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SHAPE FIBRATIONS AND FIBER SHAPE EQUIVALENCES, I

By

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0. Introduction.

In [6], Coram and Duvall introduced approximate fibrations and Mardešić and Rushing [11] generalized this and defined shape fibrations. For compact ANR's, shape fibrations agree with approximate fibrations. M. Jani, analogous to fiber maps, defined fiber morphisms and fiber shape equivalences [8]. In [4], Chapman proved the Complement Theorem, i.e., if X and Y are Z-sets in the Hilbert cube Q, then X and Y have the same shape (i.e., $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$, see [2]) iff $Q-X$ and $Q-Y$ are homeomorphic.

In this paper, we define notions of fiber fundamental sequences and fiber shape equivalences and prove that if a fiber fundamental sequences between approximate fibrations is a shape equivalence, then it is a fiber shape equivalence. Also, we prove the following: Let E, E' and B be compacta in the Hilbert cube Q and let $E, E'\subset Q$ be Z-sets. Then a map $p: E\rightarrow B$ over B is fiber shape equivalent to a map $p': E'\rightarrow B$ over B if and only if there is a homeomorphism $h: Q-E\cong Q-E'$ such that for each $b\in B$ and each neighborhood W' of $p'^{-1}(b)$ in Q, there is a neighborhood W of $p^{-1}(b)$ in Q such that $h(W-E)\subset W'-E'$.

All spaces considered will be metrizable. If x and y are points of a metric space, $d(x, y)$ denotes the distance from x to y. A proper map $p: E\rightarrow B$ between locally compact, separable metric ANR's is an *approximate fibration* [6] if given an open cover $\mathcal{U}$ of B, a space X and maps $h: X\rightarrow E$, $H: X\times I\rightarrow B$ such that $ph=H_0$, then there is a homotopy $\tilde{H}: X\times I\rightarrow E$ such that $\tilde{H}_0=h$ and H and $p\tilde{H}$ are $\mathcal{U}$-close, where $H_t(x)=H(x, t)$. Let $\underline{E}=(E_i, q_{ij})$ and $\underline{B}=(B_i, r_{ij})$ be inverse sequences of compacta and let $\underline{p}=(p_i)$ be a sequence of maps $p_i: E_i\rightarrow B_i$. Then $\underline{p}: \underline{E}\rightarrow\underline{B}$ is a *level map* if for any i and $j\geqq i$, $p_i q_{ij}=r_{ij}p_j$. A map $p: E\rightarrow B$ between compacta is a *shape fibration* [11] if there is a level map $\underline{p}: \underline{E}\rightarrow\underline{B}$ of compact ANR-sequences with $\underleftarrow{\lim}\,\underline{E}=E$, $\underleftarrow{\lim}\,\underline{B}=B$ and $\underleftarrow{\lim}\,\underline{p}=p$ satisfying the following property; for each i and $\varepsilon>0$ there is $j\geqq i$ and $\delta>0$ such that for any space X and any $h: X\rightarrow E_j$, $H: X\times I\rightarrow B_j$ with $d(p_j h, H_0)=\sup\{d(p_j h(x), H_0(x))|x\in X\}<\delta$, there is a homotopy

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$\widetilde{H}: X\times I\to E_i$ such that $d(\widetilde{H}_0, q_{ij}h)<\varepsilon$ and $d(p_i\widetilde{H}, r_{ij}H)<\varepsilon$. Such (E_j, δ) is called a *lifting pair* for (E_i, ε).

1. Fiber fundamental sequences.

In [8], M. Jani introduced the notions of fiber morphisms and fiber shape equivalences. In this section, we conveniently give the following definitions (compare [8, Definition 4.1, 4.2 and 4.3]). It is assumed that E, E' and B are compacta contained in the Hilbert cube Q and maps $\tilde{p}: Q\to Q$ and $\tilde{p}': Q\to Q$ are extensions of maps $p: E\to B$ and $p': E'\to B$, respectively.

DEFINITION 1.1. A fundamental sequence (see [2]) $\underline{f}=\{f_n, E, E'\}_{Q,Q}$ is a *fiber fundamental sequence over* B if for any $\varepsilon>0$ and any neighborhood U' of E' in Q there is a neighborhood U of E in Q and an integer n_0 such that for each $n\geqq n_0$ there is a homotopy $H: U\times I\to U'$ satisfying

(1) $H_0=f_{n_0}|U$ and $H_1=f_n|U$,

(2) $d(\tilde{p}'H(x, t), \tilde{p}(x))<\varepsilon$, $x\in U, t\in I$.

REMARK 1.2. Definition 1.1 is independent of the choices of the extensions $\tilde{p}$ and $\tilde{p}'$ of p and p', respectively.

DEFINITION 1.3. A fiber fundamental sequence $\underline{f}=\{f_n, E, E'\}_{Q,Q}$ over B is *fiber homotopic* to a fiber fundamental sequence $\underline{g}=\{g_n, E, E'\}_{Q,Q}$ over B ($\underline{f}\underset{B}{\simeq}\underline{g}$) if for any $\varepsilon>0$ and any neighborhood U' of E' in Q there is a neighborhood U of E in Q and an integer n_0 such that for any $n\geqq n_0$ there is a homotopy $K: U\times I\to U'$ satisfying

(1) $K_0=f_n|U$ and $K_1=g_n|U$,

(2) $d(\tilde{p}'K(x, t), \tilde{p}(x))<\varepsilon$, $x\in U, t\in I$.

REMARK 1.4. If $f: E\to E'$ is a fiber map over B (i.e. $p'f=p$), f induces a fiber fundamental sequence $\underline{f}=\{f_n, E, E'\}_{Q,Q}$, where $f_n=\tilde{f}: Q\to Q$ is an extension of f. Also, we can easily see that the composition of fiber fundamental sequences over B is a fiber fundamental sequence over B.

PROPOSITION 1.5. *Let $p: E\to B$, $p': E'\to B$ and $p'': E''\to B$ be maps over B and let $\underline{f}_i=\{f_{i,n}, E, E'\}_{Q,Q}$ and $\underline{g}_i=\{g_{i,n}, E', E''\}_{Q,Q}$ $(i=1,2)$ be fiber fundamental sequences over B. If $\underline{f}_1\underset{B}{\simeq}\underline{f}_2$ and $\underline{g}_1\underset{B}{\simeq}\underline{g}_2$, then $\underline{g}_1\underline{f}_1\underset{B}{\simeq}\underline{g}_2\underline{f}_2$.*

DEFINITION 1.6. A map $p: E\to B$ over B is *fiber shape equivalent to* a map

$p':E'\to B$ over B if there are fiber fundamental sequences over B $f=\{f_n, E, E'\}_{Q,Q}$ and $g=\{g_n, E', E\}_{Q,Q}$ such that $gf \underset{B}{\simeq} 1_E$ and $fg \underset{B}{\simeq} 1_{E'}$, where 1_E denotes a fiber fundamental sequence induced by the identity $1_E: E\to E$. Such f is called a *fiber shape equivalence over B.*

PROPOSITION 1.7. *If a fiber fundamental sequence $f=\{f_n, E, E'\}_{Q,Q}$ over B is a fiber shape equivalence over B, then for any compactum $B_1\subset B$ the restriction $f|p^{-1}(B_1) =\{f_n, p^{-1}(B_1), p'^{-1}(B_1)\}_{Q,Q}$ is a fiber shape equivalence over B_1.*

A map $p: E\to B$ between compacta is *shape shrinkable* if p induces a fiber shape equivalence from $p: E\to B$ to the identity $1_B: B\to B$. Let $p: S^1\times S^1\to S^1$ be the same as [6, p. 277, Example]. Then it is easily seen that p is fiber shape equivalent to the projection $q: p^{-1}(b)\times S^1\to S^1$ for $b\in S^1$, but p is not fiber homotopy equivalent to the projection q.

2. Fiber shape equivalences.

In this section, we shall show that if a fiber fundamental sequence from a shape fibration to an approximate fibration is a shape equivalence, then it is a fiber shape equivalence. By using this result, we see that a map $p: E\to B$ between compact ANR's is shape shrinkable if and only if p is a CE-map.

We need the following lemma.

LEMMA 2.1. *Let E, E' and B be compacta and let $p: E\to B$ and $p': E'\to B$ be shape fibrations. If a fiber fundamental sequence $f=\{f_n, E, E'\}_{Q,Q}$ over B is a weak domination* (see [7, p. 8]) *in shape category, then f is a fiber weak domination, i.e., for any $\varepsilon>0$ and any neighborhood U' of E' in Q, there is a neighborhood U of E in Q and an integer n_0 satisfying the conditions of Definition* 1.1 *such that for any $\eta>0$ and any neighborhood $W\subset U$ of E in Q there is a neighborhood $W'\subset U'$ of E' in Q, a map $g: W'\to W$ and a homotopy $R: W'\times I\to U'$ such that*

(1) $d(\tilde{p}g(e'), \tilde{p}'(e'))<\eta$, $e'\in W'$,

(2) $R(e',0)=e'$, $R(e',1)=f_{n_0}g(e')$, $e'\in W'$ and

(3) $d(\tilde{p}'R(e',t), \tilde{p}'(e'))<\varepsilon$, $e'\in W', t\in I$.

PROOF. Since $f=\{f_n, E, E'\}_{Q,Q}$ is a fiber fundamental sequence, there is a neighborhood U of E in Q and an integer n_0 such that for each $n\geqq n_0$ there is a homotopy $F_{n_0,n}: U\times I\to U'$ such that $F_{n_0,n}(e,0)=f_{n_0}(e)$, $F_{n_0,n}(e,1)=f_n(e)$ and $d(\tilde{p}'F_{n_0,n}(e,t), \tilde{p}(e))<\varepsilon/2$ for $e\in U, t\in I$. Let $\eta>0$ and W be any neighborhood of E in Q with $W\subset U$. Since $p: E\to B$ and $p': E'\to B$ are shape fibrations, by [11, Theorem 1], inductively we can find compact ANR's E_i, E_i', B_i $(i=1,2,3)$ and $\varepsilon_i>0$, $\delta_i>0$ $(i=1,2)$ and an integer $n_1 \geqq n_0$ such that

(1) $W \supset E_1 \supset E_2 \supset E_3 \supset \mathrm{Int}_Q E_3 \supset E$, $U' \supset E_1' \supset E_2' \supset E_3' \supset \mathrm{Int}_Q E_3' \supset E'$,
$B_1 \supset B_2 \supset B_3 \supset \mathrm{Int}_Q B_3 \supset B$ and $\tilde{p}(E_i) \subset B_i$, $\tilde{p}'(E_i') \subset B_i$ $(i=1,2,3)$,

(2) (E_2', δ_1) is a lifting pair for (E_1', ε_1) and (E_3, δ_2) is a lifting pair for (E_2, ε_2),

(3) any $2\varepsilon_2$-near maps to B_2 are ε_1-homotopic and

(4) $\varepsilon_1 < \varepsilon/2$, $\varepsilon_2 < \mathrm{Min}\{\eta, \varepsilon/2\}$ and $f_{n_1}(E_i) \subset E_i'$ $(i=1,2,3)$,
$d(\tilde{p}' f_{n_1}|E_3, \tilde{p}|E_3) < \delta_2$ and $d(\tilde{p}' f_{n_1}|E_2, \tilde{p}|E_2) < \varepsilon_2$.

Since f is a weak domination in shape category, we may assume that there is a neighborhood W' of E' in Q with $W' \subset E_3'$, a map $g': W' \to E_3$ and a homotopy $H: W' \times I \to E_3'$ such that

(5) $H(e', 0) = e'$, $H(e', 1) = f_{n_1} g'(e')$, $e' \in W'$.

By (4), $d(\tilde{p} g'(e'), \tilde{p}' H(e', 1)) = d(\tilde{p} g'(e'), \tilde{p}' f_{n_1} g'(e')) < \delta_2$, $e' \in W'$. By (2) and [11, Proposition 1], there is a homotopy $\tilde{H}: W' \times I \to E_2$ such that

(6) $\tilde{H}(e', 1) = g'(e')$, $e' \in W'$ and

(7) $d(\tilde{p}\tilde{H}, \tilde{p}' H) < \varepsilon_2$.

Define a map $g: W' \to E_2 \subset W$ by

(8) $g(e') = \tilde{H}(e', 0)$, $e' \in W'$.

By (4), (5), (7) and (8) we have

(9) $d(\tilde{p} g(e'), \tilde{p}'(e')) < \varepsilon_2 < \mathrm{Min}\{\eta, \varepsilon/2\}$.

Define a homotopy $L: W' \times [0,2] \to E_2'$ by

(10) $$L(e', s) = \begin{cases} H(e', s), & e' \in W', \quad 0 \leqq s \leqq 1, \\ f_{n_1}\tilde{H}(e', 2-s), & e' \in W', \quad 1 \leqq s \leqq 2. \end{cases}$$

Then $L(e', 0) = e'$ and $L(e', 2) = f_{n_1} g(e')$, $e' \in W'$. By (4), (7) and (10),

(11) $$\begin{aligned} & d(\tilde{p}' L(e', s), \tilde{p}' L(e', 2-s)) = d(\tilde{p}' H(e', s), \tilde{p}' f_{n_1}\tilde{H}(e', s)) \\ & \leqq d(\tilde{p}' H(e', s), \tilde{p}\tilde{H}(e', s)) + d(\tilde{p}\tilde{H}(e', s), \tilde{p}' f_{n_1}\tilde{H}(e', s)) \\ & < \varepsilon_2 + \varepsilon_2 = 2\varepsilon_2, \quad 0 \leqq s \leqq 1. \end{aligned}$$

By (3), there is a homotopy $K: W' \times [0,2] \times [0,1] \to B_2$ such that

(12) $K(e', s, t) = \tilde{p}' L(e', s)$, $t \leqq 1-s$ or $t \leqq s-1$,

(13) $d(\tilde{p}'(e'), K(e', s, 1)) < \varepsilon_1$, $0 \leqq s \leqq 2$.

Define a map $L': W'\times(0\times[0,1]\cup[0,2]\times 0\cup 2\times[0,1])\to E_2'$ by

$$(14)\quad L'(e',s,t)=\begin{cases} L(e',0), & s=0,\ 0\leqq t\leqq 1, \\ L(e',s), & 0\leqq s\leqq 2,\ t=0, \\ L(e',2), & s=2,\ 0\leqq t\leqq 1. \end{cases}$$

Then $\tilde{p}'L'=K|W'\times(0\times[0,1]\cup[0,2]\times 0\cup 2\times[0,1])$. By (2), there is a map $\tilde{K}: W'\times[0,2]\times[0,1]\to E_1'$ such that

(15) $\tilde{K}|W'\times(0\times[0,1]\cup[0,2]\times 0\cup 2\times[0,1])=L'$ and

(16) $d(\tilde{p}'\tilde{K},K)<\varepsilon_1$.

Define a homotopy $G: W'\times[0,2]\to E_1'$ by

(17) $G(e',s)=\tilde{K}(e',s,1)$.

By (4), (13), (16) and (17) we have

$$(18)\quad d(\tilde{p}'G(e',s),\tilde{p}'(e'))\leqq d(\tilde{p}'\tilde{K}(e',s,1),K(e',s,1))+d(K(e',s,1),\tilde{p}'(e')) < \varepsilon_1+\varepsilon_1<\varepsilon.$$

Then $G(e',0)=e'$ and $G(e',2)=f_{n_1}g(e')$, $e'\in W'$. Define a homotopy $R: W'\times[0,3]\to U'$ by

$$(19)\quad R(e',t)=\begin{cases} G(e',t), & 0\leqq t\leqq 2, \\ F_{n_0,n_1}(g(e'),3-t), & 2\leqq t\leqq 3. \end{cases}$$

Then $R(e',0)=e'$, $R(e',3)=f_{n_0}g(e')$ and $d(\tilde{p}'R(e',t),\tilde{p}'(e'))<\varepsilon$ for $e'\in W'$, $t\in[0,3]$. Hence $\underline{f}$ is a fiber weak domination.

COROLLARY 2.2. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta. If a fiber fundamental sequence $\underline{f}=\{f_n, E, E'\}_{Q,Q}$ is a weak domination in shape category, then for any compactum $B_1\subset B$, the restriction*

$$\underline{f}|p^{-1}(B_1)=\{f_n, p^{-1}(B_1), p'^{-1}(B_1)\}_{Q,Q}$$

is a fiber weak dominatian, hence we have the following.

(1) *If $p^{-1}(B_1)$ is movable* (see [3]), *then $p'^{-1}(B_1)$ is movable.*
(2) *If $p^{-1}(B_1)\in AC^n$* (see [3]), *then $p'^{-1}(B_1)\in AC^n$.*
(3) *If $p^{-1}(B_1)$ is an FAR* (see [3]), *then $p'^{-1}(B_1)$ is an FAR.*
(4) $\mathrm{Fd}(p^{-1}(B_1))\geqq\mathrm{Fd}(p'^{-1}(B_1))$ (see [3]).

THEOREM 2.3. *Let $p: E\to B$ be a shape fibration from a compactum E to a compact ANR B and let $p': E'\to B$ be an approximate fibration between compact*

ANR's. Then a fiber fundamental sequence $f=\{f_n, E, E'\}_{Q,Q}$ over B is a fiber shape equivalence if and only if it is a shape equivalence.

PROOF. It is enough to give the proof of sufficiency. Since E' and B are ANR's, we may assume that there is a neighborhood U of E in Q and an extension $\tilde{p}: Q\to Q$ of $p: E\to B$ such that $f_n(U)\subset E'$ for all n and $\tilde{p}(U)\subset B$. Let $\varepsilon_1>\varepsilon_2>\varepsilon_3>\cdots$, be a sequence of positive numbers with $\lim \varepsilon_i=0$. Since p is a shape fibration, inductively we can find a sequence $U\supset E_1\supset E_{1+1/2}\supset E_2\supset E_{2+1/2}\supset\cdots$, of compact ANR's, an increasing sequence $k_1<k_2<k_3<\cdots$, of natural numbers and a sequence $\delta_1>\delta_2>\delta_3>\cdots$, $(\delta_i<\varepsilon_i)$ of positive numbers such that

(1) $\mathrm{Int}_Q E_i\supset E$ and $\bigcap_{i=1}^{\infty} E_i=E$,

(2) $(E_{i+1/2}, 2\delta_i)$ is a lifting pair for $(E_i, \varepsilon_i/2)$, $i=1,2,\cdots$, and

(3) for each $k\geqq k_i$, there is a homotopy $F_{k_i,k}: E_i\times I\to E'$ such that $F_{k_i,k}(e,0)=f_{k_i}(e)$, $F_{k_i,k}(e,1)=f_k(e)$ and $d(p'F_{k_i,k}(e,t), \tilde{p}((e))<\varepsilon_i$, $e\in E_i, t\in I$.

Since $p': E'\to B$ is an approximate fibration, there is a sequence $\delta_1'>\delta_2'>\delta_3'>\cdots$, $(\delta_i'<\delta_i)$ of positive numbers such that (E', δ_i') is a lifting pair for (E', δ_i). By Lemma 2.1, we may assume that there is a map $g_i: E'\to E_{i+1/2}$ and a homotopy $R_i: E'\times I\to E'$ for each i such that

(4) $d(\tilde{p}g_i(e'), p'(e'))<\delta_i'$, $e'\in E'$,

(5) $R_i(e',0)=e'$, $R_i(e',1)=f_{k_i}g_i(e')$, $e'\in E'$ and

(6) $d(p'R_i(e',t), p'(e'))<\varepsilon_i$, $e'\in E', t\in I$.

Since f is a shape equivalence, by the construction of g_i (see the proof of Lemma 2.1) we may assume that there is a homotopy $L_i: E_{i+1}\times I\to E_{i+1/2}$ with $L_i(e,0)=e$, $L_i(e,1)=g_i f_{k_{i+1}}(e)$, $e\in E_{i+1}$. By (4), we have

(7) $d(\tilde{p}L_i(e,1), p'f_{k_{i+1}}(e))=d(\tilde{p}g_i f_{k_{i+1}}(e), p'f_{k_{i+1}}(e))<\delta_i'$, $e\in E_{i+1}$.

Hence, by (2) and the same way as the proof of Lemma 2.1, there is a map $f'_{k_{i+1}}: E_{i+1}\to E'$ and a homotopy $M_i: E_{i+1}\times I\to E_i$ such that

(8) $d(p'f'_{k_{i+1}}(e), \tilde{p}(e))<\delta_i$, $e\in E_{i+1}$,

(9) $M_i(e,0)=e$, $M_i(e,1)=g_i f'_{k_{i+1}}(e)$, $e\in E_{i+1}$ and

(10) $d(\tilde{p}M_i(e,t), \tilde{p}(e))<\varepsilon_i$, $e\in E_{i+1}$.

By (3), (5) and (9), we can define a homotopy $G_i: E_{i+1}\times[0,3]\to E'$ by

$$(11)\quad G_i(e,t)=\begin{cases} R_i(f'_{k_{i+1}}(e),t)\,, & 0\leqq t\leqq 1 \\ f_{k_i}M_i(e,2-t)\,, & 1\leqq t\leqq 2\,, \\ F_{k_i,k_{i+1}}(e,t-2)\,, & 2\leqq t\leqq 3\,. \end{cases}$$

By (3), (6), (10) and (11), we have

(12) $G_i(e,0)=f'_{k_{i+1}}(e)$, $G_i(e,3)=f_{k_{i+1}}(e)$, $e\in E_{i+1}$ and

(13) $d(p'G_i(e,t),\tilde{p}(e))<2\varepsilon_i$, $e\in E_{i+1}$, $0\leqq t\leqq 3$.

By (12) and (13), we obtain a fiber fundamental sequence f' over B induced by $\{f'_{k_i}:E_{k_i}\to E'\}$ such that $f'\underset{B}{\simeq}f$. By (5), (9) and (12), we can define a homotopy $S_i:E'\times[0,5]\to E_i$ by

$$(14)\quad S_i(e',t)=\begin{cases} M_i(g_{i+1}(e'),t)\,, & 0\leqq t\leqq 1\,, \\ g_iG_i(g_{i+1}(e'),t-1)\,, & 1\leqq t\leqq 4\,, \\ g_iR_{i+1}(e',5-t)\,, & 4\leqq t\leqq 5\,. \end{cases}$$

Then $S_i(e',0)=g_{i+1}(e')$, $S_i(e',5)=g_i(e')$, $e'\in E'$. Also by (4), (6), (10), (13) and (14), we have $d(\tilde{p}S_i(e',t),p'(e'))<4\varepsilon_i$ for $e'\in E'$ $0\leqq t\leqq 5$. Hence we obtain a fiber fundamental sequence g over B induced by $\{g_i:E'\to E_i\}$. By (9) and (10), we conclude that $gf\underset{B}{\simeq}gf'\underset{B}{\simeq}1_E$. Also by (5) and (6), $fg\underset{B}{\simeq}1_{E'}$. Hence f is a fiber shape equivalence over B.

COROLLARY 2.4. *Let $p:E\to B$ and $p':E'\to B$ be approximate fibrations between compact ANR's. If a fiber fundamental sequence $f=\{f_n,E,E'\}_{Q,Q}$ over B is a shape equivalence, then it is a fiber shape equivalence. In particular, if a fiber map $f:E\to E'$ over B is a homotopy equivalence, it is a fiber shape equivalence.*

The next result follows from Vietoris-Smale theorem, [10, Lemma 2.3 or 11, Theorem 4], Corollary 1.7 and 2.4.

COROLLARY 2.5. *Let $p:E\to B$ be a map between compact ANR's. Then the following are equivalent.*

(1) *p is a CE-map.*
(2) *p is a homotopy equivalence and an approximate fibration.*
(3) *p is shape shrinkable.*
(4) *p is a hereditary shape equivalence.*

3. The Complement Theorem of fiber shape equivalences.

In this section, we prove the following theorem.

THEOREM 3.1. *Let E, E' and B be compacta in the Hilbert cube Q and let $E, E' \subset Q$ be Z-sets. Then a map $p: E \to B$ over B is fiber shape equivalent to a map $p': E' \to B$ over B if and only if there is a homeomorphism $h: Q-E \cong Q-E'$ such that for each $b \in B$ and each neighborhood W' of $p'^{-1}(b)$ in Q, there is a neighborhood W of $p^{-1}(b)$ in Q such that $h(W-E) \subset W'-E'$.*

COROLLARY 3.2. *Let E, E' and B be compacta in the Hilbert cube Q and let $E, E' \subset Q$ be Z-sets. Then a map $p: E \to B$ over B is fiber shape equivalent to a map $p': E' \to B$ over B if and only if there is a homeomorphism $h: Q-E \cong Q-E'$ such that for any extension $\tilde{p}': Q \to Q$ of p' there is the extension $\tilde{p}: Q \to Q$ of p such that $\tilde{p}|Q-E = \tilde{p}'h$.*

COROLLARY 3.3. *Let E and B be Z-sets in the Hilbert cube Q. Then a map $p: E \to B$ is shape shrinkable if and only if there is an extension $\tilde{p}: Q \to Q$ of p such that $\tilde{p}|Q-E: Q-E \cong Q-B$ is a homeomorphism.*

Let $\mathcal{U}$ be a collection of subsets of a space Y. A map $f: X \to Y$ is *$\mathcal{U}$-close* to a map $g: X \to Y$ if for each $x \in X$, there is $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. A homotopy $H: X \times I \to Y$ is *$\mathcal{U}$-limited* if for each $x \in X$ there is $U \in \mathcal{U}$ such that $H(\{x\} \times I) \subset U$. A closed subset A in a space X is a *Z-set* in X if for each open cover $\mathcal{U}$ of X there is a map of X into $X-A$ which is $\mathcal{U}$-close to the identity 1_X. A map $f: A \to X$ is a *Z-embedding* if f is an embedding and $f(A)$ is a Z-set in X.

The proof of Theorem 3.1 is analogous to one of Chapman's [4], but much sharper results will be used. We need the followings.

LEMMA 3.4 (see [1, Theorem 3.1] or [5, Theorem 11.2]). *If (A, A_0) is a compact pair and $f: A \to Q$ is a map such that $f|A_0$ is a Z-embedding, then for any open cover $\mathcal{U}$ of Q there is a Z-embedding $g: A \to Q$ such that $g|A_0 = f|A_0$ and g is $\mathcal{U}$-close to f.*

LEMMA 3.5 (see [1, Theorem 6.1] or [5, Theorem 19.4]). *Let M be a Q-manifold, A be a compactum and let $F: A \times I \to M$ be a map such that F_0 and F_1 are Z-embeddings. If F is $\mathcal{U}$-limited for an open cover $\mathcal{U}$ of M, then there is an isotopy $H: M \times I \to M$ such that $H_0 = id$, $H_1F_0 = F_1$ and H is $\mathcal{U}$-limited.*

PROOF OF THEOREM 3.1. Let $h: Q-E \cong Q-E'$ be a homeomorphism satisfying the condition as above. Note that for each $b \in B$ and each neighborhood W of $p^{-1}(b)$ in Q there is a neighborhood W' of $p'^{-1}(b)$ in Q such that $h^{-1}(W'-E') \subset W-E$. In fact, suppose, on the contrary, that there is a sequence $\{x_i'\}_{i=1,2,\ldots}$ such that $x_i' \in$

$Q-E'$, $\lim_{i\to\infty} x_i'=x'\in p'^{-1}(b)$ and $h^{-1}(x_i')\in Q-W$ for each i. Choose a subsequence $\{x'_{n_i}\}$ of $\{x_i'\}$ such that $\lim_{i\to\infty} h^{-1}(x'_{n_i})=x\in E-p(b)$. Let W_1' and W_2' be neighborhoods of $p'^{-1}(b)$ and $p'^{-1}(p(x))$ in Q, respectively, such that $W_1'\cap W_2'=\phi$. Since there is a neighborhood W_2 of $p^{-1}(p(x))$ in Q such that $h(W_2-E)\subset W_2'-E'$ and $h^{-1}(x'_{n_i})\in W_2$ for almost all i, $h(h^{-1}(x'_{n_i}))=x'_{n_i}\in W_2'$, which implies the contradiction.

Since $E\subset Q$ is a Z-set, there is a homotopy $F: Q\times I\to Q$ such that $F(q,0)=q$, $F(q,t)\in Q-E$, for $q\in Q$, $0<t\leqq 1$. Similarly there is a homotopy $G: Q\times I\to Q$ such that $G(q,0)=q$, $G(q,t)\in Q-E'$, for $q\in Q$, $0<t\leqq 1$. Define maps $f_n: Q\to Q$ and $g_n: Q\to Q$ for each integer n by $f_n(q)=h(F(q,1/n))$, $g_n(q)=h^{-1}(G(q,1/n))$, for $q\in Q$. Consider $f=\{f_n,E,E'\}_{Q,Q}$ and $g=\{g_n,E',E\}_{Q,Q}$. Then we shall show that f and g are fiber fundamental sequences over B such that $gf \underset{B}{\simeq} 1_E$ and $gf \underset{B}{\simeq} 1_{E'}$. Let $\tilde{p}: Q\to Q$ and $\tilde{p}': Q\to Q$ be extensions of p and p', respectively. Let U' be a neighborhood of E' in Q and let $\varepsilon>0$. For each $b\in B$, choose a neighborhood C_b of b in Q such that $\operatorname{diam} C_b<\varepsilon/2$. Then there is a neighborhood W_b of $p^{-1}(b)$ in Q such that $h(W_b-E)\subset[U'\cap\tilde{p}'^{-1}(C_b)]-E'$ and $\tilde{p}(W_b)\subset C_b$. Choose a finite collection $\{W_{b_1}, W_{b_2},\cdots, W_{b_m}\}$ such that $\bigcup_{i=1}^{m} W_{b_i}\supset E$. Also choose a neighborhood U of E in Q and an integer N_0 such that $F(U\times[0,1/N_0])\subset h^{-1}(U'-E')\cup E$ and for any $q\in U$, $F(\{q\}\times[0,1/N_0])\subset W_{b_i}$ for some i. For each $n\geqq N_0$, define a homotopy $H: U\times[1/n+1,1/n]\to U'$ by $H(q,t)=h(F(q,t))$, for $q\in U$, $1/n+1\leqq t\leqq 1/n$. Then $H(q,1/n+1)=f_{n+1}(q)$, $H(q,1/n)=f_n(q)$ and $d(\tilde{p}(q),\tilde{p}'H(q,t))<\varepsilon$, for $q\in U$, $1/n+1\leqq t\leqq 1/n$. Hence f is a fiber fundamental sequence over B. Similarly, g is a fiber fundamental sequence over B. To see that $gf \underset{B}{\simeq} 1_E$, choose a neighborhood U of E in Q and $\varepsilon>0$. By the same way as above, we can choose a neighborhood V' of E' in Q and $\varepsilon_1>0$ such that $h^{-1}(V'-E')\subset U-E$ and $h^{-1}G(q,t)\in U$, for $q\in V'$, $0<t\leqq\varepsilon_1$ and $d(\tilde{p}h^{-1}G(q,t),\tilde{p}'(q))<\varepsilon/2$, for $q\in V'$, $0<t\leqq\varepsilon_1$. Choose a small neighborhood V of E in Q ($V\subset U$) and $\varepsilon_2>0$ ($\varepsilon_2<\varepsilon_1$) such that $hF(q,t)\in V'$, for $q\in V$, $0<t\leqq\varepsilon_2$ and $d(\tilde{p}'hF(q,t),\tilde{p}(q))<\varepsilon/2$, for $q\in V$, $0<t\leqq\varepsilon_2$. Also, we may assume that $d(\tilde{p}F(q,t),\tilde{p}(q))<\varepsilon/2$, for $q\in V$, $0\leqq t\leqq\varepsilon_2$. Let N_1 be an integer sucn that $\varepsilon_2>1/N_1$. Then for each $n\geqq N_1$, we can define a homotopy $H: V\times I\to U$ by

$$H(q,t)=\begin{cases} h^{-1}G(hF(q,1/n),1/n-t), & q\in V,\ 0\leqq t\leqq 1/n, \\ F(q,(t-1)/(1-n)), & q\in V,\ 1/n\leqq t\leqq 1. \end{cases}$$

Then $H(q,0)=g_nf_n(q)$, $H(q,1)=q$ for $q\in V$ $d(\tilde{p}(q),\tilde{p}H(q,t))<\varepsilon$, for $q\in V$, $t\in I$. Hence, $gf \underset{B}{\simeq} 1_E$. Similarly, $fg \underset{B}{\simeq} 1_{E'}$. Thus p is fiber shape equivalent to p' over B.

Conversely, we shall construct a homeomorphism $h: Q-E\cong Q-E'$ satisfying the condition of Theorem 3.1. Let $\tilde{p}: Q\to Q$ and $\tilde{p}': Q\to Q$ be extensions of p and p', respectively. Let $f=\{f_n,E,E'\}_{Q,Q}$ and $g=\{g_n,E',E\}_{Q,Q}$ be fiber fundamental sequences over B such that $gf \underset{B}{\simeq} 1_E$ and $fg \underset{B}{\simeq} 1_{E'}$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. We inductively construct sequences $\{U_n\}$ and $\{V_n\}$ of open sets in Q and a sequence $\{h_n\}$ of homeomorphisms of Q onto itself satisfying the following properties.

(1) $E=\bigcap_{n=1}^{\infty} U_n$ and $U_{n+1} \subset U_n$, for each $n \geqq 1$,

(2) $E'=\bigcap_{n=1}^{\infty} V_n$ and $V_{n+1} \subset V_n$, for each $n \geqq 1$,

(3) $h_i \cdots h_2 h_1(U_n) \subset V_n$, for $i \geqq 2n-1$,

(4) $h_i \cdots h_2 h_1(U_n) \supset V_{n+1}$, for $i \geqq 2n$,

(5) $h_i | Q - V_n = id$, for $i \geqq 2n$,

(6) $h_i | Q - h_{2n} \cdots h_2 h_1(U_n) = id$, for $i \geqq 2n+1$,

(7) $d(\tilde{p}' h_i(q), \tilde{p}'(q)) < 4\varepsilon_n$, for $q \in Q$, $i \geqq 2n$,

(8) $d(\tilde{p}' h_i \cdots h_2 h_1(z), b) < 2\varepsilon_n$, for $b \in B$, $z \in p^{-1}(b)$ and $i \geqq 2n-1$.

First, we will construct a homeomorphism $h_1 : Q \to Q$. Let V_1 be a small neighborhood of E' in Q. Since f is a fiber fundamental sequence over B, there is an integer N_1 and a neighborhood E_1 of E in Q such that for $m \geqq n \geqq N_1$ there is a homotopy $F_{n,m} : E_1 \times I \to V_1$ such that $F_{n,m}(q, 0) = f_n(q)$, $F_{n,m}(q, 1) = f_m(q)$ and $d(\tilde{p}' F_{n,m}(q, t), \tilde{p}(q)) < \varepsilon_1/2$, for $q \in E_1$, $t \in I$. By Lemma 3.4, there is a Z-embedding $\alpha_1 : E \to V_1$ such that there is a homotopy $H : E \times I \to V_1$ such that $H_0 = \alpha_1$ and $H_1 = f_{N_1}|E$ and $d(\tilde{p}' H(q, t), \tilde{p}(q)) < \varepsilon_1/2$, for $q \in E$, $t \in I$. By Lemma 3.5, there is a homeomorphism $h_1 : Q \to Q$ such that $h_1|E = \alpha_1$. Since V_1 is an ANR, we may assume that there is an extension $\tilde{H} : E_1 \times I \to V_1$ of H such that $\tilde{H}_0 = h_1|E_1$, $\tilde{H}_1 = f_{N_1}|E_1$ and $d(\tilde{p}' \tilde{H}(q, t), \tilde{p}(q)) < \varepsilon_1/2$, for $q \in E_1$, $t \in I$.

Next, we will construct a homeomorphism $h_2 : Q \to Q$. Let $U_1 \subset E_1$ be a small neighborhood of E in Q such that $h_1(U_1) \subset V_1$. Since g is a fiber fundamental sequence over B, there is an integer $N_2 \geqq N_1$ and a neighborhood E_1' of E' in Q such that for $m \geqq n \geqq N_2$ there is a homotopy $G_{n,m} : E_1' \times I \to U_1$ such that $G_{n,m}(q, 0) = g_n(q)$, $G_{n,m}(q, 1) = g_m(q)$ and $d(\tilde{p} G_{n,m}(q, t), \tilde{p}'(q)) < \varepsilon_1/2$, for $q \in E_1'$, $t \in I$. By Lemma 3.4, there is a Z-embeding $\alpha_2 : E' \to U_1$ and a homotopy $K : E' \times I \to U_1$ such that $K_0 = \alpha_2$, $K_1 = g_{N_2}|E'$ and $d(\tilde{p} K(q, t), \tilde{p}'(q)) < \varepsilon_1/2$, for $q \in E'$, $t \in I$. By choosing N_2 sufficiently large, we may assume that there is a homotopy $L : E' \times I \to V_1$ such that $L_0 = f_{N_2} g_{N_2}|E'$, $L_1 = id$ and $d(\tilde{p}' L(q, t), \tilde{p}'(q)) < \varepsilon_1$, for $q \in E'$, $t \in I$. Define a homotopy $M : E' \times I \to V_1$ by

$$M(q,t)=\begin{cases} h_1K(q,4t), & q\in E',\ 0\leqq t\leqq 1/4, \\ \tilde{H}(g_{N_2}(q),4t-1), & q\in E',\ 1/4\leqq t\leqq 1/2, \\ F_{N_1,N_2}(g_{N_2}(q),4t-2), & q\in E',\ 1/2\leqq t\leqq 3/4, \\ L(q,4t-3), & q\in E',\ 3/4\leqq t\leqq 1. \end{cases}$$

Then $M_0=h_1\alpha_2$, $M_1=id$ and $d(\tilde{p}'M(q,t),\tilde{p}'(q))<\varepsilon_1$, for $q\in E'$, $t\in I$. By Lemma 3.5, we may assume that there is a homeomorphism $h_2':Q\to Q$ such that $h_2'|E'=h_1\alpha_2$, $h_2'|Q-V_1=id$ and $d(\tilde{p}'h_2'(q),\tilde{p}'(q))<\varepsilon_1$, for $q\in Q$. Let $h_2=(h_2')^{-1}$.

Also, we will construct a homeomorphism $h_3:Q\to Q$. Since $h_1(U_1)$ is an ANR, there is a neighborhood E_2' ($E_2'\subset E_1'$) of E' in Q such that $E'\subset E_2'\subset h_2h_1(U_1)$ and there is an extension $\tilde{K}:E_2'\times I\to h_1(U_1)$ of h_1K such that $\tilde{K}_0=h_2'|E_2'$, $\tilde{K}_1=h_1g_{N_2}|E_2'$ and $d(\tilde{p}'\tilde{K}(q,t),\tilde{p}'(q))<\varepsilon_1$, for $q\in E_2'$, $t\in I$. Let V_2 be a small neighborhood of E' in Q such that $V_2\subset V_1$, $V_2\subset E_2'$. Since f is a fiber fundamental sequence over B, there is an integer $N_3\geqq N_2$ and a neighborhood $E_2\subset E_1$ of E in Q such that for $m\geqq n\geqq N_3$ there is a homotopy $F_{n,m}:E_2\times I\to V_2$ such that $F_{n,m}(q,0)=f_n(q)$, $F_{n,m}(q,1)=f_m(q)$ and $d(\tilde{p}'F_{n,m}(q,t),\tilde{p}(q))<\varepsilon_2/2$, for $q\in E_2$, $t\in I$. Choose a Z-embedding $\alpha_3:E\to V_2$ and a homotopy $R:E\times I\to V_2$ such that $R_0=\alpha_3$, $R_1=f_{N_3}|E$ and $d(\tilde{p}'R(q,t),\tilde{p}(q))<\varepsilon_2/2$, for $q\in E$, $t\in I$. By choosing N_3 large, there is a homotopy $D:E\times I\to U_1$ such that $D_0=g_{N_3}f_{N_3}|E$, $D_1=id$ and $d(\tilde{p}D(q,t),\tilde{p}(q))<\varepsilon_1$, for $q\in E$, $t\in I$. Then we can define a homotopy $T:E\times I\to h_2h_1(U_1)$ by

$$T(q,t)=\begin{cases} R(q,4t), & q\in E,\ 0\leqq t\leqq 1/4, \\ h_2\tilde{K}(f_{N_3}(q),4t-1), & q\in E,\ 1/4\leqq t\leqq 1/2, \\ h_2h_1G_{N_2,N_3}(f_{N_3}(q),4t-2), & q\in E,\ 1/2\leqq t\leqq 3/4, \\ h_2h_1D(q,4t-3), & q\in E,\ 3/4\leqq t\leqq 1. \end{cases}$$

Then $T_0=\alpha_3$, $T_1=h_2h_1|E$ and $d(\tilde{p}'T(q,t),\tilde{p}(q))<4\varepsilon_1$, for $q\in E$, $t\in I$. By Lemma 3.5, there is a homeomorphism $h_3:Q\to Q$ such that $h_3|Q-h_2h_1(U_1)=id$, $h_3h_2h_1|E=\alpha_3$ and $d(\tilde{p}'h_3(q),\tilde{p}'(q))<4\varepsilon_1$, for $q\in Q$.

If we continue the process as above, we have desired sequences $\{U_n\}$, $\{V_n\}$ and $\{h_n\}$ satisfying the properties (1)—(8) as we wanted. Define a map $h:Q-E\to Q-E'$ by $h(q)=\lim_{j\to\infty}h_j\cdots h_2h_1(q)$ for $q\in Q-E$. By (1)—(6), h is a homeomorphism (see [4]). To prove that h is a desired homeomorphism, for each $b\in B$ choose a neighborhood W' of $p'^{-1}(b)$ in Q. Let N_0 be an integer and $\varepsilon>0$ such that $V_{N_0}\cap\tilde{p}'^{-1}(B(b;\varepsilon))\subset W'$, where $B(b;\varepsilon)=\{x\in Q|d(x,b)<\varepsilon\}$. Choose an integer n_0 such that $\sum_{n=n_0}^{\infty}4\varepsilon_n<\varepsilon/2$. By (8), $h_i\cdots h_2h_1(p^{-1}(b))\subset\tilde{p}'^{-1}(B(b;\varepsilon/2))$ for $i\geqq 2n_0-1$. Let $n_1=\mathrm{Max}\,\{N_0,n_0\}$. Choose a neighborhood W of $p^{-1}(b)$ in Q such that $W\subset U_{n_1}$ and $h_{2n_1-1}\cdots h_2h_1(W)\subset\tilde{p}'^{-1}(B(b;\varepsilon/2))$. By (3) and (7), $h_ih_{i-1}\cdots h_2h_1(W)\subset V_{N_0}\cap\tilde{p}'^{-1}(B(b;\varepsilon))$ for $i\geqq 2n_1-1$. Hence $h(W-$

$E)\subset W'-E'$. This completes the proof.

COROLLARY 3.4. *Let $p: E\to B$ be a CE-map from a compact ANR E to a compactum B. Then p is shape shrinkable if and only if B is an ANR.*

PROOF. Sufficiency follows from Corollory 2.5. Suppose that p is shape shrinkable. By Corollary 3.3, there is an extension $\tilde{p}: Q\to Q$ of p such that $\tilde{p}|Q-E: Q-E\cong Q-B$ is a homeomorphism. Since E is an ANR, there is a neighborhood U of E in Q and a retraction $r: U\to E$. Clearly, there is a retraction $r': \tilde{p}(U)\to B$ such that $\tilde{p}r(x)=r'\tilde{p}(x)$ for $x\in U$. Hence B is an ANR.

In [9], Kozłowski proved the following. If E and B are Z-sets in the Hilbert cube Q, then a map $p: E\to B$ between compacta is a hereditary shape equivalence iff there is an extension $\tilde{p}: Q\to Q$ of p such that $\tilde{p}|Q-E: Q-E\cong Q-B$ is a homeomorphism. Hence by Corollary 3.3 we have the following.

COROLLARY 3.5. *Let $p: E\to B$ be a map between compacta. Then p is shape shrinkable if and only if p is a hereditary shape equivalence.*

By Theorem 3.1 and Corollary 3.5, we can easily see the following.

COROLLARY 3.6. *Fiber shape equivalences preserve shape fibrations. In particular, hereditary shape equivalences are shape fibrations.*

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