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COMPACT CARDINALS AND ABELIAN GROURS

By

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Some properties about abelian groups are known to be related to large cardinals. Among them a certain property of the radical R_{Z}, i.e., $R_{Z}(A)=\cap\{\mathrm{Ker}(h): h\in \mathrm{Hom}(A, \boldsymbol{Z})\}$ for an abelian group A, has been known to be related to the existence of a compact cardinal and a measurable cardinal. To state it more precisely, let $R_{Z}^{[\kappa]}(A)=\sum\{R_{Z}(B): B$ is a subgroup of A of cardinality less than $\kappa\}$ for a cardinal κ. The radical R_{Z} satisfies the cardinal condition, if there exists a cardinal κ such that $R_{Z}(A)=R_{Z}^{[\kappa]}(A)$ for every abelian group A. M. Dugas and R. Göbel [4] proved that if there exists no measurable cardinal, then the condition does not hold. On the other hand M. Dugas [5] showed that if there exists a strongly compact cardinal, then the condition holds. Using subgroups of $\boldsymbol{Z}^{\kappa}/\boldsymbol{Z}^{<\kappa}(\simeq \boldsymbol{Z}^{(B\kappa)})$, which itself was also used in [5], B. Wald [15] got some result relating to a weakly compact cardinal.

In the present paper we show that their results can be unified under the notion of λ-$L_{\omega_1\omega}$-compactness and using it we improve their results, e.g. the radical R_{Z} satisfies the cardinal condition iff a strongly $L_{\omega_1\omega}$-compact cardinal exists, where the last property has been studied by J. Bell [2].

First we state definitions. $\boldsymbol{Z}$ is the additive group of integers and N is the set of natural numbers. In this paper κ always stands for an infinite cardinal and in most cases is regular. The word "of cardinality $\leq\lambda$" is an abbreviation of "of cardinality less than or equal to λ". $L_{\mu\nu}$ is the infinitary language which admits α-sequences of disjunctions and conjunctions and β-sequences of quantifiers for $\alpha<\mu$ and $\beta<\nu$. See [3] for a precise definition. A cardinal κ is λ-$L_{\mu\nu}$-compact, if the following hold: For a set T of $L_{\mu\nu}$-sentences of cardinality λ, if any subset of T of cardinality less than κ has a model, then T itself has a model. κ is strongly $L_{\mu\nu}$-compact, if κ is λ-$L_{\mu\nu}$-compact for any λ. $P_{\kappa}\lambda$ is the set of all subsets of λ whose cardinalities are less than κ. Let $U_{x}=\{y\in P_{\kappa}\lambda : x\subseteq y\}$ for $x\in P_{\kappa}\lambda$ and $F_{\kappa}\lambda =\{x\subseteq P_{\kappa}\lambda : U_{x}\subseteq X$ for some $x\in P_{\kappa}\lambda\}$. Then, $F_{\kappa}\lambda$ is a κ-complete filter on $P_{\kappa}\lambda$ for a regular cardinal κ. Let $B_{\kappa\lambda}$ be the quotient algebra $P(P_{\kappa}\lambda)/F_{\kappa\lambda}$. (We use filters instead of ideals when constructing quotient algebras, differing from [13].) Then, a filter on $P_{\kappa}\lambda$ which contains U_{x} for all $x\in P_{\kappa}\lambda$ corresponds to a filter of $B_{\kappa\lambda}$.

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Moreover, a countably complete ultrafilter on $P_\kappa\lambda$ which contains Ux for all $x\in P_\kappa\lambda$ corresponds to a countably complete ultrafiter of $B_{\kappa\lambda}$. In case that κ is regular, by B_κ, we denote the κ-complete quotient Boolean algebra $P(\kappa)/F_\kappa$, where $F_\kappa=\{X\subseteq\kappa : |\kappa-X|<\kappa\}$. A κ-complete Boolean algebra B is κ-representable, if B is isomorphic to the quotient algebra of a κ-complete field of sets modulo a κ-complete filter [13, § 29]. (Note that "κ-complete", "κ-representable" and so on in [13] mean our "κ^+-complete", "κ^+-representable" and so on.) The symbols $\vee$, $\wedge$, $\neg$ denote least upper bound, product, complement respectively. For a countably complete Boolean algebra B, $\boldsymbol{Z}^{(B)}$ is the Boolean power of the group of integers $\boldsymbol{Z}$, i.e. $\boldsymbol{Z}^{(B)}=\{f:f:\boldsymbol{Z}\to B \ \&\ \vee_{m\in \boldsymbol{Z}} f(m)=1 \ \&\ f(m)\wedge f(n)=0 \text{ for } m\neq n\}$ and $(f+g)(m)=\wedge_{m=n+k} f(n)\wedge g(k)$. An abelian group A is torsionless, if A is a subgroup of $\boldsymbol{Z}^I$ for some I. It is equivalent to the property that for any nonzero $a\in A$ there exists a homomorphism $h: A\to \boldsymbol{Z}$ such that $h(a)\neq 0$.

Now we state the main theorem.

THEOREM 1. *Let κ be an uncountable regular cardinal and $\lambda^{<\kappa}=\lambda$. Then, the following propositions are equivalent*:

(1) *κ is λ-$L_{\omega_1\omega_1}$-compact*;

(2) *κ is λ-$L_{\omega_1\omega}$-compact*;

(3) *Any κ-complete κ-representable Boolean algebra of cardinality λ has a countably complete ultrafilter*;

(4) *If A is an abelian group of cardinality $\leq\lambda$, then $R_{\boldsymbol{Z}}(A)=R_{\boldsymbol{Z}}^{[\kappa]}(A)$ holds*;

(5) *If A is an abelian group of cardinality $\leq\lambda$ and any subgroup of A of cardinality less than κ is torsionless, then A itself is torsionless*;

(6) *Any subgroup of $\boldsymbol{Z}^{(B_{\kappa\lambda})}$ of cardinality $\leq\lambda$ is torsionless*;

(7) *For any subgroup S of $\boldsymbol{Z}^{(B_{\kappa\lambda})}$ of cardinality $\leq\lambda$,* $\mathrm{Hom}(S, \boldsymbol{Z})\neq 0$;

(8) *For any κ-complete κ-representable Boolean algebra B of cardinality $\leq\lambda$,* $\mathrm{Hom}(\boldsymbol{Z}^{(B)}, \boldsymbol{Z})\neq 0$.

To prove the theorem, we state some lemmas.

LEMMA 2. ([7, Theorem 1]) *Let B be a countably complete Boolean algebra. Then,* $\mathrm{Hom}(\boldsymbol{Z}^{(B)}, \boldsymbol{Z})=\bigoplus_{F\in\mathcal{F}}\boldsymbol{Z}$, *where $\mathcal{F}$ is the set of all countably complete ultrafilters of B. Consequently,* $\mathrm{Hom}(\boldsymbol{Z}^{(B)}, \boldsymbol{Z})\neq 0$ *iff a countably complete ultrafilter of B exists.*

LEMMA 3. ([13, 29.3]) *Let B be a κ-complete κ-representable Boolean algebra. If $b\neq 0$ and $\vee_{m\in N} b_{\alpha m}=1$ for $\alpha<\mu$ where $\mu<\kappa$, then exists an $f\in{}^{\mu}N$ such that $\{b, b_{\alpha f(\alpha)} : \alpha<\mu\}$ satisfies the finite intersection property.*

PROOF OF THEOREM 1. Our proofs go on according to the following diagram:

$$\begin{tikzcd} (1) \arrow[r] & (2) \arrow[r] \arrow[d] & (3) \arrow[r, leftrightarrow] & (8) \arrow[rrd] & & \\ & (4) \arrow[r] & (5) \arrow[r] & (6) \arrow[r] & (7) \arrow[r] & (1) \end{tikzcd}$$

(1)→(2): trivial.

(2)→(3): Let $\mathscr{F}$ be a κ-complete field and F a κ-complete fiter of $\mathscr{F}$ and $B=\mathscr{F}/F$. By the assumption of cardinality of λ, we can take a κ-complete subfield $\mathscr{F}'$ of $\mathscr{F}$ cardinality λ such that $B=\mathscr{F}'/\mathscr{F}'\cap F$. Let $\mathscr{F}'=\{P_\xi:\xi<\lambda\}$ and T be the set of the following $L_{\omega_1\omega}$-sentences:

(a) $\underline{P_\xi}(c)$ if $P_\xi\in F$:

(b) $\forall x(\wedge_{n\in N}\underline{P_{\xi n}}(x)\leftrightarrow\underline{P_\xi}(x))$ if $\bigcap_{n\in N}P_{\xi n}=P_\xi$;

(c) $\forall x(\underline{P_\xi}(x)\leftrightarrow\neg\underline{P_\eta}(x))$ if $P_\xi=P_\eta^c$.

Since F is κ-complete, any subset of T of cardinality less than κ has a model. Hence T has model $\mathscr{A}$. Let $P_\xi\in\overline{F}$ iff $\mathscr{A}\models\underline{P_\xi}(c)$. Then, $\overline{F}$ extends $\mathscr{F}'\cap F$ and is a countably complete ultrafilter of $\mathscr{F}'$. Consequently, B has a countably complete ultrafilter.

(3)↔(8): Clear by Lemma 2.

(2)→(4): To prove it by absurd, suppose the negation of (4). Then, there exists an $a^*\in R_Z(A)$ such that $a^*\notin R_Z^{[\kappa]}(A)$. Let T be the following set of $L_{\omega_1\omega}$-sentences:

(a) $\underline{a}\neq\underline{a'}$ for $a\neq a'$, $a,a'\in A$, $\underline{a}+\underline{b}=\underline{c}$ for $a+b=c$, $a,b,c\in A$;

(b) The axiom of abelian groups;

(c) $\forall x\vee_{m\in Z}(H_m(x)\ \&\ \wedge_{n\neq m,\, n\in Z}\neg H_n(x))$;

$\forall x,\ y\vee_{m,n,k\in Z,\ m+n=k}(H_m(x)\ \&\ H_n(x)\ \&\ H_k(x+y))$;

$\vee_{m\neq 0}H_m(\underline{a^*})$.

Let T' be a subset of T of cardinality less tank κ. Then, there exists a subgroup B of cardinality less than κ such that B contains a^* and if $\underline{a}$ appears in T' then a belongs to B. Since $a^*\notin R_Z^{[\kappa]}(A)$, there exists an $h\in\mathrm{Hom}(B,\boldsymbol{Z})$ such that $h(a^*)\neq 0$. Now, the group B with the homomorphism h is a model of T'. By (2) there exists a model $\mathscr{A}$ of T'. Then, A is a subgroup of the domain of $\mathscr{A}$ and $H_m(m\in\boldsymbol{Z})$ defines a homomorphism to $\boldsymbol{Z}$ which maps a^* to a nonzero element, which is a contradication.

(4)→(5): It is clear, since A is torsionless iff $R_Z(A)=0$.

(5)→(6): It is enough to show that S is torsionless for any subgroup of $\boldsymbol{Z}^{(B_{\kappa\lambda})}$ of cardinality less than κ. Let s^* be a nonzero element of S, then $s^*(m)\neq 0$ for some $m\neq 0$. By Lemma 3, there exists a map $h: S\to\boldsymbol{Z}$ such that $\{s(h(s)): s\in S\}$ satisfies the finite intersection property and $h(s^*)=m\neq 0$. If $s+t=u$ for $s,t,u\in S$, then $u(h(s)+h(t))\geq s(h(s))\wedge t(h(t))\neq 0$. Hence $u(h(s)+h(t))\wedge u(h(u))\neq 0$ and so $h(s)+h(t)=h(u)$. Now, We've gotten a desired homomorphism.

(6)→(7): Trivial.

(3)→(1) and (7)→(1): The property (1) is reduced to the existence of a countably complete ultrafilter of κ-complete subfield $\mathcal{F}$ of $P(P_\kappa\lambda)$ which extends $F_{\kappa\lambda}$[1, pp. 76–77; or 14, pp. 64–65]. By Lemma 2, both of (7) and (3) imply the existence of such an ultrafilter.

COROLLARY 4. *The radical R_Z satisfies the cardinal condition iff there exists a strongly $L_{\omega_1\omega}$-compact cardinal.*

The proof is clear by the equivalence of (2) and (4) of the theorem. Another characterization of the strongly $L_{\omega_1\omega}$-compact cardinal has been given in [2, Theorem 2]. As noted in [2, Theorem 4], the existence of a strongly $L_{\omega_1\omega}$-compact cardinal is strictly stronger than that of a measurable cardinal. However, we do not know whether it is strictly weaker than the existence of a strongly compact cardinal. (See the last remark.)

Under the assumption that κ is inaccessible, many conditions are known to be equivalent to the κ-$L_{\kappa\omega}$-compactness of κ. An observation of the proof of [14, Theorem 1] gives us

PROPOSITION 5. *Let κ be an infinite cardinal, then the following propositions are equivalent*:

(1) $\kappa\to(\kappa)^2_2$ *(See [14] or [12] for the definition.)*;

(2) *κ is $2^{<\kappa}$-$L_{\kappa\omega}$-compact*;

(3) *κ is regular and any κ-complete κ-representable Boolean algebra of cardinality $\leq 2^{<\kappa}$ has a κ-complete ultrafilter*;

(4) *κ is regular and any κ-complete subalgebra of B_κ of cardinality $\leq 2^{<\kappa}$ has a κ-complete ultrafilter.*

PROOF. Since $\kappa\to(\kappa)^2_2$ implies that κ is inaccessible, $2^{<\kappa}=\kappa$ and hence (1) →(2) is clear by [14, Theorem 1.13]. It is known that the κ-$L_{\kappa\omega}$-compactness of κ implies that κ is regular [3]. Hence, (2) implies that $2^{<\kappa}=\kappa^{<\kappa}$. The proof of implication (2)→(3) is similar to that of (2)→(3) of Theorem 1. The difference is to take $(b)'$ instead of (b), where $(b)'$ is: $\forall x(\bigwedge_{\alpha<\mu}\underline{P}_{\xi\alpha}(x)\leftrightarrow\underline{P}_\xi(x))$ if $\bigcap_{\alpha<\mu}P_{\xi\alpha}=P_\xi$ for $\mu<\kappa$. After this change the cardinality of the set of sentences does not exceed $2^{<\kappa}$. Therefore, we can prove similarly as before.

The implication (3)→(4) is clear. Though Silver's proof [14, p. 64] is essentially a proof of (4)→(1), we present the proof for reader's convenience. Suppose the negation of (1), then there exists $f:[\kappa]^2\to 2$ such that there exists no homogeneous set of cardinality κ. Let $\mathcal{F}$ be the minimal κ-complete subfield of $P(\kappa)$ generated by all singletons and $U_\alpha^i(=\{\beta : f(\{\alpha\beta\})=i\})$ for $\alpha<\kappa$, $i<2$. Then, the cardinality of $\mathcal{F}$ is $2^{<\kappa}$. Let $\pi : P(\kappa)\to B_\kappa(=P(\kappa)/F_\kappa)$ be the canonical map. Then, $\pi(\mathcal{F})$ is a κ-complete subalgebra of B_κ of cardinality $2^{<\kappa}$. Let F be a κ-complete ultrafilter of $\pi(\mathcal{F})$, then $\pi(U_\alpha^0)\in F$ or $\pi(U_\alpha^1)\in F$. Construct a sequence $\alpha_\xi(\xi<\kappa)$ and $\phi : \kappa\to 2$ such that $\alpha_\xi\in\bigcap_{\eta<\xi}U_{\alpha_\eta}^{\phi(\eta)}$and $\pi(U_{\alpha_\xi}^{\phi(\xi)})\in F$, then we can get homogeneous sets $\{\alpha_\xi : \phi(\xi)=0\}$ and $\{\alpha_\xi : \phi(\xi)=1\}$. One of them must be of cardinality κ, which is a contradiction.

As noted in [1, Corollary], if κ is less than the least measurable cardinal and $2^{<\kappa}$-$L_{\omega_1\omega}$-compact, then κ is $2^{<\kappa}$-$L_{\kappa\omega}$-compact. Any κ-complete subalgebra of a κ-comlete κ-representable Boolean algebra B is also κ-representable and any restriction $[0, b](=\{x\in B : 0\leq x\leq b\})$ for nonzero $b\in B$ is also a κ-complete κ-representable Boolean algebra. Hence, Theorem 1, Lemma 2 and Proposition 5 imply

COROLLARY 6. (B. Wald [15]) *Let κ be an uncountable regular cardinal which is less than the least measurable cardinal. Then,* the *following are equivalent*:

(1) $\kappa\to(\kappa)_2^2$ *holds*;

(2) *If A is an abelian group of cardinality $2^{<\kappa}$, then* $R_Z(A)=R_Z^{[\kappa]}(A)$;

(3) *If a subgroup S of $Z^{(B\kappa)}$ is of cardinality$\leq 2^{<\kappa}$, then* $\mathrm{Hom}(S, Z)\neq 0$.

REMARK: It is known that some results are restricted under the lest measurable cardinal and they do not hold beyond it [11, p. 161; and 5, Theorem 2.7]. However, we did not know whether the class of Fuchs-44-groups were closed under arbitrary direct products [8]. Here, we show that it is not. To treat such things it is convenient to use elementary embeddings of the universe [5, Remark 2; and 10]. Therefore, we use notions about elementary embeddings [12]. Let κ be the least measurable cardinal, F a normal ultrafilter on κ and M_F the related transitive universe. For an $f\in{}^\kappa V$, $[f]_F$ is the element of M_F corresponding to f. Let $A_\alpha(\alpha<\kappa)$ be the abelian groups such that $A_\alpha=(\bigoplus_\omega Z)^{(B\alpha)}$ if α is a regular uncountable cardinal and $A_\alpha=0$ otherwise. Since B_α has no countably complete ultrafilter, A_α is a Fuchs-44-group for each α [8, Corollary 3; and 9]. Since F is normal, $[\langle A_\alpha : \alpha<\kappa\rangle]_F=(\bigoplus_\omega Z)^{(B\kappa)}$ holds in M_F. Since $B_\kappa=(B_\kappa)^{M_F}$, $\Pi_{\alpha<\kappa}A_\alpha/F\simeq(\bigoplus_\omega Z)^{(B\kappa)}$. On the other hand, B_κ has a countably complete ultrafilter and hence there exists a surjective homomorphism from $\Pi_{\alpha<\kappa}A_\alpha/F$ to $\bigoplus_\omega Z$. This implies that $\Pi_{\alpha<\kappa}A_\alpha$ contains a direct summand isomorphic to $\bigoplus_\omega Z$. Hence, $\Pi_{\alpha<\kappa}A_\alpha$ is not a Fuchs-44-group.

As we have referred it before, Dugas and Göbel proved that the radical R_Z

does not commute with a measurable direct product [5, Theorem 2.7]. Here we show,

PROPOSITION 7. *Let κ be a cardinal less than the least measurable cardinal. If the cardinality of A_i is less than κ for every $i \in I$, then $R_Z(\Pi_{i\in I} A_i) = \Pi_{i\in I} R_Z(A_i)$ holds.*

PROOF. Since $R_Z(\Pi_{i\in I} A_i) \subseteq \Pi_{i\in I} R_Z(A_i)$ clearly, we show the other inclusion. $\mathrm{Hom}(\Pi_{i\in I} A_i, \boldsymbol{Z}) = \bigoplus_{F\in\mathcal{F}} \mathrm{Hom}(\Pi_{i\in I} A_i/F, \boldsymbol{Z})$ where $\mathcal{F}$ is the set of all countably complete ultrafilters on I [6, Corollary 2] and hence what we must show is that $h\cdot\pi_F(f) = 0$ holds for $f \in \Pi_{i\in I} R_Z(A_i)$, $h \in \mathrm{Hom}(\Pi_{i\in I} A_i/F, \boldsymbol{Z})$ and $F \in \mathcal{F}$, where $\pi_F : \Pi_{i\in I} A_i \to \Pi_{i\in I} A_i/F$ is the canonical homomorphism. By the fundamental theorem of ultraproducts [12], $V^I/F \models \forall h \in \mathrm{Hom}(\Pi_{i\in I} A_i/F, \ \Pi_I \boldsymbol{Z}/F)(h(\pi_F(f)) = 0)$. Since the cardinaity of $\Pi_{i\in I} A_i/F$ is less the least measurable cardinal and $\Pi_I \boldsymbol{Z}/F \simeq \boldsymbol{Z}$, $h\cdot\pi_F(f) = 0$ for each $h \in \mathrm{Hom}(\Pi_{i\in I} A_i/F, \boldsymbol{Z})$.

ADDED IN PROOF

1. There is another radical R_Z^∞, i.e. $R_Z^\infty A = \Sigma\{X \le A : \mathrm{Hom}(X, \boldsymbol{Z}) = 0\}$. The purpose of this addendum is to answer a question in [17]. Therefore, we use their notion.

We show,

PROPOSITION 8.

(1) *The radical R_Z^∞ satisfies the cardiual condition (iff R_Z^∞ is a singly generated socle) iff there exists a strongly $L_{\omega_1\omega}$-compact cardinal.*

(2) *R_Z^∞ is not a singly generated radical.*

PROOF. First observe the following fact: For a cardinal κ of uncountable cofinality, $A = \Sigma\{X \le A : \mathrm{Hom}(X, \boldsymbol{Z}) = 0 \ \& \ |X| < \kappa\}$ iff $A = \Sigma\{R_Z X : X \le A \ \& \ |X| < \kappa\}$. This can be shown by a closure argument. If there exists a strongly $L_{\omega_1\omega}$-compact cardinal, let κ be a regular strongly $L_{\omega_1\omega}$-compact cardinal. Suppose that $R_Z^\infty A \neq \Sigma\{R_Z^\infty X : X \le A \ \& \ |X| < \kappa\}$. Since $R_Z^\infty Y$ is the largest subgroup X of Y such that $\mathrm{Hom}(X, \boldsymbol{Z}) = 0$, $R_Z^\infty A \neq \Sigma\{R_Z X : X \le R_Z^\infty A \ \& \ |X| < \kappa\}$ by the above fact. Hence, there exists an $a^* \in R_Z^\infty A$ such that $a^* \notin R_Z X$ for any subgroup X of $R_Z^\infty A$ of cardinality less than κ. As the proof of (2)$\to$(4) of Theorem 1, we get a nonzero homomorphism $R_Z^\infty A$ to $\boldsymbol{Z}$, which is a contradiction.

Suppose that a regular cardinal κ is not strongly $L_{\omega_1\omega}$-compact. Then, there exists a λ such that $\lambda = \lambda^{<\kappa}$ and κ is not λ-$L_{\omega_1\omega}$-compact. By Theorem 1 (7) and a fact in the proof of (5)$\to$(6) of Theorem 1, there exists a group S such that $R_Z^\infty S = S$ and $\Sigma\{X \le S : \mathrm{Hom}(X, \mathrm{A}) = 0 \ \& \ |X| < \kappa\} = 0$. Hence, the cardinal condi-

tion does not hold. Another equivalence is easy to show.
(2) (The same reasoning as [17, Proposition 2.8]) Suppose that R_Z^∞ is a singly generated radical, i.e. $R_Z^\infty A = R_Y A = \cap \{\mathrm{Ker}(h) : h \in \mathrm{Hom}(A, Y)\}$. Then, $R_Z^\infty = R_Y Y = 0$. Let α be an ordinal such that $R_Z^\alpha Y = 0$. By [16, Corollary 3.10] (due to Mines), there exists a group A such that $R_Z^\infty A = 0$ and $R_Z^\alpha A \neq 0$. Since A is isomorphic to a subgroup of the direct product Y^I for some I, $R_Z^\alpha A \leq R_Z^\alpha Y^I \leq (R_Z^\alpha Y)^I = 0$, which is a contradiction.

2. Recently, G. Bergman and R. M. Solovay [18] announced a similar result to Theorem 1, i.e. The class of all torsionless groups is characterized by a set of generalized Horn sentences, iff there exists a strongly $L_{\omega_1\omega}$-compact cardinal. They also commented that M. Magidor had shown that the existence of a strongly $L_{\omega_1\omega}$-compact cardinal is strictly weaker than that of a strongly compact cardinal, which answers our question after Corollary 4.

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