locally finite refinement $\bigcup_{m\in\omega}\mathcal{R}_m$ of $\mathcal{O}^F$, consisting of elements of $\mathcal{R}$. Fix $m\in\omega$. For each $R=E_R\times\prod_{i\in\omega} R_i\in\mathcal{R}_m$, let $\{O(R, k):$

$k=0, \cdots, j(R)\}$ be a finite subcollection of $\mathcal{O}$ such that $R\subset\bigcup_{k=0}^{j(R)} O(R, k)$. Let $O(R, k)=U_{R,k}\times\prod_{i\in\omega} O(R, k)_i$ for each $k\leqq j(R)$, and let $n=\max\{n(R), n(O(R, k)): k\leqq j(R)\}$. Put $R(n)=E_R\times\prod_{i=0}^{n} R_i$ and $O(R, k, n)=U_{R,k}\times\prod_{i=0}^{n} O(R, k)_i$ for each $k\leqq j(R)$. Let $\mathcal{O}(R)=\{O(R, k, n): k\leqq j(R)\}$. Then $R(n)\subset\bigcup\mathcal{O}(R)$. Rince $Z\times\prod_{i=0}^{n} Y_i$ is subparacompact and $R(n)$ is a closed subspace of $Z\times\prod_{i=0}^{n} Y_i$, $R(n)$ is subparacompact. Thus there is a σ-discrete closed refinement $\bigcup_{t\in\omega}\mathcal{D}_t(R)$ of $\mathcal{O}(R)|R(n)$. For each $t\in\omega$, let $\mathcal{D}'_t(R)=\{D\times\prod_{i>n} Y_i: D\in\mathcal{D}_t\}$. Put $\mathcal{G}_{m,t}=\bigcup\{\mathcal{D}'_t(R): R\in\mathcal{R}_m\}$ for each $m, t\in\omega$. Then $\bigcup_{m,t\in\omega}\mathcal{G}_{m,t}$ is a σ-locally finite closed refinement of $\mathcal{O}$. It follows from Lemma 3.1 that $Z\times\prod_{i\in\omega} Y_i$ is subparacompact. The proof is completed.

In order to study metacompactness and metalindelöf property of countable products of C-scattered spaces, we need the following.

LEMMA 3.3. *Let X be a regular C-scattered metacompact (metalindelöf) space. For every open over $\mathcal{U}$ of X, there is a point finite (point countable) open cover $\mathcal{V}$ of X such that: For each $V\in\mathcal{V}$,*

(a) *clV is contained in some member of $\mathcal{U}$,*

(b) *$(clV)^{(\alpha)}$ is compact for some $\alpha<\varepsilon(X)$.*

PROOF. We prove this lemma by induction on the C-scattered height $\varepsilon(X)$ for the sake of completeness. Let X be a locally compact metacompact (metalindelöf) Hausdorff space (i.e. $\varepsilon(X)=1$). Thus there is a point finite (point countable) open cover $\mathcal{V}$ of X satisfying the condition (a) such that for each $V\in\mathcal{V}$, clW is compact. Clearly $\mathcal{V}$ satisfies the condition (b). Let X be a regular C-scattered metacompact (metalindelöf) space and $\varepsilon=\varepsilon(X)$, and assume that for each $\alpha<\varepsilon$, the lemma holds. Then there is a point finite (point countable) open cover $\mathcal{W}$ of X such that (cf. R. Telgársky [18, Theorem 1.6]): Let $W\in\mathcal{W}$.

(i) clW is contained in some member of $\mathcal{U}$,

(ii) If ε is a successor ordinal, then $(clW)^{(\varepsilon-1)}$ is compact,

(iii) If ε is a limit ordinal, then $(clW)^{(\alpha)}=\emptyset$ for some $\alpha<\varepsilon$.

Case 1. ε is a limit ordinal. By induction hypothesis, for each $W\in\mathcal{W}$, there is a point finite (point countable) open collection $\mathcal{V}'(W)$ in clW such that $\mathcal{V}'(W)$ covers clW and for each $V\in\mathcal{V}'(W)$, $(clV)^{(\alpha)}$ is compact for some $\alpha<\varepsilon$. Put $\mathcal{V}(W)=\mathcal{V}'(W)|W$ for each $W\in\mathcal{W}$ and $\mathcal{V}=\bigcup\{\mathcal{V}(W): W\in\mathcal{W}\}$. Then $\mathcal{V}$ satisfies the conditions (a) and (b).

Case 2. ε is a successor ordinal. Let $\mathcal{W}_0=\{W\in\mathcal{W}: \varepsilon(clW)=\varepsilon\}$, and $\mathcal{W}_1=\mathcal{W}-\mathcal{W}_0$. Take a $W\in\mathcal{W}_1$. Then $\varepsilon(clW)<\varepsilon$. By induction hypothesis, there is a point finite (point countable) open collection $\mathcal{V}'(W)$ in clW such that $\mathcal{V}'(W)$ covers clW and for each $V\in\mathcal{V}'(W)$, $(clV)^{(\alpha)}$ is compact for some $\alpha<\varepsilon$. Put Put $\mathcal{V}(W)=\mathcal{V}'(W)|W$ for each $W\in\mathcal{W}_1$. Take a $W\in\mathcal{W}_0$. Since $\varepsilon(clW)=\varepsilon$, $(clW)^{(\varepsilon-1)}$ is compact. Let $\mathcal{V}=\mathcal{W}_0\cup(\cup\{\mathcal{V}(W): W\in\mathcal{W}_1\})$. Then $\mathcal{V}$ satisfies the conditions (a) and (b).

The proof is completed.

4. Subparacompactness.

We firstly study subparacompactness of $Z\times\prod_{i\in\omega}Y_i$.

THEOREM 4.1. *If Z is a perfect subparacompact space and Y_i is a regular subparacompact $\mathcal{DC}$-like space for each $i\in\omega$,, then $Z\times\prod_{i\in\omega}Y_i$ is subparacompact.*

PROOF. Without loss of generality, we may assume that $Y_i=X$ for each $i\in\omega$ and there is an isolated point a in X. Indeed, put $X=\bigoplus_{i\in\omega}Y_i\cup\{a\}$, where $a\notin\bigcup_{i\in\omega}Y_i$. The topology of X is as follows: Every Y_i is an open-and-closed subspace of X and a is an isolated point in X. Since every Y_i is a regular subparacompact $\mathcal{DC}$-like space, by Lemma 2.2, X is a regular subparacompact $\mathcal{DC}$-like space. $Z\times\prod_{i\in\omega}Y_i$ is a closed subspace of $Z\times X^\omega$. Therefore, if $Z\times X^\omega$ is subparacompact, then $Z\times\prod_{i\in\omega}Y_i$ is subparacompact.

Let $\mathcal{O}$ be an open cover of $Z\times X^\omega$. Put $\mathcal{O}'=\{B\in\mathcal{B}: B\subset O$ for some $O\in\mathcal{O}^F\}$. For each $z\in Z$ and $K\in\mathcal{K}$, there is an $O\in\mathcal{O}^F$ such that $K_{(z,K)}\subset O$. Then, by Wallace theorem in R. Engelking [8], there is a $B\in\mathcal{B}$ such that $K_{(z,K)}\subset B\subset O$. Thus we have $B\in\mathcal{O}'$. Define $n(K_{(z,K)})=\inf\{n(O): O\in\mathcal{O}'$ and $K_{(z,K)}\subset O\}$.

Let s be a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$. Let $R=E_R\times\prod_{i\in\omega}R_i\in\mathcal{R}$ such that for each $i\leqq n(R)$, we have already obtained a compact set $C_{\lambda(R,i)}$ of R_i. ($C_{\lambda(R,n(R))}=\emptyset$. $C_{\lambda(R,i)}=\emptyset$ may be occur for $i<n(R)$.) Fix $i\leqq n(R)$. If $C_{\lambda(R,i)}\neq\emptyset$, let $F_{\gamma(R,i,m)}=R_i$ for each $m\in\omega$. Put $\Lambda(R, i)=\{\lambda(R, i)\}$ and $\Gamma(R, i, m)=\{\gamma(R, i, m)\}$ for each $m\in\omega$. Let $\mathcal{C}(R, i)=\{C_\lambda: \lambda\in\Lambda(R, i)\}=\{C_{\lambda(R,i)}\}$ and $\mathcal{F}(R, i, m)=\{F_\gamma: \gamma\in\Gamma(R, i, m)\}=\{F_{\gamma(R,i,m)}\}$ for each $m\in\omega$. Put $\mathcal{F}(R, i)=\bigcup_{m\in\omega}\mathcal{F}(R, i, m)$. Assume that $C_{\lambda(R,i)}=\emptyset$. Then there is a discrete collection $\mathcal{C}(R, i)=\{C_\lambda: \lambda\in\Lambda(R, i)\}$ of compact subsets of X such that $s(R_i)=\cup\mathcal{C}(R, i)$. Since R_i is a closed subspace of X, R_i is a subparacompact space.

Then there is a family $\mathcal{F}(R, i)=\bigcup_{m\in\omega}\mathcal{F}(R, i, m)$, where $\mathcal{F}(R, i, m)=\{F_\gamma:\gamma\in\Gamma(R, i, m)\}$, of collections of closed subsets in R_i (and hence, in X), satisfying

(1) $\mathcal{F}(R, i)$ covers R_i,

(2) Every member of $\mathcal{F}(R, i)$ meets at most one member of $\mathcal{C}(R, i)$,

(3) $\mathcal{F}(R, i, m)$ is discrete in X for each $m\in\omega$.

In each case, for $\gamma\in\bigcup_{m\in\omega}\Gamma(R, i, m)$, let $K_\gamma=F_\gamma\cap C_\lambda$ if $F_\gamma\cap C_\lambda\neq\emptyset$ for some (unique) C_λ. If $F_\gamma\cap(\cup\mathcal{C}(R, i))=\emptyset$, then we take a point $p_\gamma\in F_\gamma$ and let $K_\gamma=\{p_\gamma\}$. Thus, if $C_{\lambda(R,i)}\neq\emptyset$, then $K_{\gamma(R,i,m)}=F_{\gamma(R,i,m)}\cap C_{\lambda(R,i)}=C_{\lambda(R,i)}$ for each $m\in\omega$. For $\eta=(m_0, \cdots, m_{n(R)})\in\omega^{n(R)+1}$, let $\Delta_{R,\eta}=\Gamma(R, 0, m_0)\times\cdots\times\Gamma(R, n(R), m_{n(R)})$. For each $\eta\in\omega^{n(R)+1}$ and $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(R)))\in\Delta_{R,\eta}$, let $K(\delta)=K_{\gamma(\delta,0)}\times\cdots\times K_{\gamma(\delta,n(R))}\times\{a\}\times\cdots\times\{a\}\times\cdots$, and let $\mathcal{K}_{R,\eta}=\{K(\delta):\delta\in\Delta_{R,\eta}\}$. Then $\mathcal{K}_{R,\eta}\subset\mathcal{K}$. For each $z\in E_R$, $\eta\in\omega^{n(R)+1}$ and $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(R)))\in\Delta_{R,\eta}$, let $r(K_{(z,K(\delta))})=\max\{n(K_{(z,K(\delta))}), n(R)\}$. Fix $z\in E_R$, $\eta\in\omega^{n(R)+1}$ and $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(R)))\in\Delta_{R,\eta}$. Take an $O_{z,\delta}=U_{z,\delta}\times\prod_{i\in\omega}O_{z,\delta,i}\in\mathcal{O}'$ such that $K_{(z,K(\delta))}\subset O_{z,\delta}$ and $n(K_{(z,K(\delta))})=n(O_{z,\delta})$. Then we can take a subset $H_{(z,K(\delta))}=H_{z,\delta}\times\prod_{i\in\omega}H_{(z,K(\delta)),i}$ of $Z\times X^\omega$ such that

(4) $H_{z,\delta}$ is an open neighborhood of z in E_R such that $H_{z,\delta}\subset U_{z,\delta}$,

(5) $H_{z,\delta}\times\prod_{i=0}^{n(K(z,K(\delta)))-1}clH_{(z,K(\delta)),i}\times X\times\cdots\times X\times\cdots\subset O_{z,\delta}$,

(6-1) For each i with $n(K_{(z,K(\delta))})\leqq i\leqq r(K_{z,K(\delta)})$, let $H_{(z,K(\delta)),i}=F_{\gamma(\delta,i)}$,

(6-2) For each $i<n(K_{(z,K(\delta))})$ with $i\leqq n(R)$, $H_{(z,K(\delta)),i}$ be an open subset of $F_{\gamma(\delta,i)}$ such that $K_{\gamma(\delta,i)}\subset H_{(z,K(\delta)),i}\subset clH_{(z,K(\delta)),i}\subset O_{z,\delta,i}$,

(6-3) For each i with $n(R)<i<n(K_{(z,K(\delta))})$, let $H_{(z,K(\delta)),i}=\{a\}$,

(6-4) In case of that $r(K_{(z,K(\delta))})=n(R)$, let $H_{(z,K(\delta)),i}=X$ for $n(R)<i$. In case of that $r(K_{(z,K(\delta))})=n(K_{(z,K(\delta))})>n(R)$, let $H_{(z,K(\delta)),i}=X$ for $n(K_{(z,K(\delta))})\leqq i$.

Then we have $K_{(z,K(\delta))}\subset H_{(z,K(\delta))}$. For each $j\in\omega$, let $V_j(K(\delta))=\{z\in E_R: n(K_{(z,K(\delta))})=j\}$ and $\mathcal{H}_j(K(\delta))=\{H_{z,\delta}: n(K_{(z,K(\delta))})=j\}$. Fix $j\in\omega$. Then $\bigcup_{k=0}^{j}V_k(K(\delta))=\cup\{H_{z,\delta}: n(K_{(z,K(\delta))})\leqq j\}=\bigcup_{k=0}^{j}(\cup\mathcal{H}_k(K(\delta)))$. Since Z is a perfect space, $V_j(K(\delta))$ is an F_σ-set in E_R. Since E_R is subparacompact, there is a family $\mathcal{D}_{\eta,\delta,j}=\bigcup_{k\in\omega}\mathcal{D}_{\eta,\delta,j,k}$, where $\mathcal{D}_{\eta,\delta,j,k}=\{D_\xi:\xi\in\Xi_{\eta,\delta,j,k}\}$, of collections of closed subsets in E_R (and hence, in Z) satisfying

(7) Every member of $\mathcal{D}_{\eta,\delta,j}$ is contained in some member of $\mathcal{H}_j(K(\delta))|V_j(K(\delta))$,

(8) $\mathcal{D}_{\eta,\delta,j}$ covers $V_j(K(\delta))$,

(9) $\mathcal{D}_{\eta,\delta,j,k}$ is discrete in Z for each $k\in\omega$.

For $k\in\omega$ and $\xi\in\Xi_{\eta,\delta,j,k}$, take a $z(\xi)\in V_j(K(\delta))$ such that $D_\xi\subset H_{z(\xi),\delta}\cap V_j(K(\delta))$.

Put $F_\delta=\prod_{i=0}^{n(R)} F_{\gamma(\delta,i)}\times X\times\cdots X\times\cdots$ and $D_{\xi,\delta}=D_\xi\times F_\delta$. Then $\{D_{\xi,\delta}: \eta\in\omega^{n(R)+1}, \delta\in\Delta_{R,\eta}, j, k\in\omega$ and $\xi\in\Xi_{\eta,\delta,j,k}\}$ is a collection of elements of $\mathcal{R}$ such that for each $\eta\in\omega^{n(R)+1}$, $\delta\in\Delta_{R,\eta}$, j, $k\in\omega$ and $\xi\in\Xi_{\eta,\delta,j,k}$, $D_{\xi,\delta}\subset R$ and $\{D_{\xi,\delta}: \eta\in\omega^{n(R)+1}, \delta\in\Delta_{R,\eta}, j, k\in\omega$ and $\xi\in\Xi_{\eta,\delta,j,k}\}$ covers R.

(10) For each $\eta\in\omega^{n(R)+1}$ and j, $k\in\omega$, $\{D_{\xi,\delta}: \delta\in\Delta_{R,\eta}$ and $\xi\in\Xi_{\eta,\delta,j,k}\}$ is discrete in $Z\times X^\omega$.

Fix $\eta\in\omega^{n(R)+1}$ and j, $k\in\omega$. Let $(z, x)\in Z\times X^\omega$ and $x=(x_i)_{i\in\omega}$. For each $i\leqq n(R)$, since R_i is a closed subset of X, we may assume that $x_i\in R_i$. Then, for each $i\leqq n(R)$, there is an open neighborhood $B(x_i)$ of x_i in X such that $|\{\delta\in\Delta_{R,\eta}: \prod_{i=0}^{n(R)} B(x_i)\cap F_\delta(n(R))\neq\emptyset\}|\leqq 1$, where $F_\delta(n(R))=\prod_{i=0}^{n(R)} F_{\gamma(\delta,i)}$ for each $\delta\in\Delta_{R,\eta}$. Put $B'(x)=\prod_{i=0}^{n(R)} B(x_i)$ and $B(x)=B'(x)\times\prod_{i>n(R)} X_i$, where X_i is a copy of X for $i>n(R)$. If $B'(x)\cap F_\delta(n(R))=\emptyset$ for each $\delta\in\Delta_{R,\eta}$, then $Z\times B(x)\in\mathcal{B}$ and $(Z\times B(x))\cap D_{\xi,\delta}=\emptyset$ for each $\delta\in\Delta_{R,\eta}$ and $\xi\in\Xi_{\eta,\delta,j,k}$. Otherwise, take a unique $\delta\in\Delta_{R,\eta}$ such that $B'(x)\cap F_\delta(n(R))\neq\emptyset$. Since $\mathcal{D}_{\eta,\delta,j,k}$ is discrete in Z, there is an open neighborhood U of z in Z such that $|\{\xi\in\Xi_{\eta,\delta,j,k}: U\cap D_\xi\neq\emptyset\}|\leqq 1$. Then $U\times B(x)\in\mathcal{B}$ and $|\{D_{\xi,\delta'}: D_{\xi,\delta'}\cap(U\times B(x))\neq\emptyset, \delta'\in\Delta_{R,\eta}$ and $\xi\in\Xi_{\eta,\delta',j,k}\}|\leqq 1$. Thus $\{D_{\xi,\delta}: \delta\in\Delta_{R,\eta}$ and $\xi\in\Xi_{\eta,\delta,j,k}\}$ is discrete in $Z\times X^\omega$.

For each $\eta\in\omega^{n(R)+1}$, $\delta\in\Delta_{R,\eta}$, j, $k\in\omega$ and $\xi\in\Xi_{\eta,\delta,j,k}$, let $G_{\xi,\delta}=D_\xi\times\prod_{i\in\omega} clH_{(z(\xi),K(\delta)),i}\subset D_{\xi,\delta}$ and $\mathcal{G}_{\eta,\delta,j,k}(R)=\{G_{\xi,\delta}: \xi\in\Xi_{\eta,\delta,j,k}\}$. Define $\mathcal{G}_{\eta,j,k}(R)=\bigcup\{\mathcal{G}_{\eta,\delta,j,k}(R): \delta\in\Delta_{R,\eta}\}$ for each $\eta\in\omega^{n(R)+1}$ and j, $k\in\omega$. Then we have

(11) For each $\eta\in\omega^{n(R)+1}$, j, $k\in\omega$, every member of $\mathcal{G}_{\eta,j,k}(R)$ is contained in some member of $\mathcal{O}'$.

(12) For each $\eta\in\omega^{n(R)+1}$, j, $k\in\omega$, $\mathcal{G}_{\eta,j,k}(R)$ is discrete in $Z\times X^\omega$.

This is clear from (10).

(13) For each $\eta\in\omega^{n(R)+1}$, j, $k\in\omega$, every element of $\mathcal{G}_{\eta,j,k}$ has the length $\max\{j, n(R)+1\}$.

Fix $\eta\in\omega^{n(R)+1}$, $\delta=(\gamma(\delta), 0), \cdots, \gamma(\delta, n(R)))\in\Delta_{R,\eta}$, j, $k\in\omega$ and $\xi\in\Xi_{\eta,\delta,j,k}$. Then $n(K_{(z(\xi),K(\delta))})=j$ and hence, $r(K_{(z(\xi),K(\delta))})=\max\{j, n(R)\}$. Let $A\in\mathcal{P}(\{0, 1, \cdots, r(K_{(z(\xi),K(\delta))})\})$. In case of that $r(K_{(z(\xi),K(\delta))})=n(R)$, i.e., $n(R)\geqq j$. For each $i\in A$, let $R_{\xi,A,i}=F_{\gamma(\delta,i)}-H_{(z(\xi),K(\delta)),i}$. For each $i\notin A$ with $i\leqq n(R)$, let $R_{\xi,A,i}=clH_{(z(\xi),K(\delta)),i}$. For each $i>n(R)$, let $R_{\xi,A,i}=X$. Put $R_{\xi,A}=D_\xi\times\prod_{i\in\omega} R_{\xi,A,i}$. In case of that $j>n(R)$. For each $i\in A$ with $i\leqq n(R)$, let $R_{\xi,A,i}=F_{\gamma(\delta,i)}-H_{(z(\xi),K(\delta)),i}$. For each $i\notin A$ with $i\leqq n(R)$, let $R_{\xi,A,i}=clH_{(z(\xi),K(\delta)),i}$. Let $n(R)<i<j$. If $i\in A$, let $R_{\xi,A,i}=X-H_{(z(\xi),K(\delta)),i}=X-\{a\}$. If $i\notin A$, let $R_{\xi,A,i}=clH_{(z(\xi),K(\delta)),i}=\{a\}$. For $i\geqq j$, let $R_{\xi,A,i}=X$. Put $R_{\xi,A}=D_\xi\times\prod_{i\in\omega} R_{\xi,A,i}$. In each

case, $R_{\xi,A,i}\subset R_i$ for each $i\in\omega$. Notice that if $R_{\xi,A}\neq\emptyset$, then $n(R)<n(R_{\xi,A})$. By the definition, $D_{\xi,\delta}=G_{\xi,\delta}\cup(\cup\{R_{\xi,A}: A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(R)\}\})\})$. For each $A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(R)\}\})$, let $\mathcal{R}_{\eta,\delta,j,k,A}(R)=\{R_{\xi,A}: \xi\in\Xi_{\eta,\delta,j,k}$ and $R_{\xi,A}\neq\emptyset\}$. For $j,k\in\omega$ and $A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(R)\}\})$, define $\mathcal{R}_{\eta,j,k,A}(R)=\cup\{\mathcal{R}_{\eta,\delta,j,k,A}(R): \delta\in\Delta_{R,\eta}\}$. Then, by (10), we have

(14) Every $\mathcal{R}_{\eta,j,k,A}(R)$ is discrete in $Z\times X^{\omega}$.

Let $\mathcal{R}_{\eta,j,k}(R)=\cup\{\mathcal{R}_{\eta,j,k,A}(R): A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(R)\}\})\}$. Then, by (14),

(15) For each $\eta\in\omega^{n(R)+1}$, $j,k\in\omega$, $\mathcal{R}_{\eta,j,k}(R)$ is locally finite in $Z\times X^{\omega}$.

(16) For each $\eta\in\omega^{n(R)+1}$ and $j,k\in\omega$ with $\mathcal{R}_{\eta,j,k}\neq\emptyset$, every element of $\mathcal{R}_{\eta,j,k}$ has the length $\max\{j,n(R)+1\}$.

Fix a $R_{\xi,A}=D_{\xi}\times\prod_{i\in\omega}R_{\xi,A,i}\in\mathcal{R}_{\eta,\delta,j,k,A}(R)$ for $\eta\in\omega^{n(R)+1}$, $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_{R,\eta}$, $j,k\in\omega$, $\xi\in\Xi_{\eta,\delta,j,k}$ and $A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(R)\}\})$.

(17) For each $i\in A$ with $i\leqq n(R)$ such that $C_{\lambda(R,i)}=\emptyset$, $s(R_i)\cap R_{\xi,A,i}=\emptyset$.

Since $R_{\xi,A,i}=F_{\gamma(\delta,i)}-H_{(z(\xi),K(\delta)),i}$, $s(R_i)\cap R_{\xi,A,i}=(\cup C(R,i))\cap(F_{\gamma(\delta,i)}-H_{(z(\xi),K(\delta)),i})=K_{\gamma(\delta,i)}-H_{(z(\xi),K(\delta)),i}=\emptyset$.

For each $i\notin A$ with $i\leqq n(R)$, a compact set $K_{\gamma(\delta,i)}$ is contained in $R_{\xi,A,i}=clH_{(z(\xi),K(\delta)),i}$. Let $C_{\lambda(R_{\xi,A},i)}=K_{\gamma(\delta,i)}$. For each $i\notin A$ with $n(R)<i<j$, let $C_{\lambda(R_{\xi,A},i)}=\{a\}$. For each $i\in A$, let $C_{\lambda(R_{\xi,A},i)}=\emptyset$.

For $t\in\omega$, we shall inductively construct an index set Φ_t and two collections $\mathcal{G}_{\tau}$ and $\mathcal{R}_{\tau}$ for each $\tau\in\Phi_t$ satisfying

(18) For $t\geqq 1$ and $\tau\in\Phi_t$, $\tau_-\in\Phi_{t-1}$,

(19) For $t\in\omega$ and $\tau\in\Phi_t$, $\mathcal{G}_{\tau}$ and $\mathcal{R}_{\tau}$ are collections of elements of $\mathcal{R}$,

(20) For $t\in\omega$ and $\tau\in\Phi_t$ with $\mathcal{R}_{\tau}\neq\emptyset$, elements of $\mathcal{R}_{\tau}$ have the same length.

Let $\Phi_0=\omega^3$. For each $\tau=(m,j,k)\in\Phi_0$, let $\mathcal{G}_{\tau}=\mathcal{G}_{\tau}(Z\times X^{\omega})=\mathcal{G}_{m,j,k}(Z\times X^{\omega})$ and $\mathcal{R}_{\tau}=\mathcal{R}_{\tau}(Z\times X^{\omega})=\mathcal{R}_{m,j,k}(Z\times X^{\omega})$. Let $\tau=(m,j,k)\in\Phi_0$. By the construction, $\mathcal{G}_{\tau}$ and $\mathcal{R}_{\tau}$ are collections of elements of $\mathcal{R}$. Assume that $\mathcal{R}_{\tau}\neq\emptyset$. By (16), elements of $\mathcal{R}_{\tau}$ have the same length. Thus $\mathcal{G}_{\tau}$ and $\mathcal{R}_{\tau}$, $\tau\in\Phi_0$, satisfy the conditions (19) and (20). Assume that for $t\in\omega$, we have already obtained an index set Φ_i, for $i\leqq t$, and families $\left\{\mathcal{G}_{\tau}: \tau\in\bigcup_{i=0}^{t}\Phi_i\right\}$, $\left\{\mathcal{R}_{\tau}: \tau\in\bigcup_{i=0}^{t}\Phi_i\right\}$ satisfying the conditions (18), (19) and (20). Take a $\tau\in\Phi_t$ with $\mathcal{R}_{\tau}\neq\emptyset$. By (20), elements of $\mathcal{R}_{\tau}$ have the same length. So we denote this length by $n(\tau)$. Let $\Phi_{\tau}=\{\tau\oplus(\eta,j,k): \eta\in\omega^{n(\tau)+1}, j,k\in\omega\}$. For each $R\in\mathcal{R}_{\tau}$ and $\eta\in\omega^{n(\tau)+1}$, $j,k\in\omega$, we denote $\mathcal{G}_{\eta,j,k}(R)$ and $\mathcal{R}_{\eta,j,k}(R)$ by $\mathcal{G}_{\tau\oplus(\eta,j,k)}(R)$ and $\mathcal{R}_{\tau\oplus(\eta,j,k)}(R)$ respectively. Define $\mathcal{G}_{\tau\oplus(\eta,j,k)}=\cup\{\mathcal{G}_{\tau\oplus(\eta,j,k)}(R): R\in\mathcal{R}_{\tau}\}$ and $\mathcal{R}_{\tau\oplus(\eta,j,k)}=\cup\{R_{\tau\oplus(\eta,j,k)}(R): R\in\mathcal{R}_{\tau}\}$. Let $\Phi_{t+1}=\cup\{\Phi_{\tau}: \tau\in\Phi_t$ and $\mathcal{R}_{\tau}\neq\emptyset\}$. Then, by (16) and the construction, Φ_{t+1}, families $\{\mathcal{G}_{\mu}: \mu\in\Phi_{t+1}\}$ and $\{\mathcal{R}_{\mu}: \mu\in\Phi_{t+1}\}$ satisfy the conditions (18),

(19) and (20). Thus, for each $t\in\omega$, we heve an index set Φ_t, families $\{\mathcal{G}_\tau: \tau\in\Phi_t\}$ and $\{\mathcal{R}_\tau: \tau\in\Phi_t\}$ satisfying the conditions (18), (19) and (20). Let $\Phi=\cup\{\Phi_t: t\in\omega\}$. Then $|\Phi|\leqq\omega$.

By Lemmas 2.4 and 3.2, our proof is complete if we show

CLAIM. *$\cup\{\mathcal{G}_\tau: \tau\in\Phi\}$ is a σ-locally finite closed refinement of $\mathcal{O}'$.*

PROOF OF CLAIM. Let $\tau\in\Phi$. By (19), $\mathcal{G}_\tau\subset\mathcal{R}$. By (11), every member of $\mathcal{G}_\tau$ is contained in some member of $\mathcal{O}'$. By (12), (15) and induction, $\mathcal{G}_\tau$ is locally finite in $Z\times X^\omega$. Assume that $\cup\{\mathcal{G}_\tau: \tau\in\Phi\}$ does not cover $Z\times X^\omega$. Take a point $(z, x)\in Z\times X^\omega-\cup\{\cup\mathcal{G}_\tau: \tau\in\Phi\}$. Let $x=(x_i)_{i\in\omega}$. Take an $\eta(0)=m(0)\in\omega$ and $\delta(0)=\gamma(\delta(0), 0)\in\Delta_{Z\times X^\omega,\eta(0)}=\Gamma(Z\times X^\omega, 0, m(0))$ such that $x\in F_{\delta(0)}$. Put $\mathcal{F}(0)=\{F_{\gamma(\delta(0),0)}\}$. Let $K(0)=K(\delta(0))\in\mathcal{K}_{Z\times X^\omega,\eta(0)}$ and let $j(0)=n(K_{(z,K(0))})$. Choose a $k(0)\in\omega$ such that $(z, x)\in\cup\mathcal{G}_{\eta(0),j(0),k(0)}(Z\times X^\omega)\cup(\cup\mathcal{R}_{\eta(0),j(0),k(0)}(Z\times X^\omega))$. Let $\tau(0)=(\eta(0), j(0), k(0))\in\Phi_0$. Take a $\xi(0)\in\Xi_{\eta(0),\delta(0),j(0),k(0)}$ such that $z\in D_{\xi(0)}$. Put $\mathcal{H}(0)=\{H_{(z(\xi(0)),K(0)),i}: i\leqq j(0)\}$. Since $(z, x)\notin\cup\mathcal{G}_{\tau(0)}$, there is an $A(0)\in\mathcal{P}(\{0, 1, \cdots, j(0)\})$ such that $(z, x)\in R_{\xi(0),A(0)}$, $R_{\xi(0),A(0)}\in\mathcal{R}_{\tau(0)}(Z\times X^\omega)$. By the definition, if $0\in A(0)$, then $R_{\xi(0),A(0),0}=F_{\gamma(\delta(0),0)}-H_{(z(\xi(0)),K(0)),0}$. We have $0=n(Z\times X^\omega)<n(R_{\xi(0),A(0)})$. For $R_{\xi(0),A(0)}$, take $\eta(1)\in\omega^{n(R_{\xi(0),A(0)})+1}$, $\delta(1)=(\gamma(\delta(1)), 0), \cdots, \gamma(\delta(1), n(R_{\xi(0),A(0)})))\in\Delta_{R_{\xi(0),A(0)},\eta(1)}$ such that $x\in F_{\delta(1)}$. Put $\mathcal{F}(1)=\{F_{\gamma(\delta(1),i)}: i\leqq n(R_{\xi(0),A(0)})\}$. Let $K(1)=K(\delta(1))\in\mathcal{K}_{R_{\xi(0),A(0)},\eta(1)}$ and $j(1)=n(K_{(z,K(1))})$. Take a $k(1)\in\omega$ such that $(z, x)\in\cup\mathcal{G}_{\eta(1),j(1),k(1)}(R_{\xi(0),A(0)})\cup(\cup\mathcal{R}_{\eta(1),j(1),k(1)}(R_{\xi(0),A(0)}))$. Let $\tau(1)=((\eta(0), j(0), k(0)), (\eta(1), j(1), k(1)))\in\Phi_1$. Take a $\xi(1)\in\Xi_{\eta(1),\delta(1),j(1),k(1)}$ such that $z\in D_{\xi(1)}$. Put $\mathcal{H}(1)=\{H_{(z(\xi(1)),K(1)),i}: i\leqq\max\{j(1), n(R_{\xi(0),A(0)})\}\}$. Since $(z, x)\notin\cup\mathcal{G}_{\tau(1)}$, there is an $A(1)\in\mathcal{P}(\{0, 1, \cdots, \max\{j(1), n(R_{\xi(0),A(0)})\}\})$ such that $(z, x)\in R_{\xi(1),A(1)}$, $R_{\xi(1),A(1)}\in\mathcal{R}_{\tau(1)}(R_{\xi(0),A(0)})$. Then, if $i\in A(1)$ with $i\leqq n(R_{\xi(0),A(0)})$, then $R_{\xi(1),A(1),i}=F_{\gamma(\delta(1),i)}-H_{(z(\xi(1)),K(1)),i}$. We have $n(R_{\xi(0),A(0)})<n(R_{\xi(1),A(1)})$. Continuing this matter, we can choose a sequence $\{\eta(t): t\in\omega\}$ of elements of $\omega^{<\omega}$, a sequence $\{\delta(t): t\in\omega\}$, a sequence $\{\mathcal{F}(t): t\in\omega\}$ of collections, a sequence $\{K(t): t\in\omega\}$ of compact subsets in X^ω, where $K(t)=\prod_{i\in\omega}K(t)_i\in\mathcal{K}$, sequences $\{j(t): t\in\omega\}$, $\{k(t): t\in\omega\}$ of natural numbers, a sequence $\{\tau(t): t\in\omega\}$ of elements of Φ, where $\tau(t)=((\eta(0), j(0), k(0)), \cdots, (\eta(t), j(t), k(t)))$, a sequence $\{\xi(t): t\in\omega\}$, a sequence $\{\mathcal{H}(t): t\in\omega\}$ of collections, a sequence $\{A(t): t\in\omega\}$ of finite subsets of ω, a sequence $\{R_{\xi(t),A(t)}: t\in\omega\}$ of elements of $\mathcal{R}$ containing (z, x), where $R_{\xi(t),A(t)}=D_{\xi(t)}\times\prod_{i\in\omega}R_{\xi(t),A(t),i}$, satisfying the following: Let $t\in\omega$. Assume that we have already obtained sequences $\{\eta(i): i\leqq t\}$, $\{\delta(i): i\leqq t\}$, $\{\mathcal{F}(i): i\leqq t\}$, $\{K(i): i\leqq t\}$, $\{j(i): i\leqq t\}$, $\{k(i): i\leqq t\}$, $\{\tau(i): i\leqq t\}$ $\{\xi(i): i\leqq t\}$, $\{\mathcal{H}(i): i\leqq t\}$, $\{A(i): i\leqq t\}$ and $\{R_{\xi(i),A(i)}: i\leqq t\}$. Then

(21) $\eta(t+1)\in\omega^{n(R_{\xi(t),A(t)})+1}$,

(22) $\delta(t+1)=(\gamma(\delta(t+1), 0), \cdots, \gamma(\delta(t+1), n(R_{\xi(t),A(t)})))\in\Delta_{R_{\xi(t),A(t)},\eta(t+1)}$ such that $x\in F_{\delta(t+1)}$, and $\mathcal{F}(t+1)=\{F_{\gamma(\delta(t+1),i)}: i\leqq n(R_{\xi(t),A(t)})\}$,

(23) $K(t+1)=K(\delta(t+1))\in\mathcal{K}_{R_{\xi(t),A(t)},\eta(t+1)}$,

(24) $j(t+1)=n(K_{(z,K(t+1))})$, $k(t+1)\in\omega$ and $\tau(t+1)=((\eta(0), j(0), k(0)), \cdots, (\eta(t+1), j(t+1), k(t+1)))\in\Phi_{t+1}$,

(25) $\xi(t+1)\in\Xi_{\xi(t+1),\delta(t+1),j(t+1),k(t+1)}$, $\mathcal{H}(t+1)=\{H_{(z(\xi(t+1)),K(t+1)),i}: i\leqq \max\{j(t+1), n(R_{\xi(t),A(t)})\}\}$ and $A(t+1)\in\mathcal{P}(\{0, 1, \cdots, \max\{j(t+1), n(R_{\xi(t),A(t)})\}\})$,

(26) If $i\in A(t+1)$ with $i\leqq n(R_{\xi(t),A(t)})$, then $R_{\xi(t+1),A(t+1),i}=F_{\gamma(\delta(t+1),i)}-H_{(z(\xi(t+1)),K(t+1)),i}$,

(27) $(z, x)\in R_{\xi(t+1),A(t+1)}=D_{\xi(t+1)}\times\prod_{i\in\omega}R_{\xi(t+1),A(t+1),i}$, $R_{\xi(t+1),A(t+1)}\in\mathcal{R}_{\tau(t+1)}(R_{\xi(t),A(t)})$, and $n(R_{\xi(t),A(t)})<nR_{\xi(t+1),A(t+1)})$,

(28) For each $i\leqq n(R_{\xi(t),A(t)})$ with $i\in A(t+1)$ such that $C_{\lambda(R_{\xi(t),A(t)},i)}=\emptyset$, $s(R_{\xi(t),A(t),i})\cap R_{\xi(t+1),A(t+1),i}=\emptyset$,

(29) For each $i\leqq n(R_{\xi(t),A(t)})$ with $i\notin A(t+1)$ such that $C_{\lambda(R_{\xi(t),A(t)},i)}\neq\emptyset$, $K(t+1)_i=C_{\lambda(R_{\xi(t),A(t)},i)}$.

The rest of the proof is similar to that of Theorem 3.2 in the author [17]. However we include it here, because the method of it plays the fundamental role in this paper.

Assume that for each $i\in\omega$, $|\{t\in\omega: i\in A(t)\}|<\omega$. Then for each $i\in\omega$, there is a $t_i\in\omega$ such that $i\leqq t_i$ and if $t\geqq t_i$, then $i\notin A(t)$. Then, by (29),

(30) For each $i\in\omega$ and $t\geqq t_i$, $K(t)_i=K(t_i)_i$.

Let $K=\prod_{i\in\omega}K(t_i)_i\in\mathcal{K}$. There is an $O\in\mathcal{O}'$ such that $K_{(z,K)}\subset O$. By (27) and (30), take a $t\geqq 1$ such that $n(O)\leqq n(R_{\xi(t-1),A(t-1)})$ and if $i\leqq n(O)$, then $K(t)_i=K(t_i)_i$. Then we have $K_{(z,K(t))}\subset O$ and hence, $j(t)=n(K_{(z,K(t))})\leqq n(O)$. Since $\xi(t)\in\Xi_{\eta(t),\delta(t),j(t),k(t)}$, $n(K_{(z(\xi(t)),K(t))})=j(t)$. For i with $n(O)\leqq i\leqq n(R_{\xi(t-1),A(t-1)})$, by the definition, $H_{(z(\xi(t)),K(t)),i}=F_{\gamma(\delta(t),i)}$. Hence $A(t)\cap\{n(O), \cdots, n(R_{\xi(t-1),A(t-1)})\}=\emptyset$. Since $(z, x)\in R_{\xi(t),A(t)}$ and $R_{\xi(t),A(t)}\in\mathcal{R}_{\tau(t)}(R_{\xi(t-1),A(t-1)})$, there is an $i\in A(t)$ such that $x_i\notin H_{(z(\xi(t)),K(t)),i}$. Thus $i<n(O)$ and $x_i\in R_{\xi(t),A(t),i}=F_{\gamma(\delta(t),i)}-H_{(z(\xi(t)),K(t)),i}$. Since $i\in A(t)$, $t<t_i$. For each $t'>t$, $K(t')_i\subset R_{\xi(t),A(t),i}$. Thus $K(t_i)_i\subset R_{\xi(t),A(t),i}$. Since $K(t)_i\subset H_{(z(\xi(t)),K(t)),i}$, we have $K(t)_i\neq K(t_i)_i$. This is a contradiction. Therefore there is an $i\in\omega$ such that $|\{t\in\omega: i\in A(t)\}|=\omega$. Let $\{t\in\omega: i\in A(t)$ and $i\leqq n(R_{\xi(t),A(t)})\}=\{t_\rho: \rho\in\omega\}$. Let $\rho\in\omega$. Since $C_{\lambda(R_{\xi(t_\rho),A(t_\rho)},i)}=\emptyset$, if $t_{\rho+1}=t_\rho+1$, then, by (28), $s(R_{\xi(t_\rho),A(t_\rho),i})\cap R_{\xi(t_{\rho+1}),A(t_{\rho+1}),i}=\emptyset$. Assume that $t_{\rho+1}>t_\rho+1$. Since $K_{\gamma(\delta(t_\rho+1),i)}=C_{\lambda(R_{\xi(t_\rho+1),A(t_\rho+1)},i)}=C_{\lambda(R_{\xi(t_{\rho+1}-1),A(t_{\rho+1}-1)},i)}\subset H_{(z(\xi(t_{\rho+1})),K(t_{\rho+1})),i}$, by the definition, we have $s(R_{\xi(t_\rho),A(t_\rho),i})\cap R_{\xi(t_{\rho+1}),A(t_{\rho+1}),i}=\emptyset$. Since s is a stationary winning strategy for Player I in $G(\mathscr{DC}, X)$, $\bigcap_{\rho\in\omega} R_{\xi(t_\rho), A(t_\rho), i}=\emptyset$. But $x_i\in\bigcap_{\rho\in\omega} R_{\xi(t_\rho), A(t_\rho), i}$, which is a contradiction. It follows that $\cup\{\mathscr{G}_\tau : \tau\in\Phi\}$ is a cover of $Z\times X^\omega$. The proof is completed.

COROLLARY 4.2. *If Z is a perfect subparacompact space and Y_i is a regular subparacompact space with a σ-closure-preserving cover by compact sets for each $i\in\omega$, then $Z\times\prod_{i\in\omega} Y_i$ is subparacompact.*

PROOF. This immediately follows from Theorem 4.1 and Lemma 2.3 (a).

Similarly, by Theorem 4.1 and Lemma 2.3 (b), we have

COROLLARY 4.3. *If Z is a perfect subparacompact space and Y_i is a regular subparacompact, $\sigma-C$-scattered space for each $i\in\omega$, then $Z\times\prod_{i\in\omega} Y_i$ is subparacompact.*

REMARK 4.4. Let M be the Michael line and let $\boldsymbol{P}$ be the space of irrationals. $\boldsymbol{P}$ is homeomorphic to ω^ω. The following are well-known (see D. K. Burke [4]).

(a) M is hereditarily paracompact but $M\times\boldsymbol{P}$ is not normal and hence, not paracompact.

(b) $M\times\boldsymbol{P}$ is hereditarily subparacompact and hereditarily metacompact (see also P. Nyikos [15]).

5. Metacompactness, orthocompactness and metalindelöf property.

THEOREM 5.1. *If Y_i is a regular metacompact $\mathscr{DC}$-like space for each $i\in\omega$, then $\prod_{i\in\omega} Y_i$ is metacompact.*

PROOF. We may assume that $Y_i=X$ for each $i\in\omega$ and there is an isolated point a in X. Let $\mathcal{O}$ be an open cover of $Z\times X^\omega$. Similarly, let $\mathcal{O}'=\{B\in\mathscr{B} : B\subset O$ for some $O\in\mathcal{O}^F\}$. For $K\in\mathscr{K}$, there is an $O\in\mathcal{O}^F$ such that $K\subset O$. Then there is a $B\in\mathscr{B}$ such that $K\subset B\subset O$. Define $n(K)=\inf\{n(O) : O\in\mathcal{O}'$ and $K\subset O\}$. It suffices to prove that $\mathcal{O}'$ has a point finite open refinement.

Let s be a stationary winning strategy for Player I in $G(\mathscr{DC}, X)$. Let $B=\prod_{i\in\omega} B_i\in\mathscr{B}$ such that for each $i\leqq n(B)$, we have already obtained a compact set $C_{\lambda(B,i)}$ of clB_i. ($C_{\lambda(B,n(B))}=\emptyset$. $C_{\lambda(B,i)}=\emptyset$ may be occur for $i<n(B)$.) We define $\mathscr{G}(B)$ and $\mathscr{B}(B)$ of collections of elements of $\mathscr{B}$. Fix $i\leqq n(B)$. If $C_{\lambda(B,i)}$

$\neq\emptyset$, let $W_{\gamma(B,i)}=B_i$. Put $\Lambda(B, i)=\{\lambda(B, i)\}$ and $\Gamma(B, i)=\{\gamma(B, i)\}$. Let $\mathcal{C}(B, i)=\{C_\lambda : \lambda\in\Lambda(B, i)\}=\{C_{\lambda(B,i)}\}$, and $\mathcal{W}(B, i)=\{W_\gamma : \gamma\in\Gamma(B, i)\}=\{W_{\gamma(B,i)}\}$. Assume that $C_{\lambda(B,i)}=\emptyset$. Then there is a discrete collection $\mathcal{C}(B, i)=\{C_\lambda : \lambda\in\Lambda(B, i)\}$ of compact subsets of X such that $s(clB_i)=\cup\mathcal{C}(B, i)$. Since X is a regular metacompact space, there is a collection $\mathcal{W}(B, i)=\{W_\gamma : \gamma\in\Gamma(B, i)\}$ of open subsets in B_i (and hence, in X) satisfying

(1) $\mathcal{W}(B, i)$ covers B_i,

(2) For each $\gamma\in\Gamma(B, i)$, clW_γ meets at most one member of $\mathcal{C}(B, i)$,

(3) $\mathcal{W}(B, i)$ is point finite in B_i and hence, point finite in X.

In each case, for $\gamma\in\Gamma(B, i)$, $K_\gamma=clW_\gamma\cap C_\lambda$ if $clW_\gamma\cap C_\lambda\neq\emptyset$ for some (unique) C_λ. If $clW_\gamma\cap(\cup\mathcal{C}(B, i))=\emptyset$, then we take a point $p\in W_\gamma$ and let $K_\gamma=\{p_\gamma\}$. Thus, if $C_{\lambda(B,i)}\neq\emptyset$, then $K_{\gamma(B,i)}=clW_{\gamma(B,i)}\cap C_{\lambda(B,i)}=C_{\lambda(B,i)}$. Put $\Delta_B=\Gamma(B, 0)\times\cdots\times\Gamma(B, n(B))$. For each $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$, let $K(\delta)=K_{\gamma(\delta,0)}\times\cdots\times K_{\gamma(\delta,n(B))}\times\{a\}\times\cdots\times\{a\}\times\cdots$, and let $\mathcal{K}_B=\{K(\delta) : \delta\in\Delta_B\}$. Then $\mathcal{K}_B\subset\mathcal{K}$. For each $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$, let $r(K(\delta))=\max\{(n(K(\delta)), n(B)\}$. Fix a $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$. Take an $O(\delta)=\prod_{i\in\omega}O(\delta)_i\in\mathcal{O}'$ such that $K(\delta)\subset O(\delta)$ and $n(K(\delta))=n(O(\delta))$. Since X is a regular space, there is an $H(\delta)=\prod_{i\in\omega}H(\delta)_i\in\mathcal{B}$ such that:

(4) $\prod_{i=0}^{n(K(\delta))-1} clH(\delta)_i\times X\times\cdots\times X\times\cdots\subset O(\delta)$,

(5-1) For each i with $n(K(\delta))\leqq i\leqq r(K(\delta))$, let $H(\delta)_i=X$,

(5-2) For each $i<n(K(\delta))$ with $i\leqq n(B)$, let $H(\delta)_i$ be an open subset of X such that $K_{\gamma(\delta,i)}\subset H(\delta)_i\subset clH(\delta)_i\subset O(\delta)_i$,

(5-3) For each i with $n(B)<i<n(K(\delta))$, let $H(\delta)_i=\{a\}$,

(5-4) In case of that $r(K(\delta))=n(B)$, let $H(\delta)_i=X$ for $n(B)<i$. In case of that $r(K(\delta))=n(K(\delta))>n(B)$, let $H(\delta)_i=X$ for $n(K(\delta))\leqq i$.

Then we have $K(\delta)(\subset H\delta)$. Put $W(\delta)=\prod_{i=0}^{n(B)}W_{\gamma(\delta,i)}\times X\times\cdots\times X\times\cdots$. Then $\{W(\delta) : \delta\in\Delta_B\}$ is a collection of elements of $\mathcal{B}$ such that for each $\delta\in\Delta_B$, $W(\delta)\subset B$ and $\{W(\delta) : \delta\in\Delta_B\}$ covers B. By the definition, we have

(6) $\{W(\delta) : \delta\in\Delta_B\}$ is point finite in X^ω.

Fix a $\delta=(\gamma(\delta, , 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$. In case of that $r(K(\delta))=n(B)$. For each $i\leqq n(B)$, let $G(\delta)_i=O(\delta)_i\cap W_{\gamma(\delta,i)}$. For each $i>n(B)$, let $G(\delta)_i=X$. Put $G(\delta)=\prod_{i\in\omega}G(\delta)_i$. In case of that $r(K(\delta))=n(K(\delta))>n(B)$. For each $i\leqq n(B)$, let $G(\delta)_i=O(\delta)_i\cap W_{\gamma(\delta,i)}$. For each i with $n(B)<i<n(K(\delta))$, let $G(\delta)_i=H(\delta)_i=\{a\}$. For each $i>n(K(\delta))$, let $G(\delta)_i=X$. Put $G(\delta)=\prod_{i\in\omega}G(\delta)_i$. Then we have $G(\delta)\subset W(\delta)$. Define $\mathcal{G}(B)=\{G(\delta) : \delta\in\Delta_B\}$. Then

(7) Every member of $\mathcal{G}(B)$ is contained in some member of $\mathcal{O}'$.

(8) $\mathcal{G}(B)$ is point finite in X^{ω}.

This is clear from (6).

Fix $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$. Let $A\in\mathcal{P}(\{0, 1, \cdots, r(K(\delta))\})$. In case of that $r(K(\delta))=n(B)$. For each $i\in A$, let $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$. For each $i\notin A$ with $i\leqq n(B)$, let $B_{\delta,A,i}=O(\delta)_i\cap W_{\gamma(\delta,i)}$. For each $i>n(B)$, let $B_{\delta,A,i}=X$. Put $B_{\delta,A}=\prod_{i\in\omega}B_{\delta,A,i}$. In case of that $r(K(\delta))=n(K(\delta))>n(B)$. For each $i\in A$ with $i\leqq n(B)$, let $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$. For each $i\notin A$ with $i\leqq n(B)$, let $B_{\delta,A,i}=O(\delta)_i\cap W_{\gamma(\delta,i)}$. Let $n(B)<i<n(K(\delta))$. If $i\in A$, let $B_{\delta,A,i}=X-clH(\delta)_i=X-\{a\}$. If $i\notin A$, let $B_{\delta,A,i}=H(\delta)_i=\{a\}$. For $i\geqq n(K(\delta))$, let $B_{\delta,A,i}=X$. Put $B_{\delta,A}=\prod_{i\in\omega}B_{\delta,A,i}$. In each case, $B_{\delta,A,i}\subset B_i$ for each $i\in\omega$. We have that if $B_{\delta,A}\neq\emptyset$, then $n(B)<n(B_{\delta,A})$. Let $\mathcal{B}_\delta(B)=\{B_{\delta,A}: A\in\mathcal{P}(\{0, 1, \cdots, r(K(\delta))\})$ and $B_{\delta,A}\neq\emptyset\}$. By the definition, $W(\delta)=G(\delta)\cup(\cup\mathcal{B}_\delta(B))$. Define $\mathcal{B}(B)=\cup\{\mathcal{B}_\delta(B): \delta\in\Delta_B\}$. Then, by (6), we have

(9) $\mathcal{B}(B)$ is point finite in X^{ω}.

Fix a $B_{\delta,A}=\prod_{i\in\omega}B_{\delta,A,i}\in\mathcal{B}_\delta(B)$ for $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$ and $A\in\mathcal{P}(\{0, 1, \cdots, r(K(\delta))\})$.

(10) For each $i\in A$ with $i\leqq n(B)$ such that $C_{\lambda(B,i)}=\emptyset$, $s(clB_i)\cap clB_{\delta,A,i}=\emptyset$.

Since $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$, $s(clB_i)\cap clB_{\delta,A,i}\subset(\cup\mathcal{C}(B, i))\cap(clW_{\gamma(\delta,i)}-H(\delta)_i)=K_{\gamma(\delta,i)}-H(\delta)_i=\emptyset$.

For each $i\notin A$ with $i\leqq n(B)$, since $clB_{\delta,A,i}=cl(O(\delta)_i\cap W_{\gamma(\delta,i)})\supset O(\delta)_i\cap clW_{\gamma(\delta,i)}$, a compact set $K_{\gamma(\delta,i)}$ is contained in $clB_{\delta,A,i}$. Let $C_{\lambda(B_{\delta,A},i)}=K_{\gamma(\delta,i)}$. For each $i\notin A$ with $n(B)<i<n(K(\delta))$, let $C_{\lambda(B_{\delta,A},i)}=\{a\}$. For each $i\in A$, let $C_{\lambda(B_{\delta,A},i)}=\emptyset$.

Now we define $\mathcal{G}_j$ and $\mathcal{B}_j$ for each $j\in\omega$. Let $\mathcal{G}_0=\mathcal{G}_0(X^{\omega})=\mathcal{G}(X^{\omega})$ and $\mathcal{B}_0=\mathcal{B}_0(X^{\omega})=\mathcal{B}(X^{\omega})$. Assume that for $j\in\omega$, we have already obtained $\mathcal{G}_j$ and $\mathcal{B}_j$. For each $B\in\mathcal{B}_j$, we denote $\mathcal{G}(B)$ and $\mathcal{B}(B)$ by $\mathcal{G}_{j+1}(B)$ and $\mathcal{B}_{j+1}(B)$ respectively. Define $\mathcal{G}_{j+1}=\cup\{\mathcal{G}_{j+1}(B): B\in\mathcal{B}_j\}$ and $\mathcal{B}_{j+1}=\cup\{\mathcal{B}_{j+1}(B): B\in\mathcal{B}_j\}$.

Our proof is complete if we show

CLAIM. *$\cup\{\mathcal{G}_j: j\in\omega\}$ is a point finite open refinement of $\mathcal{O}'$.*

PROOF OF CLAIM. Let $j\in\omega$. By the construction, $\mathcal{G}_j\subset\mathcal{B}$. By (7), every member of $\mathcal{G}_j$ is contained in some member of $\mathcal{O}'$. By (8), (9) and induction, $\mathcal{G}_j$ is point finite in X^{ω}. Take a $x=(x_i)_{i\in\omega}\in X^{\omega}$. Let $\Delta(0)=\{\delta\in\Delta_{X^{\omega}}: x\in W(\delta)\}$. Then, by (6), $1\leqq|\Delta(0)|<\omega$. Let $\mathcal{K}(0)=\{K(\delta): \delta\in\Delta(0)\}$. Put $\mathcal{H}(0)=\{H(\delta): \delta\in\Delta(0)\}$, $\mathcal{W}(0)=\{W(\delta): \delta\in\Delta(0)\}$ and $\mathcal{G}(0)=\{G(\delta): \delta\in\Delta(0)\}\subset\mathcal{G}_0$. For each $\delta\in\Delta(0)$, let $\mathcal{A}(\delta)=\mathcal{P}(\{0, 1, \cdots, r(K(\delta))\})$, and let $\mathcal{A}(0)=\cup\{\mathcal{A}(\delta): \delta\in\Delta(0)\}$. Let $\mathcal{B}(0)=\cup\{\mathcal{B}_\delta(X^{\omega}): \delta\in\Delta(0)\}$. Then $\mathcal{B}(0)\subset\mathcal{B}_0$. By the definition, for each $\delta=\gamma(\delta, 0)\in$

$\Delta(0)$ and $A\in\mathcal{A}(\delta)$ with $0\in A$, $B_{\delta,A,0}=W_{\gamma(\delta,0)}-clH(\delta)_0$. Since $W(\delta)=G(\delta)\cup(\cup\mathcal{B}_\delta(X^\omega))$ for each $\delta\in\Delta(0)$, $1\leqq|\mathcal{G}(0)\cup\mathcal{B}(0)|<\omega$. Observe that $(\mathcal{G}_0\cup\mathcal{B}_0)_x\subset\mathcal{G}(0)\cup\mathcal{B}(0)$. Take a $B\in\mathcal{B}(0)$. Let $\Delta(B)=\{\delta'\in\Delta_B: x\in W(\delta')\}$ and let $\Delta(1)=\cup\{\Delta(B): B\in\mathcal{B}(0)\}$. Let $\mathcal{K}(1)=\{K(\delta): \delta\in\Delta(1)\}$. Put $\mathcal{H}(1)=\{H(\delta): \delta\in\Delta(1)\}$, $\mathcal{W}(1)=\{W(\delta): \delta\in\Delta(1)\}$ and $\mathcal{G}(1)=\{G(\delta): \delta\in\Delta(1)\}\subset\mathcal{G}_1$. Define $\mathcal{A}(\delta)$ for each $\delta\in\Delta(1)$, and $\mathcal{A}(1)$ as before. Let $\mathcal{B}(1)=\cup\{\mathcal{B}_\delta(B): B\in\mathcal{B}(0)$ and $\delta\in\Delta(B)\}\subset\mathcal{B}_1$. Let $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta(B)$ and $B\in\mathcal{B}(0)$. For each $A\in\mathcal{A}(\delta)$, if $i\in A$ with $i\leqq n(B)$, then $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$. We have $|\mathcal{G}(1)\cup\mathcal{B}(1)|<\omega$ and $(\mathcal{G}_1\cup\mathcal{B}_1)_x\subset\mathcal{G}(1)\cup\mathcal{B}(1)$. Continuing this matter, we can choose a collection $\{\Delta(j): j\in\omega\}$, a family $\{\mathcal{K}(j): j\in\omega\}$ of collections of compact subsets of X^ω, where for each $K\in\mathcal{K}(j)$ and $j\in\omega$, $K=\prod_{i\in\omega}K_i\in\mathcal{K}$, families $\{\mathcal{H}(j): j\in\omega\}$, $\{\mathcal{W}(j): j\in\omega\}$, $\{\mathcal{G}(j): j\in\omega\}$ of collections of elements of $\mathcal{B}$, a family $\{\mathcal{A}(j): j\in\omega\}$ of collections of finite subsets of ω and a family $\{\mathcal{B}(j): j\in\omega\}$ of collections of elements of $\mathcal{B}$ such that for $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta(B)$, $B\in\mathcal{B}(j-1)$, where $B_{\delta(-1),A(-1)}=X^\omega$, and $\mathcal{B}_{-1}=\mathcal{B}(-1)=\{X^\omega\}$, and $A\in\mathcal{A}(\delta)$, if $i\in A$ with $i\leqq n(B)$, then $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$, and for each $j\in\omega$, $|\mathcal{G}(j)\cup\mathcal{B}(j)|<\omega$ and $(\mathcal{G}_j\cup\mathcal{B}_j)_x\subset\mathcal{G}(j)\cup\mathcal{B}(j)$. Assume that $x\in\cup\mathcal{B}_j$ for each $j\in\omega$. Then, by the construction, $x\in\cup\mathcal{B}(j)$ for each $j\in\omega$. Since $\mathcal{B}(j)_x$ is non-empty and finite for each $j\in\omega$, it follows from König's lemma (cf. K. Kunen [13]) that there are a sequence $\{\delta(j): j\in\omega\}$, a sequence $\{K(j): j\in\omega\}$ of compact subsets of X^ω, sequences $\{H(\delta(j)): j\in\omega\}$, $\{W(\delta(j)): j\in\omega\}$ of elements of $\mathcal{B}$, a sequence $\{A(j): j\in\omega\}$ of finite subsets of ω, a sequence $\{B_{\delta(j),A(j)}: j\in\omega\}$ of elements of $\mathcal{B}$ such that: For each $j\in\omega$,

(11) $\delta(j)=(\gamma(\delta(j),0),\cdots,\gamma(\delta(j),n(B_{\delta(j-1),A(j-1)})))\in\Delta(j)$,

(12) $K(j)=K(\delta(j))$,

(13) $A(j)\in\mathcal{A}(\delta(j))$,

(14) For each $i\in A(j)$ with $i\leqq n(B_{\delta(j-1),A(j-1)})$, $B_{\delta(j),A(j),i}=W_{\gamma(\delta(j),i)}-clH(\delta(j))_i$,

(15) $x\in B_{\delta(j),A(j)}$ and $B_{\delta(j),A(j)}\in\mathcal{B}(B_{\delta(j-1),A(j-1)})$.

Furthermore we have

(16) $n(B_{\delta(j),A(j)})<n(B_{\delta(j+1),A(j+1)})$ for each $j\in\omega$,

(17) For each $i\leqq n(B_{\delta(j),A(j)})$ with $i\in A(j+1)$ such that $C_{\lambda(B_{\delta(j),A(j)},i)}=\emptyset$, $s(clB_{\delta(j),A(j)})\cap clB_{\delta(j+1),A(j+1)}=\emptyset$,

(18) For each $i\leqq n(B_{\delta(j),A(j)})$ with $i\notin A(j+1)$ such that $C_{\lambda(B_{\delta(j),A(j)},i)}\neq\emptyset$, $K(j+1)=C_{\lambda(B_{\delta(j),A(j)},i)}$.

By the similar proof of Claim in Theorem 4.1, we can show that there is an $i\in\omega$ such that $|\{j\in\omega: i\in A(j)\}|=\omega$. Let $\{j\in\omega: i\in A(j)$ and $i\leqq n(B_{\delta(j),A(j)})\}$

$=\{j_k: k\in\omega\}$. Then we can prove that $s(clB_{\delta(j_k),A(j_k)})\cap clB_{\delta(j_{k+1}),A(j_{k+1})}=\emptyset$ for each $k\in\omega$. Since s is a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$, $\bigcap_{k\in\omega} clB_{\delta(j_k),A(j_k)}=\emptyset$. But $x_i\in\bigcap_{k\in\omega} B_{\delta(j_k),A(j_k)}$, which is a contradiction. Thus there is a $k\in\omega$ such that $x\notin\bigcup\mathcal{B}_k$. Let $j=\inf\{k\in\omega: x\notin\bigcup\mathcal{B}_k\}$. Since $x\in\bigcup\mathcal{B}_{j-1}$, we have $x\in\bigcup\mathcal{G}_j$. For each $k>j$, every element of $\mathcal{G}_k$ is contained in some member of $\mathcal{B}_j$. Therefore $(\bigcup\{\mathcal{G}_k: k\in\omega\})_x\subset\bigcup\{\mathcal{G}_k: k\leqq j\}$. Since every $\mathcal{G}_k$ is point finite in X^ω, it follows that $\bigcup\{\mathcal{G}_k: k\in\omega\}$ is a point finite open refinement of $\mathcal{O}'$. The proof is completed.

COROLLARY 5.2. *If Y_i is a regular metacompact space with a σ-closure-preserving cover by compact sets for each $i\in\omega$, then $\prod_{i\in\omega} Y_i$ is metacompact.*

PROOF. This follows from Theorem 5.1 and Lemma 2.3(a).

For a T_1-space X, let $\mathcal{F}[X]$ denote the Pixley-Roy hyperspace of X (cf. E.K. van Douwen [7]). Every Pixley-Roy hyperspace is a hereditarily metacompact Tychonoff space and has a closure-preserving cover by finite sets. In [17], the author proved that if Z is a perfect paracompact Hausdorff space and Y_i is a T_1-space such that $\mathcal{F}[Y_i]$ is paracompact for each $i\in\omega$, then $Z\times\prod_{i\in\omega}\mathcal{F}[Y_i]$ is paracompact.

COROLLARY 5.3. *If Y_i is a T_1-space for each $i\in\omega$, then $\prod_{i\in\omega}\mathcal{F}[Y_i]$ is metacompact.*

By D.K. Burke [4] and M.M. Čoban [6], every perfect metacompact (metalindelöf) space is hereditarily metacompact (hereditarily metalindelöf). Next, we show the following result.

THEOREM 5.4. *Let Z be a hereditarily metacompact space and Y_i be a regular metacompact $\mathcal{DC}$-like space for each $i\in\omega$. Then the following are equivalent.*

(a) $Z\times\prod_{i\in\omega} Y_i$ *is metacompact,*

(b) $Z\times\prod_{i\in\omega} Y_i$ *is countably metacompact,*

(c) $Z\times\prod_{i\in\omega} Y_i$ *is orthocompact.*

PROOF. (a)→(c) Obvious.

(c)→(b) We shall modify the proof of Theorem 2.1 in N. Kemoto and Y. Yajima [12]. Assume that $Z\times\prod_{i\in\omega} Y_i$ is orthocompct. Let $\mathcal{O}=\{O_j: j\in\omega\}$ be a countable open cover of $Z\times\prod_{i\in\omega}Y_i$. By their proof, it suffices to prove that there is a countable open refinement $\mathcal{V}$ of $\mathcal{O}$ such that for every infinite subcollection $\mathcal{V}'$ of $\mathcal{V}$, $\mathrm{int}(\cap\mathcal{V}')=\emptyset$. Applying their technique to $Z\times\prod_{i\in\omega}Y_i$, we have a countable collection $\{G_{j,t}: j\in\omega \text{ and } t=0, 1\}$, where $G_{j,t}=Z\times H_{j,t}$ for each $j\in\omega$ $t=0, 1$, of open subsets of $Z\times\prod_{i\in\omega}Y_i$ such that

(i) For each $j\in\omega$, $\prod_{i\in\omega}Y_i=H_{j,0}\cup H_{j,1}$ and hence, $Z\times\prod_{i\in\omega}Y_i=G_{j,0}\cup G_{j,1}$,

(ii) For each infinite subset M of ω and each $t=0, 1$, $\mathrm{int}\{\cap\{H_{j,t}: j\in M\})=\emptyset$ and hence, $\mathrm{int}(\cap\{G_{j,t}: j\in M\})=Z\times\mathrm{int}(\cap\{H_{j,t}: j\in M\})=\emptyset$.

Let $\mathcal{V}=\{O_j\cap G_{j,t}: j\in\omega \text{ and } t=0, 1\}$. Then $\mathcal{V}$ is a countable open refinement of $\mathcal{O}$ such that for every infinite subcollection $\mathcal{V}'$ of $\mathcal{V}$, $\mathrm{int}(\cap\mathcal{V}')=\emptyset$.

(b)→(a) Assume that $Z\times\prod_{i\in\omega}Y_i$ is countably metacompact. For each $i\in\omega$, take a point a_i in Y_i. Let $\mathcal{O}$ be an open cover of $Z\times\prod_{i\in\omega}Y_i$ and let $\mathcal{O}'=\{B\in\mathcal{B}: B\subset O \text{ for som } O\in\mathcal{O}^F\}$. For each $z\in Z$ and $K\in\mathcal{K}$, define $n(K_{(z,K)})$ as the proof of Theorem 4.1.

Let s_i be a stationary winning strategy for Player I in $G(\mathcal{DC}, Y_i)$ for $i\in\omega$. As Theorem 5.1, take a $B=U_B\times\prod_{i\in\omega}B_i\in\mathcal{B}$ satisfying the following condition: For each $i\leqq n(B)$, we have already obtained a compact set $C_{\lambda(B,i)}$ of clB_i. ($C_{\lambda(B,n(B))}=\emptyset$. $C_{\lambda(B,i)}=\emptyset$ may be occur for $i<n(B)$.) Fix $i\leqq n(B)$. If $C_{\lambda(B,i)}\neq\emptyset$, take the same $W_{\gamma(B,i)}$, $\Lambda(B, i)$, $\Gamma(B, i)$, $\mathcal{C}(B, i)$ and $\mathcal{W}(B, i)$ in Theorem 5.1. Assume that $C_{\lambda(B,i)}=\emptyset$. Then we take a discrete collection $\mathcal{C}(B, i)=\{C_\lambda: \lambda\in\Lambda(B, i)\}$ of comoact subset of Y_i such that $s_i(clB_i)=\cup\mathcal{C}(B, i)$, and a collectiom $\mathcal{W}(B, i)=\{W_\gamma: \gamma\in\Gamma(B, i)\}$ of open subsets in B_i (and hence, in Y_i) satisfying the condition (1')=(1), (2')=(2) in the proof of Theorem 5.1 and

(3') $\mathcal{W}(B, i)$ is point finite in B_i and hence, point finite in Y_i.

Define the same K_γ for $\gamma\in\Gamma(B, i)$ and Δ_B in Theorem 5.1. For $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$, let $K(\delta)=K_{\gamma(\delta,0)}\times\cdots\times K_{\gamma(\delta,n(B))}\times\{a_{n(B)+1}\}\times\cdots\times\{a_k\}\times\cdots$. Define $\mathcal{K}_B$ as before. For each $z\in U_B$ and $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, , n(B)))\in\Delta_B$, let $r(K_{(z,K(\delta))})=\max\{n(K_{(z,K(\delta))}), n(B)\}$. Fix $z\in U_B$ and $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$. Take an $O_{z,\delta}=U_{z,\delta}\times\prod_{i\in\omega}O_{z,\delta,i}\in\mathcal{O}'$ such that $K_{(z,K(\delta))}\subset O_{z,\delta}$ and $n(K_{(z,K(\delta))})=n(O_{z,\delta})$. Since Y_i is a regular space, there is an $H_{(z,K(\delta))}=H_{z,\delta}\times\prod_{i\in\omega}H_{(z,K(\delta)),i}\in\mathcal{B}$ such that:

(4') $H_{z,\delta}\times\prod_{i=0}^{n(K_{(z,K(\delta))})-1}clH_{(z,K(\delta)),i}\times Y_{n(K_{(z,K(\delta))})}\times\cdots\times Y_k\times\cdots\subset O_{z,\delta}$ and $z\in H_{z,\delta}\subset U_B\cap U_{z,\delta}$,

(5'-1) For each i with $n(K_{(z,K(\delta))})\leqq i\leqq r(K_{(z,K(\delta))})$, let $H_{(z,K(\delta)),i}=Y_i$,

(5'-2) For each $i<n(K_{(z,K(\delta))})$ with $i\leqq n(B)$, let $H_{(z,K(\delta)),i}$ be an open subset

of Y_i such that $K_{\gamma(\delta,i)}\subset H_{(z,K(\delta)),i}\subset clH_{(z,K(\delta)),i}\subset O_{z,\delta,i}$,

(5′-3) For each i with $n(B)<i<n(K_{(z,K(\delta))})$, let $H_{(z,K(\delta)),i}$ be an open subset of Y_i such that $a_i\in H_{(z,K(\delta)),i}\subset clH_{(z,K(\delta)),i}\subset O_{z,\delta,i}$,

(5′-4) In case of that $r(K_{(z,K(\delta))})=n(B)$, let $H_{(z,K(\delta)),i}=Y_i$ for $n(B)<i$. In case of that $r(K_{(z,K(\delta))})=n(K_{(z,K(\delta))})>n(B)$, let $H_{(z,K(\delta)),i}=Y_i$ for $n(K_{(z,K(\delta))})\leqq i$.

Then we have $K_{(z,K(\delta))}\subset H_{(z,K(\delta))}$. For each $j\in\omega$, let $\mathcal{H}_{\delta,j}=\{H_{z,\delta}: n(K_{(z,K(\delta))})\leqq j\}$. Fix $j\in\omega$ and let $V_j(K(\delta))=\{z\in U_B: n(K_{(z,K(\delta))})\leqq j\}$. Then $V_j(K(\delta))=\cup\mathcal{H}_{\delta,j}$. Since Z is a hereditarily metacompact space, there is a family $\mathcal{V}_{\delta,j}=\{V_\xi: \xi\in\Xi_{\delta,j}\}$, of collections of open sets in $V_j(K(\delta))$ (and hence, in Z) satisfying

(6′) Every member of $\mathcal{V}_{\delta,j}$ is contained in some member of $\mathcal{H}_{\delta,j}$,

(7′) $\mathcal{V}_{\delta,j}$ covers $V_j(K(\delta))$,

(8′) $\mathcal{V}_{\delta,j}$ is point finite in $V_j(K(\delta))$ and hence, point finite in Z.

For each $\xi\in\Xi_{\delta,j}$, take a $z(\xi)\in V_j(K(\delta))$ such that $V_\xi\subset H_{z(\xi),\delta}$. Put $W_\delta=\prod_{i=0}^{n(B)} W_{\gamma(\delta,i)}\times Y_{n(B)+1}\times\cdots\times Y_k\times\cdots$ and $V_{\xi,\delta}=V_\xi\times W_\delta$. Then $\{V_{\xi,\delta}: \delta\in\Delta_B,\ j\in\omega$ and $\xi\in\Xi_{\delta,j}\}$ is a collection of elements of $\mathcal{B}$ such that for each $\delta\in\Delta_B$, $j\in\omega$ and $\xi\in\Xi_{\delta,j}$, $V_{\xi,\delta}\subset B$ and $\{V_{\xi,\delta}: \delta\in\Delta_B,\ j\in\omega$ and $\xi\in\Xi_{\delta,j}\}$ covers B. Clearly we have

(9′) For each $j\in\omega$, $\{V_{\xi,\delta}: \delta\in\Delta_B$ and $\xi\in\Xi_{\delta,j}\}$ is point finite in $Z\times\prod_{i\in\omega}Y_i$.

Fix a $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$, $j\in\omega$ and $\xi\in\Xi_{\delta,j}$. In case of that $r(K_{(z(\xi),K(\delta))})=n(B)$. For each $i\leqq n(B)$, let $G_{(z(\xi),K(\delta)),i}=O_{z(\xi),\delta,i}\cap W_{\gamma(\delta,i)}$. For each $i>n(B)$, let $G_{(z(\xi),K(\delta)),i}=Y_i$. Put $G_{(z(\xi),K(\delta))}=V_\xi\times\prod_{i\in\omega}G_{(z(\xi),K(\delta)),i}$. In case of that $r(K_{(z(\xi),K(\delta))})=n(K_{(z(\xi),K(\delta))})>n(B)$. For each $i\leqq n(B)$, let $G_{(z(\xi),K(\delta)),i}=O_{z(\xi),\delta,i}\cap W_{\gamma(\delta,i)}$. For each i with $n(B)<i<n(K_{(z(\xi),K(\delta))})$, let $G_{(z(\xi),K(\delta)),i}=O_{z(\xi),\delta,i}$. For each $i\geqq n(K_{(z(\xi),K(\delta))})$, let $G_{(z(\xi),K(\delta)),i}=Y_i$. Put $G_{(z(\xi),K(\delta))}=V_\xi\times\prod_{i\in\omega}G_{(z(\xi),K(\delta)),i}$. Then we have $G_{(z(\xi),K(\delta))}\subset V_{\xi,\delta}$. Define $\mathcal{G}_{\delta,j}(B)=\{G_{(z(\xi),K(\delta))}: \xi\in\Xi_{\delta,j}\}$ and $\mathcal{G}_j(B)=\cup\{\mathcal{G}_{\delta,j}(B): \delta\in\Delta_B\}$. Then, by (9′) and definition,

(10′) For each $j\in\omega$, every member of $\mathcal{G}_j(B)$ is contained in some member of $\mathcal{O}'$.

(11′) For each $j\in\omega$, $\mathcal{G}_j(B)$ is point finite in $Z\times\prod_{i\in\omega}Y_i$.

Fix $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$, $j\in\omega$ and $\xi\in\Xi_{\delta,j}$. Let $A\in\mathcal{P}(\{0,1,\cdots,r(K_{(z(\xi),K(\delta))})\})$. In case of that $r(K_{(z(\xi),K(\delta))})=n(B)$. For each $i\in A$, let $B_{\xi,A,i}=W_{\gamma(\delta,i)}-clH_{(z(\xi),K(\delta)),i}$. For each $i\notin A$ with $i\leqq n(B)$, let $B_{\xi,A,i}=O_{z,\delta,i}\cap W_{\gamma(\delta,i)}$. For each $i>n(B)$, let $B_{\xi,A,i}=Y_i$. Put $B_{\xi,A}=V_\xi\times\prod_{i\in\omega}B_{\xi,A,i}$. In case of that $r(K_{(z(\xi),K(\delta))})=n(K_{(z(\xi),K(\delta))})>n(B)$. For each $i\in A$ with $i\leqq n(B)$, let $B_{\xi,A,i}=W_{\gamma(\delta,i)}-clH_{(z(\xi),K(\delta)),i}$. For each $i\notin A$ with $i\leqq n(B)$, let $B_{\xi,A,i}=O_{z(\xi),\delta,i}\cap W_{\gamma(\delta,i)}$.

Let $n(B)<i<n(K_{(z(\xi),K(\delta))})$. If $i\in A$, let $B_{\xi,A,i}=Y_i-clH_{(z(\xi),K(\delta)),i}$. If $i\notin A$, let $B_{\xi,A,i}=O_{z(\xi),\delta,i}$. For $i>n(K_{(z(\xi),K(\delta))})$, let $B_{\xi,A,i}=Y_i$. Put $B_{\xi,A}=V_\xi\times\prod_{i\in\omega}B_{\xi,A,i}$. We have that $B_{\xi,A,i}\subset B_i$ for each $i\in\omega$ and if $B_{\xi,A}\neq\emptyset$, then $n(B)<n(B_{\xi,A})$. Since $n(K_{(z(\xi),K(\delta))})\leqq j$, for a subset $A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(B)\}\})$, let $\mathcal{B}_{\delta,j,A}(B)=\{B_{\xi,A}:\xi\in\Xi_{\delta,j},\ B_{\xi,A}$ is defined and $B_{\xi,A}\neq\emptyset\}$. For $j\in\omega$, let $\mathcal{B}_j(B)=\cup\{\mathcal{B}_{\delta,j,A}(B):\delta\in\Delta_B$ and $A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(B)\}\})\}$. Then we have

(12′) Every $\mathcal{B}_j(B)$ is point finite in $Z\times\prod_{i\in\omega}Y_i$.

Fix a $B_{\xi,A}=V_\xi\times\prod_{i\in\omega}B_{\xi,A,i}\in\mathcal{B}_{\delta,j,A}(B)$ for $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$, $j\in\omega$, $\xi\in\Xi_{\delta,j}$ and $A\in\mathcal{P}(\{0,1,\cdots,\max\{j,n(B)\}\})$. Then

(13′) For each $i\in A$ with $i\leqq n(B)$ such that $C_{\lambda(B,i)}=\emptyset$, $s_i(clB_i)\cap clB_{\xi,A,i}=\emptyset$.

For each $i\leqq n(B_{\xi,A})$, define a compact set $C_{\lambda(B_{\xi,A},i)}$ in $clB_{\xi,A,i}$ as Theorem 5.1.

Now we define $\mathcal{G}_\tau$ and $\mathcal{B}_\tau$ for each $\tau\in\omega^{<\omega}$ with $\tau\neq\emptyset$. For each $j\in\omega$, let $\mathcal{G}_j=\mathcal{G}_j(Z\times\prod_{i\in\omega}Y_i)$ and $\mathcal{B}_j=\mathcal{B}_j(Z\times\prod_{i\in\omega}Y_i)$. Assume that for $\tau\in\omega^{<\omega}$ with $\tau\neq\emptyset$, we have already obtained $\mathcal{G}_\tau$ and $\mathcal{B}_\tau$. For each $B\in\mathcal{B}_\tau$ and $j\in\omega$, we denote $\mathcal{G}_j(B)$ and $\mathcal{B}_j(B)$ by $\mathcal{G}_{\tau\oplus j}(B)$ and $\mathcal{B}_{\tau\oplus j}(B)$ respectively. Define $\mathcal{G}_{\tau\oplus j}=\cup\{\mathcal{G}_{\tau\oplus j}(B):B\in\mathcal{B}_\tau\}$ and $\mathcal{B}_{\tau\oplus j}=\cup\{\mathcal{B}_{\tau\oplus j}(B):B\in\mathcal{B}_\tau\}$.

Firstly we show that $\cup\{\mathcal{G}_\tau:\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset\}$ is a σ-point finite open refinement of $\mathcal{O}'$. Let $\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset$. By (10′), every element of $\mathcal{G}_\tau$ is contained in some member of $\mathcal{O}'$. By (11′), (12′) and induction, for each $\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset$, $\mathcal{G}_\tau$ is point finite. Thus, it suffices to prove that $\cup\{\mathcal{G}_\tau:\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset\}$ is a cover of $Z\times\prod_{i\in\omega}Y_i$. However, the proof is similar to that of Claim in Theorem 4.1. Let $G_\tau=\cup\mathcal{G}_\tau$ for each $\tau\in\omega^{<\omega}$ with $\tau\neq\emptyset$. Then $\{G_\tau:\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset\}$ is a countable open cover of $Z\times\prod_{i\in\omega}Y_i$. Since $Z\times\prod_{i\in\omega}Y_i$ is countably metacompact, there is a point finite open refinement $\{G'_\tau:\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset\}$ such that $G'_\tau\subset G_\tau$ for each $\tau\in\omega^{<\omega}$ with $\tau\neq\emptyset$. Then $\{G'_\tau\cap G:G\in\mathcal{G}_\tau,\tau\in\omega^{<\omega}$ and $\tau\neq\emptyset\}$ is a point finite open refinement of $\mathcal{O}'$. It follows that $Z\times\prod_{i\in\omega}Y_i$ is metacompact. The proof is completed.

REMARK 5.5. B. Scott [16] showed that if Y is orthocompact and Z is compact, metric and infinite, then $Y\times Z$ is orthocompact if and only if Y is countably metacompact. J. Chaber [5] constructed a scattered hereditarily orthocompact space Y which is not countably metacompact. Thus, for J. Chaber's space Y, $Y\times(\omega+1)$ is not orthocompact, even though both factors are hereditarily orthocompact and scattered (cf. Lemma 2.4).

COROLLARY 5.6. *Let Z be a hereditarily metacompact space and Y_i be a regular metacompact space with a σ-closurepreserving cover by compact sets for each $i\in\omega$. Then the following are equivalent.*

(a) $Z\times\prod_{i\in\omega}Y_i$ *is metacompact,*

(b) $Z\times\prod_{i\in\omega}Y_i$ *is countably metacompact,*

(c) $Z\times\prod_{i\in\omega}Y_i$ *is orthocompact.*

Since every σ-point countable collection of $Z\times\prod_{i\in\omega}Y_i$ is point countable, by the proof of the implication (b)→(a) in Theorem 5.4, we have

THEOREM 5.7. *If Z is a hereditarily metalindelöf space and Y_i is a regular metalindelöf $\mathcal{DC}$-like space for each $i\in\omega$, then $Z\times\prod_{i\in\omega}Y_i$ is metalindelöf.*

COPOLLARY 5.8. *If Z is a hereditarily metalindelöf space and Y_i is a regular metalindelöf space with a σ-closure-preserving cover by compact sets for each $i\in\omega$, then $Z\times\prod_{i\in\omega}Y_i$ is metalindelöf.*

We consider metacompactness, orthocompactness and metalindelöf property of countable products using C-scattered spaces.

THEOREM 5.9. *If Y_i is a regular C-scattered metacompact space for each $i\in\omega$, then $\prod_{i\in\omega}Y_i$ is metacompact.*

PROOF. We also assume that $Y_i=X$ for each $i\in\omega$ and there is an isolated point a in X. We shall modify the proof of Theorem 5.1. Let $\mathcal{O}$ be an open cover of X^ω. Define the same $\mathcal{O}'$ and $n(K)$ for each $K\in\mathcal{K}$. We take a $B=\prod_{i\in\omega}B_i\in\mathcal{B}$ satisfying the condition of the proof of Theorem 5.1. Fix $i\leqq n(B)$. If $C_{\lambda(B,i)}\neq\emptyset$, then we take the same $W_{\gamma(B,i)}$, $\Lambda(B, i)$, $\Gamma(B, i)$, $\mathcal{C}(B, i)$, and $\mathcal{W}(B, i)$. Assume that $C_{\lambda(B,i)}=\emptyset$. Since clB_i is a regular C-scattered metacompact space, by Lemma 3.3, there is a collection $\mathcal{W}(B, i)=\{W_\gamma: \gamma\in\Gamma(B, i)\}$ of open subsets in B_i satisfying the conditions (1″)=(1) and (2″)=(3) in the proof of Theorem 5.1 and

(3″) For each $\gamma\in\Gamma(B, i)$, $(clW_\gamma)^{(\alpha(\gamma))}$ is compact for some $\alpha(\gamma)$.

Let $\Lambda(B, i)=\Gamma(B, i)$ and $\mathcal{C}(B, i)=\{(clW_\lambda)^{(\alpha(\lambda))}: \lambda\in\Lambda(B, i)\}$.

Let $K_\gamma=(clW_\gamma)^{(\alpha(\gamma))}$ for $\gamma\in\Gamma(B, i)$ and take Δ_B, $K(\delta)$ for $\delta\in\Delta_B$, $\mathcal{K}_B$, $r(K(\delta))$, $H(\delta)$, $W(\delta)$ and $G(\delta)$ for $\delta\in\Delta_B$, $\mathcal{G}(B)$, $B_{\delta,A}$, $\mathcal{B}_\delta(B)$ and $\mathcal{B}(B)$ for $\delta\in\Delta(B)$, $A\in\mathcal{P}(\{0, 1, \cdots, r(K(\delta))\})$ as before satisfying the conditions (4″)=(4), (5″-i)=(5-i) for i=1, 2, 3 and 4, (6″)=(6), (7″)=(7), (8″)=(8) and (9″)=(9). Furthermore, we take the same $\mathcal{G}_j$ and $\mathcal{B}_j$ for each $j\in\omega$, and show that $\cup\{\mathcal{G}_j: j\in\omega\}$ is a point finite open refinement of $\mathcal{O}'$. Let $x=(x_i)_{i\in\omega}$. Take the same $\{\Delta(j): j\in\omega\}$, $\{\mathcal{K}(j): j\in\omega\}$, $\{\mathcal{H}(j): j\in\omega\}$, $\{\mathcal{W}(j): j\in\omega\}$, $\{\mathcal{G}(j): j\in\omega\}$, $\{\mathcal{A}(j): j\in\omega\}$ and $\{\mathcal{B}(j): j\in\omega\}$. Assuming $x\in\cup\mathcal{B}_j$ for each $j\in\omega$, we similarly choose a sequence $\{\delta(j): j\in\omega\}$, a sequence $\{K(j): j\in\omega\}$ of compact subsets of X^ω, where for each $j\in\omega$, $K(j)=\prod_{i\in\omega} K(j)_i\in\mathcal{K}$, sequences $\{H(\delta(j)): j\in\omega\}$, $\{W(\delta(j)): j\in\omega\}$ of elements of $\mathcal{B}$, a sequence $\{A(j): j\in\omega\}$ of finite subsets of ω, a sequence $\{B_{\delta(j),A(j)}: j\in\omega\}$ of elements of $\mathcal{B}$ satisfying the conditions $(10'')=(11)$, $(11'')=(12)$, $(12'')=(13)$, $(13'')=(14)$, $(14'')=(15)$, $(15'')=(16)$ and $(16'')=(18)$. Then there is an $i\in\omega$ such such that $|\{j\in\omega: i\in A(j)\}|=\omega$. Let $\{j\in\omega: i\in A(j)$ and $i\leqq n(B_{\delta(j),A(j)})\}=\{j_k: k\in\omega\}$. We have

(17″) For each $k\in\omega$, $\varepsilon(clW_{\gamma(\delta(j_{k+1}+1),i)})<\varepsilon(clW_{\gamma(\delta(j_k+1),i)})$.

Fix $k\in\omega$ and take a $y\in clW_{\gamma(\delta(_{k+1}+1),i)}$. Since $W_{\gamma(\delta(j_{k+1}+1),i)}\subset W_{\gamma(\delta(j_k+1),i)}$, $\alpha clW_{\gamma(\delta(j_{k+1}+1),i)}(y)\leqq\alpha clW_{\gamma(\delta(j_k+1),i)}(y)$. Assume that $j_{k+1}=j_k+1$. Then $W_{\gamma(\delta(j_{k+1}+1),i)}\subset B_{\delta(j_{k+1}),A(j_{k+1}),i}$ and

$$K(j_{k+1})_i=K_{\gamma(\delta(j_{k+1}),i)}=(clW_{\gamma(\delta(j_{k+1}),i)})^{\alpha(\gamma(\delta(j_{k+1}),i))}\subset H(\delta(j_{k+1}))_i.$$

Assume that $j_{k+1}>j_k+1$. Then

$$K(j_k+1)_i=K_{\gamma(\delta(j_k+1),i)}=C_{\lambda(B_{\delta(j_k+1),A(j_k+1)},i)}=C_{\lambda(B_{\delta(j_{k+1}-1),A(j_{k+1}-1)},i)}\subset H(\delta(j_{k+1}))_i.$$

In each case, we have $\alpha clW_{\gamma(\delta(j_k+1),i)}(y)<\alpha(\gamma(\delta(j_k+1),i))$. Hence $\alpha clW_{\gamma(\delta(j_{k+1}+1),i)}(y)<\alpha(\gamma(\delta(j_k+1),i))$. Therefore $\varepsilon(clW_{\gamma(\delta(j_{k+1}+1),i)})\leqq\alpha(\gamma(\delta(j_k+1),i)$. Since $\varepsilon(clW_{\gamma(\delta(j_k+1),i)})=\alpha(\gamma(\delta(j_k+1),i)+1$, we have $\varepsilon(clW_{\gamma(j_{k+1}+1),i)})<\varepsilon(clW_{\gamma(\delta(j_k+1),i)})$.

Thus $\{\varepsilon(clW_{\gamma(\delta(j_k+1),i)}): k\in\omega\}$ is an infinite decreasing sequence of ordinals, which is a contradiction. Thus there is a $k\in\omega$ such that $x\notin\cup\mathcal{B}_k$. Similarly, it follows that $\cup\{\mathcal{G}_j: j\in\omega\}$ is a point finite open refinement of $\mathcal{O}'$. The proof is completed.

Similarly, we have

THEOREM 5.10. *Let Z be a hereditarily metacompact space and Y_i be a regular C-scattered metacompact space for each $i\in\omega$. Then the following are equivalent.*

(a) $Z\times\prod_{i\in\omega}Y_i$ *is metacompact,*

(b) $Z\times\prod_{i\in\omega}Y_i$ *is countably metacompact,*

(c) $Z\times\prod_{i\in\omega}Y_i$ *is orthocompact.*

THEOREM 5.11. *If Z is a hereditarily metalindelöf space and Y_i is a regular C-scattered metalindelöf space for each $i\in\omega$, then $Z\times\prod_{i\in\omega}Y_i$ is metalindelöf.*

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