TSUKUBA J. MATH.
Vol. 3, No. 2 (1979), 41–51

TITS' SYSTEMS IN CHEVALLEY GROUPS OVER LAURENT POLYNOMIAL RINGS

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0. Introduction.

Our aim is to show that the elementary subgroup of a Chevalley group over a Laurent polynomial ring has the structure of a Tits' system with an affine Weyl group (as for Tits' system, see [2]).

We let denote $\boldsymbol{Z}$ the rational integers.

Let Δ be a (reduced) root system (cf. [2], [4]). Then there is a finite dimensional complex semisimple Lie algebra $L=L(\Delta)$, unique up to isomorphism, whose root system is Δ. Let ρ be a finite dimensional complex faithful representation of L.

Let G be a Chevalley-Demazure group scheme associated with L and ρ (as for the definition, see [1], [8]). Since G is a representable covariant functor from the category of commutative rings with 1 to the category of groups, we get a group $G(R)$ of the points of a commutative ring R, with 1. We call $G(R)$ a Chevalley group over R. For each root $\alpha\in\Delta$, there is a group isomorphism of the additive group R^{+} of R onto a subgroup X_{α} of $G(R)$ (cf. [1], [8]). The elementary subgroup $E(R)$ is defined to be the subgroup of $G(R)$ generated by X_{α} for all $\alpha\in\Delta$.

If Δ is of type A_{l} and ρ is of universal type (cf. [4]), then $G(R)=SL_{l+1}(R)$ and $E(R)$ is the subgroup $E_{l+1}(R)$ of $SL_{l+1}(R)$ generated by $I_{l+1}+ae_{ij}$ for all $a\in R$ and $1\leq i\neq j\leq l+1$, where I_{l+1} is the $(l+1)\times(l+1)$ identity matrix and e_{ij} is a matrix unit (1 in the i, j position, 0 elsewhere).

If R is a field, then $E(R)$ has the structure of a Tits' system associated with the Weyl group of Δ (cf. [9]). If R is a field with a discrete valuation, then $E(R)$ has the structure of a Tits' system associated with the affine Weyl group of Δ (cf. [5]). Let $K[T, T^{-1}]$ be the ring of Laurent polynomials in T and T^{-1} with coefficients in a field K. In this paper, we will show that $E(K[T, T^{-1}])$ has the structure of a Tits' system associated with the affine Weyl group of Δ. Let $L_{\boldsymbol{Z}}$ be a Chevalley lattice in L (cf. [4]) and set $\mathfrak{g}_{K}=K[T, T^{-1}]\otimes_{\boldsymbol{Z}}L_{\boldsymbol{Z}}$. Then $\mathfrak{g}_{K}$ is isomorphic to a Euclidean Lie algebla (cf. [6]). Thus, if ρ is of adjoint type, and if

Received April 24, 1979

char $K=0$ or ≥ 5, then our result correspons to the special case of [7].

Let x and y be elements of a group, then the symbol $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$ of x and y. For two subgroups G_2 and G_3 of a group G_1, let $[G_2, G_3]$ be the subgroup of G_1 generated by $[x, y]$ for all $x \in G_2$ and $y \in G_3$. We shall write $G_1 = G_2 \cdot G_3$ when a group G_1 is a semidirect product of two groups G_2 and G_3, and G_3 normalizes G_2.

The author wishes to express his hearty gratitude to Professor Eiichi Abe for his valuable advice.

1. Characterization of affine Weyl groups.

Let Δ be a (reduced) root system of rank l, W the Weyl group of Δ, and W^* the affine Weyl group of Δ (cf. [2], [4], [5]). Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a simple system of Δ, and Δ^+ (resp. Δ^-) the positive system (resp. negative system) of Δ with respect to Π. Let α and β be in Δ, then we abbreviate $2(\beta, \alpha)/(\alpha, \alpha)$ by $\langle \beta, \alpha \rangle$, where $(\ ,\ )$ is a scalar product (cf. [4]). For each $\alpha \in \Delta$, w_α denotes the reflection with respect to α. Set $\Delta_1 = \Delta \times \boldsymbol{Z}$, then an element of Δ_1 is represented by $\alpha^{(n)}$, where $\alpha \in \Delta$ and $n \in \boldsymbol{Z}$. For each $\alpha^{(n)} \in \Delta_1$, let $w_\alpha^{(n)}$ be a permutation on Δ defined by

$$w_\alpha^{(n)} \beta^{(m)} = (w_\alpha \beta)^{(m - \langle \beta, \alpha \rangle n)}$$

for any $\beta^{(m)} \in \Delta_1$. Let W_1 be the permutation group on Δ_1 generated by $w_\alpha^{(n)}$ for all $\alpha^{(n)} \in \Delta_1$. We shall identify W with the subgroup of W_1 generated by $w_\alpha^{(0)}$ for all $\alpha \in \Delta$. Set $h_\alpha^{(n)} = w_\alpha^{(n)} w_\alpha^{(0)-1}$ and let H_1 be the subgruop of W_1 generated by $h_\alpha^{(n)}$ for all $\alpha^{(n)} \in \Delta_1$.

LEMMA 1.1

(1) *Let $\alpha^{(n)}$ and $\beta^{(m)}$ be in Δ_1. Then*

$$h_\alpha^{(n)} \beta^{(m)} = \beta^{(m + \langle \beta, \alpha \rangle n)}.$$

(2) *H_1 is a free abelian group generated by $h_{\alpha_i}^{(1)}$ for all $\alpha_i \in \Pi$.*

(3) *Let $\alpha^{(n)}$ and $\beta^{(m)}$ be in Δ_1, and set $\gamma = w_\alpha \beta$. Then*

$$w_\alpha^{(n)} h_\beta^{(m)} w_\alpha^{(n)-1} = h_\gamma^{(m)}.$$

PROOF. (1) and (3) are confirmed by direct calculation. We will show (2). Set $\alpha^* = 2\alpha/(\alpha, \alpha)$ for each $\alpha \in \Delta$, then $\Delta^* = \{\alpha^*;\ \alpha \in \Delta\}$ is also a root system and $\Pi^* = \{\alpha_i^*;\ \alpha_i \in \Pi\}$ a simple system of Δ^*. Let α be in Δ and write $\alpha^* = \sum_{i=1}^{l} c_i \alpha_i^*$ $(c_i \in \boldsymbol{Z})$, then we have $h_\alpha^{(1)} = h_{\alpha_1}^{(c_1)} h_{\alpha_2}^{(c_2)} \cdots h_{\alpha_l}^{(c_l)}$. On the other hand, $h_\alpha^{(n)} = (h_\alpha^{(1)})^n$. Hence H_1 is generated by $h_{\alpha_i}^{(1)}$ for all $1 \leq i \leq l$. Next assume $h_{\alpha_1}^{(m_1)} \cdots h_{\alpha_l}^{(m_l)} = 1$ $(m_j \in \boldsymbol{Z},\ 1 \leq j \leq l)$. This yields $\sum_{j=1}^{l} \langle \beta, \alpha_j \rangle m_j = 0$ for all $\beta \in \Delta$. Thus $m_j = 0$ for all j. q.e.d.

PROPOSITION 1.2 *Let W, W^*, W_1 and H_1 be as above. Then $W_1=H_1\cdot W$. In particular, $W_1\simeq W^*$.*

PROOF Lemma 1.1 implies $H_1\lhd W_1$ and $H_1\cap W=1$. For any $\alpha^{(n)}\in\Delta_1$, $w_\alpha^{(n)}=h_\alpha^{(n)}w_\alpha^{(0)}\in H_1W$. q. e. d.

LEMMA 1.3 *Let $\alpha^{(m)}$ be in Δ_1 and w in W_1, and set $\beta^{(n)}=w\alpha^{(m)}$. Then $ww_\alpha^{(m)}w^{-1}=w_\beta^{(n)}$.*

PROOF We can assume $w=w_\gamma^{(k)}$ for some $\gamma^{(k)}\in\Delta_1$. For any $\delta^{(c)}\in\Delta_1$, we have $w_\gamma^{(k)}w_\alpha^{(m)}w_\gamma^{(k)-1}\delta^{(c)}=w_\beta^{(n)}\delta^{(c)}$ by the following formula:

$$\langle\delta,\gamma\rangle+\langle w_\alpha w_\gamma\delta,\gamma\rangle+\langle\delta,w_\gamma\alpha\rangle\langle\alpha,\gamma\rangle=0.$$

q. e. d.

Let $\Delta=\Delta^{(1)}\cup\Delta^{(2)}\cup\cdots\cup\Delta^{(r)}$ be the irreducible decomposition of Δ (cf. [2], [4]), and set $\Pi^{(j)}=\Delta^{(j)}\cap\Pi$ for each j $(1\leq j\leq r)$. Let β_j be the unique highest root of $\Delta^{(j)}$ with respect to $\Pi^{(j)}$ for each j. Set $\Pi_1=\{-\alpha_i^{(0)},\ \beta_j^{(1)};\ 1\leq i\leq l,\ 1\leq j\leq r\}$ and $Y=\{w_\alpha^{(n)};\ \alpha^{(n)}\in\Pi_1\}$.

PROPOSITION 1.4 *Let W_1 and Y be as above. Then Y generates W_1.*

PROOF We can assume Δ is irreducible. Let X be the subgroup of W_1 generated by Y. If Δ has only one root length, then $w_\alpha^{(1)}\in X$ for all $\alpha\in\Delta$ by Lemma 1.3. Thus $h_\alpha^{(1)}\in X$ for all $\alpha\in\Delta$, and $X=W_1$. Assume that Δ has two root lengths. Then we can choose α and β in Π such that α is short, β long, and $\langle\alpha,\beta\rangle=-1$. By Lemma 1.3, $w_\beta^{(1)}w_\alpha^{(0)}w_\beta^{(1)-1}=w_\gamma^{(1)}\in X$, where $\gamma=w_\beta\alpha$. Hence $w_\alpha^{(1)}\in X$ for all $\alpha\in\Delta$, which yields $X=W_1$. q. e. d.

When $w\in W_1$ is written as $w_1w_2\cdots w_k$ $(w_j\in Y,\ k$ minimal), we write $l(w)=k$: this is the length of w. Set $\Delta_1^+=(\Delta^+\times\boldsymbol{Z}_{>0})\cup(\Delta^-\times\boldsymbol{Z}_{\geq0})$ and $\Delta_1^-=\Delta_1-\Delta_1^+$. For each $w\in W_1$, set $\Gamma(w)=\{\alpha^{(n)}\in\Delta_1^+;\ w\alpha^{(n)}\in\Delta_1^-\}$ and $N(w)=Card\ \Gamma(w)$. We will show $N(w)=l(w)$. The following proposition is easily verified.

PROPOSITION 1.5 *Let $\alpha^{(n)}$ be in Π_1 and w in W_1. Then:*

(1) $\Gamma(w_\alpha^{(n)})=\{\alpha^{(n)}\}$,

(2) $w_\alpha^{(n)}(\Gamma(w)-\{\alpha^{(n)}\})=\Gamma(ww_\alpha^{(n)})-\{\alpha^{(n)}\}$,

(3) *$\alpha^{(n)}$ is in precisely one of $\Gamma(w)$ or $\Gamma(ww_\alpha^{(n)})$,*

(4) *$N(ww_\alpha^{(n)})=N(w)-1$ if $\alpha^{(n)}\in\Gamma(w)$, $N(ww_\alpha^{(n)})=N(w)+1$ if $\alpha^{(n)}\notin\Gamma(w)$.*

LEMMA 1.6 *Let t be in $\boldsymbol{Z}_{>1}$ and $\alpha^{(n)}$ in Π_1. Let w_j be in Y $(j=1,2,\cdots,t-1)$ and set $w_t=w_\alpha^{(n)}$. Suppose $w_1w_2\cdots w_{t-1}\alpha^{(n)}$ is in Δ_1^-. Then $w_1\cdots w_t=w_1\cdots w_{s-1}w_{s+1}\cdots w_{t-1}$ for some index $1\leq s\leq t-1$.*

PROOF Write $\gamma_k=w_{k+1}w_{k+2}\cdots w_{t-1}\alpha^{(n)}$ $(0\leq k\leq t-2)$, $\gamma_{t-1}=\alpha^{(n)}$. Since $\gamma_0\in\Delta_1^-$ and

$\gamma_{t-1}\in\Delta_1^+$, we can find a smallest index s for which $\gamma_s\in\Delta_1^+$. Then $w_s\gamma_s=\gamma_{s-1}\in\Delta_1^-$, so $\gamma_s\in\Pi_1$. Thus $w_s=w_\gamma^{(m)}$, where $\gamma^{(m)}=\gamma_s$. By Lemma 1.3, $w_s=(w_{s+1}\cdots w_{t-1})w_t(w_{t-1}\cdots w_{s+1})$, which yields the lemma. q.e.d.

COROLLARY 1.7 *If* $w=w_1w_2\cdots w_t$ $(w_j\in Y,\ 1\leq j\leq t)$ *is a reduced expression* (i.e. $l(w)=t$), *and if* $w_t=w_\alpha^{(n)}$ *for some* $\alpha^{(n)}\in\Pi_1$, *then* $w\alpha^{(n)}\in\Delta_1^-$.

PROPOSITION 1.8 *Let* w *be in* W_1. *Then* $N(w)=l(w)$.

PROOF Proceed by induction on $l(w)$. If $l(w)=0$, then $w=1$, so $N(w)=0$. Assume $l(w)>0$, and write $w=w_1w_2\cdots w_t$ as a reduced expression, where $w_j\in Y$, $1\leq j\leq t$. For some $\alpha^{(n)}\in\Pi_1$, $w_t=w_\alpha^{(n)}$. By Corollary 1.7, $w\alpha^{(n)}\in\Delta_1^-$ and $\alpha^{(n)}\in\Gamma(w)$. Thus $N(ww_\alpha^{(n)})=N(w)-1$ by Proposition 1.5(4). On the other hand, $l(ww_\alpha^{(n)})=l(w)-1$. By induction, $N(ww_\alpha^{(n)})=l(ww_\alpha^{(n)})$, which implies $N(w)=l(w)$. q.e.d.

2. The statement of Main Theorem, some basic results.

Let Δ be a (reduced) root system of rank l and $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ a simple system of Δ (cf. [2], [4]). Let $L=L(\Delta)$ be a finite dimensional complex semisimple Lie algebra whose root system with respect to a Cartan subalgebra $\mathfrak{h}$ of L is Δ, and let ρ be a finite dimensional complex faithful representation of L. Let G be a Chevalley-Demazure group scheme associated with L and ρ (as for the definition, see [1], [8]). Let $\{h_i, e_\alpha;\ 1\leq i\leq l,\ \alpha\in\Delta\}$ be a Chevalley basis of L (cf. [3]). Then we have a Chevalley lattice $L_{\boldsymbol{Z}}=\sum_{i=1}^{l}\boldsymbol{Z}h_i+\sum_{\alpha\in\Delta}\boldsymbol{Z}e_\alpha$ in L. Let $\mathcal{U}$ be a universal enveloping algebra of L and $\mathcal{U}_{\boldsymbol{Z}}$ the subring of $\mathcal{U}$ generated by 1 and $e_\alpha^k/k!$ for all $\alpha\in\Delta$ and $k\in\boldsymbol{Z}_{>0}$. Then $L_{\boldsymbol{Z}}$ is a $\mathcal{U}_{\boldsymbol{Z}}$-module. Let V be the representation space of ρ, Λ the weights of V with respect to $\mathfrak{h}$, and $V=\amalg_{\mu\in\Lambda}V_\mu$ the weight decomposition of V. Let M be an admissible lattice in V (cf. [4], [9]), and set $M_\mu=M\cap V_\mu$. Let $K[T, T^{-1}]$ be the ring of Laurent polynomials in T and T^{-1} with coefficients in a field K. Set $M'=K[T,T^{-1}]\otimes_{\boldsymbol{Z}}M$ and $M'_\mu=K[T,T^{-1}]\otimes_{\boldsymbol{Z}}M_\mu$. For each $t\in K$, $n\in\boldsymbol{Z}$ and $\alpha\in\Delta$,

$$\exp tT^n\rho(e_\alpha)=1+tT^n\rho(e_\alpha)/1!+t^2T^{2n}\rho(e_\alpha)^2/2!+\cdots\cdots$$

induces an automorphism of M' under the following action:

$$(t^kT^{kn}\rho(e_\alpha)^k/k!)(f\otimes v)=(t^kT^{kn}f)\otimes(\rho(e_\alpha)^k/k!)v,$$

where $f\in K[T,T^{-1}]$ and $v\in M$. Then $X_\alpha=\langle\exp tT^n\rho(e_\alpha);\ t\in K,\ n\in\boldsymbol{Z}\rangle$ is a subgroup of $G(K[T,T^{-1}])$ and isomorphic to the additive group of $K[T,T^{-1}]$. Let $E(K[T,T^{-1}])$ denote the subgroup of $G(K[T,T^{-1}])$ generated by X_α for all $\alpha\in\Delta$. We shall write $x_\alpha^{(n)}(t)=\exp tT^n\rho(e_\alpha)$ for each $\alpha\in\Delta$, $n\in\boldsymbol{Z}$, and $t\in K$. Let K^* be the multiplicative group of K. For each $\alpha\in\Delta$, $n\in\boldsymbol{Z}$, and $t\in K^*$, we write

$$w_\alpha^{(n)}(t)=x_\alpha^{(n)}(t)x_{-\alpha}^{(-n)}(-t^{-1})x_\alpha^{(n)}(t),$$
$$h_\alpha^{(n)}(t)=w_\alpha^{(n)}(t)w_\alpha^{(0)}(1)^{-1}.$$

Let U be the subgroup of $E(K[T, T^{-1}])$ generated by $x_\alpha^{(n)}(t)$ for all $\alpha^{(n)}\in\Delta_1^+$ and $t\in K$, H_0 the subgroup generated by $h_\alpha^{(0)}(t)$ for all $\alpha\in\Delta$ and $t\in K^*$, B the subgroup generated by U and H_0, and N the subgroup generated by $w_\alpha^{(n)}(t)$ for all $\alpha^{(n)}\in\Delta_1$ and $t\in K^*$.

THEOREM 2.1 (*Main Theorem*) *Notation is as above. Set $E=E(K[T, T^{-1}])$ and let Y be as in* §1. *Then (E, B, N, Y) is a Tits' system.*

The proof of Theorem 2.1 will be completed in § 4.

LEMMA 2.2 *Let $\alpha^{(m)}$ and $\beta^{(n)}$ be in Δ_1, and assume $\alpha+\beta\neq 0$. Then*

$$[x_\alpha^{(m)}(t),\ x_\beta^{(n)}(u)]=\Pi x_{i\alpha+j\beta}^{(im+jn)}(c_{ij}t^iu^j)$$

for all $t, u\in K$, where the product is taken over all roots of the form $i\alpha+j\beta$, $i, j\in \mathbf{Z}_{>0}$ in some fixed order, and c_{ij} is as in [9, Lemma 15].

PROOF Let ξ and η be indeterminates, and let α and β be in Δ such that $\alpha+\beta\neq 0$, then we have

$$[\exp\xi e_\alpha, \exp\eta e_\beta]=\Pi \exp c_{ij}\xi^i\eta^j e_{i\alpha+j\beta} \text{ in } \mathcal{U}_{\mathbf{Z}}[[\xi,\eta]],$$

where $c_{ij}\in\mathbf{Z}$ (cf. [9, Lemma 15]). The representation ρ induces a map, also denoted ρ, of $\mathcal{U}_{\mathbf{Z}}$ to End(M) because M is admissible. Following this with the map $\varphi\to\mathrm{id}\otimes\varphi$ of End(M) to End (M') yields a map, again called ρ, of $\mathcal{U}_{\mathbf{Z}}$ to End(M'). Next, map $\mathcal{U}_{\mathbf{Z}}[\xi,\eta]$ to End(M') as follows: (for $t, u\in K$, and $u_{ij}\in\mathcal{U}_{\mathbf{Z}}$)

$$\sum_{i,j} u_{ij}\xi^i\eta^j \to \sum_{i,j} t^iu^jT^{im+jn}\rho(u_{ij}),$$

where in general, if $f\in K[T, T^{-1}]$, $g\in$ End(M') then fg is the element in End(M') which is "first act by g and then left multiply by f." Then our lemma is established. q. e. d.

LEMMA 2.3 *Let $\alpha^{(n)}$ and $\beta^{(m)}$ be in Δ_1 and set $\gamma=w_\alpha\beta$. Then:*

(1) $w_\alpha^{(n)}(1)h_\beta^{(m)}(t)w_\alpha^{(n)}(1)^{-1}=h_\gamma^{(m)}(t)$ *for any $t\in K^*$.*

(2) $w_\alpha^{(n)}(1)x_\beta^{(m)}(t)w_\alpha^{(n)}(1)^{-1}=x_\gamma^{(m-\langle\beta,\alpha\rangle n)}(ct)$ *for any $t\in K$, where c is as in* [9, Lemma 19].

(3) $h_\alpha^{(n)}(t)x_\beta^{(m)}(u)h_\alpha^{(n)}(t)^{-1}=x_\beta^{(m+\langle\beta,\alpha\rangle n)}(t^{\langle\beta,\alpha\rangle}u)$ *for any $t\in K^*$ and $u\in K$.*

PROOF These follw as in [9, Lemma 20].

LEMMA 2.4 (cf. [3], [9]) *Let α be in Δ, m and n in $\mathbf{Z}$, and t and u in K^*. Then:*

(1) $h_\alpha^{(n)}(t)$ *acts as multiplication on M'_μ by $t^{\langle\mu,\alpha\rangle}T^{\langle\mu,\alpha\rangle n}$.*

(2) $h_\alpha^{(m)}(t)h_\alpha^{(n)}(u)=h_\alpha^{(m+n)}(tu)$.

(3) $w_\alpha^{(n)}(t)=w_{-\alpha}^{(-n)}(-t^{-1})$.

Let N_0 be the subgroup of $E(K[T, T^{-1}])$ generated by $w_\alpha^{(0)}(t)$ for all $\alpha\in\Delta$ and $t\in K^*$, and H the subgroup generated by $h_\alpha^{(n)}(t)$ for all $\alpha^{(n)}\in\Delta_1$ and $t\in K^*$.

LEMMA 2.5

(1) $B=U\cdot H_0$.

(2) *H_0 and H are normal subgroups of N.*

(3) *$N=HN_0$ and $H\cap N_0=H_0$.*

PROOF (1): Any element of U is a superdiagonal unipotent matrix of infinite degree and any element of H_0 is a diagonal matrix of infinite degree with respect to an appropriate choice of a K-basis of M'. Hence $U\cap H_0=1$. By Lemma 2.3(3), H_0 normalizes U. Thus $B=U\cdot H_0$. (2): By Lemma 2.3(1), we see that N normalizes H_0 and H. (3): For any $\alpha^{(n)}\in\Delta_1$ and $t\in K^*$, we have $w_\alpha^{(n)}(t)=h_\alpha^{(n)}(t)w_\alpha^{(0)}(1)\in HN_0$, so $N=HN_0$. Clearly $H\cap N_0\supseteq H_0$. Conversely we take $h\in H\cap N_0$ and write $h=\prod_{i=1}^{l} h_{\alpha_j}^{(m_j)}(t_j)$ $(\alpha_j\in\Pi,\ m_j\in\boldsymbol{Z},\ t_j\in K^*)$. Then h maps $K\otimes_{\boldsymbol{Z}}M_\mu$ to itself and hence, by Lemma 2.4(1), $\sum_{j=1}^{l}\langle\mu, \alpha_j\rangle m_j=0$ for all weights μ of the module. Thus we have $m_j=0$, which implies $h\in H_0$. q.e.d.

THEOREM 2.6 *Notation is as above. Then $N/H_0\simeq W_1$.*

PROOF By Lemma 2.5, we have $N/H_0=(H/H_0)\cdot(N_0/H_0)$. Since $H/H_0\simeq H_1$ and $N_0/H_0\simeq W$, our theorem is established by Lemma 1.1(3), Proposition 1.2 and Lemma 2.3(1). q.e.d.

We sometimes identify an element of W_1 with a representative in N of N/H_0 through the isomorphism in Theorem 2.6.

3. The case of rank 1.

In this section, we assume Δ is of rank 1, i.e. $\Delta=\{\pm\alpha\}$. Then we have $\Delta_1=\{\alpha^{(n)}, -\alpha^{(n)};\ n\in\boldsymbol{Z}\}$ and $\Delta_1^+=\{\alpha^{(m)}, -\alpha^{(n)};\ m\in\boldsymbol{Z}_{>0},\ n\in\boldsymbol{Z}_{\geq 0}\}$. Set $E=E(K[T, T^{-1}])$, and for each $\beta^{(m)}\in\Delta_1$ let $X_\beta^{(m)}$ be the subgroup of E generated by $x_\beta^{(m)}(t)$ for all $t\in K$. We identify $w_\alpha^{(0)}$ (resp. $w_\alpha^{(1)}$) in W_1 with $w_\alpha^{(0)}(1)$ (resp. $w_\alpha^{(1)}(1)$) in N, and simply write $w_\lambda=w_\alpha^{(\lambda)}$ for $\lambda=0, 1$. Set $S_\lambda=B\cup Bw_\lambda B$. Our purpose in this section is to establish the following theorem.

THEOREM 3.1 *Notation is as above. Then S_λ is a subgroup of E for $\lambda=0, 1$.*

The proof of Theorem 3.1 is given by the next proposition.

PROPOSITION 3.2 *Let $\lambda=0, 1$. Then $w_\lambda Uw_\lambda^{-1}\subseteq S_\lambda$.*

We shall give the proof of this proposition after Lemma 3.7.

LEMMA 3.3 *The following statements hold.*

(1) $w_0 X_\alpha^{(n)} w_0^{-1} = X_{-\alpha}^{(n)} \subseteq B$ *if* $n \geq 1$.

(2) $w_0 X_{-\alpha}^{(n)} w_0^{-1} = X_\alpha^{(n)} \subseteq B$ *if* $n \geq 1$.

(3) $w_0 X_{-\alpha}^{(0)} w_0^{-1} = X_\alpha^{(0)} \subseteq S_0$.

(4) $w_1 X_\alpha^{(n)} w_1^{-1} = X_{-\alpha}^{(n-2)} \subseteq B$ *if* $n \geq 2$.

(5) $w_1 X_{-\alpha}^{(n)} w_1^{-1} = X_\alpha^{(n+2)} \subseteq B$ *if* $n \geq 0$.

(6) $w_1 X_\alpha^{(1)} w_1^{-1} = X_{-\alpha}^{(-1)} \subseteq S_1$.

PROOF (1), (2), (4) and (5) are clear. (3): For any $t \in K^*$, $x_\alpha^{(0)}(t) = x_{-\alpha}^{(0)}(t^{-1}) w_{-\alpha}^{(0)}(-t^{-1}) x_{-\alpha}^{(0)}(t^{-1}) \in S_0$, hence $w_0 X_{-\alpha}^{(0)} w_0^{-1} = X_\alpha^{(0)} \subseteq S_0$. (6) is similarly shown. q. e. d.

DEFINITION Let x be in E.

(1) x is called a $(QS, 0)$-*element* if x can be written as

$$x_{-\alpha}^{(0)}(t) x_\alpha^{(0)}(u) x_{\beta_1}^{(m_1)}(t_1) \cdots x_{\beta_k}^{(m_k)}(t_k) x_{-\alpha}^{(0)}(v),$$

where $\beta_j^{(m_j)} \in \Delta_1^+ - \{-\alpha^{(0)}\}$, $k \in \boldsymbol{Z}_{\geq 0}$, $t, u, t_1, \cdots, t_k \in K$, and $v \in K^*$.

(2) x is called a $(QS, 1)$-*element* if x can be written as

$$x_{\alpha}^{(1)}(t) x_{-\alpha}^{(-1)}(u) x_{\beta_1}^{(m_1)}(t_1) \cdots x_{\alpha_k}^{(m_k)}(t_k) x_\alpha^{(1)}(v),$$

where $\beta_j^{(m_j)} \in \Delta_1^+ - \{\alpha^{(1)}\}$, $k \in \boldsymbol{Z}_{\geq 0}$, $t, u, t_1, \cdots, t_k \in K$, and $v \in K^*$.

(3) x is called an $(S, 0)$-*element* (resp. $(S, 1)$-*element*) if x is a $(QS, 0)$-*element* (resp. $(QS, 1)$-*element*) with $u=0$.

LEMMA 3.4 *Let x be in E and $\lambda = 0, 1$. If x is an (S, λ)-element, then* $w_\lambda x w_\lambda^{-1} \in S_\lambda$.

PROOF Set $\lambda = 0$. We proceed by induction on k. If $t = 0$, clearly $w_0 x w_0^{-1} \in S_0$ by Lemma 3.3. Assume $t \neq 0$. If $\beta_1^{(m_1)} = -\alpha^{(m)}$, $m > 0$, then

$$\begin{aligned} w_0 x w_0^{-1} &= w_0 x_{-\alpha}^{(0)}(t) x_{-\alpha}^{(m)}(t_1) x_{\beta_2}^{(m_2)}(t_2) \cdots x_{\beta_k}^{(m_k)}(t_k) x_{-\alpha}^{(0)}(v) w_0^{-1} \\ &= w_0 x_{-\alpha}^{(m)}(t_1) x_{-\alpha}^{(0)}(t) x_{\beta_2}^{(m_2)}(t_2) \cdots x_{\beta_k}^{(m_k)}(t_k) x_{-\alpha}^{(0)}(v) w_0^{-1} \in X_\alpha^{(m)} S_0 = S_0. \end{aligned}$$

If $\beta_1^{(m_1)} = \alpha^{(m)}$, $m > 0$, then

$$\begin{aligned} w_0 x w_0^{-1} &= x_\alpha^{(0)}(-t) x_{-\alpha}^{(m)}(-t_1) x_2 \cdots x_k x_\alpha^{(0)}(-v) \\ &= x_{-\alpha}^{(0)}(-t^{-1}) w_{-\alpha}^{(0)}(t^{-1}) x_{-\alpha}^{(0)}(-t^{-1}) x_{-\alpha}^{(m)}(-t_1) x_2 \cdots x_k x_{-\alpha}^{(0)}(-v^{-1}) w_{-\alpha}^{(0)}(v^{-1}) x_{-\alpha}^{(0)}(-v^{-1}) \\ &\in B w_0 x_{-\alpha}^{(0)}(-t^{-1}) x_{-\alpha}^{(m)}(-t_1) x_2 \cdots x_k x_{-\alpha}^{(0)}(-v^{-1}) w_0^{-1} B \subseteq B S_0 B = S_0, \end{aligned}$$

where $x_j = w_0 x_{\beta_j}^{(m_j)}(t_j) w_0^{-1}$, $2 \leq j \leq k$. The case when $\lambda = 1$ is similarly shown. q. e. d.

LEMMA 3.5 *Let x be in E.*

(1) *If x is an $(S, 0)$-element, then*

$$w_0 x w_0^{-1} \in B w_0 X_{-\alpha}^{(0)} X_\alpha^{(0)} w_0^{-1}.$$

(2) *If x is an $(S, 1)$-element, then*

$$w_1 x w_1^{-1} \in B w_1 X_\alpha^{(1)} X_{-\alpha}^{(-1)} w_1^{-1}.$$

PROOF Proceed by induction on k as in Lemma 3.4. Then we have (1) and

(2). q. e. d.

Lemma 3.6 *Let x be in E and $\lambda=0, 1$. If x is a (QS, λ)-element, then $w_\lambda x w_\lambda^{-1} \in S_\lambda$.*

Proof Set $\lambda=0$. *If $t=0$, clearly* $w_0 x w_0^{-1} \in S_0$ by Lemma 3.3. Assume $t \neq 0$. Then

$$\begin{aligned} w_0 x w_0^{-1} &= x_{-\alpha}^{(0)}(-t^{-1}) w_{-\alpha}^{(0)}(t^{-1}) x_{-\alpha}^{(0)}(-t^{-1}) x_{-\alpha}^{(0)}(-u) \\ &\quad \times x_1 \cdots x_k x_{-\alpha}^{(0)}(-v^{-1}) w_{-\alpha}^{(0)}(v^{-1}) x_{-\alpha}^{(0)}(-v^{-1}) \\ &\in B w_0 x_{-\alpha}^{(0)}(-t^{-1}-u) x_1 \cdots x_k x_{-\alpha}^{(0)}(-v^{-1}) w_0 B \subseteq B S_0 B = S_0, \end{aligned}$$

where $x_j = w_0 x_{\beta_j}^{(m_j)}(t_j) w_0^{-1}$, $1 \leq j \leq k$. The case when $\lambda=1$ is similarly shown. q. e. d.

Lemma 3.7 *Let x be in E.*

(1) *If x is a $(QS, 0)$-element, then*

$$w_0 x w_0^{-1} \in B w_0 X_{-\alpha}^{(0)} X_\alpha^{(0)} w_0^{-1}.$$

(2) *If x is a $(QS, 1)$-element, then*

$$w_1 x w_1^{-1} \in B w_1 X_\alpha^{(1)} X_{-\alpha}^{(-1)} w_1^{-1}.$$

Proof Lemma 3.5 implies this lemma. q. e. d.

Proof of Proposition 3.2 Set $\lambda=0$ and let x be in U. We can assume $x = x_1 \cdots x_k$, where x_j is an $(S, 0)$-element, $1 \leq j \leq k$. If $k=1$, $w_0 x w_0^{-1} \in S_0$ by Lemma 3.4. Assume $k>1$. By Lemma 3.5, $w_0 x_1 w_0^{-1} \in B w_0 X_{-\alpha}^{(0)} X_\alpha^{(0)} w_0^{-1}$. Thus we have $w_0 x_1 x_2 w_0^{-1} = b_2 w_0 y_2 w_0^{-1}$, where $b_2 \in B$ and y_2 is a $(QS, 0)$-element. By Lemma 3.7, $w_0 x_1 x_2 x_3 w_0^{-1} = b_3 w_0 y_3 w_0^{-1}$, where $b_3 \in B$ and y_3 is a $(QS, 0)$-element. Recurrently we have $w_0 x w_0^{-1} = b w_0 y w_0^{-1}$, where $b \in B$ and y is a $(QS, 0)$-element. By Lemma 3.6, $w_0 x w_0^{-1} \in S_0$. The case when $\lambda=1$ is similarly shown. q. e. d.

4. Proof of Main Theorem.

Notation is as in §2. A quadruple (G^*, B^*, N^*, S^*) consisting of four sets G^*, B^*, N^*, and S^* is called a Tits' system if the following axioms are satisfied (cf. [2]):

$(T1)$ *G^* is a group, and B^* and N^* are subgroups of G^* such that G^* is generated by B^* and N^*, and $B^* \cap N^* \lhd N^*$;*

$(T2)$ *S^* is a subset of the group $N^*/(B^* \cap N^*)$ consisting of involutions and generates $N^*/(B^* \cap N^*)$;*

$(T3)$ *For any $\sigma \in S^*$ and $w \in N^*/(B^* \cap N^*)$, $wB^*\sigma \subseteq B^*wB^* \cup B^*w\sigma B^*$;*

$(T4)$ *For any $\sigma \in S^*$, $\sigma B^* \sigma \nsubseteq B^*$.*

To prove Theorem 2.1 we proceed in steps. For each $\alpha^{(n)} \in \Delta_1$, let $X_\alpha^{(n)}$ be the subgroup of $E(K[T, T^{-1}])$ generated by $x_\alpha^{(n)}(t)$ for all $t \in K$. Let $\alpha^{(m)}$ and $\beta^{(n)}$

be in Δ_1 such that $\alpha+\beta\neq 0$. Then, by Lemma 2.2,

(4.1) $$[X_\alpha^{(m)},\ X_\beta^{(n)}]\subseteq\langle X_\gamma^{(k)};\ \gamma=i\alpha+j\beta\in\Delta,\ k=im+jn,\ i,j\in\boldsymbol{Z}_{>0}\rangle.$$

For each $\alpha\in\Delta^+$, set $P_\alpha=\langle X_\alpha^{(m)},\ X_{-\alpha}^{(n)};\ m\in\boldsymbol{Z}_{>0},\ n\in\boldsymbol{Z}_{\geq 0}\rangle$ and $Q_\alpha=\langle X_\beta^{(m)},\ X_\beta^{(n)};\ \beta\in\Delta^+-\{\alpha\},\ m\in\boldsymbol{Z}_{>0},\ n\in\boldsymbol{Z}_{\geq 0}\rangle$. Then (4.1) implies

(4.2) $$U=P_\alpha Q_\alpha.$$

Let σ be in Y. We can write $\sigma=w_\alpha^{(n)}$ for some $\alpha\in\Delta^+$ and $n\in\boldsymbol{Z}$ because $w_\alpha^{(n)}$ coincides with $w_{-\alpha}^{(-n)}$. Then, by Theorem 3.1 and (4.2),

$$\begin{aligned}\sigma B\sigma^{-1}&=\sigma(P_\alpha Q_\alpha H_0)\sigma^{-1}\\&=(\sigma P_\alpha\sigma^{-1})(\sigma Q_\alpha\sigma^{-1})(\sigma H_0\sigma^{-1})\\&\subseteq(B\cup B\sigma B)BH_0\\&=B\cup B\sigma B.\end{aligned}$$

Hence

(4.3) $B\cup B\sigma B$ is a subgroup of E.

We see that $E(K[T,\ T^{-1}])$ acts on $\mathfrak{g}_K=K[T,\ T^{-1}]\otimes_{\boldsymbol{Z}}L_{\boldsymbol{Z}}$ naturally, i.e.

$$x_\alpha^{(n)}(t)(f\otimes v)=(\exp\ \mathrm{ad}\ tT^n e_\alpha)(f\otimes v),$$

where $\alpha^{(n)}\in\Delta_1$, $t\in K$, $f\in K[T,\ T^{-1}]$ and $v\in L_{\boldsymbol{Z}}$. For each $\beta^{(m)}\in\Delta_1$, set $e_\beta^{(m)}=T^m e_\beta$, $h_\beta=[e_\beta,\ e_{-\beta}]$ and $h_\beta^{(m)}=T^m h_\beta$ in $\mathfrak{g}_K$. Let g be in U and $\alpha^{(n)}$ in Π_1, and set $J_\alpha^{(n)}$ $\sum_{\beta^{(m)}\in\Delta_1^+-\{\alpha^{(n)}\}}Ke_\beta^{(m)}$. Write $ge_{-\alpha}^{(-n)}=e_{-\alpha}^{(-n)}+\zeta h_\alpha^{(0)}-\zeta^2 e_\alpha^{(n)}+z$, where $\zeta\in K$ and $z\in J_\alpha^{(n)}$. Let $\theta_\alpha^{(n)}$ be a map of U onto K defined by $\theta_\alpha^{(n)}(g)=\zeta$. As $gh_\alpha^{(0)}=h_\alpha^{(0)}-2\zeta e_\alpha^{(n)}+z'$ $(z'\in J_\alpha^{(n)})$ and $gJ_\alpha^{(n)}\subseteq J_\alpha^{(n)}$, the map $\theta_\alpha^{(n)}$ is a group homomorphism of U onto the additive group K^+ of K. Let $D_\alpha^{(n)}$ be the kernel of the homomorphism $\theta_\alpha^{(n)}$. By (4.3),

$$w_\alpha^{(n)}D_\alpha^{(n)}w_\alpha^{(n)-1}\subseteq B\cup Bw_\alpha^{(n)}B.$$

For any $x\in D_\alpha^{(n)}$, $(w_\alpha^{(n)}xw_\alpha^{(n)-1})e_\alpha^{(n)}=e_\alpha^{(n)}+z''$ $(z''\in J_\alpha^{(n)})$, so $w_\alpha^{(n)}xw_\alpha^{(n)-1}$ can not be in $Bw_\alpha^{(n)}B$. Thus,

(4.4) $$w_\alpha^{(n)}D_\alpha^{(n)}w_\alpha^{(n)-1}\subseteq B.$$

If g is in U, $\alpha^{(n)}\in\Pi_1$ and $\theta_\alpha^{(n)}(g)=\zeta$, then $gx_\alpha^{(n)}(-\zeta)\in D_\alpha^{(n)}$. Hence,

(4.5) $$U=D_\alpha^{(n)}\cdot X_\alpha^{(n)}.$$

Let $\alpha^{(n)}$ be in Π_1 and w in W_1, and set $\sigma=w_\alpha^{(n)}$. If $N(w\sigma)>N(w)$, then (4.4) and (4.5) imply

$$\begin{aligned}(BwB)(B\sigma B)&=Bw(X_\alpha^{(n)}D_\alpha^{(n)}H_0)\sigma B\\&=B(wX_\alpha^{(n)}w^{-1})w\sigma(\sigma^{-1}D_\alpha^{(n)}\sigma)(\sigma^{-1}H_0\sigma)B\\&=Bw\sigma B.\end{aligned}$$

Assume $N(w\sigma)<N(w)$. Set $w'=w\sigma$, then $N(w'\sigma)>N(w')$. Thus,

$$\begin{aligned}(BwB)(B\sigma B) &= (Bw'\sigma B)(B\sigma B)\\ &= (Bw'B)(B\sigma B)(B\sigma B)\\ &\subseteq (Bw'B)(B\cup B\sigma B)\\ &= (Bw'B)\cup(Bw'BB\sigma B)\\ &= (Bw\sigma B)\cup(BwB).\end{aligned}$$

In general, we have

$$(BwB)(B\sigma B)\subseteq(Bw\sigma B)\cup(Bw). \tag{4.6}$$

By the definition, $B\cap N\supseteq H_0$. Conversely let x be in $B\cap N$. Then $\bar{x}\in W_1$, where $\bar{x}$ is the image of x under the canonical homomorphism $^{-}$ of N onto N/H_0. Since x is in B, $\bar{x}\Delta_1^+\subseteq\Delta_1^+$, hence $N(\bar{x})=0$. Thus $\bar{x}=1$ and $x\in H_0$. This implies

$$B\cap N=H_0. \tag{4.7}$$

These facts show that (E, B, N, S) is a Tits' system.

REMARKS

1. There exists a canonical group homomorphism of the group G_K defined by Moody and Teo (cf. [7]) onto our group E under the following conditions: (1) G_K is defined over a 1-tiered Euclidean Cartan matrix, (2) char $K=0$ or ≥ 5, (3) ρ is of adjoint type.
2. If the scheme G is simply connected (i.e. ρ is of universal type), then $G(K[T, T^{-1}])=E(K[T, T^{-1}])$.
3. The group $E(K[T, T^{-1}])$ is not simple. Congruence subgroups, for example, are normal subgroups.
4. For 2-tiered or 3-tiered Euclidean types, the corresponding groups would be the twisted Chevalley groups over $K[T, T^{-1}]$.

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