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FUNCTION SPACES WHICH ARE STRATIFIABLE(*)

By

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Abstract. Let X be a compact metric space and Y a stratifiable space. By $C(X, Y)$, we denote the space of continuous maps from X to Y with the compact-open topology. In general, $C(X, Y)$ is not stratifiable. In this paper, we show that $C(X, Y)$ is stratifiable if Y satisfies the condition given by Mizokami [Mi]. And we construct a stratifiable space Y such that $C(X, Y)$ is not stratifiable even if X is countable and compact.

1. Introduction.

Let X and Y be topological spaces. By $\mathfrak{F}(X)$, $\mathfrak{K}(X)$ and $\mathfrak{O}(X)$, we denote the families of all nonempty finite subsets, all compact subsets and all open subsets of X, respectively. By $C(X, Y)$, we denote the space of all continuous maps of X to Y admitting the compact-open topology, whose open base is

$$\{M(K_1, \cdots, K_n; U_1, \cdots, U_n) \mid n \in \mathbf{N},$$
$$K_i \in \mathfrak{K}(X),\ U_i \in \mathfrak{O}(Y) \quad \text{for } i=1, \cdots, n\}$$

where

$$M(K_1, \cdots, K_n; U_1, \cdots, U_n)$$
$$=\{f \in C(X, Y) \mid f(K_i) \subset U_i \quad \text{for } i=1, \cdots, n\}$$

A regular space Y is *stratifiable* if it has a σ-closure preserving (abbrev. σ-CP) quasi-base $\mathcal{B}$ [Ce] (cf. [Bo₁]), where $\mathcal{B}$ is a *quasi-base* for Y if for any $y \in Y$ and each neighborhood U of y, there exists $B \in \mathcal{B}$ such that $y \in \operatorname{Int} B \subset B \subset U$. In general, $C(X, Y)$ is not stratifiable even if X is compact metric and Y is stratifiable. In fact, Borges [Bo₂] constructed a stratifiable space Y such that $C(\mathbf{I}, Y)$ is not normal, where $\mathbf{I}=[0, 1]$ is the unit interval.

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Similarly to $C(X, Y)$, the hyperspace $\mathfrak{K}(Y)$ with the Vietoris topology is not stratifiable even if Y is stratifiable (cf. [MK] and [Mi]). In [Mi], Misokami gave a condition for Y such that $\mathfrak{K}(Y)$ is stratifible. In this paper, we show that if Y satisfies this Mizokami's condition then $C(X, Y)$ is stratifiable for any compact metric space X. Cauty [Ca_3] proved that if Y is a CW-complex then $C(X, Y)$ is stratifiable for any compact space X. But any non-metrizable CW-complex Y does not satisfy the Mizokami's condition by [Mi, Theorem 4.3] (or cf. [GS, Example 3.2]). Therefore our result is independent from Cauty's result.

By $C_p(X, Y)$, we denote the space of all continuous maps from X to Y admitting the pointwise convergence topology, that is, $C_p(X, Y)$ is a subspace of the product space Y^X. Note that if Y is stratifiable then $C_p(X, Y)$ is stratifiable for a countable space X, since it can be embedded in the countable product space Y^ω of Y (cf. [Ce]). Thus it is natural to ask whether $C(X, Y)$ is stratifiable for a compact countable space X and a stratifiable space Y. However it can be seen in Section 3 that $C(X, Y)$ is not stratifiable for a compact countable space X and a stratifiable space Y which is constructed by Mizokami in [Mi, Example 2.1].

2. Main Result.

For a family $\mathcal{B}$ of subsets of Y and $A \subset Y$, let $\mathcal{B}|A = \{B \cap A \mid B \in \mathcal{B}\}$. We say that $\mathcal{B}$ is *finite on compact sets* (abbrev. *CF*) in Y if $\mathcal{B}|K$ is finite for each $K \in \mathfrak{K}(X)$. And $\mathcal{B}$ is *σ-CP-CF* if it can be written as $\mathcal{B} = \bigcup_{n \in \mathrm{N}} \mathcal{B}_n$ such that each $\mathcal{B}_n$ is CP (closure-preserving) and CF in Y. In this section, we show the following theorem.

THEOREM 2.1. *Let X be a compact metric space and Y a stratifiable space which has a σ-CP-CF quasi-base consisting of closed sets. Then $C(X, Y)$ has a σ-CP quasi-base, hence it is stratifiable.*

To prove this theorem, we need some lemmas.

LEMMA 2.2. *Let $\mathcal{U}$ be an open set in $C(X, Y)$, $f \in \mathcal{U}$ and $\mathcal{B}$ a quasi-base for Y. Then there exist $K_1, \cdots, K_n \in \mathfrak{K}(X)$ and $B_1, \cdots, B_n \in \mathcal{B}$ such that*

$$f \in M(K_1, \cdots, K_n; \mathrm{Int}\, B_1, \cdots, \mathrm{Int}\, B_n)$$
$$\subset M(K_1, \cdots, K_n; B_1, \cdots, B_n) \subset \mathcal{U},$$

that is, the family

$$\{M(K_1, \cdots, K_n\,; B_1, \cdots, B_n) \mid n \in \mathbf{N},$$
$$K_i \in \mathfrak{K}(X),\ B_i \in \mathcal{B}\ (i=1, \cdots, n)\}$$

is a quasi-base for $C(X, Y)$.

PROOF. Since $\mathcal{U}$ is open in $C(X, Y)$, we have $K_1, \cdots, K_m \in \mathfrak{K}(X)$ and $U_1, \cdots, U_m \in \mathfrak{O}(Y)$ such that

$$f \in M(K_1, \cdots, K_m\,; U_1, \cdots, U_m) \subset \mathcal{U}\,.$$

For any $i=1, \cdots, m$ and $x \in K_i$, since $f(x) \in U_i$, there is $B_i^x \in \mathcal{B}$ such that $f(x) \in \mathrm{Int}\, B_i^x \subset B_i^x \subset U_i$, whence

$$x \in f^{-1}(\mathrm{Int}\, B_i^x) \subset f^{-1}(B_i^x) \subset f^{-1}(U_i)\,.$$

By compactness of K_i, there are $x_i^1, \cdots, x_i^{n(i)} \in K_i$ such that

$$K_i \subset \bigcup_{j=1}^{n(i)} f^{-1}(\mathrm{Int}\, B_{i,j}) = f^{-1}\Big(\bigcup_{j=1}^{n(i)} \mathrm{Int}\, B_{i,j}\Big)$$
$$\subset f^{-1}\Big(\bigcup_{j=1}^{n(i)} B_{i,j}\Big) \subset f^{-1}(U_i)\,.$$

where $B_{i,j} = B_i^{x_i^j}$. Then K_i has a closed cover $\{K_{i,j}\}_{j=1}^{n(i)}$ such that $K_{i,j} \subset f^{-1}(\mathrm{Int}\, B_{i,j})$. Note that $K_{i,j} \in \mathfrak{K}(X)$. It is clear that

$$f \in M(K_{i,1}, \cdots, K_{i,n(i)}\,; \mathrm{Int}\, B_{i,1}, \cdots, \mathrm{Int}\, B_{i,n(i)})$$
$$\subset M(K_{i,1}, \cdots, K_{i,n(i)}\,; B_{i,1}, \cdots, B_{i,n(i)}) \subset M(K_i, U_i)$$

Therefore we have

$$f \in \bigcap_{i=1}^{m} M(K_{i,1}, \cdots, K_{i,n(i)}\,; \mathrm{Int}\, B_{i,1}, \cdots, \mathrm{Int}\, B_{i,n(i)})$$
$$\subset \bigcap_{i=1}^{m} M(K_{i,1}, \cdots, K_{i,n(1)}\,; B_{i,1}, \cdots, B_{i,n(i)})$$
$$\subset M(K_1, \cdots, K_m\,; U_1, \cdots, U_m) \subset \mathcal{U}\,. \qquad \square$$

LEMMA 2.3. *Let* $\mathcal{B}$ *be a CP (resp. CF) family of closed sets in* X. *Then* $\mathcal{B}^* = \{\cap \mathcal{A} \mid \mathcal{A} \in \mathfrak{F}(\mathcal{B})\}$ *is also CP (resp. CF).*

PROOF. The CF case is obvious. To see the CP case, let $\mathfrak{C} \subset \mathfrak{F}(\mathcal{B})$. We prove that

$$\bigcup_{\mathcal{A} \in \mathfrak{C}} (\cap \mathcal{A}) = \overline{\bigcup_{\mathcal{A} \in \mathfrak{C}} (\cap \mathcal{A})}\,.$$

To this end, let $x \notin \bigcup_{\mathcal{A} \in \mathfrak{C}} (\cap \mathcal{A})$. For each $\mathcal{A} \in \mathfrak{C}$, since $x \notin \cap \mathcal{A}$, we can choose $B_{\mathcal{A}} \in \mathcal{A}$ such that $x \notin B_{\mathcal{A}}$. Since $\{B_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{C}\} \subset \mathcal{B}$ and $\mathcal{B}$ is CP, we have

$$\overline{\bigcup_{\mathcal{A}\in\mathfrak{C}}(\cap\mathcal{A})}\subset\overline{\bigcup_{\mathcal{A}\in\mathfrak{C}}B_{\mathcal{A}}}=\bigcup_{\mathcal{A}\in\mathfrak{C}}B_{\mathcal{A}}.$$

Since $x\notin\bigcup_{\mathcal{A}\in\mathfrak{C}}B_{\mathcal{A}}$, $x\notin\overline{\bigcup_{\mathcal{A}\in\mathfrak{D}}(\cap\mathcal{A})}$. Therefore it follows that

$$\bigcup_{\mathcal{A}\in\mathfrak{C}}(\cap\mathcal{A})=\overline{\bigcup_{\mathcal{A}\in\mathfrak{C}}(\cap\mathcal{A})}. \qquad \square$$

REMARK. In the CP case of the above lemma, it is necessary to assume that members of $\mathcal{B}$ are closed in X. In fact, let

$$X=\{0\}\cup\left\{\left\{\frac{1}{n}\right\}\Big|\,n\in\mathbf{N}\right\} \quad \text{and} \quad \mathcal{B}=\{B\subset X\,|\,|B|=\aleph_0\}.$$

Note that $X\setminus\{0\}\in\mathcal{B}$, but it is not closed in X. For any $\emptyset\neq\mathcal{B}_0\subset\mathcal{B}$, we have

$$\bigcup\overline{\mathcal{B}_0}=\{0\}\cup\bigcup\mathcal{B}_0=\overline{\bigcup\mathcal{B}_0},$$

that is, $\mathcal{B}$ is CP. On the other hand, $\{\{1/n\}\,|\,n\in\mathbf{N}\}\subset\mathcal{B}^*$ and

$$\overline{\bigcup\left\{\left\{\frac{1}{n}\right\}\Big|\,n\in\mathbf{N}\right\}}\not\subset\bigcup\left\{\left\{\frac{1}{n}\right\}\Big|\,n\in\mathbf{N}\right\},$$

whence $\mathcal{B}^*$ is not CP.

The following lemma is easy.

LEMMA 2.4. *If $\mathcal{A}$ and $\mathcal{B}$ are CP (resp. CF) families, then $\mathcal{A}\cup\mathcal{B}$ is also CP (resp. CF).* $\square$

PROOF OF THEOREM 2.1. Let $\mathcal{B}$ be a σ-CP-CF quasi-base for Y consisting of closed sets. By Lemma 2.4, we can write $\mathcal{B}=\bigcup_{n\in N}\mathcal{B}_n$, where $\mathcal{B}_1\subset\mathcal{B}_2\subset\cdots$ are CP and CF. Using the compactness, X has a sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of finite closed covers of X such that mesh $\mathcal{C}_n\to 0$ if $n\to\infty$. For each m, $n\in\mathbf{N}$ and $(C_1, \cdots, C_m)\in(\mathcal{C}_n)^m$, we define

$$\mathcal{A}^n_{(C_1,\cdots,C_m)}=\{M(C_1, \cdots, C_m; B_1, \cdots, B_m)\,|\,B_i\in\mathcal{B}_n \ i=1, \cdots, m\}.$$

We shall show that

$$\mathcal{A}=\bigcup_{n\in\mathrm{N}}\bigcup_{m\in\mathrm{N}}\bigcup\{\mathcal{A}^n_{(C_1,\cdots,C_m)}\,|\,(C_1, \cdots, C_m)\in(\mathcal{C}_n)^m\}$$

is a quasi-base for $C(X, Y)$ and that each $\mathcal{A}^n_{(C_1,\cdots,C_m)}$ is CP. Then $\mathcal{A}$ is σ-CP quasi-base since each $(\mathcal{C}_n)^m$ is finite.

First to prove that $\mathcal{A}$ is a quasi-base for $C(X, Y)$, let $\mathcal{U}$ be open in $C(X,Y)$ and $f\in\mathcal{U}$. By Lemma 2.2, there are $K_1, \cdots, K_l\in\mathfrak{K}(X)$ and $B_1, \cdots, B_l\in\mathcal{B}$ such that

$$f \in M(K_1, \cdots, K_l; \mathrm{Int}\, B_1, \cdots, \mathrm{Int}\, B_l)$$
$$\subset M(K_1, \cdots, K_l; B_1, \cdots, B_l) \subset \mathcal{V}.$$

Let

$$\eta = \min\{\mathrm{dist}(K_i, X \setminus f^{-1}(\mathrm{Int}\, B_i)) \mid i=1, \cdots, l\} > 0,$$

where $\mathrm{dist}(A, \emptyset) = \mathrm{diam}\, X$. Since mesh $\mathcal{C}_n \to 0$ $(n \to \infty)$ and $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ we can choose $n \in \mathbf{N}$ such that mesh $\mathcal{C}_n < \eta$ and $B_1, \cdots, B_l \in \mathcal{B}_n$. For each $i=1, \cdots, l$, write

$$\{C \in \mathcal{C}_n \mid C \cap K_i \neq \emptyset\} = \{C_{i,1}, \cdots, C_{i,m_i}\}.$$

Then $K_i \subset \bigcup_{j=1}^{m_i} C_{i,j} \subset f^{-1}(\mathrm{Int}\, B_i)$, whence

$$f \in M(C_{i,1}, \cdots, C_{i,m_i}; \overbrace{\mathrm{Int}\, B_i, \cdots, \mathrm{Int}\, B_i}^{m_i})$$
$$\subset M(C_{i,1}, \cdots, C_{i,m_i}; \overbrace{B_i, \cdots, B_i}^{m_i}) \subset M(K_i, B_i).$$

Hence we have

$$f \in \bigcap_{i=1}^{l} M(C_{i,1}, \cdots, C_{i,m_i}; \overbrace{\mathrm{Int}\, B_i, \cdots, \mathrm{Int}\, B_i}^{m_i})$$
$$\subset \mathrm{Int} \bigcap_{i=1}^{l} M(C_{i,1}, \cdots, C_{i,m_i}; \overbrace{B_i, \cdots, B_i}^{m_i})$$
$$\subset \bigcap_{i=1}^{l} M(C_{i,1}, \cdots, C_{i,m_i}; \overbrace{B_i, \cdots, B_i}^{m_i})$$
$$\subset M(K_1, \cdots, K_l; B_1, \cdots, B_l) \subset \mathcal{V}.$$

Let $m = \sum_{i=1}^{l} m_i$ and

$$(C_1, \cdots, C_m) = (C_{1,1}, \cdots, C_{1,m_1}, \cdots, C_{l,1}, \cdots, C_{l,m_l}) \in (\mathcal{C}_n)^m.$$

Then

$$\bigcap_{i=1}^{l} M(C_{i,1}, \cdots, C_{i,m_i}; \overbrace{B_i, \cdots, B_i}^{m_i})$$
$$= M(C_1, \cdots, C_m; \overbrace{B_1, \cdots, B_1}^{m_1}, \cdots, \overbrace{B_l, \cdots, B_l}^{m_l})$$
$$\in \mathcal{A}^n_{(C_1, \cdots, C_m)} \subset \mathcal{A}.$$

Next to show that each $\mathcal{A}^n_{(C_1, \cdots, C_m)}$ is CP, let $\mathcal{B}' \subset (\mathcal{B}_n)^m$ and

$$\mathcal{A}' = \{M(C_1, \cdots, C_m; B_1, \cdots, B_m) \mid (B_1, \cdots, B_m) \in \mathcal{B}'\}$$
$$\subset \mathcal{A}^n_{(C_1, \cdots, C_m)}.$$

To prove that $\overline{\bigcup \mathcal{A}'} = \bigcup \mathcal{A}'$, let $g \in C(X, Y) \setminus \bigcup \mathcal{A}'$. For each $k=1, \cdots, m$, let $p_k : (\mathcal{B}_n)^m \to \mathcal{B}_n$ be the projection defined by $p_k(B_1, \cdots, B_m) = B_k$ and

$$\mathcal{B}'(k) = \{B \in p_k(\mathcal{B}') \mid g(C_k) \not\subset B\} \subset \mathcal{B}_n.$$

In case $\mathscr{B}'(k)=\emptyset$, let $M_k=C(X, Y)$. In case $\mathscr{B}'(k)\neq\emptyset$, we can write

$$\mathscr{B}'(k)\,|\,g(C_k)=\{G_{k,1}, \cdots, G_{k,m_k}\},$$

because $\mathscr{B}_n$ is CF. Note that $g(C_k)\setminus G_{k,i}\neq\emptyset$ for each $i=1, \cdots, m_k$. We can choose points $x_{k,1}, \cdots, x_{k,m_k}\in C_k$ such that $g(x_{k,i})\in g(C_k)\setminus G_{k,i}$. Then

$$V_{k,i}=Y\setminus\bigcup\{B\in\mathscr{B}_n\,|\,g(x_{k,i})\notin B\}$$

is an open neighborhood of $g(x_{k,i})$ in Y because $\mathscr{B}_n$ is CP. Let

$$M_k=M(\{x_{k,1}\}, \cdots, \{x_{k,m_k}\}\,;\,V_{k,1}, \cdots, V_{k,m_k}).$$

Then $M(g)=\bigcap_{k=1}^{m} M_k$ is an open neighbornood of g in $C(X, Y)$. And moreover $M(g)\cap(\bigcup\mathscr{A}')=\emptyset$. In fact, for any $(B_1, \cdots, B_m)\in\mathscr{B}'$,

$$g\notin M(C_1, \cdots, C_m\,;\,B_1, \cdots, B_m),$$

whence $g(C_k)\not\subset B_k$, i.e., $B_k\in\mathscr{B}'(k)$ for some $k\leqq m$. Then $B_k\cap g(C_k)=G_{k,i}$ for some $i\leqq m_k$, which implies that

$$g(x_{k,i})\in V_{k,i}\cap(g(C_k)\setminus B_k).$$

By the definition of $V_{k,i}$, we have $V_{k,i}\cap B_k=\emptyset$. Hence

$$M(g)\cap M(C_1, \cdots, C_m\,;\,B_1, \cdots, B_m)=\emptyset.$$

Thus $g\notin\overline{\bigcup\mathscr{A}'}$. □

REMARKS. In the above proof,

$$\mathscr{M}=\{M(C_1, \cdots, C_n\,;\,B_1, \cdots, B_n)\,|\,n\in\mathbf{N},$$
$$C_i\in\mathfrak{K}(X),\ B_i\in\mathscr{B} \quad \text{for } i=1, \cdots, n\}$$

is a quasi-base for $C(X, Y)$ by Lemma 2.2. Since $\mathscr{B}=\bigcup_{k=1}^{\infty}\mathscr{B}_k$, $\mathscr{M}=\bigcup_{k=1}^{\infty}\mathscr{M}_k$, where

$$\mathscr{M}_k=\{M(C_1, \cdots, C_n\,;\,B_1, \cdots, B_n)\,|\,C_i\in\mathfrak{K}(X),\ B_i\in\mathscr{B}_k \text{ and } n\in\mathbf{N}\}.$$

Although one might expect that each $\mathscr{M}_k$ is CP, this is not true. In fact, let $X=\{0\}\cup\{1/n\,|\,n\in\mathbf{N}\}$ and $Y=[0, 1]$. We inductively define families $\mathscr{B}_n$ of closed sets in Y as follows: $\mathscr{B}_1=\{[0, 1/2], [1/2, 1]\}$ and

$$\mathscr{B}_n=\mathscr{B}_{n-1}\cup\left\{\left[\frac{i-1}{n+1}, \frac{i}{n+1}\right]\,\middle|\,i=1, \cdots, n+1\right\}$$

for each $n>1$. Clearly $\mathscr{B}_1\subset\mathscr{B}_2\subset\cdots$ are CP and CF in Y and $\mathscr{B}=\bigcup_{n\in\mathbf{N}}\mathscr{B}_n$ is a quasi-base of Y. To see that $\mathscr{M}_k$ is not CP in $C(X, Y)$, let

$$\mathscr{M}_k'=\left\{M\left(\left\{\frac{1}{n}\right\}, \left[\frac{1}{k+1}, \frac{2}{k+1}\right]\right)\,\middle|\,n\in\mathbf{N}\right\}(\subset\mathscr{M}_k)$$

and define $f\in C(X, Y)$ by

$$f(x)=\frac{1}{k+1}(1-x) \qquad \text{for each } x\in X.$$

It is easy to see that $f\notin\bigcup\mathcal{M}'_k=\bigcup\overline{\mathcal{M}'_k}$. We show that $f\in\overline{\bigcup\mathcal{M}'_k}$. To this end, let $\mathfrak{U}=\bigcap_{i=1}^{l} M(C_i, U_i)$ be any basic open neighborhood of f in $C(X, Y)$, where $C_i\in\mathfrak{K}(X)$ and $U_i\in\mathfrak{O}(Y)$. In the case $0\notin\bigcup_{i=1}^{l} C_i$, there exist $N\in\mathbf{N}$ such that $1/m\notin\bigcup_{i=1}^{l} C_i$ for each $m\geqq N$. Then we have

$$M\Big(\Big\{\frac{1}{2N}\Big\}, \Big[\frac{1}{k+1}, \frac{2}{k+1}\Big]\Big)\cap\mathfrak{U}\neq\emptyset,$$

whence $(\bigcup\mathcal{M}'_k)\cap\mathfrak{U}\neq\emptyset$. In the case $0\in C_j$ for some $j\leqq l$, $1/(k+1)=f(0)\in U_j$. Let

$$U=\bigcap\{U_j\,|\,0\in C_j,\ j=1, \cdots, l\}\ (\neq\emptyset).$$

Since U is open in Y and $1/(k+1)\in U$, we can choose some $m\in\mathbf{N}$ such that

$$\Big[\frac{1}{k+1}-\frac{1}{m}, \frac{1}{k+1}+\frac{1}{m}\Big]\subset U \quad\text{and}\quad \Big[0, \frac{2}{m}\Big]\cap C_i=0 \qquad \text{if } 0\notin C_i.$$

We define $g\in C(X, Y)$ by

$$g(x)=\begin{cases} \dfrac{1}{k+1} & \text{if } x\leqq\dfrac{1}{m}, \\ \dfrac{1}{k+1}\Big(1+\dfrac{2}{m}-2x\Big) & \text{if } \dfrac{1}{m}\leqq x\leqq\dfrac{2}{m}, \\ f(x) & \text{if } x\geqq\dfrac{2}{m}. \end{cases}$$

Then $g\in M(\{1/m\}, [1/(k+1), 2/(k+1)])\cap\mathfrak{U}$, whence $(\bigcup\mathcal{M}'_k)\cap\mathfrak{U}\neq\emptyset$.

In fact, if $C_i\cap[0, 2/m]=\emptyset$ then $g(C_i)=f(C_i)\subset U_i$. If $C_i\cap[0, 2/m]\neq\emptyset$, we have $0\in C_i$, whence $U\subset U_i$. Then

$$\begin{aligned} g(C_i)&=g\Big(C_i\cap\Big[0, \frac{2}{m}\Big]\Big)\cup g\Big(C_i\setminus\Big[0, \frac{2}{m}\Big]\Big) \\ &\subset\Big[\frac{1}{k+1}-\frac{1}{m}, \frac{1}{k+1}+\frac{1}{m}\Big]\cup f\Big(C_i\setminus\Big[0, \frac{2}{m}\Big]\Big) \\ &\subset U_i. \end{aligned}$$

Therefore $f\in\overline{\bigcup\mathcal{M}'_k}$, that is, $\mathcal{M}_k$ is not CP in $C(X, Y)$.

By $\mathcal{S}$, we denote the class of stratifiable spaces. It is known that a stratifiable space is an ANR$(\mathcal{S})$ iff it is an ANE$(\mathcal{S})$. In Theorem 2.1, if Y is an ANR$(\mathcal{S})$ then $C(X, Y)$ is an ANE$(\mathcal{S})$, hence an ANR$(\mathcal{S})$. In fact, let A be a closed set in a stratifiable space Z and $\varphi\in C(A, C(X, Y))$. We define $\tilde{\varphi}: A\times X$

$\to Y$ by $\tilde{\varphi}(a, x)=\varphi(a)(x)$. By the compactness of X, we have $\tilde{\varphi}$ is continuous. Since Y is an ANE$(\mathcal{S})$ and $Z\times X$ is stratifiable, there exists a neighborhood W of $A\times X$ in $Z\times X$ and $\tilde{\Phi}\in C(W, Y)$ such that $\tilde{\Phi}|A\times X=\tilde{\varphi}$. Since X is compact, A have a neighborhood U in Y such that $A\times X\subset U\times X\subset W$. We define $\Phi: U\to C(X, Y)$ by

$$\Phi(z)(x)=\tilde{\Phi}(z, x) \quad (x\in X)$$

for each $z\in U$. Then Φ is an extension of φ on U. Thus we have the following result.

COROLLARY 2.5. *Let X be a compact metric space and Y an ANR$(\mathcal{S})$ which has a σ-CP-CF quasi-base consisting of closed sets. Then $C(X, Y)$ is an ANR$(\mathcal{S})$.*

In Theorem 2.1, it is a problem whether metrizability of X is necessary or not, that is,

PROBLEM 2.6. *Is Theorem 2.1 true for a non-metrizable compact space X?*

3. A Counterexample.

In this section, we show that $C(X, Y)$ is not stratifiable for $X=\{0\}\cup\{1/n\,|\,n\in\mathbf{N}\}$ and the stratifiable space Y which is constructed by Mizokami in [Mi, Example 2.1] (indeed, Y is a countable Lašnev space). First we show the following:

LEMMA 3.1. *Let X be compac, $y_0\in Y$ and $\mathcal{A}$ a neighborhood base of y_0 in Y. Then $\{M(X, A)\,|\,A\in\mathcal{A}\}$ is a neighborhood base of the constant map f_0 with $f_0(X)=\{y_0\}$ in $C(X, Y)$.*

PROOF. For each neighborhood $\mathfrak{N}$ of f_0 in $C(X, Y)$, there exist $C_1, \cdots, C_n \in\mathfrak{K}(X)$ and $U_1, \cdots, U_n\in\mathfrak{O}(Y)$ such that

$$f_0\in M(C_1, \cdots, C_n; U_1, \cdots, U_n)\subset\mathfrak{N}.$$

Since each U_i is an open neighborhood of y_0 in Y, there is $A\in\mathcal{A}$ such that $A\subset\bigcap_{i=1}^n U_i$, whence

$$f_0\in M(X, A)\subset M(C_1, \cdots, C_n; U_1, \cdots, U_n)\subset\mathfrak{N}. \qquad \square$$

EXAMPLE 3.2. *Let $X=\{0\}\cup\{1/n\,|\,n\in\mathbf{N}\}\subset\mathbf{R}$ be the space of a convergent sequence. There exists a countable Lašnev space Y such that $C(X, Y)$ is not stratifiable.*

PROOF. Let Y be the space of [Mi, Example 2.1], namely $Y=Y'/A$, where

$$Y'=\left[(\mathbf{Q}\cap(0,\ 1))\setminus\left\{\frac{1}{n}\,\middle|\,n\in\mathbf{N}\right\}\right]\times\left[\{0\}\cup\left\{\frac{1}{n}\,\middle|\,n\in\mathbf{N}\right\}\right]$$

is a subspace of $\mathbf{R}^2$ and $A=\{(x,\ 0)\mid(x,\ 0)\in Y'\}$. Let $p:Y'\to Y$ be the quotient map and $y_0=p(A)\in Y$. We shall show that $C(X,\ Y)$ is not stratifiable. For each $k\in\mathbf{N}$, let

$$N_k=p\left(\left(\left(\frac{1}{k+1},\ \frac{1}{k}\right)\times\left[0,\ \frac{1}{k}\right]\right)\cap Y'\right)$$

and $\tilde{N}_k=\bigcup_{i\geqq k}N_i$. For simplicity, we write $N=\tilde{N}_1$. Note that $y_0\in N$ and N has the weak topology with respect to $\{N_k\}_{k\in\mathbf{N}}$. For each $(y_1,\ \cdots,\ y_n)\in N^n$, we define $f_{(y_1,\cdots,y_n)}\in C(X,\ N)$ by

$$f_{(y_1,\cdots,y_n)}(x)=\begin{cases} y_i & \text{if } x=\dfrac{1}{i}\geqq\dfrac{1}{n}, \\ y_0 & \text{otherwise.}\end{cases}$$

In case $y_1=\cdots=y_n=y_0$, $f_{(y_0,\cdots,y_0)}$ is the constant map, which is simply denoted by f_0.

To see that $C(X,\ Y)$ is not stratifiable, it suffices to show that $C(X,\ N)$ is not stratifiable. On the contrary, assume that $C(X,\ N)$ is stratifiable. Then f_0 has a CP neighborhood base $\mathfrak{B}$ consisting of closed sets in $C(X,\ N)$ (see [Ce, Lemma 7.3]). For each $B^*\in\mathfrak{B}$, we define a subset $O(B^*)$ of N by

$$O(B^*)=\bigcup\{f(X)\mid y_0\in f(X)\in\mathfrak{F}(N) \text{ and } f\in\operatorname{Int}B^*\}.$$

Then we have

LEMMA 3.3. *$O(\mathfrak{B})=\{O(B^*)\mid B^*\in\mathfrak{B}\}$ is a neighborhood base of y_0 in N.*

PROOF. For each neighborhood V of y_0 in N, there exists $B^*\in\mathfrak{B}$ such that $B^*\subset M(X,\ V)$, whence $O(B^*)\subset\bigcup\{f(X)\mid f\in\operatorname{Int}B^*\}\subset V$.

Next we show that $O(B^*)$ is a neighborhood of y_0 in N for each $B^*\in\mathfrak{B}$. Since each B^* is a neighborhood of f_0 in $C(X,\ N)$, there are $C_1,\ \cdots,\ C_n\in\mathfrak{K}(X)$ and $U_1,\ \cdots,\ U_n\in\mathfrak{O}(N)$ such that

$$f_0\in M(C_1,\ \cdots,\ C_n;\ U_1,\ \cdots,\ U_n)\subset\operatorname{Int}B^*.$$

Since each U_i is a neighborhood of y_0 in N, $U=\bigcap_{i=1}^n U_i$ is a neighborhood of y_0 in N. Observe that for each $y\in U$,

$$f_y\in M(X,\ U)\subset M(C_1,\ \cdots,\ C_n;\ U_1,\ \cdots,\ U_n)\subset\operatorname{Int}B^*.$$

Then it follows that $U=\bigcup_{y\in U}f_y(X)\subset O(B^*)$. Thus $O(B^*)$ is a neighborhood of y_0 in Y. $\square$

Next, for each $(y_1, \cdots, y_n) \in N^n$, we define

$$\mathfrak{B}(y_1, \cdots, y_n) = \{B^* \in \mathfrak{B} \mid f_{(y_1, \cdots, y_i)} \in \operatorname{Int} B^* \text{ for each } i=1, \cdots, n\}.$$

LEMMA 3.4. *For any neighborhood V_k of y_0 in N_k,*

$$\mathfrak{B} = \bigcup \{\mathfrak{B}(y) \mid y \in V_k \setminus \{y_0\}\}.$$

And for any $(y_1, \cdots, y_n) \in N^n$ and any neighborhood V_k of y_0 in N_k,

$$\mathfrak{B}(y_1, \cdots, y_n) = \bigcup \{\mathfrak{B}(y_1, \cdots, y_n, y) \mid y \in V_k \setminus \{y_0\}\}.$$

PROOF. Because of similarity, we show only the second statement. From the definition of $\mathfrak{B}(y_1, \cdots, y_n, y)$,

$$\bigcup \{\mathfrak{B}(y_1, \cdots, y_n, y) \mid y \in V_k \setminus \{y_0\}\} \subset \mathfrak{B}(y_1, \cdots, y_n).$$

Conversely let $B^* \in \mathfrak{B}(y_1, \cdots, y_n)$. Since $f_{(y_1, \cdots, y_n)} \in \operatorname{Int} B^*$, we have $C_1, \cdots, C_l \in \mathfrak{R}(X)$ and $U_1, \cdots, U_l \in \mathfrak{O}(N)$ such that

$$f_{(y_1, \cdots, y_n)} \in M(C_1, \cdots, C_l ; U_1, \cdots, U_l) \subset \operatorname{Int} B^*.$$

Then $O_k = V_k \cap \bigcap \{U_i \mid y_0 \in U_i\}$ is an open neighborhood of y_0 in N_k. For $y' \in O_k \setminus \{y_0\} \subset V_k \setminus \{y_0\}$, $B^* \in \mathfrak{B}(y_0, \cdots, y_n, y')$. In fact, if $1/(n+1) \notin C_i$ then

$$f_{(y_1, \cdots, y_n, y')}(C_i) = f_{(y_1, \cdots, y_n)}(C_i) \subset U_i.$$

If $1/(n+1) \in C_i$ then $y_0 \in f_{(y_1, \cdots, y_n)}(C_i) \subset U_i$, which $O_k \subset U_i$. Hence

$$f_{(y_1, \cdots, y_n, y')}(C_i) \subset f_{(y_1, \cdots, y_n)}(C_i) \cup \{y'\} \subset U_i \cup O_k \subset U_i.$$

Therefore $f_{(y_1, \cdots, y_n, y')} \in M(C_1, \cdots, C_l ; U_1, \cdots, U_l) \subset \operatorname{Int} B^*$. □

LEMMA 3.5. *There exist $1 = k_0 < k_1 < \cdots \in \mathbf{N}$, open neighborhoods $N = W_0 \supset W_1 \supset \cdots$ of y_0 in N, $y_n \in (W_{n-1} \setminus W_n) \cap N_{k_{n-1}}$ and $B_n^* \in \mathfrak{B}_{n-1} = \mathfrak{B}(y_1, \cdots, y_{n-1})$ where $\mathfrak{B}_0 = \mathfrak{B}$ such that*

$(1)_n$ $O(\mathfrak{B}_n) \mid \tilde{N}_k$ is a neighborhood base of y_0 in $\tilde{N}_k$ for each $k \geq k_n$,

$(2)_n$ $f_n = f_{(y_1, \cdots, y_n)} \in M_n(W_n)$ and $M_n(W_n) \cap B_n^* = \emptyset$, where

$$M_n(W_n) = M\left(\{1\}, \cdots, \left\{\frac{1}{n}\right\}; \{y_1\}, \cdots, \{y_n\}\right) \cap M\left(X \setminus \left\{1, \cdots, \frac{1}{n}\right\}, W_n\right).$$

PROOF. Note that $k_0 = 1$ and $O(\mathfrak{B}_0) = O(\mathfrak{B})$ satisfy $(1)_0$ by Lemma 3.3. Supposing that $\{k_0, \cdots, k_{n-1}\}$, $\{W_0, \cdots, W_{n-1}\}$, $\{y_1, \cdots, y_{n-1}\}$ and $\{B_1^*, \cdots, B_{n-1}^*\}$ have been obtained, we find k_n, W_n, y_n and B_n^*.

First assume that no $y \in (W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\}$ and no $k > k_{n-1}$ satisfy $(1)_n$, that is, for each $y \in (W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\}$, $O(\mathfrak{B}(y_1, \cdots, y_{n-1}, y)) \mid \tilde{N}_m$ is not a neighborhood base of y_0 in $\tilde{N}_m$ for infinitely many $m>k_{n-1}$. Since $(W_{n-1}\cap N_{k_{n-1}})\setminus\{y_0\}$ is countable, we can write

$$(W_{n-1}\cap N_{k_{n-1}})\setminus\{y_0\}=\{z_i\,|\,i\in\mathbf{N}\}\,(\subset N),$$

where $z_i\neq z_j$ if $i\neq j$. Then we can inductively choose $k_{n-1}<m_1<m_2<\cdots$ and neighborhoods V_{m_i} of y_0 in $\tilde{N}_{m_i}$ such that

$$O(B^*)\cap\tilde{N}_{m_i}\not\subset V_{m_i}\quad\text{ for each } B^*\in\mathfrak{B}(y_1,\cdots,y_{n-1},z_i).$$

Without loss of generality, we can assume $\tilde{N}_{k_{n-1}}=V_{m_0}\supset V_{m_1}\supset\cdots$ and define $V=\bigcup_{i=0}^{\infty}(V_{m_i}\setminus\tilde{N}_{m_{i+1}})$. Then V is a neighborhood of y_0 in $\tilde{N}_{k_{n-1}}$. By $(1)_{n-1}$, $O(B^*)\cap\tilde{N}_{k_{n-1}}\subset V$ for some $B^*\in\mathfrak{B}_{n-1}$, whence

$$O(B^*)\cap\tilde{N}_{m_i}\subset V\cap\tilde{N}_{m_i}=V_{m_i}\qquad\text{for each } i\in\mathbf{N}$$

On tne other hand,

$$\mathfrak{B}_{n-1}=\mathfrak{B}(y_1,\cdots,y_{n-1})=\bigcup_{i\in\mathbf{N}}\mathfrak{B}(y_1,\cdots,y_{n-1},z_i)$$

by Lemma 3.4, whence $B^*\in\mathfrak{B}(y_1,\cdots,y_{n-1},z_i)$ for some $i\in\mathbf{N}$. This is a contradiction. Therefore we have $k_n\in\mathbf{N}$ and $y_n\in(W_{n-1}\cap N_{k_{n-1}})\setminus\{y_0\}$ satisfying $(1)_n$.

Note that $(W_{n-1}\cap N_{k_{n-1}})\setminus\{y_n\}$ is a neighborhood of y_0 in $N_{k_{n-1}}$. By $(1)_{n-1}$, we have $B_n^*\in\mathfrak{B}_{n-1}$ such that $O(B_n^*)\cap N_{k_{n-1}}\subset(W_{n-1}\cap N_{k_{n-1}})\setminus\{y_n\}$. Hence $f=f_{(y_1,\cdots,y_n)}\notin B_n^*$. Since B_n^* is closed in $C(X,N)$, there exist $C_1,\cdots,C_l\in\mathfrak{K}(X)$ and $U_1,\cdots,U_l\in\mathfrak{O}(N)$ such that

$$f_n\in M(C_1,\cdots,C_l;U_1,\cdots,U_l)\subset C(X,N)\setminus B_n^*.$$

Let

$$W_n=(W_{n-1}\cap(\bigcap\{U_i\,|\,y_0\in U_i\}))\setminus\{y_n\}\subset W_{n-1}.$$

Then W_n is an open neighborhood of y_0 in N, $y_n\in(W_{n-1}\setminus W_n)\cap N_{k_{n-1}}$ and $f\in M_n(W_n)$. To see that $M_n(W_n)\cap B_n^*=\emptyset$, it suffices to show that

$$M_n(W_n)\subset M(C_1,\cdots,C_l;U_1,\cdots,U_l).$$

Let $g\in M_n(W_n)$. If $C_i\subset\{1,\cdots,1/n\}$ then $g(C_i)=f_n(C_i)\subset U_i$. If $C_i\setminus\{1,\cdots,1/n\}\neq\emptyset$ then

$$\begin{aligned}g(C_i)&\subset g\Big(X\setminus\Big\{1,\cdots,\frac{1}{n}\Big\}\Big)\cup g\Big(C_i\cap\Big\{1,\cdots,\frac{1}{n}\Big\}\Big)\\&\subset W_n\cup f_n(C_i)\subset U_i,\end{aligned}$$

because $y_0\in f_n(C_i)\subset U_i$. Thus W_n and B_n^* satisfy $(2)_n$. □

To complete the proof of Example 3.2, let $\{k_n\,|\,n\in\mathbf{N}\}$, $\{y_n\,|\,n\in\mathbf{N}\}$, $\{W_n\,|\,n\in\mathbf{N}\}$

and $\{B_n^* \mid n \in \mathbf{N}\}$ be obtained in Lemma 3.5. We define $f \in C(X, Y)$ by

$$f(x)=\begin{cases} y_n & \text{if } x=\dfrac{1}{n}, \\ y_0 & \text{if } x=0. \end{cases}$$

Then $f_n = f_{(y_1, \cdots, y_n)}$ converges to f in $C(X, N)$ if $n \to \infty$. In fact, let $U^* = M(C_1, \cdots, C_l; U_1, \cdots, U_l)$ be a basic neighborhood of f in $C(X, N)$, where $C_i \in \mathfrak{K}(X)$ and $U_i \in \mathfrak{O}(N)$. Without loss of generality, we can assume $C_1 = \{1\}$. And let

$$n_0 = \max\left\{n \,\middle|\, \frac{1}{n} \in \bigcup \{C_i \mid 0 \notin C_i\}\right\}.$$

For each $n \geqq n_0$, $f_n(C_i) = f(C_i) \subset U_i$ if $0 \notin C_i$ and $f_n(C_i) \subset f(C_i) \subset U_i$ if $0 \in C_i$, whence $f_n \in U^*$.

Since $f \in M_n(W_n)$ by the definition, $f \notin B_n^* = \mathrm{cl}\, B_n^*$ for each $n \in \mathbf{N}$, whence $f \notin \mathrm{cl}\,(\bigcup \{B_n^* \mid n \in \mathbf{N}\})$ because $\mathfrak{B}$ is CP. Then f has a neighborhood V^* in $C(X, N)$ such that $V^* \cap B_n^* = \emptyset$ for each $n \in \mathbf{N}$. Choose $m \in \mathbf{N}$ so that $f_{k_m} \in V^*$. Then $B_{k_m+1}^* \in \mathfrak{B}_{k_m} = \mathfrak{B}(y_1, \cdots, y_{k_m})$. From the definition of $\mathfrak{B}(y_1, \cdots, y_{k_m})$, it follows that

$$f_{k_m} = f_{(y_1, \cdots, y_m)} \in \mathrm{Int}\, B_{k_m+1}^* \subset B_{k_m+1}^*.$$

Hence $f_{k_m} \in V^* \cap B_{k_m+1}^*$. This is a contradiction. The proof is completed. □

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