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APPROXIMATIVE SHAPE IV
—UV^n-MAPS AND THE VIETORIS-SMALE THEOREM—

Dedicated to Professor Yukihiro Kodama on his sixtieth birthday

By

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§ 0. Introduction.

This paper is a continuation of [35-37]. We introduced approximate shape in [35], discussed approximative shape properties of spaces and generalized ANRs in [36], and fixed point theorems in [37]. In this paper we investigate approximative shape properties of maps and show the Vietoris-Smale theorem in shape theory.

Many mathematicians studied UV^n-maps. See the references of Lacher [18] for their studies. Smale [30] gave a Vietoris type theorem for homotopy groups and UV^n-maps, called the Vietoris-Smale theorem. Kozlowski [13] gave a factorization theorem for UV^n-maps. Borsuk introduced approximatively n-connected spaces. This is a basic notion in shape theory. Various Vietoris-Smale theorems in shape theory were given by Bogatyĭ [2, 3], Dydak [4-7], Kodama [11, 12], Kuperberg [16], Kozlowski-Segal [15] and Morita [27, 28].

In this paper we discuss the following topics: In § 1 we introduce the approximative lifting property and investigate its properties. In § 2 we prove restriction and product theorems for the approximative lifting property. In § 3 we introduce approximatively n-connected maps and give their characterizations. We show the Vietoris-Smale theorem and the Whitehead theorem for approximatively n-connected maps. In § 4 we introduce the approximative extension property. We characterize approximatively n-connected spaces by this property. In § 5 we introduce partial realizations for decomposition spaces. We introduce the approximative full extension property and investigate its properties. In § 6 we show that our approximatively n-connected maps and usual UV^n-maps are equivalent. Hence by using results in § 3 we show the Vietoris-Smale theorem and the Whitehead theorem in shape theory for closed UV^n-maps between paracompacta.

We assume that the reader is familiar with theory of ANRs and shape theory. As reference books we use Hu [10] for theory of ANR's and Mardešić

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and Segal [23], quoted by MS [23], for shape theory. For undefined notations and terminology see these books. We use the same notations and terminology as in [35-37]. We quote results in [35-37] as follows: For example (I. 3.3), (II. 5.5) and (III. 1.1) denote theorem (3.3) in [35], theorem (5.5) in [36] and theorem (1.1) in [37], respectively.

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§ 1. Approximative lifting property.

In this section we introduce the approximative lifting property for maps and discuss its properties.

Let $\mathcal{K}$ be a collection of pairs of spaces. Let $\boldsymbol{C}$ be a subcategory of **TOP**. Let $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be approximative inverse systems in $\boldsymbol{C}$. Let $\boldsymbol{f}=\{f, f_b : b\in B\} : (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative system map in $\boldsymbol{C}$. We say that $\boldsymbol{f}$ has the approximative lifting property, in notation ALP, with respect to $\mathcal{K}$ provided it satisfies the following condition:

(ALP) For each admissible pair (a, b) of $\boldsymbol{f}$ there exists an admissible pair $(a_0, b_0)>(a, b)$ with the following property; for each $(K, K_0)\in\mathcal{K}$ and for any maps $g: K\to Y_{b_0}$, $h: K_0\to X_{a_0}$ with $(f_{b_0}p_{a_0,f(b_0)}h, g|K_0)<st\mathcal{V}_{b_0}$, then there exists a map $H: K\to X_a$ such that $(p_{a_0,a}h, H|K_0)<st\mathcal{U}_a$ and $(q_{b_0,b}g, f_b p_{a,f(b)}H)<st\mathcal{V}_b$.

(1.1) LEMMA. *Let $\boldsymbol{f}, \boldsymbol{g}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be approximative system maps in $\boldsymbol{C}$. If $\boldsymbol{f}\equiv:\boldsymbol{g}$ and $\boldsymbol{f}$ has ALP with respect to $\mathcal{K}$, then so does $\boldsymbol{g}$.*

PROOF. We put $\boldsymbol{g}=\{g, g_b : b\in B\}$. We may assume $\boldsymbol{f}=:\boldsymbol{g}$ without loss of generality. Take any admissible pair (a, b) of $\boldsymbol{g}$. By (AI3) there exists $b_1>b$ such that $q_{b_1,b}^{-1}\mathcal{V}_b>st\mathcal{V}_{b_1}$. Since $\boldsymbol{f}=:\boldsymbol{g}$ and $b_1>b$, there exists $a_1>a$, $f(b)$, $f(b_1)$ such that

(1) $$(g_b p_{a_1,g(b)}, f_b p_{a_1,f(b)})<\mathcal{V}_b \quad \text{and}$$

(2) $$(f_b p_{a_1,f(b)}, q_{b_1,b}f_{b_1}p_{a_1,f(b_1)})<\mathcal{V}_b.$$

By the assumption there exists an admissible pair $(a_2, b_2)>(a_1, b_1)$ satisfying ALP with respect to $\mathcal{K}$ for $\boldsymbol{f}$ and (a_1, b_1). By (AI3) there exists $b_3>b_2$ such that $q_{b_3,b_2}^{-1}\mathcal{V}_{b_2}>st^2\mathcal{V}_{b_3}$. Since $\boldsymbol{f}=:\boldsymbol{g}$, there exists $a_3>f(b_3)$, $g(b_3)$ such that

(3) $$(f_{b_3}p_{a_3,f(b_3)}, g_{b_3}p_{a_3,g(b_3)})<\mathcal{V}_{b_3}.$$

(4) $$(f_{b_2}p_{a_3,f(b_2)},\ q_{b_3,b_2}f_{b_3}p_{a_3,f(b_3)})<\mathcal{V}_{b_2}.$$

We show that (a_3, b_3) is the required admissible pair for $\boldsymbol{g}$. Take any $(K, K_0)\in\mathcal{K}$ and any maps $s: K\to Y_{b_3}$, $h: K_0\to X_{a_3}$ such that

(5) $$(g_{b_3}p_{a_3,g(b_3)}h,\ s|K_0)<st\mathcal{V}_{b_3}.$$

By (3)-(5) and the choice of b_3, $(f_{b_2}p_{a_3,f(b_2)}h,\ q_{b_3,b_2}s|K_0)<st\mathcal{V}_{b_2}$. Thus by the choice of (a_2, b_2) there exists a map $H: K\to X_{a_1}$ such that

(6) $$(p_{a_3,a_1}h,\ H|K_0)<st\mathcal{U}_{a_1} \text{ and } (f_{b_1}p_{a_1,f(b_1)}H,\ q_{b_3,b_1}s)<st\mathcal{V}_{b_1}.$$

By (1), (2), (6) and the choice of b_1, $(p_{a_3,a}h,\ p_{a_1,a}H|K_0)<st\mathcal{U}_a$ and $(g_bp_{a_1,g(b)}H,\ q_{b_3,b}s)<st\mathcal{V}_b$. Hence $\boldsymbol{g}$ has ALP with respect to $\mathcal{K}$. ■

Thus by (1.1) we may say that $[\boldsymbol{f}]$ has ALP with respect to $\mathcal{K}$ provided that $\boldsymbol{f}$ has ALP with respect to $\mathcal{K}$.

(1.2) LEMMA. *Let $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{g}: (\mathcal{Y}, \mathcal{V})\to(\mathcal{Z}, \mathcal{W})$ be approximative system maps in $\boldsymbol{C}$. If $[\boldsymbol{f}]$ and $[\boldsymbol{g}]$ have ALP with respect to $\mathcal{K}$, then so does $[\boldsymbol{g}][\boldsymbol{f}]$.*

PROOF. We put $\boldsymbol{g}=\{g, g_c: c\in C\}: (\mathcal{Y}, \mathcal{V})\to(\mathcal{Z}, \mathcal{W})=\{(Z_c, \mathcal{W}_c), r_{c',c}, C\}$ and take a 1-refinement function $u: C\to C$ of $(\mathcal{Z}, \mathcal{W})$. We show that $\boldsymbol{r}(u)(\boldsymbol{gf})$ has ALP. Take any admissible pair (a, c) of $\boldsymbol{r}(u)(\boldsymbol{gf})$ and take $c_1>c$ such that $r^{-1}_{c_1,c}\mathcal{W}_c>st\mathcal{W}_{c_1}$. Since $\boldsymbol{r}(u)(\boldsymbol{gf})$ is an approximative system map, there exists $a_1>a$, $fgu(c_1)$ such that

(1) $$(r_{u(c),c}g_{u(c)}f_{gu(c)}p_{a_1,fgu(c)},\ r_{u(c_1),c}g_{u(c_1)}f_{gu(c_1)}p_{a_1,fgu(c_1)})<\mathcal{V}_c.$$

By the assumption there exists an admissible pair $(a_2, b_1)>(a_1, gu(c_1))$ satisfying ALP for $\boldsymbol{f}$ and $(a_1, gu(c_1))$. Take any $b_2>b_1$ such that $q^{-1}_{b_2,b_1}\mathcal{V}_{b_1}>st\mathcal{V}_{b_2}$. By the assumption there exists an admissible pair $(b_3, c_2)>(b_2, u(c_1))$ satisfying ALP for $\boldsymbol{g}$ and $(b_2, u(c_1))$. Take any $c_3>c_2$ such that $r^{-1}_{c_3,c_2}\mathcal{W}_{c_2}>st^2\mathcal{W}_{c_3}$. By (AM2) there exist $b_4>b_3$, $gu(c_3)$ and $a_4>a_2$, $fgu(c_3)$, $f(b_3)$ such that

(2) $$(g_{c_2}q_{b_4,g(c_2)},\ r_{u(c_3),c_2}g_{u(c_3)}q_{b_4,gu(c_3)})<\mathcal{W}_{c_2},$$

(3) $$(f_{gu(c_3)}p_{a_4,fgu(c_3)},\ q_{b_4,gu(c_3)}f_{b_4}p_{a_4,f(b_4)})<\mathcal{V}_{gu(c_3)} \text{ and}$$

(4) $$(f_{b_1}p_{a_4,f(b_1)},\ q_{b_4,b_1}f_{b_4}p_{a_4,f(b_4)})<\mathcal{V}_{b_1}.$$

We show that the admissible pair (a_4, c_3) of $\boldsymbol{r}(u)(\boldsymbol{gf})$ has the required property. Take any $(K, K_0)\in\mathcal{K}$ and maps $s: K\to Z_{c_3}$, $h: K_0\to X_{a_4}$ such that

$(r_{u(c_3),c_3}g_{u(c_3)}f_{gu(c_3)}p_{a_4,fgu(c_3)}h,\ s|K_0)<st\mathcal{W}_{c_3}$. Then by (2), (3), (AM1) for $\boldsymbol{g}$ and the choice of c_3 $(g_{c_2}q_{b_4,g(c_2)}f_{b_4}p_{a_4,f(b_4)}h,\ r_{c_3,c_2}s|K_0)<st\mathcal{W}_{c_2}$. Thus by the choice of (b_3, c_2) there exists a map $H_1: K\to Y_{b_2}$ such that

(5) $$(q_{b_4,b_2}f_{b_4}p_{a_4,f(b_4)}h,\ H_1|K_0)<st\mathcal{V}_{b_2},\ \text{and}$$

(6) $$(g_{u(c_1)}q_{b_2,gu(c_1)}H_1,\ r_{c_3,u(c_1)}s)<st\mathcal{W}_{u(c_1)}.$$

By (4), (5) and the choice of b_2 $(f_{b_1}p_{a_4,f(b_1)}h,\ q_{b_2,b_1}H_1|K_0)<st\mathcal{V}_{b_1}$. Then by the choice of (a_2, b_1) there exists a map $H_2: K\to X_{a_1}$ such that

(7) $$(p_{a_4,a_1}h,\ H_2|K_0)<st\mathcal{U}_{a_1}\ \text{and}$$

(8) $$(f_{gu(c_1)}p_{a_1,fgu(c_1)}H_2,\ q_{b_2,gu(c_1)}H_1)<st\mathcal{V}_{gu(c_1)}.$$

By (1), (6), (8) and the choice of c_1 $(r_{u(c),c}g_{u(c)}f_{gu(c)}p_{a_1,fgu(c)}H_2,\ r_{c_3,c}s)<st\mathcal{W}_c$ and by (7) $(p_{a_4,a}h,\ H_2|K_0)<st\mathcal{U}_a$. Thus $\boldsymbol{r}(u)(\boldsymbol{gf})$ has ALP with respect to $\mathcal{K}$ and hence $[\boldsymbol{g}][\boldsymbol{f}]$ has the required property ALP. ■

(1.3) LEMMA. *If $[\boldsymbol{f}]:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is an isomorphism in Appro-**C**, then $[\boldsymbol{f}]$ has ALP with respect to any collection of pairs.*

(1.3) follows from (I.2.16). ■

Let X and Y be spaces and $f: X\to Y$ a map. Let $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$, $\boldsymbol{p}': X\to(\mathcal{X}, \mathcal{U})'$, $\boldsymbol{q}: Y\to(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{q}': Y\to(\mathcal{Y}, \mathcal{V})'$ be approximative **AP**-resolutions. Let $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{f}':(\mathcal{X}, \mathcal{U})'\to(\mathcal{Y}, \mathcal{V})'$ be approximative resolutions of f with respect to $\boldsymbol{p}$, $\boldsymbol{q}$ and with respect to $\boldsymbol{p}'$, $\boldsymbol{q}'$, respectively. By (I.5.1) $[\boldsymbol{f}']=[1_Y]_{\boldsymbol{q},\boldsymbol{q}'}[\boldsymbol{f}][1_X]_{\boldsymbol{p}',\boldsymbol{p}}$. Since $[1_X]_{\boldsymbol{p}',\boldsymbol{p}}$ and $[1_Y]_{\boldsymbol{q},\boldsymbol{q}'}$ are isomorphisms in Appro-**AP** by (I.5.1), we have the following from (1.2) and (1.3):

(1.4) LEMMA. *If $[\boldsymbol{f}]$ has ALP with respect to $\mathcal{K}$, then so does $[\boldsymbol{f}']$.* ■

Let $\boldsymbol{p}=\{p_a: a\in A\}: X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ and $\boldsymbol{q}=\{q_b: b\in B\}: Y\to\mathcal{Y}=\{Y_b, q_{b',b}, B\}$ be **AP**-resolutions. Let $\boldsymbol{f}=\{f, f_b: b\in B\}: \mathcal{X}\to\mathcal{Y}$ be a resolution of $f: X\to Y$ with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. We say that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ has the approximative lifting property, in notation ALP, with respect to $\mathcal{K}$ provided it satisfies the following condition:

(ALP)* For each admissible pair (a, b) of $\boldsymbol{f}$ and for any $\mathcal{U}\in Cov(X_a)$, $\mathcal{V}\in Cov(Y_b)$ there exist an admissible pair $(a_0, b_0)>(a, b)$ and $\mathcal{V}_0\in Cov(Y_{b_0})$ with the following property; for any $(K, K_0)\in\mathcal{K}$ and any maps $h: K_0\to X_{a_0}$, $g: K\to Y_{b_0}$ with $(f_{b_0}p_{a_0,f(b_0)}h,\ g|K_0)<\mathcal{V}_0$, there exists a map $H: K\to X_a$ such that $(p_{a_0,a}h,\ H|K_0)<\mathcal{U}$ and $(f_bp_{a,f(b)}H,\ q_{b_0,b}g)<\mathcal{V}$.

(1.5) LEMMA. *Let $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ and $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$ be* **AP**-*resolutions of $f: X \to Y$. If $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ has ALP with respect to $\mathcal{K}$, then so does $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$.*

PROOF. We put $\boldsymbol{r}=\{r_c : c \in C\}: X \to \mathcal{R}=\{R_c, r_{c',c}, C\}$, $\boldsymbol{s}=\{s_d : d \in D\}: Y \to \mathcal{S}=\{S_d, s_{d',d}, D\}$ and $\boldsymbol{g}=\{g, g_d : d \in D\}$. We assume that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ satisfies (ALP)* and show that $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$ satisfies (ALP)*.

Take any admissible pair (c, d) of $\boldsymbol{g}$ and $\mathcal{W} \in \mathcal{C}ov(R_c)$, $\mathcal{L} \in \mathcal{C}ov(S_d)$. There exist $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 \in \mathcal{C}ov(S_d)$ such that $st\mathcal{L}_1 < \mathcal{L}$, $\mathcal{L}_2$ satisfies (R2) for $\boldsymbol{p}$ and $\mathcal{L}_1$, $\mathcal{L}_3$ satisfies (R2) for $\boldsymbol{s}$ and $\mathcal{L}_1$, and $st\mathcal{L}_4 < \mathcal{L}_1 \wedge \mathcal{L}_2 \wedge \mathcal{L}_3$. Also there exist $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in \mathcal{C}ov(R_c)$ such that $st\mathcal{W}_1 < \mathcal{W}$, $\mathcal{W}_2$ satisfies (R2) for $\boldsymbol{r}$ and $\mathcal{W}_1$, $st\mathcal{W}_3 < \mathcal{W}_1 \wedge \mathcal{W}_2 \wedge (g_d r_{c,g(d)})^{-1}\mathcal{L}_4$. By (R1) there exist $b_1 \in B$ and a map $t: Y_{b_1} \to S_d$ such that

$$(1) \qquad (s_d, tq_{b_1}) < \mathcal{L}_4.$$

By (R1) there exist $a_1 > f(b_1)$ and a map $u: X_{a_1} \to R_c$ such that

$$(2) \qquad (r_c, up_{a_1}) < \mathcal{W}_3.$$

Then by the choice of $\mathcal{W}_3$

$$(3) \qquad (g_d r_{g(d)}, g_d r_{c,g(d)} up_{a_1}) < \mathcal{L}_4.$$

Since $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ and $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$ are resolutions of f, $tq_{b_1}f = tf_{b_1}p_{f(b_1)}$ and $s_d f = g_d r_{g(d)}$. Then since $(s_d f, tq_{b_1}f) < \mathcal{L}_4$ by (1), (3) follows $(g_d r_{c,g(d)} up_{a_1}, tf_{b_1}p_{a_1,f(b_1)}p_{a_1}) < st\mathcal{L}_4 < \mathcal{L}_2$. Thus by the choice of $\mathcal{L}_2$ there exists $a_2 > a_1$ such that

$$(4) \qquad (g_d r_{c,g(d)} up_{a_2,a_1}, tf_{b_1}p_{a_2,f(b_1)}) < \mathcal{L}_1.$$

By the assumption there exist an admissible pair $(a_3, b_2) > (a_2, b_1)$ and $\mathcal{V}' \in \mathcal{C}ov(Y_{b_2})$ satisfying (ALP)* for $\boldsymbol{f}$, (a_2, b_1), $(up_{a_2,a_1})^{-1}\mathcal{W}_3 \in \mathcal{C}ov(X_{a_2})$ and $t^{-1}\mathcal{L}_4 \in \mathcal{C}ov(Y_{b_1})$. There exist $\mathcal{V}_1', \mathcal{V}_2', \mathcal{V}_3' \in \mathcal{C}ov(Y_{b_2})$ such that $st\mathcal{V}_1' < \mathcal{V}'$, $\mathcal{V}_2'$ satisfies (R2) for $\boldsymbol{r}$ and $\mathcal{V}_1'$, and $st\mathcal{V}_3' < \mathcal{V}_2' \wedge (tq_{b_2,b_1})^{-1}\mathcal{L}_4$. By (R1) there exist $d_1 > d$ and a map $v: S_{d_1} \to Y_{b_2}$ such that

$$(5) \qquad (vs_{d_1}, q_{b_2}) < \mathcal{V}_3'.$$

By (5) and the choice of $\mathcal{V}_3'$, $(tq_{b_2,b_1}vs_{d_1}, tq_{b_1}) < \mathcal{L}_4$ and then by (1), $(tq_{b_2,b_1}vs_{d_1}, s_{d_1,d}s_{d_1}) < st\mathcal{L}_4 < \mathcal{L}_3$. By the choice of $\mathcal{L}_3$ there exists $d_2 > d_1$ such that

$$(6) \qquad (tq_{b_2,b_1}vs_{d_2,d_1}, s_{d_2,d}) < \mathcal{L}_1.$$

By (R1) there exist $c_1 > c$, $g(d_2)$ and a map $w: R_{c_1} \to X_{a_3}$ such that

$$(7) \qquad (wr_{c_1}, p_{a_3}) < (up_{a_3,a_1})^{-1}\mathcal{W}_3 \wedge (f_{b_2}p_{a_3,f(b_2)})^{-1}\mathcal{V}_3'.$$

By (5), $(vs_{d_1}f,\ q_{b_2}f)<\mathcal{V}_3'$. By (7), $(f_{b_2}p_{a_3,f(b_2)}wr_{c_1},\ f_{b_2}p_{f(b_2)})<\mathcal{V}_3'$. Since $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ and $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$ are resolutions of f, $f_{b_2}p_{f(b_2)}=q_{b_2}f$ and $s_{d_1}f=s_{d_2,d_1}g_{d_2}r_{g(d_2)}$. Then $(f_{b_2}p_{a_3,f(b_2)}wr_{c_1},\ vs_{d_2,d_1}g_{d_2}r_{c_1,g(d_2)}r_{c_1})<st\mathcal{V}_3'<\mathcal{V}_2'$. By the choice of $\mathcal{V}_2'$ there exists $c_2>c_1$ such that

(8) $$(f_{b_2}p_{a_3,f(b_2)}wr_{c_2,c_1},\ vs_{d_2,d_1}g_{d_2}r_{c_2,g(g_2)})<\mathcal{V}_1'.$$

By (7), $(up_{a_3,a_1}wr_{c_1},\ up_{a_1})<\mathcal{W}_3$ and by (2), $(up_{a_3,a_1}wr_{c_2,c_1}r_{c_2},\ r_{c_2,c}r_{c_2})<st\mathcal{W}_3<\mathcal{W}_2$. By the choice of $\mathcal{W}_2$, there exists $c_3>c_2$ such that

(9) $$(up_{a_3,a_1}wr_{c_3,c_1},\ r_{c_3,c})<\mathcal{W}_1.$$

We show that the admissible pair (c_3, d_2) for $\boldsymbol{g}$ and $(vs_{d_2,d_1})^{-1}\mathcal{V}_1'\in\mathcal{C}ov(S_{d_2})$ are the required ones. Take any $(K, K_0)\in\mathcal{K}$ and any maps $k:K_0\to R_{c_3}$, $h:K\to S_{d_2}$ such that $(g_{d_2}r_{c_3,g(d_2)}k,\ h|K_0)<(vs_{d_2,d_1})^{-1}\mathcal{V}_1'$, and then $(vs_{d_2,d_1}g_{d_2}r_{c_3,g(d_2)}k,\ vs_{d_2,d_1}h|K_0)<\mathcal{V}_1'$. Then by (8), $(f_{b_2}p_{a_3,f(b_2)}wr_{c_3,c_1}k,\ vs_{d_2,d_1}h|K_0)<st\mathcal{V}_1'<\mathcal{V}'$. By the choice of (a_3, b_2) there exists a map $H:K\to X_{a_2}$ such that

(10) $$(p_{a_3,a_2}wr_{c_3,c_1}k,\ H|K_0)<(up_{a_2,a_1})^{-1}\mathcal{W}_3 \quad\text{and}$$

(11) $$(f_{b_1}p_{a_2,f(b_1)}H,\ q_{b_2,b_1}vs_{d_2,d_1}h)<t^{-1}\mathcal{L}_4.$$

By (10), $(up_{a_3,a_1}wr_{c_3,c_1}k,\ up_{a_2,a_1}H|K_0)<\mathcal{W}_3$ and by (9), $(up_{a_3,a_1}wr_{c_3,c_1}k,\ r_{c_3,c_1}k)<\mathcal{W}_1$. Then

(12) $$(r_{c_3,c}k,\ up_{a_2,a_1}H|K_0)<st\mathcal{W}_1<\mathcal{W}.$$

By (11), $(tf_{b_1}p_{a_2,f(b)}H,\ tq_{b_2,b_1}vs_{d_2,d_1}h)<\mathcal{L}_4<\mathcal{L}_1$, by (4), $(g_dr_{c,g(d)}up_{a_2,a_1}H,\ tf_{b_1}p_{a_2,f(b_1)}H)<\mathcal{L}_1$ and by (6), $(tq_{b_2,b_1}vs_{d_2,d_1}h,\ s_{d_2,d}h)<\mathcal{L}_1$. Thus

(13) $$(g_dr_{c,g(d)}up_{a_2,a_1}H,\ s_{d_2,d}h)<st\mathcal{L}_1<\mathcal{L}.$$

Hence, by (12) and (13), $\boldsymbol{g}$ satisfies (ALP)* with respect to $\mathcal{K}$. ■

By (I. 4.9) there exist approximative **ANR**-resolutions $\boldsymbol{p}:X\to(\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}:Y\to(\mathcal{Y}, \mathcal{V})$ and an approximative resolution $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ such that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an **ANR**-resolution of f.

(1.6) LEMMA. *Under the above conditions,* $\boldsymbol{f}$ *satisfies (ALP) with respect to* $\mathcal{K}$ *iff* $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ *satisfies (ALP)* with respect to* $\mathcal{K}$.

PROOF. We assume that $\boldsymbol{f}$ satisfies (ALP) and show that it satisfies (ALP)*. Take any admissible pair (a, b) of $\boldsymbol{f}$, $\mathcal{U}\in\mathcal{C}ov(X_a)$ and $\mathcal{V}\in\mathcal{C}ov(Y_b)$. There exist $b_1>b$, $a_1>a$, $f(b_1)$ such that $p_{a_1,a}^{-1}\mathcal{U}>st\mathcal{U}_{a_1}$, $q_{b_1,b}^{-1}\mathcal{V}>st\mathcal{V}_{b_1}$ and $f_bp_{a_1,f(b)}=q_{b_1,b}f_{b_1}p_{a_1,f(b_1)}$. There exists an admissible pair $(a_2, b_2)>(a_1, b_1)$ satisfying

(ALP) for (a_1, b_1). Thus it is easy to show that (a_2, b_2) and $st\mathcal{V}_{b_2}$ satisfy (ALP)* for (a, b), $\mathcal{U}$ and $\mathcal{V}$.

Next we assume that $\boldsymbol{f}$ satisfies (ALP)* and show that it satisfies (ALP). Take any admissible pair (a, b) of $\boldsymbol{f}$. Then there exist an admissible pair $(a_1, b_1)>(a, b)$ and $\mathcal{V}'\in Cov(Y_{b_1})$ satisfying (ALP)* for (a, b), $\mathcal{U}_a$ and $\mathcal{V}_b$. Take $b_3>b_1$ and $a_2>a_1$, $f(b_2)$ such that $q_{b_2,b_1}^{-1}\mathcal{V}'>st\mathcal{V}_{b_2}$ and $f_{b_1}p_{a_2,f(b_1)}=q_{b_2,b_1}f_{b_2}p_{a_2,f(b_2)}$. It is easy to show that (a_2, b_2) satisfies (ALP) for (a, b). ■

(1.7) THEOREM. *Let $\mathcal{K}$ be a class of pairs. Let $f: X\to Y$ be a map. Then the following statements are equivalent:*

(i) *Any/some approximative* **AP**-*resolution of f has ALP with respect to $\mathcal{K}$.*

(ii) *Any/some* **AP**-*resolution of f has ALP with respect to $\mathcal{K}$.*

(1.7) follows from (1.4)-(1.6). ■

Thus by (1.7) we say that a map $f: X\to Y$ has the approximative lifting property, in notation ALP, with respect to $\mathcal{K}$ provided that it satisfies any one of the conditions in (1.7).

(1.8) COROLLARY. *Let $f: X\to Y$ and $g: Y\to Z$ be maps.*

(i) *If $CT(f): CT(X)\to CT(Y)$ is a homeomorphism, then f has ALP with respeɔt to any $\mathcal{K}$.*

(ii) *If f is a homeomorphism, then f has ALP with respect to any $\mathcal{K}$.*

(iii) *If f and g have ALP with respect to $\mathcal{K}$, then so does gf.*

(1.9) COROLLARY. *A map $f: X\to Y$ has ALP with respect to $\mathcal{K}$ iff so does $CT(f): CT(X)\to CT(Y)$.*

(i) in (1.8) follows from (I. 7.10) and (1.3). (ii) follows from (i). (iii) follows from (1.2). (1.9) follows from (I. 6.9), (I. 6.11), (I. 7.10), (1.2) and (1.3.). ■

Let **TOP**(ALP, $\mathcal{K}$) be the subcategory of **TOP** consisting of all spaces and all maps having ALP with respect to $\mathcal{K}$. Thus by (1.8) we have the following:

(1.10) THEOREM. **TOP**(ALP, $\mathcal{K}$) *forms a subcategory of* **TOP** *including all homeomorphisms for any collection $\mathcal{K}$ of pairs.* ■

Let X_0 be a subspace of a space X. We say that (X, X_0) is a closed pair provided that X_0 is closed in X. We recall that coverings are always normal open coverings (see [1]). We say that X_0 is P-embedded in X provided that for any covering $\mathcal{U}_0$ of X_0 there exists a covering $\mathcal{U}$ of X such that $\mathcal{U}|X_0=$

$\{U\cap X_0 : U\in\mathcal{V}\}<\mathcal{V}_0$. In MS [23] "normally embedded" is used instead of "P-embedded".

Let $\mathbf{TOP}_{\text{pairs}}$ be the category consisting of all pairs and all map between pairs. $\mathbf{TOP}_{C\text{-pairs}}$ and $\mathbf{TOP}_{P\text{-pairs}}$ denote the full subcategories of $\mathbf{TOP}_{\text{pairs}}$ consisting of all closed pairs and all P-embedded pairs, respectively. $\mathbf{TOP}_{CP\text{-pairs}}$ denotes the full subcategory consisting of all closed P-embedded pairs. $\mathbf{TOP}^n_{\text{pairs}}$ and ${}^n\mathbf{TOP}_{\text{pairs}}$ denote the full subcategories of $\mathbf{TOP}_{\text{pairs}}$ consisting of all pairs (X, X_0) with $\dim X\leqq n$ and $\dim(X-X_0)\leqq n$, respectively.

We say that (X, X_0) is a polyhedral pair provided that there exists a simplicial complex K and a subcomplex K_0 of K such that $|K|=X$ and $|K_0|=X_0$. Here $|K|$ denotes the realization of K endowed with CW-topology. We say that (X, X_0) is an ANR-pair provided that X_0 is closed in X and X, X_0 are ANRs. $\mathbf{POL}_{\text{pairs}}$ and $\mathbf{ANR}_{\text{pairs}}$ denote the full subcategories of $\mathbf{TOP}_{CP\text{-pairs}}$ consisting of all polyhedral pairs and ANR-pairs, respectively. Similarly we may define categories $\mathbf{POL}^n_{\text{pairs}}$, ${}^n\mathbf{POL}_{\text{pairs}}$, $\mathbf{ANR}^n_{\text{pairs}}$, ${}^n\mathbf{ANR}_{\text{pairs}}$ and so on.

Let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be subcategories of $\mathbf{TOP}_{\text{pairs}}$. We say that $\boldsymbol{K}_1$ is expandable by $\boldsymbol{K}_2$, in notation $\boldsymbol{K}_1<_e\boldsymbol{K}_2$, provided that any $(X, X_0)\in Ob\boldsymbol{K}_1$ admits a $\boldsymbol{K}_2$-resolution. We say that $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are expansively equivalent, in notation $\boldsymbol{K}_1=_e\boldsymbol{K}_2$, provided that both $\boldsymbol{K}_1<_e\boldsymbol{K}_2$ and $\boldsymbol{K}_2<_e\boldsymbol{K}_1$. We say that $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ and $f: X\to Y$ have ALP with respect to $\boldsymbol{K}_1$ provided that they have ALP with respect to $Ob\boldsymbol{K}_1$, respectively. (I. 3.15) and (I. 3.16) (see [21] and [23]) mean the following:

(1.11) LEMMA. $\mathbf{TOP}_{\text{pairs}}=_e\mathbf{TOP}_{P\text{-pairs}}=_e\mathbf{POL}_{\text{pairs}}=_e\mathbf{ANR}_{\text{pairs}}$, $\mathbf{TOP}=_e\mathbf{POL}=_e\mathbf{ANR}$, $\mathbf{TOP}^n_{\text{pairs}}=_e\mathbf{TOP}^n_{P\text{-pairs}}=_e\mathbf{POL}^n_{\text{pairs}}$ *and* $\mathbf{TOP}^n=_e\mathbf{POL}^n$. ■

(1.12) THEOREM. *Let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be subcategories of $\mathbf{TOP}_{P\text{-pairs}}$. We assume that $\boldsymbol{K}_1<_e\boldsymbol{K}_2$. If a map f has ALP with respect to $\boldsymbol{K}_2$, then f has ALP with respect to $\boldsymbol{K}_1$.*

To prove (1.12) we need the following lemma.

(1.13) LEMMA. *Let Z be an AP. Then for each $\mathcal{V}\in Cov(Z)$ there exists $\mathcal{V}'\in Cov(Z)$ satisfying* (R2) *for any resolution and $\mathcal{V}$.*

PROOF. For any $\mathcal{V}\in Cov(Z)$ we take $\mathcal{V}'\in Cov(Z)$ with $st\mathcal{V}'<\mathcal{V}$. We show that $\mathcal{V}'$ has the required one. Take any resolution $\boldsymbol{p}: X\to\mathcal{X}$ of any space X. Take any $a\in A$ and any maps $g, h: X_a\to Z$ such that $(gp_a, hp_a)<\mathcal{V}'$. Put $\mathcal{W}=g^{-1}\mathcal{V}'\wedge h^{-1}\mathcal{V}'\in Cov(X_a)$.

(1) $\qquad (g|st(p_a(X), \mathcal{W}), h|st(p_a(X), \mathcal{W}))<\mathcal{U}.$

Take any $y\in st(p_a(X), \mathcal{W})$ and then there exist $U_1', U_2'\in\mathcal{U}'$ and $x\in X$ such that $p_a(x)\in p_a(X)\cap g^{-1}U_1'\cap h^{-1}U_2'$ and $y\in g^{-1}U_1'\cap h^{-1}U_2'$. Thus $gp_a(x)$, $g(y)\in U_1'$, $hp_a(x)$, $h(y)\in U_2'$. Since $(gp_a, hp_a)<\mathcal{U}'$, there exists $U_3'\in\mathcal{U}'$ such that $gp_a(x)$, $hp_a(x)\in U_3'$. Thus $g(y)$, $h(y)\in st(U_3', \mathcal{U}')\subset U$ for some $U\in\mathcal{U}$. Hence we have (1).

By (I. 3.3) $\boldsymbol{p}$ satisfies (B4) and then there exists $a'>a$ such that $p_{a',a}(X_{a'})\subset st(p_a(X), \mathcal{W})$. Hence by (1) $(gp_{a',a}, hp_{a',a})<\mathcal{U}$. ■

PROOF OF (1.12). Take any approximative **AP**-resolutions $\boldsymbol{p}: X\to\mathcal{X}$ and $\boldsymbol{q}: Y\to\mathcal{Y}$. Let $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. We assume that $\boldsymbol{f}$ has ALP with respect to $\boldsymbol{K}_2$ and we show that it has ALP with respect to $\boldsymbol{K}_1$.

Take any admissible pair (a, b) of $\boldsymbol{f}$. Then there exist $b_1>b$ such that $q_{b_1,b}^{-1}\mathcal{V}_b>st\mathcal{V}_{b_1}$. By (AM1)-(AM2) there exists $a_2>a_1>a$, $f(b_1)$ such that $(f_bp_{a_1,f(b)}, q_{b_1,b}f_{b_1}p_{a_1,f(b_1)})<\mathcal{V}_b$ and $p_{a_2,a_1}^{-1}\mathcal{U}_{a_1}>st\mathcal{U}_{a_2}$. By the assumption there exists an admissible pair $(a_3, b_3)>(a_2, b_1)$ satisfying (ALP) for $\boldsymbol{K}_2$ and (a_2, b_1). By (1.13) there exists $\mathcal{V}\in Cov(Y_{b_3})$ satisfying (R2) for any resolution and $\mathcal{V}_{b_3}$. Take any $b_4>b_3$ and $a_4>a_3$, $f(b_4)$ such that $q_{b_4,b_3}^{-1}\mathcal{V}>st^2\mathcal{V}_{b_4}$ and $(f_{b_3}p_{a_4,f(b_3)}, q_{b_4,b_3}f_{b_4}p_{a_4,f(b_4)})<\mathcal{V}_{b_3}$.

We show that the admissible pair (a_4, b_4) has the required property. Take any $(Z, Z_0)\in\boldsymbol{K}_1$ and maps $g: Z\to Y_{b_4}$, $h: Z_0\to X_{a_4}$ such that

(1) $\qquad (f_{b_4}p_{a_4,f(b_4)}h, g|Z_0)<st\mathcal{V}_{b_4}.$

By the assumption there exists a $\boldsymbol{K}_2$-resolution $\boldsymbol{r}=\{r_c: c\in C\}:(Z, Z_0)\to(\mathcal{X}, \mathcal{X}_0)=\{(Z_c, Z_{0c}), r_{c',c}, C\}$ of (Z, Z_0). Then by Theorems 2-3 of Mardešić [21] $\boldsymbol{r}_Z=\{r_c: c\in C\}: Z\to\mathcal{Z}=\{Z_c, r_{c',c}, C\}$ and $\boldsymbol{r}_{Z_0}=\{r_c^0: c\in C\}: Z_0\to\mathcal{Z}_0=\{Z_{0c}, r_{c,c}^0, C\}$ form resolutions. Here $r_c^0: Z_0\to Z_{0c}$ and $r_{c',c}^0: Z_{0c'}\to Z_{0c}$ are induced by r_c and $r_{c',c}$. By (R1) there exist $c_1\in C$ andmaps $h': Z_{0c_1}\to X_{a_4}$, $g': Z_{c_1}\to Y_{b_4}$ such that

(2) $\qquad (h, h'r_{c_1}^0)<\mathcal{U}_{a_4}$ and $(g, g'r_{c_1})<\mathcal{V}_{b_4}$.

By (1), (2) and the choice of b_4, $(q_{b_4,b_3}f_{b_4}p_{a_4,f(b_4)}h'r_{c_1}^0, q_{b_4,b_3}g'r_{c_1}^0)<\mathcal{V}$. By the choice of $\mathcal{V}$ there exists $c_2>c_1$ such that $(q_{b_4,b_3}f_{b_4}p_{a_4,f(b_4)}h'r_{c_2,c_1}^0, q_{b_4,b_3}g'r_{c_2,c_1}^0)<\mathcal{V}_{b_3}$. Thus $(f_{b_3}p_{a_4,f(b_3)}h'r_{c_2,c_1}^0, q_{b_4,b_3}g'r_{c_2,c_1}|Z_{0c_2})<st\mathcal{V}_{b_3}$. By the choice of (a_3, b_3) there exists a map $H: Z_{c_2}\to X_{a_2}$ such that

(3) $\qquad (H|Z_{0c_2}, p_{a_4,a_2}h'r_{c_2,c_1}^0)<st\mathcal{U}_{a_2}$ and

(4) $\qquad (f_{b_1}p_{a_2,f(b_1)}H, q_{b_4,b_1}g'r_{c_2,c_1})<st\mathcal{V}_{b_1}.$

By (2), (3) and the choice of a_2

(5) $\quad (p_{a_2,a}Hr_{c_2}|Z_0,\ p_{a_4,a}h)<st\mathcal{U}_a.$

By (2), (4) and the choices of a_1, b_1

(6) $\quad (f_b p_{a_2,f(b)}Hr_{c_2},\ q_{b_4,b}g)<st\mathcal{V}_b.$

By (5) and (6) (a_4, b_4) satisfies the required conditions. Hence f has ALP with respect to $\boldsymbol{K}_1$. ∎

(1.14) COROLLARY. *Let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be subcategories of $\mathbf{TOP}_{P\text{-pairs}}$ and $\boldsymbol{K}_1=_e\boldsymbol{K}_2$. A map f has ALP with respect to $\boldsymbol{K}_1$ iff f has ALP with respect to $\boldsymbol{K}_2$.* ∎

(1.15) COROLLARY. *The following statements are equivalent:*

(i) *A map f has ALP with respect to $\mathbf{TOP}_{P\text{-pairs}}$.*

(ii) *f has ALP with respect to $\mathbf{ANR}_{\text{pairs}}$.*

(iii) *f has ALP with respect to $\mathbf{POL}_{\text{pairs}}$.*

(1.16) COROLLARY. *For each integer n the following statements are equivalent:*

(i) *A map f has ALP with respect to $\mathbf{TOP}^n_{P\text{-pairs}}$.*

(ii) *f has ALP with respect to $\mathbf{POL}^n_{\text{pairs}}$.*

(1.15) and (1.16) follow from (1.11) and (1.14). ∎

(1.16) REMARK. Let $\mathcal{K}$ be a collection of pairs of spaces. Let $t(\mathcal{K})$ be the full subcategory of $\mathbf{TOP}_{\text{pairs}}$ consisting of $\mathcal{K}$ as objects. Then $\mathcal{K}$ uniquely determines $t(\mathcal{K})$. Since $Obt(\mathcal{K})=\mathcal{K}$, $t(\mathcal{K})$ uniquely determines $\mathcal{K}$. Hence we may identify $\mathcal{K}$ and $t(\mathcal{K})$. For example, for collections $\mathcal{K}_1$ and $\mathcal{K}_2$ of pairs of spaces we say that $\mathcal{K}_1$ is expandable by $\mathcal{K}_2$, in notation $\mathcal{K}_1<_e\mathcal{K}_2$, provided that $t(\mathcal{K}_1)<_e t(\mathcal{K}_2)$. Similarly we can define $\mathcal{K}_1=_e\mathcal{K}_2$. Thus (1.12) and (1.14) hold for collections.

(1.17) REMARK. Approximative homotopy lifting property was introduced for maps between compacta by Mardešić and Rushing [**22**] and for maps between arbitrary spaces by Mardešić [**20**]. In the sequel we shall show that the approximative homotopy lifting property is a special case of our approximative lifting property.

§ 2. Restrictions and products of the approximative lifting property.

Let (X, X_0) be a pair of spaces. Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative resolution. We put $X_{0a}=st(p_a(X_0), \mathcal{U}_a)$ and $\mathcal{U}_{0a}=\mathcal{U}_a|X_{0a}$ for $a\in A$. Then $p_a : X\to X_a$ and $p_{a',a} : X_{a'}\to X_a$ induce maps $p_a^0 : X_0\to X_{0a}$ and $p_{a',a}^0 : X_{0a'}\to X_{0a}$ for $a'>a$.

(2.1) LEMMA. *If X_0 is P-embedded in X and all X_a are paracompact, then $\boldsymbol{p}^0=\{p_a^0 : a\in A\} : X_0\to(\mathcal{X}_0, \mathcal{U}_0)=\{(X_{0a}, \mathcal{U}_{0a}), p_{a',a}^0, A\}$ is an approximative resolution.*

PROOF. First we show that $(\mathcal{X}_0, \mathcal{U}_0)$ forms an approximative inverse system. (AI1) and (AI2) for $(\mathcal{X}_0, \mathcal{U}_0)$ follows from (AI1) and (AI2) for $(\mathcal{X}, \mathcal{U})$.

We show (AI3). Take any $a\in A$ and any $\mathcal{V}\in \mathcal{C}ov(X_{0a})$. Since $\overline{p_a(X_0)}\subset X_{0a}$ and X_a is paracompact, $\mathcal{U}=\mathcal{V}\cup\{X_a-\overline{p_a(X_0)}\}$ is an open normal covering of X_a. Take any $\mathcal{U}'\in \mathcal{C}ov(X_a)$ such that $st\mathcal{U}'<\mathcal{U}$. By (AI3) for $(\mathcal{X}, \mathcal{U})$ there exists $a'>a$ such that $p_{a',a}^{-1}\mathcal{U}'>\mathcal{U}_{a'}$.

We show that $(p_{a',a}^0)^{-1}\mathcal{V}>\mathcal{U}_{0a'}$. Take any non-empty $U_{01}\in\mathcal{U}_{0a'}$ and then $U_{01}=st(p_{a'}(X_0), \mathcal{U}_{a'})\cap U_1$ for some $U_1\in\mathcal{U}_{a'}$. Thus there exists $U_2\in\mathcal{U}_{a'}$ such that $U_2\cap p_{a'}(X_0)\neq\emptyset$ and $U_1\cap U_2\neq\emptyset$. By the choice of a' there exist $U_1', U_2'\in\mathcal{U}'$ such that $p_{a',a}^{-1}U_i'\supset U_i$, $i=1, 2$. Then $U_2'\cap p_a(X_0)\neq\emptyset$ and $U_1'\cap U_2'\neq\emptyset$. By the choice of $\mathcal{U}'$ there exists $U\in\mathcal{U}$ such that $st(U_1', \mathcal{U}')\subset U$. Since $U\cap p_a(X_0)\neq\emptyset$, by the definition of $\mathcal{U}$, $U\in\mathcal{V}$ and $(p_{a',a}^0)^{-1}U=X_{0a'}\cap p_{a',a}^{-1}U\supset X_{0a'}\cap U_1=U_{01}$. We thus have the required property. Hence $(\mathcal{X}_0, \mathcal{U}_0)$ forms an approximative inverse system.

Next we show that $\boldsymbol{p}^0 : X_0\to(\mathcal{X}_0, \mathcal{U}_0)$ is an approximative resolution. We show (AR1). Take any $\mathcal{V}\in\mathcal{C}ov(X_0)$. Since X_0 is P-embedded, there exists $\mathcal{U}\in\mathcal{C}ov(X)$ such that $\mathcal{V}>\mathcal{U}|X_0$. By (AR1) for $\boldsymbol{p}$ there exists $a\in A$ such that $p_a^{-1}\mathcal{U}_a<\mathcal{U}$. It is easy to show that $(p_a^0)^{-1}\mathcal{U}_{0a}<\mathcal{V}$. We show (AR2). Take any $a\in A$ and any $\mathcal{V}\in\mathcal{C}ov(X_{0a})$. Put $\mathcal{U}, \mathcal{U}'\in\mathcal{C}ov(X_a)$ and $a'>a$ as in the second paragraph. Since $(p_{a',a}^0)^{-1}\mathcal{V}>\mathcal{U}_{0a'}$, $p_{a',a}^0(X_{0a'})\subset st(p_{0a}(X_0), \mathcal{V})$. Hence $\boldsymbol{p}^0$ is an approximative resolution. ■

(2.2) COROLLARY. *If $\boldsymbol{p} : X\to(\mathcal{X}, \mathcal{U})$ is an approximative* **ANR**-*resolution and X_0 is P-embedded in X, then $\boldsymbol{p}^0 : X_0\to(\mathcal{X}_0, \mathcal{U}_0)$ is an approximative* **ANR**-*resolution.* ■

(2.3) COROLLARY. *Any space X admits an approximative* **ANR**-*resolution $\boldsymbol{p} : X\to(\mathcal{X}, \mathcal{U})$ such that $st(p_a(X), \mathcal{U}_a)=X_a$ for all $a\in A$.* ■

Let (X, X_0), (Y, Y_0) be P-pairs and $f:(X, X_0)\to(Y, Y_0)$ a map. Then f induces a map $f_0: X_0\to Y_0$. Let $\boldsymbol{q}=\{q_b: b\in B\}: Y\to(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be an approximative resolution. We assume that all X_a and all Y_b are paracompact. We put $Y_{0b}=st(q_b(Y_0), \mathcal{V}_b)$ and $\mathcal{V}_{0b}=\mathcal{V}_b|Y_{0b}\in \mathit{Cov}(Y_{0b})$ for all $b\in B$. Then $q_b: Y\to Y_b$ and $q_{b',b}: Y_{b'}\to Y_b$ induce maps $q_b^0: Y_0\to Y_{0b}$ and $q_{b',b}^0: Y_{0b'}\to Y_{0b}$ for $b'>b$. By (2.1) $\boldsymbol{q}^0=\{q_b^0: b\in B\}: Y_0\to(\mathcal{Y}_0, \mathcal{V}_0)=\{(Y_{0b}, \mathcal{V}_{0b}), q_{b',b}^0, B\}$ is an approximative resolution. Let $\boldsymbol{f}=\{f, f_b: b\in B\}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Let $k: B\to B$ be a 1-refinement function of $(\mathcal{Y}, \mathcal{V})$. We put $f_b^0=q_{k(b),b}f_{k(b)}|X_{0fk(b)}: X_{0fk(b)}\to Y_{0b}$ for $b\in B$. In a straightforward way we can easily show the following:

(2.4) LEMMA. *Under the above conditions* $\boldsymbol{f}_0=\{fk, f_b^0: b\in B\}:(\mathcal{X}_0, \mathcal{U}_0)\to(\mathcal{Y}_0, \mathcal{V}_0)$ *is an approximative resolution of* $f_0: X_0\to Y_0$ *with respect to* $\boldsymbol{p}^0$ *and* $\boldsymbol{q}^0$.

(2.5) THEOREM. *Let* $f:(X, X_0)\to(Y, Y_0)$ *be a map and* $\mathcal{K}$ *a collection of pairs of spaces. We assume the following two conditions:*

(i) X_0 *and* Y_0 *are P-embedded in* X *and* Y*, respectively.*

(ii) *For each* $\mathcal{U}\in \mathit{Cov}(X)$ *there exists* $\mathcal{V}\in \mathit{Cov}(Y)$ *such that* $f^{-1}st(Y_0, \mathcal{V})\subset st(X_0, \mathcal{U})$.

If $f: X\to Y$ *has ALP with respect to* $\mathcal{K}$*, then so does* $f_0: X_0\to Y_0$*. Here* f_0 *is induced by* f.

PROOF. By (2.3) there exist approximative **ANR**-resolutions $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$ and $\boldsymbol{q}: Y\to(\mathcal{Y}, \mathcal{V})$ satisfying $X_a=st(p_a(X), \mathcal{U}_a)$ and $Y_b=st(q_b(Y), \mathcal{V}_b)$ for $a\in A$ and $b\in B$. By (i) and (2.2) $\boldsymbol{p}^0: X_0\to(\mathcal{X}_0, \mathcal{U}_0)$ and $\boldsymbol{q}^0: Y_0\to(\mathcal{Y}_0, \mathcal{V}_0)$ are approximative **ANR**-resolutions. Let $\boldsymbol{f}:(\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. By (2.4) $\boldsymbol{f}_0:(\mathcal{X}_0, \mathcal{U}_0)\to(\mathcal{Y}_0, \mathcal{V}_0)$ is an approximative resolution of $f_0: X_0\to Y_0$ with respect to $\boldsymbol{p}^0$ and $\boldsymbol{q}^0$.

By the assumption $\boldsymbol{f}$ has ALP with respect to $\mathcal{K}$. We show that also $\boldsymbol{f}_0$ has ALP with respect to $\mathcal{K}$. Take any admissible pair (a, b) of $\boldsymbol{f}_0$, that is, $a>fk(b)$. Here $k: B\to B$ is a 1-refinement function of $(\mathcal{Y}, \mathcal{V})$. By (AI3) there exists $a_1>a$ such that $p_{a_1,a}^{-1}\mathcal{U}_a>st\mathcal{U}_{a_1}$. By (ii) there exist $\mathcal{V}\in \mathit{Cov}(Y)$ such that $f^{-1}st(Y_0, \mathcal{V})\subset st(X_0, p_{a_1}^{-1}\mathcal{U}_{a_1})$. By (AR1) and (AI3) there exist $b_2>b_1>k(b)$ such that $q_{b_1}^{-1}\mathcal{V}_{b_1}<\mathcal{V}$ and $q_{b_2,b_1}^{-1}\mathcal{V}_{b_1}>st\mathcal{V}_{b_2}$. By (AM2) there exists $a_2>a_1$, $fk(b_2)$ such that

$$(1) \qquad (f_{k(b)}p_{a_2,fk(b)},\ q_{k(b_2),k(b)}f_{k(b_2)}p_{a_2,fk(b_2)})<\mathcal{V}_{k(b)}.$$

By the assumption there exists an admissible pair $(a_3, b_3)>(a_2, k(b_2))$ satisfying (ALP) for $\boldsymbol{f}$ and $(a_2, k(b_2))$. Take $b_4>b_3$ such that $q_{b_4,b_3}^{-1}\mathcal{V}_{b_3}>st\mathcal{V}_{b_4}$. By (AM2) there exists $a_4 > a_3,\ fk(b_4)$ such that

(2) $$(f_{b_3}p_{a_4, f(b_3)},\ q_{k(b_4), b_3}f_{k(b_4)}p_{a_4, fk(b_4)}) < \mathcal{V}_{b_3} \text{ and}$$
$$(q_{k(b_4), b_4}f^0_{k(b_4)}p^0_{a_4, fk(b_4)},\ f^0_{b_4}p^0_{a_4, fk(b_4)}) < \mathcal{V}_{b_4}.$$

We show that (a_4, b_4) has the required properties. Clearly (a_4, b_4) is an admissible pair of $\boldsymbol{f}_0$ and $a_4 > a$, $b_4 > b$. Take any $(K, K_0) \in \mathcal{K}$ and take maps $g: K_0 \to X_{0a_4}$, $h: K \to Y_{0b_4}$ such that $(h|K_0,\ f^0_{b_4}p^0_{a_4, fk(b_4)}g) < st\mathcal{V}_{0b_4}$. By the choice of b_4, $(q_{b_4, b_3}j_1h|K_0,\ q_{k(b_4), b_3}f_{k(b_4)}p_{a_4, fk(b_4)}j_2g) < \mathcal{V}_{b_3}$. Here $j_1: Y_{0b_4} \to Y_{b_4}$ and $j_2: X_{0a_4} \to X_{a_4}$ are inclusion maps. By (2) $(f_{b_3}p_{a_4, f(b_3)}j_2g,\ q_{k(b_4), b_3}f_{k(b_4)}p_{a_4, fk(b_4)}j_2g) < \mathcal{V}_{b_3}$ and then $(f_{b_3}p_{a_4, f(b_3)}j_2g,\ q_{b_4, b_3}j_1h|K_0) < st\mathcal{V}_{b_3}$. By the choice of (a_3, b_3) there exists a map $H: K \to X_{a_2}$ such that

(3) $$(H|K_0,\ p_{a_4, a_2}j_2g) < st\mathcal{U}_{a_2} \text{ and}$$

(4) $$(f_{k(b_2)}p_{a_2, fk(b_2)}H,\ q_{b_4, k(b_2)}j_1h) < st\mathcal{V}_{k(b_2)}.$$

We show that

(5) $$p_{a_2, a}H(K) \subset X_{0a}.$$

Take any $k \in K$. Since $st(p_a(X), \mathcal{U}_a) = X_a$ for all $a \in A$, there exist $x \in X$ and $U_1 \in \mathcal{U}_{a_2}$ such that

(6) $$H(k),\ p_{a_2}(x) \in U_1.$$

By (AM1) and (AI2) there exists $V_1 \in \mathcal{V}_{k(b_2)}$ such that $(f_{k(b_2)}p_{a_2, fk(b_2)})^{-1}(V_1) \supset U_1$. Then by (6)

(7) $$f_{k(b_2)}p_{a_2, fk(b_2)}H(k),\ f_{k(b_2)}p_{fk(b_2)}(x) \in V_1.$$

Since $\boldsymbol{f}$ is an approximative resolution of f, there exists $V_2 \in \mathcal{V}_{k(b_2)}$ such that

(8) $$f_{k(b_2)}p_{fk(b_2)}(x),\ q_{k(b_2)}f(x) \in V_2.$$

By (4) there exists $V_3 \in \mathcal{V}_{k(b_2)}$ such that $f_{k(b_2)}p_{a_2, fk(b_2)}H(k),\ q_{b_4, k(b_2)}j_1h(k) \in st(V_3, \mathcal{V}_{k(b_2)})$. Since $q_{b_4, k(b_2)}j_1h(k) \in Y_{0k(b_2)}$, there exist $y_0 \in Y_0$ and $V_4 \in \mathcal{V}_{k(b_2)}$ such that $q_{b_4, k(b_2)}j_1h(k),\ q_{k(b_2)}(y_0) \in V_4$. Thus by (7) and (8)

(9) $$q_{k(b_2)}f(x),\ q_{k(b_2)}(y_0) \in st(st(V_1, \mathcal{V}_{k(b_2)}), st\mathcal{V}_{k(b_2)}).$$

Since k is a 1-refinement function of $(\mathcal{Y}, \mathcal{V})$, by the choice of b_2 $q^{-1}_{k(b_2), b_1}\mathcal{V}_{b_1} > st^2\mathcal{V}_{k(b_2)}$. Then by (9) there exists $V' \in \mathcal{V}_{b_1}$ such that $q_{b_1}f(x),\ q_{b_1}(y_0) \in V'$. By the choice of b_1 there exists $V \in \mathcal{V}$ such that $f(x),\ y^0 \in q^{-1}_{b_1}V' \subset V$. Since $y_0 \in Y_0$, $f(x) \in st(Y_0, \mathcal{V})$. By the choice of $\mathcal{V}$, $x \in f^{-1}st(Y_0, \mathcal{V}) \subset st(X_0, p^{-1}_{a_1}\mathcal{U}_{a_1})$ and then there exist $x_0 \in X_0$ and $U_2 \in \mathcal{U}_{a_1}$ such that $p_{a_1}(x),\ p_{a_1}(x_0) \in U_2$. By (6) there exists $U_3 \in \mathcal{U}_{a_1}$ such that $p_{a_2,a_1}H(k)$, $p_{a_1}(x) \in U_3$. Thus $p_{a_2,a_1}H(k)$, $p_{a_1}(x_0) \in st(U_2, \mathcal{U}_{a_1})$ and then by the choice of a_1 there exists $U \in \mathcal{U}_a$ such that $p_{a_2,a}H(k)$, $p_a(x_0) \in U$. This means $p_{a_2,a}H(k) \in st(p_a(X_0), \mathcal{U}_a) = X_{0a}$. Hence we have (5).

By (3) and the choice of a_1

$$(10) \qquad (p_{a_2,a}H | K_0, p^0_{a_4,a}g) < \mathcal{U}_{0a} < st\mathcal{U}_{0a}.$$

Since k is a 1-refinement function, by (1) and (4) it is easy to show that $(q_{k(b),b} f_{k(b)} p_{a_2,f k(b)} H, q_{b_4,b} j_1 h) < st\mathcal{V}_b$. By (5) this means

$$(11) \qquad (f^0_b p^0_{a,f k(b)} p_{a_2,a} H, q^0_{b_4,b} h) < st\mathcal{V}_{0b}.$$

(5), (10) and (11) mean that $p_{a_2,a}H: K \to X_{0a}$ has the required properties. Then $\boldsymbol{f}_0$ has ALP with respect to $\mathcal{K}$ and hence so does f_0. ■

(2.6) LEMMA. *Let $f: X \to Y$ be a map and let Y be a normal space. Let X_0 and Y_0 be subspaces of X and Y, respectively. If f is a closed map and $f^{-1}(\bar{Y}_0) = \bar{X}_0$, then for each $\mathcal{U} \in Cov(X)$ there exists $\mathcal{V} \in Cov(Y)$ such that $f^{-1}st(Y_0, \mathcal{V}) \subset st(X_0, \mathcal{U})$.*

PROOF. Take any $\mathcal{U} \in Cov(X)$. Since $st(X_0, \mathcal{U}) = st(\bar{X}_0, \mathcal{U}) = st(f^{-1}(\bar{Y}_0), \mathcal{U})$ and f is closed, $W = Y - f(X - st(\bar{X}_0, \mathcal{U}))$ is an open set in Y and $W \supset \bar{Y}_0$. Since Y is normal, $\mathcal{V} = \{W, Y - \bar{Y}_0\}$ forms a normal open covering of Y by Theorem 1 of MS [23, p. 324]. Hence $f^{-1}st(Y_0, \mathcal{V}) = f^{-1}st(\bar{Y}_0, \mathcal{V}) = f^{-1}W \subset st(\bar{X}_0, \mathcal{U}) = st(X_0, \mathcal{U})$. ■

(2.7) COROLLARY. *Let (X, X_0) and (Y, Y_0) be P-pairs and $f: (X, X_0) \to (Y, Y_0)$ a map. Let $f: X \to Y$ be a closed map, Y a normal space and $f^{-1}(\bar{Y}_0) = \bar{X}_0$. If $f: X \to Y$ has ALP with respect to $\mathcal{K}$, then so does $f_0: X_0 \to Y_0$. Here f_0 is induced by f.*

(2.8) COROLLARY. *Let $f: X \to Y$ be a closed map. We assume any one of the conditions* (i)-(iii) *below. If f has ALP with respect to $\mathcal{K}$, then so does $f | f^{-1}(Y_0): f^{-1}(Y_0) \to Y_0$ for any closed subset Y_0 of Y.*

(i) *X and Y are collection-wise normal.*

(ii) *X and Y are paracompact.*

(iii) *X and Y are pseudo-compact normal.*

(2.7) follows from (2.5) and (2.6). (2.8) follows from (2.7) and the fact that every closed subset of a space satisfying one of the conditions (i)-(iii) is P-embedded (see Corollary 15.7, Theorem 15.11 and Corollary 15.15 of Aló-Shapiro [1]). ■

(2.9) THEOREM. *Let all* $f_c: X_c \to Y_c$ *be maps between compact spaces for* $c \in C$. *Then* $f = \Pi\{f_c: c \in C\}: X = \Pi\{X_c: c \in C\} \to Y = \Pi\{Y_c: c \in C\}$ *has ALP with respect to* $\mathcal{K}$ *iff all* f_c *have ALP with respect to* $\mathcal{K}$.

PROOF. We use the same notations as in the proof of (III. 2.1).

First we assume that all f_c have ALP with respect to $\mathcal{K}$. Then all $\boldsymbol{f}^c$ satisfy (ALP) with respect to $\mathcal{K}$. We show that $\boldsymbol{f} = \{f, f_h: h \in F(B)\}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ has ALP with respect to $\mathcal{K}$. Take any admissible pair (g, h) of $\boldsymbol{f}$, that is $g > f(h)$. Put $h: m = \{c_1, c_2, \cdots, c_k\} \to B$ and $g: m' = m \cup \{c_{k+1}, \cdots, c_{k'}\} \to A$. Since $g > f(h)$, $g(c_i) > f^{ci} h(c_i)$ and then $(g(c_i), h(c_i))$ is an admissible pair of $\boldsymbol{f}^{ci}$ for $1 \leqq i \leqq k$. By the assumption there exist admissible pairs (a_i, b_i) of $\boldsymbol{f}^{ci}$, $a_i > g(c_i)$, $b_i > h(c_i)$, satisfying (ALP) for $\boldsymbol{f}^{ci}$ and $(g(c_i), h(c_i))$ with respect to $\mathcal{K}$, $1 \leqq i \leqq k$. We define choice functions $h': m \to B$ and $g': m' \to A$ as follows: $h'(c_i) = b_i$ for $1 \leqq i \leqq k$, and $g'(c_i) = a_i$ for $1 \leqq i \leqq k$, $g'(c_i) = g(c_i)$ for $k+1 \leqq i \leqq k'$. Then $g' > g$, $h' > h$ and (g', h') is an admissible pair of $\boldsymbol{f}$.

We show that (g', h') has the required properties. Take any $(K, K_0) \in \mathcal{K}$ and maps $u = (u_{c1}, u_{c2}, \cdots, u_{ck}): K \to Y_{h'}$, $v = (v_{c1}, v_{c2}, \cdots, v_{ck}, \cdots, v_{ck'}): K_0 \to X_{g'}$ such that $(f_{h'} p_{g', f(h')} v, u | K_0) < st \mathcal{V}_{h'}$. Then $(f^{ci}_{h'(ci)} p^{ci}_{g'(ci), f^{ci} h'(ci)} v_{ci}, u_{ci} | K_0) < st \mathcal{V}^{ci}_{h'(ci)}$ for $1 \leqq i \leqq k$. By the choice of (a_i, b_i) for i, $1 \leqq i \leqq k$, there exists a map $U_{ci}: K \to X^{ci}_{g(ci)}$ satisfying

$$(1) \qquad (p^{ci}_{g'(ci), g(ci)} v_{ci}, U_{ci} | K_0) < st \mathcal{U}^{ci}_{g(ci)} \quad \text{and}$$

$$(2) \qquad (f^{ci}_{h(ci)} p^{ci}_{g(ci), f^{ci} h(ci)} U_{ci}, q^{ci}_{h'(ci), h(ci)} u_{ci}) < st \mathcal{V}^{ci}_{h(ci)}.$$

We define $U = (U_{c1}, U_{c2}, \cdots, U_{ck}, v_{ck+1}, \cdots, v_{ck'}): K \to X_g$. By (1) and (2) it is easy to show that $(p_{g', g} v, U | K_0) < st \mathcal{U}_g$ and $(f_h p_{g, f(h)} U, q_{h', h} u) < st \mathcal{V}_h$. Then $\boldsymbol{f}$ has ALP with respect to $\mathcal{K}$.

Next we assume that $\boldsymbol{f}$ has ALP with respect to $\mathcal{K}$ and show that all $\boldsymbol{f}^c$ have ALP with respect to $\mathcal{K}$. Take any $c \in C$ and any admissible pair (a, b) of $\boldsymbol{f}^c$. We define choice functions $g: m = \{c\} \to A$ and $h: m \to B$ by $g(c) = a$ and $h(c) = b$. Then (g, h) is an admissible pair of $\boldsymbol{f}$. By the assumption there exists an admissible pair (g', h') of $\boldsymbol{f}$, $g' > g$, $h' > h$, satisfying (ALP) for $\boldsymbol{f}$ and (g, h). Put $h': m_1 \to B$ and $g': m_2 \to A$. Since $g' > g$, $h' > h$ and $g' > f(h')$, $m \subset m_1$, $m \subset m_2$ and $m_1 \subset m_2$. Then we may put $m_1 = \{c, c_1, c_2, \cdots, c_k\}$ and $m_2 = m_1 \cup \{c_{k+1}, \cdots, c_{k'}\}$.

We put $a' = g'(c)$ and $b' = h'(c)$. Trivially (a', b') is an admissible pair of $\boldsymbol{f}^c$. We show that it has the required property. Take any $(K, K_0) \in \mathcal{K}$ and maps $v: K_0 \to X^c_{a'}$, $u: K \to Y^c_{b'}$ such that

(3) $\qquad (f_{b'}^c p_{a',f^c(b')}^c v,\ u|K_0)<st\mathcal{V}_{b'}^c$.

Take any point $x=(x_c, x_{c1}, \cdots, x_{ck'})\in X_{g'}$ and put $f_{h'}p_{g',h'}(x)=(y_c, y_{c1}, \cdots, y_{ck})\in Y_{h'}$. We define $v': K_0\to X_{g'}$ and $u': K\to Y_{h'}$ as follows: $v'(z)=(v(z), x_{c1}, x_{c2}, \cdots, x_{ck'})$ and $u'(z)=(u(z), y_{c1}, y_{c2}, \cdots, y_{ck})$ for $z\in K$. By (3) it is easy to show that $(f_{h'}p_{g',f(h')}v', u'|K_0)<st\mathcal{V}_{h'}$. By the choice of (g', h') there exists a map $U: K\to X_g=X_a^c$ such that $(p_{g',g}v', U|K_0)<st\mathcal{U}_g$ and $(f_h p_{g,f(h)}U, q_{h',h}u')<st\mathcal{V}_h$. This means that $(p_{a',a}^c v, U|K_0)<st\mathcal{U}_a^c$ and $(f_b^c p_{a,f^c(b)}^c U, q_{b',b}^c u)<st\mathcal{V}_b^c$. Thus $\boldsymbol{f}^c$ has ALP with respect to $\mathcal{K}$ and hence so does each f_c. ■

§3. Approximatively n-connected maps.

Using the approximative lifting property we introduce approximatively n-connected maps and investigate their properties.

Let $\mathcal{K}$ be a collection of pairs of spaces. Let $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be approximative inverse systems. Let $\boldsymbol{f}=\{f, f_b: b\in B\}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ be an approximative system maps.

We say that $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ has the $*$-approximative lifting property, in notation ALP*, with respect to $\mathcal{K}$ provided that it satisfies the following:

(ALP*) For each admissible pair (a, b) of $\boldsymbol{f}$ there exists an admissible pair $(a_0, b_0)>(a, b)$ with the following property; for each $(K, K_0)\in\mathcal{K}$ and any maps $g: K\to Y_{b_0}$, $h: K_0\to X_{a_0}$ with $(f_{b_0}p_{a_0,f(b_0)}h, g|K_0)<st\mathcal{V}_{b_0}$, there exists a map $H: K\to X_a$ such that $p_{a_0,a}h=H|K_0$ and $(q_{b_0,b}g, f_b p_{a,f(b)}H)<st\mathcal{V}_b$.

We say that $\boldsymbol{f}$ has the $**$-approximative lifting property, in notation ALP**, with respect to $\mathcal{K}$ provided that it satisfies the following:

(ALP**) For each admissible pair (a, b) of $\boldsymbol{f}$ there exists an admissible pair $(a_0, b_0)>(a, b)$ with the following property; for each $(K, K_0)\in\mathcal{K}$ and any maps $g: K\to Y_{b_0}$, $h: K_0\to X_{a_0}$ with $f_{b_0}p_{a_0,f(b_0)}h=g|K_0$, there exists a map $H: K\to X_{a_0}$ such that $p_{a_0,a}h=H|K_0$ and $(q_{b_0,b}g, f_b p_{a,f(b)}H)<st\mathcal{V}_b$.

(3.1) LEMMA. *Let $\mathcal{K}$ be a collection of polyhedral pairs. Let $(\mathcal{X}, \mathcal{U})$ and $(\mathcal{Y}, \mathcal{V})$ be approximative inverse systems in* **POL** *or in* **ANR**.

(i) *An approximative system map $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ has ALP with respect to $\mathcal{K}$ iff $\boldsymbol{f}$ has ALP* with respect to $\mathcal{K}$.*

(ii) *Let $\boldsymbol{f}$ be commutative. $\boldsymbol{f}$ has ALP* with respect to $\mathcal{K}$ iff $\boldsymbol{f}$ has ALP** with respect to $\mathcal{K}$.*

PROOF. We show that (ALP) implies (ALP*). Take any admissible pair (a, b) of $\boldsymbol{f}$. Since X_a is a polyhedron or an ANR, there exists $\mathcal{U}\in Cov(X_a)$ satisfying $(*)$ in (I. 5.5) for $\mathcal{U}_a$. There exist $b_1>b$ and $a_1>a, f(b_1)$ such that

$q_{b_1,b}^{-1}\mathcal{V}_b > st\mathcal{V}_{b_1}$, $p_{a_1,a}^{-1}\mathcal{U} > st\mathcal{U}_{a_1}$ and

(1) $\qquad (f_b p_{a_1,f(b)},\ q_{b_1,b} f_{b_1} p_{a_1,f(b_1)}) < \mathcal{V}_b.$

By the assumption there exists an admissible pair $(a_2, b_2) > (a_1, b_1)$ satisfying (ALP) for $\boldsymbol{f}$ and (a_1, b_1).

We show that (a_2, b_2) has the required property. To do so take any $(K, K_0) \in \mathcal{K}$ and maps $g: K \to Y_{b_2}$, $h: K_0 \to X_{a_2}$ such that $(f_{b_2} p_{a_2,f(b_2)} h,\ g|K_0) < st\mathcal{V}_{b_2}$. By the choice of (a_2, b_2) there exists a map $H: K \to X_{a_1}$ such that

(2) $\qquad (p_{a_2,a_1} h,\ H|K_0) < st\mathcal{U}_{a_1}$ and

(3) $\qquad (f_{b_1} p_{a_1,f(b_1)} H,\ q_{b_2,b_1} g) < st\mathcal{V}_{b_1}.$

By the choice of a_1 and (2) $(p_{a_2,a} h,\ p_{a_1,a} H|K_0) < \mathcal{U}$. By the choice of $\mathcal{U}$ there exists a homotopy $F: K_0 \times I \to X_a$ such that F is a $\mathcal{U}_a$-homotopy, $F_0 = p_{a_1,a} H|K_0$ and $F_1 = p_{a_2,a} h$. By the homotopy extension theorem (see Th. 3 of MS [**23**, p. 291]) there exists a homotopy $\bar{F}: K \times I \to X_a$ such that $\bar{F}_0 = p_{a_1,a} H$ and $\bar{F}|K_0 \times I = F$. Since F is a $\mathcal{U}_a$-homotopy, there exists an open neighborhood U of K_0 in K such that $\bar{F}|U \times I: U \times I \to X_a$ is a $\mathcal{U}_a$-homotopy. Take a map $v: K \to I$ such that $v(z) = 1$ for $z \in K_0$ and $v(z) = 0$ for $z \in K - U$. We define a map $\bar{H}: K \to X_a$ as follows: $\bar{H}(z) = \bar{F}(z, v(z))$ for $z \in K$. It is easy to show that

(4) $\qquad \bar{H}|K_0 = p_{a_1,a} h$ and

(5) $\qquad (p_{a_1,a} H,\ \bar{H}) < \mathcal{U}_a.$

By (1), (3), (5) and the choice of b_1 we have that $(f_b p_{a,f(b)} \bar{H},\ q_{b_2,b} g) < st\mathcal{V}_b$. Thus by this and (4) $\boldsymbol{f}$ satisfies (ALP*). Thus (ALP) implies (ALP*). The converse is trivial and hence we have (i).

We assume that $\boldsymbol{f}$ is commutative and satisfies (ALP**). We show that $\boldsymbol{f}$ satisfies (ALP*). Take any admissible pair (a, b) of $\boldsymbol{f}$. Since $\boldsymbol{f}$ is commutative, there exist $b_1 > b$ and $a_1 > a$, $f(b_1)$ such that $q_{b_1,b}^{-1}\mathcal{V}_b > st\mathcal{V}_{b_1}$ and $f_b p_{a_1,f(b)} = q_{b_1,b} f_{b_1} p_{a_1,f(b_1)}$. There exists an admissible pair $(a_2, b_2) > (a_1, b_1)$ satisfying (ALP**) for $\boldsymbol{f}$ and (a_1, b_1). Since Y_{b_2} is a polyhedron or an ANR, there exists $\mathcal{V} \in Cov(Y_{b_2})$ satisfying (*) in (I. 5.5) for $\mathcal{V}_{b_2}$. There exist $b_3 > b_2$ and $a_3 > a_2$, $f(b_3)$ such that $q_{b_3,b_2}^{-1}\mathcal{V} > st\mathcal{V}_{b_3}$ and $f_{b_2} p_{a_3,f(b_2)} = q_{b_3,b_2} f_{b_3} p_{a_3,f(b_3)}$.

We show that (a_3, b_3) has the required properties. Take any $(K, K_0) \in \mathcal{K}$ and any maps $g: K \to Y_{b_3}$, $h: K_0 \to X_{a_3}$ such that $(f_{b_3} p_{a_3,f(b_3)} h,\ g|K_0) < st\mathcal{V}_{b_3}$. Then by the choices of a_3 and b_3, $(f_{b_2} p_{a_3,f(b_2)} h,\ q_{b_3,b_2} g|K_0) < \mathcal{V}$. Then by the choice of $\mathcal{V}$ there exists a $\mathcal{V}_{b_2}$-homotopy $G: K_0 \times I \to Y_{b_2}$ such that $G_0 = q_{b_3,b_2} g|K_0$ and $G_1 = f_{b_2} p_{a_3,f(b_2)} h$. By the same way as in (i) there exist a homotopy $\bar{G}: K \times I \to Y_{b_2}$, an open neighborhood V of K_0 in K and a map $v: K \to I$ such that $\bar{G}_0=q_{b_3,b_2}g$, $\bar{G}|K_0\times I=G$, $\bar{G}|V\times I: V\times I\to Y_{b_2}$ is a $\mathcal{V}_{b_2}$-homotopy and $v(z)=0$ for $z\in K-V$, $v(z)=1$ for $z\in K_0$. We define a map $\bar{g}: K\to Y_{b_2}$ by $\bar{g}(z)=\bar{G}(z, v(z))$ for $z\in K$. It is easy to show that

(6) $\bar{g}|K_0=f_{b_2}p_{a_3,f(b_2)}h$ and $(\bar{g}, q_{b_3,b_2}g)<\mathcal{V}_{b_2}$.

By (6) and the choice of (a_2, b_2) there exists a map $H: K\to X_{a_1}$ such that $H|K_0=p_{a_3,a_1}h$ and $(f_{b_1}p_{a_1,f(b_1)}H, q_{b_2,b_1}\bar{g})<st\mathcal{V}_{b_1}$. Thus $p_{a_1,a}H|K_0=p_{a_3,a}h$. By the choices of a_1 and b_1, $(f_b p_{a_1,f(b)}H, q_{b_2,b}\bar{g})<\mathcal{V}_b$ and then by (6), $(f_b p_{a_1,f(b)}H, q_{b_3,b}g)<st\mathcal{V}_b$. Thus $\boldsymbol{f}$ satisfies (ALP*). Hence (ALP**) implies (ALP*). Trivially (ALP*) implies (ALP**). Hence we have (ii). ■

Let Δ^k be the standard k-simplex. $\partial\Delta^k$ denotes the boundary of Δ^k. We put $(D, S)_n=\{(\Delta^k, \partial\Delta^k): k=0, 1, \cdots, n+1\}$ and $(D, S)_\infty=\{(\Delta^k, \partial\Delta^k): k=0, 1, 2, \cdots\}$ for each $n=-1, 0, 1, 2, \cdots$.

(3.2) LEMMA. *The following statements are equivalent for each n, $n=-1, 0, 1, 2, \cdots$:*

(i) *$\boldsymbol{f}$ has ALP* with respect to $(D, S)_n$.*

(ii) *$\boldsymbol{f}$ has ALP* with respect to ${}^{n+1}\mathbf{POL}_{\mathrm{pairs}}$.*

PROOF. Trivially (ii) implies (i). By induction on n we will show that (i) implies (ii).

First we take $n=-1$. Take any admissible pair (a, b) of $\boldsymbol{f}$. There exist $b_1>b$ and $a_1>a$, $f(b_1)$ such that $q_{b_1,b}^{-1}\mathcal{V}_b>st^2\mathcal{V}_{b_1}$ and $(f_b p_{a_1,f(b)}, q_{b_1,b}f_{b_1}p_{a_1,f(b_1)})<\mathcal{V}_b$. There exists an admissible pair $(a_2, b_2)>(a_1, b_1)$ satisfying (ALP*) with respect to $(S, D)_{-1}$ for (a_1, b_1). There exists $a_3>a_2$ such that $(f_{b_1}p_{a_3,f(b_1)}, q_{b_2,b_1}f_{b_2}p_{a_3,f(b_2)})<\mathcal{V}_{b_1}$.

We show that the admissible pair (a_3, b_2) has the required property. Take any $(P, P_0)\in{}^0\mathbf{POL}_{\mathrm{pairs}}$ and maps $g: P\to Y_{b_2}$, $h: P_0\to X_{a_3}$ such that $(f_{b_2}p_{a_3,f(b_2)}h, g|P_0)<st\mathcal{V}_{b_2}$. There exist a simplicial complex K and a subcomplex L of K such that $|K|=P$, $|L|=P_0$ and $\dim(K-L)\leqq 0$. Take any vertex $v\in K-L$. Since $(\Delta^0, \partial\Delta^0)=(\Delta^0, \phi)$, by the choice of (a_2, b_2) there exists a map $g_v: \{v\}\to X_{a_1}$ such that $(f_{b_1}p_{a_1,f(b_1)}g_v, q_{b_2,b_1}g|\{v\})<st\mathcal{V}_{b_1}$. Now we define a map $H: P\to X_a$ as follows: $H(z)=p_{a_3,a}h(z)$ for $z\in|L|$ and $H(v)=p_{a_1,a}g_v(v)$ for $v\in|K|-|L|$. Clearly H is continuous. It is easy to show that $H|P_0=p_{a_3,a}h$ and $(f_b p_{a,f(b)}H, q_{b_2,b}g)<st\mathcal{V}_b$. Hence $\boldsymbol{f}$ has the required property.

Next we assume that (i) implies (ii) for each k, $k\leqq n$, and show it for $n+1$. Take any admissible pair (a, b) of $\boldsymbol{f}$. There exist $b_1>b$ and $a_1>a$, $f(b_1)$ such that $q_{b_1,b}^{-1}\mathcal{V}_b>st\mathcal{V}_{b_1}$ and $(f_b p_{a_1,f(b)}, q_{b_1,b}f_{b_1}p_{a_1,f(b_1)})<\mathcal{V}_b$. There exists an admissible pair $(a_2, b_2)>(a_1, b_1)$ satisfying (ALP*) with respect to $(S, D)_{n+1}$ for (a_1, b_1). There exists $a_3>a_2$ such that $(f_{b_1}p_{a_3,f(b_1)}, q_{b_2,b_1}f_{b_2}p_{a_3,f(b_2)})<\mathcal{V}_{b_1}$. By the inductive assumption there exists an admissible pair $(a_4, b_4)>(a_3, b_2)$ satisfying (ALP*) with respect to ${}^{n+1}\mathbf{POL}_{\text{pairs}}$ for (a_3, b_2).

We show that (a_4, b_4) is the required one. Take any $(P, P_0)\in {}^{n+2}\mathbf{POL}_{\text{pairs}}$ and maps $g: P\to Y_{b_4}$, $h: P_0\to X_{a_4}$ such that $(f_{b_4}p_{a_4,f(b_4)}h, g|P_0)<st\mathcal{V}_{b_4}$. There exist a simplicial complex K and a subcomplex L of K such that $|K|=P$, $|L|=P_0$ and $\dim(K-L)\leqq n+2$. Let K^{n+1} be the $(n+1)$-skeleton of K. We put $P'=|K^{n+1}|\cup|L|$ and then $(P', P_0)\in {}^{n+1}\mathbf{POL}_{\text{pairs}}$. By the choice of (a_4, b_4) there exists a map $H': P'\to X_{a_3}$ such that

$$H'|P_0=p_{a_4,a_3}h \text{ and } (f_{b_2}p_{a_3,f(b_2)}H', q_{b_4,b_2}g|P')<st\mathcal{V}_{b_2}. \tag{1}$$

Take any $(n+2)$-simplex $\sigma\in K-(K^{n+1}\cup L)$ and then $\partial\sigma\subset P'$. By (1) $(f_{b_2}p_{a_2,f(b_2)}p_{a_3,a_2}H'|\partial\sigma, q_{b_4,b_2}g|\partial\sigma)<st\mathcal{V}_{b_2}$ and then by the choice of (a_2, b_2) there exists a map $g_\sigma: \sigma\to X_{a_1}$ such that

$$g_\sigma|\partial\sigma=p_{a_3,a_1}H'|\partial\sigma \text{ and } (f_{b_1}p_{a_1,f'(b_1)}g_\sigma, q_{b_4,b_1}g|\sigma)<st\mathcal{V}_{b_1}. \tag{2}$$

Now we define a map $H: P\to X_a$ as follows: $H(z)=p_{a_3,a}H'(z)$ for $z\in P'$ and $H(z)=p_{a_1,a}g_\sigma(z)$ for $z\in\sigma\in K-(K^{n+1}\cup L)$. It is easy to show that H is well-defined, continuous, $H|P_0=p_{a_4,a}h$ and $(f_bp_{a,f(b)}H, q_{b_4,b}g)<st\mathcal{V}_b$ by (1) and (2). Hence f has the required property. ■

(3.3) THEOREM. *The following statements are equivalent for each n, $n=-1, 0, 1, \cdots$:*

(i) *A map f has ALP with respect to $(D, S)_n$.*

(ii) *f has ALP with respect to $\mathbf{POL}^{n+1}_{\text{pairs}}$.*

(iii) *f has ALP with respect to ${}^{n+1}\mathbf{POL}_{\text{pairs}}$.*

(iv) *f has ALP with respect to $\mathbf{TOP}^{n+1}_{P\text{-pairs}}$.*

PROOF. By (1.16) (ii) and (iv) are equivalent. Trivially (iii)→(ii)→(i) hold. By (3.1) and (3.2) (i) implies (iii). ■

We say that an approximative system map $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is approximatively n-connected, in notation $\boldsymbol{f}\in AC^n$, provided $\boldsymbol{f}$ has ALP with respect to $(D, S)_n$. We say that $\boldsymbol{f}$ is approximatively ∞-connected, in notation $\boldsymbol{f}\in AC^\infty$, provided $\boldsymbol{f}\in AC^n$ for each integer $n\geqq -1$. We say that $\boldsymbol{f}$ is approximatively $\infty\infty$-connected, in notation $\boldsymbol{f}\in AC^{\infty\infty}$, provided $\boldsymbol{f}$ has ALP with respect to $(D, S)_\infty$.

We say that a map $f: X\to Y$ is approximatively n-connected, in notation

$f \in AC^n$, provided that f has ALP with respect to $(D, S)_n$. Thus $f \in AC^n$ iff it satisfies any one of the conditions (i)-(iv) in (3.3). Similarly we may define approximative ∞-connectedness, in notation $f \in AC^\infty$, and approximative $\infty\infty$-connectedness, in notation $f \in AC^{\infty\infty}$.

(3.4) THEOREM. *If a map $f: X \to Y$ is approximatively n-connected, then for each $x \in X$ pro-$\pi_k(f)$: pro-$\pi_k(X, x) \to$ pro-$\pi_k(Y, f(x))$ is an isomorphism in pro-groups for $k \leqq n$ and an epimorphism for $k = n+1$.*

PROOF. By (I. 4.9) there exist approximative **ANR**-resolutions $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$ and an approximative resolution $\boldsymbol{f}$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying (RM1) and (RM2). Take any point $x \in X$. Then by (I. 3.16) $\boldsymbol{p}: (X, x) \to (\mathcal{X}, x, \mathcal{U}) = \{(X_a, x_a, \mathcal{U}_a), p_{a',a}, A\}$ and $\boldsymbol{q}: (Y, f(x)) \to (\mathcal{Y}, f(x), \mathcal{V}) = \{(Y_b, f(x)_b, \mathcal{V}_b), q_{b',b}, B\}$ form approximative **ANR**-resolutions. Here $x_a = p_a(x)$ and $f(x)_b = q_b f(x)$ for $a \in A$ and $b \in B$. By (RM2) $f_b: X_{f(b)} \to Y_b$ induces $f_b: (X_{f(b)}, x_{f(b)}) \to (Y_b, f(x)_b)$ for $b \in B$. Then $\boldsymbol{f}: (\mathcal{X}, x, \mathcal{U}) \to (\mathcal{Y}, f(x), \mathcal{V})$ forms an approximative resolution of $f: (X, x) \to (Y, f(x))$. By (I. 5.10) $H(\boldsymbol{p}): (X, x) \to H(\mathcal{X}, x)$ and $H(\boldsymbol{q}): (Y, f(x)) \to H(\mathcal{Y}, f(x))$ form $\mathbf{HANR}_0$-expansions and $H(\boldsymbol{f}): H(\mathcal{X}, x) \to H(\mathcal{Y}, f(x))$ forms a $\mathbf{HANR}_0$-expansion of $f: (X, x) \to (Y, f(x))$. Here $\mathbf{ANR}_0$ is the pointed category of ANRs. By (I. 4.4) and (I. 5.5) we may assume that each $\mathcal{V}_b$ has the property:

(*) if $r, s: (Z, z) \to (Y_b, f(x)_b)$ are $st\mathcal{V}_b$-near, then $r \simeq s$ rel. z for any pointed space (Z, z).

By (3.2) and (3.3) $\boldsymbol{f}$ has ALP* with respect to $\mathbf{POL}^{n+1}_{\text{pairs}}$. Take any admissible pair (a, b) of $\boldsymbol{f}$. Then there exists an admissible pair $(a_1, b_1) > (a, b)$ satisfying (ALP*) with respect to $\mathbf{POL}^{n+1}_{\text{pairs}}$ for (a, b).

When we take the polyhedral pair $(\Delta^{k+1}, \partial\Delta^{k+1})$, by the choice of (a_1, b_1) it is easy to show that

$$\text{(1)} \qquad \operatorname{Ker} \pi_k(f_{b_1} p_{a_1, f(b_1)}) \subset \operatorname{Ker} \pi_k(p_{a_1, a}) \text{ for } k \leqq n.$$

When we take a polyhedral pair $(\partial\Delta^{k+1}, v)$ where v is a vertex of $\partial\Delta^{k+1}$, by the choice of (a_1, b_1) and (*) it is easy to show that

$$\text{(2)} \qquad \operatorname{Im} \pi_k(q_{b_1, b}) \subset \operatorname{Im} \pi_k(f_b p_{a, f(b)}) \text{ for } k \leqq n+1.$$

Here Ker(h) and Im(h) denote the kernel of h and the image of h for any homomorphism h.

By Th. 2 of MS [**23**, p. 108] (1) means that pro-$\pi_k(f)$ is a monomorphism for $k \leqq n$. Also by Th. 4 of MS [**23**, p. 112] (2) means that pro-$\pi_k(f)$ is an epimorphism for $k \leqq n+1$. Hence by Th. 6 of MS [**23**, p. 114] pro-$\pi_k(f)$ is an isomorphism for $k \leqq n$. ■

(3.5) COROLLARY. *Let $f: X \to Y$ be a map. If $f \in AC^{\infty}$, then pro-$\pi_k(f)$: pro-$\pi_k(X, x) \to$ pro-$\pi_k(Y, f(x))$ is an isomorphism for all k and for any $x \in X$.* ■

(3.6) COROLLARY. *Let $f: X \to Y$ be a map.*

(i) *If $f \in AC^n$, then $\check{\pi}_n(f): \check{\pi}_n(X, x) \to \check{\pi}_n(Y, f(x))$ is an isomorphism of shape groups for any $x \in X$.*

(ii) *If $f \in AC^{\infty}$, then $\check{\pi}_k(X, x) \to \check{\pi}_k(Y, f(x))$ is an isomorphism for all k and for any $x \in X$.* ■

For a space X, $\mathrm{sd}\, X$ denotes the shape dimension of X. This notion was introduced by Dydak and he showed that $\mathrm{sd}\, X \leqq \dim X$ and $\mathrm{sd}(X, x) = \mathrm{sd}\, X$ (see Th. 7 of MS [**23**, p. 103]).

(3.7) THEOREM. *Let $f: X \to Y$ be a map and $n+1 = Max(\mathrm{sd}\, Y, \mathrm{sd}\, X+1) < \infty$. If $f \in AC^n$, then for each $x \in X$ $f: (X, x) \to (Y, f(x))$ induces a shape isomorphism.*

PROOF. Let $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ be as in the proof of (3.4). Then $(\mathcal{Y}, \mathcal{V})$ satisfies $(*)$ in the proof of (3.4). By (3.2) and (3.3) $\boldsymbol{f}$ has ALP* with respect to ${}^{n+1}\mathbf{POL}_{\mathrm{pairs}}$. Take any admissible pair (a, b) of $\boldsymbol{f}$ and then there exists an admissible pair $(a_1, b_1) > (a, b)$ satisfying (ALP*) with respect to ${}^{n+1}\mathbf{POL}_{\mathrm{pairs}}$ for (a, b). Since $\mathrm{sd}(X, f(x)) = \mathrm{sd}\, Y$, by Theorem 2 of MS [**23**, p. 96] there exist $b_2 > b_1$, a pointed polyhedron (P, p) and maps $r_1: (P, p) \to (Y_{b_1}, f(x)_{b_1})$, $s_1: (Y_{b_2}, f(x)_{b_2}) \to (P, p)$ such that

(1) $$\dim P \leqq \mathrm{sd}\, Y \text{ and } r_1 s_1 \simeq q_{b_2, b_1} \ rel.\ f(x)_{b_2}.$$

Since $\boldsymbol{f}$ satisfies (RM2), there exists $a_2 > a_1$, $f(b_2)$ such that

(2) $$f_{b_1} p_{a_2, f(b_1)} = q_{b_2, b_1} f_{b_2} p_{a_2, f(b_2)}.$$

Since $\mathrm{sd}(X, x) = \mathrm{sd}\, X$, by Theorem 2 of MS [**23**, p. 96] there exist $a_3 > a_2$, a pointed polyhedron (Q, q) and maps $r_2: (Q, q) \to (X_{a_2}, x_{a_2})$, $s_2: (X_{a_3}, x_{a_3}) \to (Q, q)$ such that

(3) $$\dim Q \leqq \mathrm{sd}\, X \text{ and } r_2 s_2 \simeq p_{a_3, a_2} \ rel.\ x_{a_3}.$$

By the simplicial approximation theorem (see [**32, 33**]) there exist simplicial complexes K, L, vertexes $q \in K$, $p \in L$, and a simplicial map $k: K \to L$ such that $|K| = Q$, $|L| = P$ and

(4) $$s_1 f_{b_2} p_{a_2, f(b_2)} r_2 \simeq k \ rel.\ q.$$

Let $M(k)$ be the mapping cylinder of k. Let $u:|K|\times I\oplus|L|\to M(k)$ be the identification map. We identify $(x, 1)$ and $k(x)$ for $x\in|K|$. Let $T=|K|\times\{0\}\cup\{q\}\times I$. Since k is simplicial, $(M(k), u(T))$ is a polyhedral pair by Theorem 6 of MS [**23**, p. 295]. By (1) and (3) $(M(k), u(T))\in{}^{n+1}\mathbf{POL}_{\text{pairs}}$.

By (1) and (4) there exist pointed homotopies $H^1: Q\times I\to P$ and $H^2: Y_{b_2}\times I\to Y_{b_1}$ such that $H_1^1=k$, $H_0^1=s_1 f_{b_2}p_{a_2, f(b_2)}r_2$, $H^1(q\times I)=p$ and $H_1^2=r_1 s_1$, $H_0^2=q_{b_2, b_1}$ and $H^2(f(x)_{b_2}\times I)=f(x)_{b_1}$. We define a map $H: |K|\times I\to Y_{b_1}$ as follows: $H(x, t)=H^2(f_{b_2}p_{a_2, f(b_2)}r_2(x), 2t)$ for $(x, t)\in|K|\times[0, 1/2]$ and $H(x, t)=r_1 H^1(x, 2t-1)$ for $(x, t)\in|K|\times[1/2, 1]$. Then $H\oplus r_1: |K|\times I\oplus|L|\to Y_{b_1}$ induces a map $H': M(k)\to Y_{b_1}$ such that $H\oplus r_1=H'u$. We define a map $h: u(T)\to X_{a_1}$ as follows: $h(u(x, 0))=p_{a_2, a_1}r_2(x)$ for $(x, 0)\in T$ and $h(u(q, t))=x_{a_1}$ for $(q, t)\in T$. Since $u||K|\times\{0\}: |K|\times\{0\}\to u(|K|\times\{0\})$ is a homeomorphism, h is well-defined and continuous. By (2) $f_{b_1}p_{a_1, f(b_1)}h=H'|u(T)$. By the choice of (a_1, b_1) there exists a map $G: M(k)\to X_a$ such that

(5) $$G|u(T)=p_{a_1, a}h \text{ and } (f_b p_{a, f(b)}G,\ q_{b_1, b}H')<st\mathcal{V}_b.$$

Let $m=Gu||L|: |L|\to X_a$. By the definition of H' and (5) $(f_b p_{a, f(b)}m, q_{b_1, b}r_1)<st\mathcal{V}_b$. Then by (*) in the proof of (3.4) $f_b p_{a, f(b)}ms_1\simeq q_{b_1, b}r_1 s_1$ *rel.* $f(x)_{b_2}$ and hence by (1)

(6) $$f_b p_{a, f(b)}ms_1\simeq q_{b_2, b}\ rel.\ f(x)_{b_2}.$$

The homotopy $Gu||K|\times I: |K|\times I\to X_a$ gives that $p_{a_2, a}r_2\simeq mk$ *rel.* q and hence by (3) and (4)

(7) $$p_{a_3, a}\simeq ms_1 f_{b_2}p_{a_3, f(b_2)}\ rel.\ x_{a_3}.$$

By Morita's diagonal lemma (see MS [**23**, p. 112]) (6) and (7) mean that $H(\boldsymbol{f}): H(\mathcal{X}, \boldsymbol{x})\to H(\mathcal{Y}, \boldsymbol{f}(\boldsymbol{x}))$ induces an isomorphism in pro-$\mathbf{HANR}_0$. Hence $f:(X, x)\to(Y, f(x))$ induces a shape isomorphism. ■

(3.8) COROLLARY. *Let $f: X\to Y$ be a map,* $\mathrm{sd}\,X<\infty$ *and* $\mathrm{sd}\,Y<\infty$. *If $f\in AC^\infty$, then $f:(X, x)\to(Y, f(x))$ induces a shape isomorphism for each $x\in X$.* ■

In the same way as in (3.7) without the dimension condition, we can show the following:

(3.9) LEMMA. *If $f: X\to Y$ has ALP with respect to* $\mathbf{POL}_{\text{pairs}}$, *then $f:(X, x)\to(Y, f(x))$ induces a shape isomorphism for each $x\in X$.* ■

(3.10) REMARK. (i) When X and Y are connected, (3.7) follows from (3.4) and Th. 7 of MS [**23**, p. 152]. (ii) Since any pointed shape equivalence induces a shape equivalence, (3.7)-(3.9) hold for unpointed case.

Kozlowski [14] introduced the notation of hereditary shape equivalence. We say that a map $f: X \to Y$ is a hereditary shape equivalence provided that for each closed subset Y_0 of Y, $f|f^{-1}(Y_0): f^{-1}(Y_0) \to Y_0$ is a shape equivalence.

By (2.8), (3.9) and (ii) of (3.10) we have the following:

(3.11) THEOREM. *Let $f: X \to Y$ be a closed map between paracompact spaces. If f has ALP with respect to* $\mathbf{POL}_{\text{pairs}}$, *then f is a hereditary shape equivalence.* ∎

§ 4. Approximative extension property.

We introduce the approximative extension property, and investigate relations between it and approximative n-connectedness.

Let $\mathcal{K}$ be a collection of pairs of spaces. Let $(\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative inverse system in **TOP**. We say that $(\mathcal{X}, \mathcal{U})$ has the approximative extension property, in notation AEP, with respect to $\mathcal{K}$ provided it satisfies the following condition:

(AEP) For each $a \in A$ there exists $a_1 > a$ satisfying that for any $(K, K_0) \in \mathcal{K}$ and for any maps $h: K_0 \to X_{a_1}$, there exists a map $H: K \to X_a$ such that $(H|K_0, p_{a_1,a}h) < \mathcal{U}_a$.

Let $\mathcal{X} = \{X_a, p_{a',a}, A\}$ be an inverse system in **TOP**. We say that $\mathcal{X}$ has the approximative extension property, in notation AEP, with respect to $\mathcal{K}$ provided it satisfies the following condition:

(AEP*) For each $a \in A$ and for each $\mathcal{U} \in Cov(X_a)$ there exists $a_1 > a$ such that for any $(K, K_0) \in \mathcal{K}$ and for any map $h: K_0 \to X_{a_1}$ there exists a map $H: K \to X_a$ such that $(H|K_0, p_{a_1,a}h) < \mathcal{U}$.

(4.1) LEMMA. *Let $(\mathcal{Y}, \mathcal{V})$ be an approximative inverse system. We assume that $(\mathcal{Y}, \mathcal{V})$ is dominated by $(\mathcal{X}, \mathcal{U})$ in Appro-***TOP**. *If $(\mathcal{X}, \mathcal{U})$ has ALP with respect to $\mathcal{K}$, then so does $(\mathcal{Y}, \mathcal{V})$.*

(4.2) COROLLARY. *The property AEP with respect to $\mathcal{K}$ is invariant in Appro-***TOP**.

(4.3) LEMMA. *Let $(\mathcal{X}, \mathcal{X})$ be an approximative inverse system. Then $(\mathcal{X}, \mathcal{U})$ has AEP with respect to $\mathcal{K}$ iff so dose $\mathcal{X}$.*

By the way similar to (II. 1.1) or (II. 1.6) we can easily show (4.1) and (4.3). (4.2) follows from (4.1). ∎

We say that $(\mathscr{X}, \mathscr{V})$ and $\mathscr{X}$ have the extension property, in notation EP, with respect to $\mathscr{K}$ provided they satisfy the following condition:

(EP) For each $a \in A$ there exists $a_1 > a$ such that for each $(K, K_0) \in \mathscr{K}$ and for any map $h: K_0 \to X_{a_1}$ there exists a map $H: K \to X_a$ with $H|K_0 = p_{a_1, a} h$.

(4.4) LEMMA. *Let $\mathscr{K}$ be a collection of polyhedral pairs. Let $(\mathscr{X}, \mathscr{V})$ be an approximative inverse system in* **POL** *or* **ANR**. *Then $(\mathscr{X}, \mathscr{V})$ has AEP with respect to $\mathscr{K}$ iff it has EP with respect to $\mathscr{K}$.*

Using (I. 5.7), by the way similar to (3.2), we can show (4.4) for approximative inverse systems in **POL** or **ANR**. ■

Let $\boldsymbol{p} = \{p_a : a \in A\} : X \to (\mathscr{X}, \mathscr{V})$ be an approximative resolution and $\boldsymbol{p}: X \to \mathscr{X}$ a resolution of a space X. We say that $\boldsymbol{p}: X \to (\mathscr{X}, \mathscr{V})$ has AEP with respect to $\mathscr{K}$ provided that $(\mathscr{X}, \mathscr{V})$ has AEP with respect to $\mathscr{K}$. Similarly we define EP for $\boldsymbol{p}: X \to (\mathscr{X}, \mathscr{V})$. Similarly we define AEP and EP for $\boldsymbol{p}: X \to \mathscr{X}$. By the way similar to (II. 1.3) and (II. 1.7) we can show the following:

(4.5) LEMMA. (i) *Let $\boldsymbol{p}: X \to (\mathscr{X}, \mathscr{V})$ and $\boldsymbol{p}': X \to (\mathscr{X}, \mathscr{V})'$ be approximative* **AP**-*resolutions. If $\boldsymbol{p}$ has AEP with respect to $\mathscr{K}$, then so does $\boldsymbol{p}'$.*

(ii) *Let $\boldsymbol{p}: X \to \mathscr{X}$ and $\boldsymbol{p}': X \to \mathscr{X}'$ be* **AP**-*resolutions. If $\boldsymbol{p}$ has AEP with respect to $\mathscr{K}$, then so does $\boldsymbol{p}'$.*

From (4.3)-(4.5), by the way similar to (II. 1.8) we have the following:

(4.6) THEOREM. *Let X be a space and $\mathscr{K}$ a collection of pairs of spaces. Then* (i) *and* (ii) *below are equivalent, and moreover if $\mathscr{K}$ is a polyhedral pair, then* (i)-(iv) *below are equivalent.*

(i) *Any/some approximative* **AP**-*resolution has AEP with respect to $\mathscr{K}$.*

(ii) *Any/some* **AP**-*resolution has AEP with respect to $\mathscr{K}$.*

(iii) *Any/some approximative* **ANR**- *or* **POL**-*resolution has EP with respect to $\mathscr{K}$.*

(iv) *Any/some* **ANR**- *or* **POL**-*resolution has EP with respect to $\mathscr{K}$.* ■

Thus by (4.6) we say that a space X has the approximative extension property, in notation AEP, with respect to $\mathscr{K}$ provided it satisfies any one of the conditions in (4.6).

By the way similar to (1.12) we have the following:

(4.7) LEMMA. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be collections of P-pairs of spaces and $\mathcal{K}_1 <_e \mathcal{K}_2$. If a space X has AEP with respect to $\mathcal{K}_2$, then so does X with respect to $\mathcal{K}_1$.* ■

Borsuk introduced the notion of approximative n-connectedness for spaces. We say that a space X is approximatively n-connected, in notation $X \in AC^n$, provided pro-$\pi_k(X, x)=0$ for each $x \in X$ and for each k, $0 \leqq k \leqq n$. We say that X is approximatively ∞-connected, in notation $X \in AC^\infty$, provided that $X \in AC^n$ for each n, $n=0, 1, 2, \cdots$.

(4.8) LEMMA. *pro-$\pi_0(X, x)=0$ for some $x \in X$ iff X is connected.*

PROOF. Let $\boldsymbol{p}: X \to \mathcal{X}$ be an **ANR**-resolution and x be a point of X. We put $x_a = p_a(x)$ for $a \in A$. Then $\boldsymbol{p}: (X, x) \to (\mathcal{X}, x) = \{(X_a, x_a), p_{a',a}, A\}$ is an **ANR**$_0$-resolution of (X, x).

First we assume that X is connected and show that pro-$\pi_0(X, x)=0$. Take any $a \in A$. Since X_a is an ANR, X_a is locally path-connected. Then all path-connected components are open and closed. Since X is connected, there exists a path-connected component T of X_a such that $\overline{p_a(X)} \subset T$. Since T is open, by (B3) there exists $a' > a$ such that $p_{a',a}(X_{a'}) \subset T$.

We show that $p_{a',a}$ induces the zero homomorphism $p_{a',a*}: \pi_0(X_{a'}, x_{a'}) \to \pi_0(X_a, x_a)$. Take any map $r: (\partial I, 0) \to (X_{a'}, x_{a'})$. By the choice of a', $p_{a',a} r(\partial I) \subset T$. Since T is path-connected, there exists an extension $r': I \to T \subset X_a$ of $p_{a',a} r$. Thus $p_{a',a*}$ is the zero homomorphism and hence pro-$\pi_0(X, x)=0$.

Next we assume that X is not connected and show that pro-$\pi_0(X, x) \neq 0$. By the assumption there exist open subsets X_0, X_1 of X such that $X_0 \cup X_1 = X$, $X_0 \cap X_1 = \emptyset$, $X_0, X_1 \neq \emptyset$. There exists a map $f: X \to \partial I = \{0, 1\}$ such that $f(X_0)=0$ and $f(X_1)=1$. By (R1) there exist $a \in A$ and a map $g: X_a \to \partial I$ such that $(f, gp_a) < \mathcal{V}$, where $\mathcal{V} = \{\{0\}, \{1\}\} \in \mathrm{Cov}(\partial I)$. By the choice of $\mathcal{V}$, $f = gp_a$ and then $p_a(X_0) \subset g^{-1}(0)$ and $p_a(X_1) \subset g^{-1}(1)$. We assume that pro-$\pi_0(X, x)=0$. Then there exists $a' > a$ such that $p_{a',a}$ induces the zero homomorphism.

We may assume that $x \in X_0$. Take any point $x_1 \in X_1$. We define a map $k: (\partial I, 0) \to (X_{a'}, x_{a'})$ by $k(0) = x_{a'}$ and $k(1) = p_{a'}(x_1)$. By the choice of a' there exists an extension $k': I \to X_a$ of $p_{a',a} k$. Then $gk': I \to \partial I$ is an extension of $gp_{a',a} k$. Since $p_{a',a} k(0) = x_a \in g^{-1}(0)$ and $p_{a',a} k(1) = p_a(x_1) \in g^{-1}(1)$, $gp_{a',a} k: \partial I \to \partial I$ is the identity map. Thus $1_{\partial I}: \partial I \to \partial I$ is extendable to $gk': I \to \partial I$. This is a contradiction and hence pro-$\pi_0(X, x) \neq 0$. ■

(4.9) LEMMA. *Let (X, x) be a pointed space. Then pro-$\pi_0(X, x)=0$ iff any* **POL**- *or* **ANR**-*resolution of X has EP with respect to $\{(\Delta^1, \partial\Delta^1)\}$.*

PROOF. We use the same notation as in the proof of (4.8). We assume that pro-$\pi_0(X, x)=0$, and show the property EP. Take any $a\in A$. Since pro-$\pi_0(X, x)=0$, there exists $a'>a$ such that $p_{a',a*}: \pi_0(X_{a'}, x_{a'})\to\pi_0(X_a, x_a)$ is the zero homomorphism. Take any map $f:\partial I=\{0, 1\}\to X_{a'}$. Take another point 2. We define maps $f_0:\{0, 1\}\to X_{a'}$ and $f_1:\{1, 2\}\to X_{a'}$ as follows: $f_0(0)=f(0)$, $f_0(1)=x_{a'}=f_1(1)$ and $f_1(2)=f(1)$. By the choice of a' there exist maps $g_0:[0, 1]\to X_a$ and $g_1:[1, 2]\to X_a$ such that g_0 and g_1 are extensions of $p_{a',a}f_0$ and $p_{a',a}f_1$, respectively. Thus we define a map $g: I\to X_a$ as follows: $g(t)=g_0(2t)$ for $0\leqq t\leqq 1/2$ and $g(t)=g_1(2t)$ for $1/2\leqq t\leqq 1$. It is easy to show that g is well-defined and is an extension of $p_{a',a}f$. Hence $\boldsymbol{p}$ has the required property. The converse is trivial. ■

(4.10) LEMMA. *Let (X, x) be a pointed space and let n be an integer. Then pro-$\pi_0(X, x)=0$ and pro-$\pi_n(X, x)=0$ iff any* **POL**- *or* **ANR**-*resolution of X has EP with respect to $\{(\Delta^1, \partial\Delta^1), (\Delta^{n+1}, \partial\Delta^{n+1})\}$.*

PROOF. We use the same notations as in the proof of (4.8). We assume that pro-$\pi_0(X, x)=0$ and pro-$\pi_n(X, x)=0$. Take any $a\in A$. Since pro-$\pi_n(X, x)=0$, there exists $a_1>a$ such that $p_{a_1,a*}: \pi_n(X_{a_1}, x_{a_1})\to\pi_n(X_a, x_a)$ is the zero homomorphism. By (4.9) there exists $a_2>a_1$ satisfying (EP) with respect to $\{(\Delta^1, \partial\Delta^1)\}$ for $\boldsymbol{p}$ and a.

We show that a_2 has the required property. Take any map $f:\partial\Delta^{n+1}\to X_{a_2}$ and any point $v\in\partial\Delta^{n+1}$. By the choice of a_2 there exists a map $h: I\to X_{a_1}$ such that $h(0)=x_{a_1}$ and $h(1)=p_{a_2,a_1}f(v)$. Put $T=\partial\Delta^{n+1}\times\{1\}\cup\{v\}\times I\subset\partial\Delta^{n+1}\times I$. We define a map $k: T\to X_{a_1}$ as follows: $k(x, 1)=p_{a_2,a_1}f(x)$ for $x\in\partial\Delta^{n+1}$ and $k(v, t)=h(t)$ for $t\in I$. By the homotopy extension theorem there exists a map $K:\partial\Delta^{n+1}\times I\to X_{a_1}$ such that $K|T=k$. We define a map $w:(\partial\Delta^{n+1}, v)\to(X_{a_1}, x_{a_1})$ by $w(x)=K(x, 0)$ for $x\in\partial\Delta^{n+1}$. By the choice of a_1 there exists a map $W:\Delta^{n+1}\to X_a$ such that $W|\partial\Delta^{n+1}=p_{a_1,a}w$. Put $S=\Delta^{n+1}\times\{0\}\cup\partial\Delta^{n+1}\times I\subset\Delta^{n+1}\times I$. We can define a map $m: S\to X_a$ as follows: $m(x, 0)=W(x)$ for $(x, 0)\in\Delta^{n+1}\times\{0\}$ and $m(x, t)=p_{a_1,a}K(x, t)$ for $(x, t)\in\partial\Delta^{n+1}\times I$. By the homotopy extension theorem there exists a map $M:\Delta^{n+1}\times I\to X_a$ such that $M|S=m$. We define a map $g:\Delta^{n+1}\to X_a$ by $g(x)=M(x, 1)$ for $x\in\Delta^{n+1}$. It is easy to show that $g|\partial\Delta^{n+1}=p_{a_2,a}f$.

By the choice of a_2, for any map $f:\partial I\to X_{a_2}$, $p_{a_2,a}f$ is extendable to I. Hence $\boldsymbol{p}$ has the required property. The converse is trivial. ■

(4.11) LEMMA. *Let (X, x) be a pointed space. Then pro-$\pi_k(X, x)=0$ for k, $0 \leqq k \leqq n$ iff any* **POL**- *or* **ANR**-*resolution of X has EP with respect to $\{(\Delta^{k+1}, \partial\Delta^{k+1})\colon k=0, 1, \cdots, n\}$.*

(4.12) LEMMA. *Let (X, x) be a pointed space. Then pro-$\pi_k(X, x)=0$ for k, $0 \leqq k \leqq n$ iff any* **POL**- *or* **ANR**-*resolution of X has EP with respect to $(D, S)_n$.*

By induction on n we easily show (4.11) from (4.9) and (4.10). (4.12) follows from (4.11) and the fact that any resolution has EP with respect to ${}^0\mathbf{POL}_{\text{pairs}}$. ■

(4.13) COROLLARY. *Let X be a space and let n be an integer.*

(i) *For any points x_1, x_2 of a connected space X, pro-$\pi_n(X, x_1)=0$ iff pro-$\pi_n(X, x_2)=0$.*

(ii) *For any $x_1, x_2 \in X$, pro-$\pi_k(X, x_1)=0$ for k, $0 \leqq k \leqq n$ iff pro-$\pi_k(X, x_2)=0$ for k, $0 \leqq k \leqq n$.*

(4.13) follows from (4.8)-(4.12). ■

(4.14) THEOREM. *For each integer $n \geqq 0$ the following statements are equivalent:*

(i) *pro-$\pi_k(X, x)=0$ for any $x \in X$ and for k, $0 \leqq k \leqq n$.*

(ii) *pro-$\pi_k(X, x)=0$ for some $x \in X$ and for k, $0 \leqq k \leqq n$.*

(iii) *X has AEP with respect to $(D, S)_n$.*

(iv) *X has AEP with respect to $\mathbf{POL}^{n+1}_{\text{pairs}}$.*

(v) *X has AEP with respect to ${}^{n+1}\mathbf{POL}_{\text{pairs}}$.*

(vi) *X has AEP with respect to $\mathbf{TOP}^{n+1}_{P\text{-pairs}}$.*

PROOF. (i) and (ii) are equivalent by (4.13). (ii) and (iii) are equivalent by (4.6) and (4.12). (iv) and (vi) are equivalent by (I. 3.16) and (4.7). Trivially (v)→(iv)→(iii). By the way similar to (4.12) we can show that the property EP with respect to $(D, S)_n$ implies the property EP with respect to ${}^{n+1}\mathbf{POL}_{\text{pairs}}$ for polyhedral resolutions. Hence (iii) implies (v). ■

Approximative n-connectedness can be defined by (i) in (4.14). Thus it is equivalent to the other conditions in (4.14). We say that X is approximatively $\infty\infty$-connected, in notation $X \in AC^{\infty\infty}$, provided that it has AEP with respect to $(D, S)_\infty$. Also we need to consider spaces having AEP with respect to $\mathbf{POL}_{\text{pairs}}$. These properties are characterized as follows:

(4.15) THEOREM. *The following statements are equivalent:*

(i) *A space X has trivial shape.*

(ii) *X has AEP with respect to $\mathbf{POL}_{\text{pairs}}$.*

(iii) *X has AEP with respect to $\mathbf{TOP}_{P\text{-pairs}}$.*

(ii) and (iii) are equivalent by (I. 3.16) and (4.3). By the way similar to (4.16) below we can show the equivalence of (i) and (ii). ∎

(4.16) THEOREM. *Let X be a space and $\operatorname{sd} X=n<\infty$. Then X is approximatively n-connected iff X has trivial shape.*

PROOF. Let $\boldsymbol{p}: X\to\mathcal{X}$ be an $\mathbf{POL}$-resolution. We assume that $X\in AC^n$. Take any $a\in A$. By (4.4) and (4.14) there exists $a'>a$ satisfying EP with respect to $\mathbf{POL}^{n+1}_{\text{pairs}}$ for a. Since $\operatorname{sd} X=n$, by Theorem 2 of MS [**23**, p. 96] there exist $a''>a'$, a polyhedron P and maps $f: X_{a''}\to P$, $g: P\to X_{a'}$ such that $p_{a'',a'}\simeq gf$ and $\dim P\leqq n$. Put $T=P\times\{0, 1\}\subset P\times I$ and take any point $x_{a'}\in X_{a'}$. We define a map $h: T\to X_{a'}$ as follows: $h(x, 0)=g(x)$ and $h(x, 1)=x_{a'}$ for $x\in P$. Since $(P\times I, T)\in\mathbf{POL}^{n+1}_{\text{pairs}}$, by the choice of a' there exists a map $H: P\times I\to X_a$ such that $H|T=p_{a',a}h$. Thus $p_{a',a}g$ is homotopic to a constant map. Since $p_{a'',a'}\simeq gf$, $p_{a'',a}$ is homotopic to a constant map. Hence X has trivial shape.

We assume that X has trivial shape. We show that $\boldsymbol{p}$ has EP with respect to $\mathbf{POL}_{\text{pairs}}$. By the assumption for any $a\in A$ there exists $a'>a$ such that $p_{a',a}$ is homotopic to a constant map $k: X_{a'}\to X_a$. We put $k(X_{a'})=x_a$. Let $H: X_{a'}\times I\to X_a$ be a homotopy such that $H_0=p_{a',a}$ and $H_1=k$. Take any polyhedral pair (P, P_0) and any map $f: P_0\to X_{a'}$. We define a map $G: P\times\{1\}\cup P_0\times I\to X_a$ by $G(x, 1)=x_a$ for $x\in P$ and $G(x, t)=H(f(x), t)$ for $(x, t)\in P_0\times I$. By the homotopy extension theorem (see MS [**23**, p. 291]) there exists an extension $\bar{G}: P\times I\to X_a$ of G. We put $g: P\to X_a$ by $g(x)=\bar{G}(x, 0)$ and then g is an extension of $p_{a',a}f$. Thus $\boldsymbol{p}$ has EP with respect to $\mathbf{POL}_{\text{pairs}}$. Hence $X\in AC^n$ by (4.14). ∎

(4.17) COROLLARY. *Let X be a space and $\operatorname{sd} X<\infty$. Then $X\in AC^\infty$ iff X has trivial shape.* ∎

§ 5. Partial realizations for decompositions.

In this section we introduced partial realizations for decompositions and the approximative full extension property.

We say that (K, L) is a simplicial pair provided L is a subcomplex of a simplicial complex K. $\mathbf{SC}_{\text{pairs}}$ denotes the category of all simplicial pairs and simplicial maps among them. Let n be an integer. ${}^n\mathbf{SC}_{\text{pairs}}$ denotes the full subcategory of $\mathbf{SC}_{\text{pairs}}$ consisting of all simplicial pairs (K, L) with $\dim(K-L)\leqq n$. $\mathbf{SC}^n_{\text{pairs}}$ denotes the full subcategory of $\mathbf{SC}_{\text{pairs}}$ consisting of all simplicial pairs (K, L) with $\dim K\leqq n$. K^n denotes the n-skeleton of K. ${}_0\mathbf{SC}_{\text{pairs}}$ is the full subcategory of $\mathbf{SC}_{\text{pairs}}$ consisting of all simplicial pairs (K, L) with $K^0\subset L$. We put ${}^n_0\mathbf{SC}_{\text{pairs}}={}^n\mathbf{SC}_{\text{pairs}}\cap{}_0\mathbf{SC}_{\text{pairs}}$ and ${}_0\mathbf{SC}^n_{\text{pairs}}=\mathbf{SC}^n_{\text{pairs}}\cap{}_0\mathbf{SC}_{\text{pairs}}$.

Let X be a space and $\mathcal{V}$ a collection of subsets of X. Let $(K, L)\in Ob_0\mathbf{SC}_{\text{pairs}}$. We say that a map $g: |L|\to X$ is a partial realization of (K, L) in X relative to $\mathcal{V}$ provided for each (closed) simplex s of K there exists $U\in\mathcal{V}$ such that $g(|s\cap L|)\subset U$. In case $L=K$, we say that g is a full realization of (K, L) in X relative to $\mathcal{V}$. Sometimes we identify a simplicial complex K and its geometrical realization $|K|$ (endowed with the CW-topology).

Hereafter we assume that Y is a paracompact space and $f: X\to Y$ is a closed onto map with the following property:

(♯) $f^{-1}(Y_0)$ is P-embedded in X for each closed subset Y_0 of Y.

Let $\mathcal{K}$ be a subcollection of $Ob_0\mathbf{SC}_{\text{pairs}}$. Let $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X}, \mathcal{V})=\{(X_a, \mathcal{V}_a), p_{a',a}, A\}$ be an approximative resolution. We put $D_y=f^{-1}(y)$, $D(y, \mathcal{V})=st(p_a(D_y), \mathcal{V})$ for any $a\in A$, $y\in Y$ and $\mathcal{V}\in\mathcal{C}ov(X_a)$, and thus $\mathcal{D}(\mathcal{V})=\{D(y, \mathcal{V}): y\in Y\}\in\mathcal{C}ov(X_a)$. We put $D(y, a)=D(y, \mathcal{V}_a)$ and $\mathcal{D}_a=\mathcal{D}(\mathcal{V}_a)$ for $y\in Y$ and $a\in A$.

We say that $\boldsymbol{p}$ has the approximative full extension property, in notation AFEP, with respect to $\mathcal{K}$ and f provided it satisfies the following property:

(AFEP) For each $a\in A$ there exists $a'>a$ such that for each $(K, L)\in\mathcal{K}$ and each partial realization $g: L\to X_{a'}$ of (K, L) in $X_{a'}$ relative to $\mathcal{D}_{a'}$ there exists a full realization $G: K\to X_a$ of (K, L) in X_a relative to $\mathcal{D}_a$ such that $(G|L, p_{a',a}g)<\mathcal{V}_a$.

We say that $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{V})$ has the full extension property, in notation FEP, with respect to $\mathcal{K}$ and f provided that it satisfies the following condition:

(FEP) For each $a\in A$ there exists $a'>a$ such that for each $(K, L)\in\mathcal{K}$ and each partial realization $g: L\to X_{a'}$ of (K, L) in $X_{a'}$ relative to $\mathcal{D}_{a'}$ there exists a full realization $G: K\to X_a$ of (K, L) relative to $\mathcal{D}_a$ such that $G|L=p_{a',a}g$.

(5.1) LEMMA. *Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be approximative* **AP**-*resolutions of X. If $\boldsymbol{p}$ has AFEP with respect to $\mathcal{K}$ and f, then so does $\boldsymbol{p}'$.*

(5.2) LEMMA. *Let* $\boldsymbol{p}$ *be an approximative* **POL**- *or* **ANR**-*resolution of* X. *Then* $\boldsymbol{p}$ *has AFEP with respect to* $\mathcal{K}$ *and* f *iff* $\boldsymbol{p}$ *has FEP with respect to* $\mathcal{K}$ *and* f.

Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ be a resolution. We say that $\boldsymbol{p}$ has the approximative full extension property, in notation AFEP, with respect to $\mathcal{K}$ f provided it satisfies the following condition:

(AFEP)* For each $a\in A$ and each $\mathcal{V}\in Cov(X_a)$ there exist $a'>a$ and $\mathcal{V}'\in Cov(X_{a'})$ such that for each $(K, L)\in\mathcal{K}$ and each partial realization $g: L\to X_{a'}$ of (K, L) in $X_{a'}$ relative to $\mathcal{D}(\mathcal{V}')$ there exists a full realization $G: K\to X_a$ of (K, L) in X_a relative to $\mathcal{D}(\mathcal{V})$ such that $(G|L, p_{a',a}g)<\mathcal{V}$.

We say that $\boldsymbol{p}$ has the full extension property, in notation FEP, with respect to $\mathcal{K}$ and f provided it satisfies the following property:

(FEP)* For each $a\in A$ and each $\mathcal{V}\in Cov(X_a)$ there exist $a'>a$ and $\mathcal{V}'\in Cov(X_{a'})$ such that for each $(K, L)\in\mathcal{K}$ and each partial realization $g: L\to X_{a'}$ of (K, L) in $X_{a'}$ relative to $\mathcal{D}(\mathcal{V}')$ there exists a full realization $G: K\to X_a$ of (K, L) in X_a relative to $\mathcal{D}(\mathcal{V})$ such that $G|L=p_{a',a}g$.

(5.3) LEMMA. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{p}'$ *be* **AP**-*resolutions of* X. *If* $\boldsymbol{p}$ *has AFEP with respect to* $\mathcal{K}$ *and* f, *then so does* $\boldsymbol{p}'$.

(5.4) LEMMA. *Let* $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{V})$ *be an approximative* **AP**-*resolution. Then* $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{V})$ *has AFEP with respect to* $\mathcal{K}$ *and* f *iff* $\boldsymbol{p}: X\to\mathcal{X}$ *has AFEP with respect to* $\mathcal{K}$ *and* f.

(5.5) LEMMA. *Let* $\boldsymbol{p}$ *be an* **ANR**- *or* **POL**-*resolution of* X. *Then* $\boldsymbol{p}$ *has AFEP with respect to* $\mathcal{K}$ *and* f *iff* $\boldsymbol{p}$ *has FEP with respect to* $\mathcal{K}$ *and* f.

(5.6) THEOREM. *The following statements are equivalent:*

(i) *Any/some approximative* **AP**-*resolution of* X *has AFEP with respect to* $\mathcal{K}$ *and* f.

(ii) *Any/some* **AP**-*resolution of* X *has AFEP with respect to* $\mathcal{K}$ *and* f.

(iii) *Any/some approximative* **ANR**- *or* **POL**-*resolution has FEP with respect to* $\mathcal{K}$ *and* f.

(iv) *Any/some* **ANR**- *or* **POL**-*resolution has FEP with respect to* $\mathcal{K}$ *and* f.

Proofs of (5.1)-(5.6). (5.6) follows from (5.1)-(5.5). In a way similar to (4.4) we can show (5.2) and (5.5). (5.4) is an easy consequence of the definitions. (5.1) follows from (I. 3.3), (5.3) and (5.4).

We show (5.3). We need an elementary fact:

Claim. Let $p: X\to R$, $q: X\to S$ and $g: R\to S$ be maps. Let $\mathcal{R}\in Cov(R)$ and $\mathcal{S}\in Cov(S)$. If $g^{-1}(\mathcal{S})>\mathcal{R}$ and $(gp, q)<\mathcal{S}$, then $g(st(p(D_y), \mathcal{R}))\subset st(q(D_y), st\mathcal{S})$ for each $y\in Y$.

Let $\boldsymbol{p}: X\to\mathcal{X}$ and $\boldsymbol{q}=\{q_b: b\in B\}: X\to\mathcal{Y}=\{Y_b, q_{b',b}, B\}$ be **AP**-resolutions. We assume that $\boldsymbol{p}$ has AFEP with respect to $\mathcal{K}$ and f, and show that $\boldsymbol{q}$ has also AFEP.

Take any $b\in B$ and any $\mathcal{V}\in Cov(Y_b)$. Then there exist $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\in Cov(Y_b)$ such that $st\mathcal{V}_1<\mathcal{V}$, $\mathcal{V}_2<\mathcal{V}_1$, $st\mathcal{V}_3<\mathcal{V}_2$ and $\mathcal{V}_2$ satisfies (R2) for $\boldsymbol{q}$ and $\mathcal{V}_1$. By (R1) there exist $a\in A$ and a map $g: X_a\to Y_b$ such that

(1) $$(q_b, gp_a)<\mathcal{V}_3.$$

By the assumption there exist $a_1>a$ and $\mathcal{U}\in Cov(X_{a_1})$ satisfying (AFEP) for a and $g^{-1}\mathcal{V}_3$. Take $\mathcal{U}_1\in Cov(X_{a_1})$ such that $st\mathcal{U}_1<\mathcal{U}\wedge(gp_{a_1,a})^{-1}\mathcal{V}_3$. By (R1) there exist $b_1>b$ and a map $h: Y_{b_1}\to X_{a_1}$ such that

(2) $$(p_{a_1}, hq_{b_1})<\mathcal{U}_1.$$

By the choice of $\mathcal{U}_1$ and (2) $(gp_a, gp_{a_1,a}hq_{b_1})<\mathcal{V}_3$. Then by (1) $(q_{b_1,b}q_{b_1}, gp_{a_1,a}hq_{b_1})<st\mathcal{V}_3<\mathcal{V}_2$. By the choice of $\mathcal{V}_2$ there exists $b_2>b_1$ such that

(3) $$(q_{b_2,b}, gp_{a_1,a}hq_{b_2,b_1})<\mathcal{V}_1.$$

We show that b_2 and $\mathcal{V}'=(hq_{b_2,b_1})^{-1}\mathcal{U}_1\in Cov(Y_{b_2})$ have the required properties. Take any $(K, L)\in\mathcal{K}$ and any partial realization $t: L\to Y_{b_2}$ of (K, L) in Y_{b_2} relative to $\mathcal{D}(\mathcal{V}')$. By (2) and the Claim, $hq_{b_2,b_1}t: L\to X_{a_1}$ is a partial realization of (K, L) relative to $\mathcal{D}(st\mathcal{U}_1)$ and then relative to $\mathcal{D}(\mathcal{U})$, because $st\mathcal{U}_1<\mathcal{U}$. By the choice of a_1 and $\mathcal{U}$ there exists a full realization $T': K\to X_a$ of (K, L) relative to $\mathcal{D}(g^{-1}\mathcal{V}_3)$ such that

(4) $$(T'|L, p_{a_1,a}hq_{b_2,b_1}t)<g^{-1}\mathcal{V}_3.$$

By (3) and (4) $(gT'|L, q_{b_2,b}t)<st\mathcal{V}_1<\mathcal{V}$. By the Claim and (1), $gT': K\to Y_b$ is a full realization of (K, L) relative to $\mathcal{D}(st\mathcal{V}_3)$ and then relative to $\mathcal{D}(\mathcal{V})$, because $st\mathcal{V}_3<\mathcal{V}$. Hence $\boldsymbol{q}$ has AFEP. ∎

Thus by (5.6) we may say that f and the decomposition $\boldsymbol{D}=\{D_y: y\in Y\}$ of X have the approximative full extension property, in notation AFEP, with respect to $\mathcal{K}$ provided they satisfy any one of the conditions in (5.6). Let $\boldsymbol{K}$ be a subcategory of ${}_0\mathbf{SC}_{\text{pairs}}$. We say that f and $\boldsymbol{D}$ have AFEP with respect to $\boldsymbol{K}$ provided that they have AFEP with respect to $Ob\boldsymbol{K}$, respectively.

Hereafter we assume that $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ is an approximative **ANR**-resolution such that $st(p_a(X), \mathcal{U}_a) = X_a$ for all $a \in A$. By (2.3) any space admits an **ANR**-resolution satisfying the above condition.

(5.7) LEMMA. *For each $a \in A$ there exists $a_1 > a$ such that $p_{a_1, a}^{-1}\mathcal{D}_a > st\mathcal{D}_{a_1}$.*

PROOF. Take any $a \in A$ and then there exists $a_1 > a$ such that $p_{a_1, a}^{-1}\mathcal{U}_a > st\mathcal{U}_{a_1}$. Since $f: X \to Y$ is closed, for each $y \in Y$ there exists an open neighborhood V_y of y in Y such that $D_y \subset f^{-1}(V_y) \subset st(D_y, p_{a_1}^{-1}(\mathcal{U}_{a_1}))$. Since Y is paracompact, $\mathcal{V} = \{V_y : y \in Y\} \in \mathcal{C}ov(Y)$ and then there exists $\mathcal{V}_1 \in \mathcal{C}ov(Y)$ such that $\mathcal{V} > st^2\mathcal{V}_1$. There exists $a_2 > a_1$ such that $f^{-1}(\mathcal{V}_1) > p_{a_2}^{-1}(\mathcal{U}_{a_2})$. Put $\mathcal{U}_{a_2} = \{U_e : e \in E\}$. Since X_{a_2} is metrizable, by Theorem 2 of Kuratowski [17, p. 226] for each $e \in E$ there exists an open subset $G(U_e)$ of X_{a_2} satisfying

(1) $\quad U_e \cap \overline{p_{a_2}(X)} = G(U_e) \cap \overline{p_{a_2}(X)}$ and

(2) $\quad$ for each finite set $\{e_1, e_2, \cdots, e_n\} \subset E$, $U_{e_1} \cap U_{e_2} \cap \cdots \cap U_{e_n} \cap \overline{p_{a_2}(X)} = \emptyset$ implies $G(U_{e_1}) \cap G(U_{e_2}) \cap \cdots \cap G(U_{e_n}) = \emptyset$.

By (1) $\mathcal{W} = \{X_{a_2} - \overline{p_{a_2}(X)}\} \cup \{G(U_e) : e \in E\}$ is an open covering of X_{a_2}. Then there exists $a_3 > a_2$ such that $p_{a_3, a_2}^{-1}(\mathcal{W}) > \mathcal{U}_{a_3}$.

We show that a_3 has the required property. Take any $y \in Y$ and then there exists $V_0 \in \mathcal{V}_1$ such that $y \in V_0$. By the choice of $\mathcal{V}_1$ there exists $V_{y_0} \in \mathcal{V}$ such that $st(V_0, st(\mathcal{V}_1)) \subset V_{y_0}$. Take any $y' \in Y$ such that $D(y, a_3) \cap D(y', a_3) \neq \emptyset$. Thus there exist $U_1, U_2 \in \mathcal{U}_{a_3}$ such that $p_{a_3}(D_y) \cap U_1 \neq \emptyset$, $p_{a_3}(D_{y'}) \cap U_2 \neq \emptyset$ and $U_1 \cap U_2 \neq \emptyset$. By the choice of a_3 there exist $e_1, e_2 \in E$ such that $p_{a_3, a_2}^{-1}(G(U_{e_1})) \supset U_1$ and $p_{a_3, a_2}^{-1}(G(U_{e_2})) \supset U_2$. Then

(3) $\quad p_{a_2}(D_y) \cap G(U_{e_1}) \neq \emptyset$, $p_{a_2}(D_{y'}) \cap G(U_{e_2}) \neq \emptyset$ and $G(U_{e_1}) \cap G(U_{e_2}) \neq \emptyset$.

By (2) and (3), $U_{e_1} \cap U_{e_2} \cap \overline{p_{a_2}(X)} \neq \emptyset$ and then $U_{e_1} \cap U_{e_2} \cap p_{a_2}(X) \neq \emptyset$. This means that

(4) $\quad p_{a_2}^{-1}(U_{e_1}) \cap p_{a_2}^{-1}(U_{e_2}) \neq \emptyset$.

By (1) and (3), $p_{a_2}(D_y) \cap U_{e_1} = p_{a_2}(D_y) \cap \overline{p_{a_2}(X)} \cap U_{e_1} = p_{a_2}(D_y) \cap \overline{p_{a_2}(X)} \cap G(U_{e_1}) = p_{a_2}(D_y) \cap G(U_{e_1}) \neq \emptyset$ and $p_{a_2}(D_{y'}) \cap U_{e_2} \neq \emptyset$. This means that $D_y \cap p_{a_2}^{-1}(U_{e_1}) \neq \emptyset$ and $D_{y'} \cap p_{a_2}^{-1}(U_{e_2}) \neq \emptyset$. By the choice of a_2 there exist $V_1, V_2 \in \mathcal{V}_1$ such that $f^{-1}(V_1) \supset p_{a_2}^{-1}(U_{e_1})$ and $f^{-1}(V_2) \supset p_{a_2}^{-1}(U_{e_2})$. Then by (4), $D_y \cap f^{-1}(V_1) \neq \emptyset$, $D_{y'} \cap f^{-1}(V_2) \neq \emptyset$ and $f^{-1}(V_1) \cap f^{-1}(V_2) \neq \emptyset$. That is, $f(D_y) \in V_1$, $f(D_{y'}) \in V_2$ and $V_1 \cap V_2 \neq \emptyset$. This means that $y' \in st(y, st\mathcal{V}_1)$. Since $y \in V_0$, $y' \in st(V_0, st\mathcal{V}_1) \subset V_{y_0}$. By the choice of V_{y_0}, $D_{y'} \subset f^{-1}(V_{y_0}) \subset st(D_{y_0}, p_{a_1}^{-1}\mathcal{U}_{a_1})$ and then $D(y', a_1)$

$\subset st(p_{a_1}(D_{y_0}), st\mathcal{U}_{a_1})$. By the choices of a_1 and a_3, $p_{a_3,a_1}D(y', a_3)\subset D(y', a_1)$, $p_{a_1,a}st(p_{a_1}(D_{y_0}), st\mathcal{U}_{a_1})\subset D(y_0, a)$ and then $D(y', a_3)\subset p_{a_3,a}^{-1}D(y_0, a)$. By the choice of $D_{y'}$ $st(D(y, a_3), \mathcal{D}_{a_3})\subset p_{a_3,a}^{-1}D(y_0, a)$ and hence $p_{a_3,a}^{-1}\mathcal{D}_a>st\mathcal{D}_{a_3}$. ■

Let $\mathcal{K}$ be a collection of polyhedral pairs. Let Y_0 be a closed subset of Y. Let $a\in A$ and $\mathcal{U}\in Cov(X_a)$. We say that Y_0 has the extension property with respect to a, $\mathcal{U}$ and $\mathcal{K}$, in notation EP$(a, \mathcal{U}, \mathcal{K})$, provided it satisfies the following condition:

EP$(a, \mathcal{U}, \mathcal{K})$: There exists $a_1>a$ with the property; for each point y of Y_0, for any $(K, K_0)\in\mathcal{K}$ and for each map $g: K_0\to st(p_{a_1}(D_y), \mathcal{U}_{a_1})$ there exist a point y' of Y_0 and a map $G: K\to st(p_a(D_{y'}), \mathcal{U})$ such that $G|K_0=p_{a_1,a}g$.

(5.8) LEMMA. *Let Y_1 and Y_2 be closed subsets of Y. If Y_1 has EP$(a, \mathcal{U}, \mathcal{K})$ and $Y_2\subset Y_1$, then Y_2 has EP$(a, st\{st(p_a(D_y), \mathcal{U}): y\in Y\}, \mathcal{K})$.*

PROOF. By the assumption there exists $a_1>a$ satisfying EP$(a, \mathcal{U}, \mathcal{K})$ for Y_1. By (AI3) there exists $a_2>a_1$ such that $p_{a_2,a}^{-1}\mathcal{U}>\mathcal{U}_{a_2}$. We show that a_2 has the required property. Take any point y of Y_2 and any map $g: K_0\to st(p_{a_2}(D_y), \mathcal{U}_{a_2})$. Since $p_{a_2,a_1}g: K_0\to st(p_{a_1}(D_y), \mathcal{U}_{a_1})$, by the choice of a_1 there exist a point y' of Y_1 and a map $G: K\to st(p_a(D_{y'}), \mathcal{U})$ such that $p_{a_2,a}g=G|K_0$. By the choice of a_2, $p_{a_2,a}g: K_0\to st(p_a(D_y), \mathcal{U})$. Thus $G(K)\subset st(p_a(D_{y'}), \mathcal{U})$ and $G(K_0)=p_{a_2,a}g(K_0)\subset st(p_a(D_y), \mathcal{U})$. Thus $G(K)\subset st(p_a(D_y), st\{st(p_a(D_y), \mathcal{U}): y\in Y\})$. Hence Y_2 has the required property. ■

(5.9) LEMMA. *Let $Y_1, Y_2, \cdots, Y_n$ be closed subsets of Y. If all Y_i have EP$(a, \mathcal{U}, \mathcal{K})$, then so does $Y_0=\cup\{Y_i: i=1, 2, \cdots, n\}$.*

(5.9) is an easy consequence of the definition. ■

(5.10) LEMMA. *Let $\{Y_s: s\in S\}$ be a discrete family of closed subsets in Y. If all Y_s have EP$(a, \mathcal{U}, \mathcal{K})$, then $Y_0=\cup\{Y_s: s\in S\}$ has EP$(a, st\mathcal{U}, \mathcal{K})$.*

PROOF. Since Y is paracompact, then by Theorem 5.1.17 of Engelking [**9**, p. 379] there exists a collection $\{V_s: s\in S\}$ such that $Y_s\subset V_s$, V_s are open subsets of Y, $V_s\cap V_{s'}=\emptyset$ for $s, s'\in S$ with $s\neq s'$. Since $\{Y_s: s\in S\}$ is discrete, Y_0 is closed and then $\mathcal{V}=\{V_s: s\in S\}\cup\{Y-Y_0\}\in Cov(Y)$. There exists $a_1>a$ such that $f^{-1}\mathcal{V}>p_{a_1}^{-1}\mathcal{U}_{a_1}$. Put $\mathcal{U}_{a_1}=\{U_e: e\in E\}$ and $E'=\{e\in E: U_e\cap\overline{p_{a_1}f^{-1}(Y_0)}\neq\emptyset\}$. By Theorem 2 of Kuratowski [**17**, p. 226] there exists a collection $\{G(U_e): e\in E'\}$ such that

(1) $G(U_e)$ are open in X_{a_1}, $G(U_e)\cap\overline{p_{a_1}f^{-1}(Y_0)}=U_e\cap\overline{p_{a_1}f^{-1}(Y_0)}$ for all $e\in E'$ and

(2) for each finite subset $\{e_1, e_2, \cdots, e_n\}$ of E',
$U_{e_1}\cap\cdots\cap U_{e_n}\cap\overline{p_{a_1}f^{-1}(Y_0)}=\emptyset$ implies $G(U_{e_1})\cap G(U_{e_2})\cap\cdots\cap G(U_{e_n})=\emptyset$.

By the choice of E' and (1) $\mathcal{V}_1=\{G(U_e): e\in E'\}\cup\{X_{a_1}-\overline{p_{a_1}f^{-1}(Y_0)}\}\in\mathit{Cov}(X_{a_1})$. By (AI3) there exists $a_2>a_1$ such that $p_{a_2,a_1}^{-1}\mathcal{V}_1>\mathcal{V}_{a_2}$.

We put $X_a^s=st(p_af^{-1}(Y_s), \mathcal{V}_a)$ for $a\in A$ and $s\in S$. We show that

(3) $X_{a_2}^s\cap X_{a_2}^{s'}=\emptyset$ for $s\neq s'$.

To prove (3) we assume that for some $s\neq s'$ (3) does not hold. Then there exist $U_1, U_2\in\mathcal{V}_{a_2}$ such that $U_1\cap p_{a_2}f^{-1}(Y_s)\neq\emptyset$, $U_2\cap p_{a_2}f^{-1}(Y_{s'})\neq\emptyset$ and $U_1\cap U_2\neq\emptyset$. By the choice of a_2 there exist $e_1, e_2\in E'$ such that $G(U_{e_1})\supset p_{a_2,a_1}(U_1)$ and $G(U_{e_2})\supset p_{a_2,a_1}(U_2)$. Then $G(U_{e_1})\cap G(U_{e_2})\neq\emptyset$, $G(U_{e_1})\cap p_{a_1}f^{-1}(Y_s)\neq\emptyset$ and $G(U_{e_2})\cap p_{a_1}f^{-1}(Y_{s'})\neq\emptyset$. Thus by (1) and (2)

(4) $U_{e_1}\cap p_{a_1}f^{-1}(Y_s)\neq\emptyset$, $U_{e_2}\cap p_{a_1}f^{-1}(Y_{s'})\neq\emptyset$ and
$U_{e_1}\cap U_{e_2}\cap p_{a_1}f^{-1}(Y_0)\neq\emptyset$.

By the choice of $\mathcal{V}$ and (4), $V_s\supset fp_{a_1}^{-1}(U_{e_1})$, $V_{s'}\supset fp_{a_1}^{-1}(U_{e_2})$ and $fp_{a_1}^{-1}(U_{e_1})\cap fp_{a_1}^{-1}(U_{e_2})\neq\emptyset$. Thus $V_s\cap V_{s'}\supset fp_{a_1}^{-1}(U_{e_1})\cap fp_{a_1}^{-1}(U_{e_2})\neq\emptyset$ and then $s=s'$ by the choice of $\{V_s\}$. However $s\neq s'$ by the assumption. This is a contradiction. Hence we have (3).

We put $X_a^0=st(p_af^{-1}(Y_0), \mathcal{V}_a)$, $\mathcal{V}_a^0=\mathcal{V}_a|X_a^0$ for $a\in A$. The maps p_a and $p_{a',a}$ induce $p_a^0: f^{-1}(Y_0)\to X_a^0$ and $p_{a',a}^0: X_{a'}^0\to X_a^0$ for $a'>a$. By (2.1) $\boldsymbol{p}^0=\{p_a^0: a\in A\}: f^{-1}(Y_0)\to(\mathcal{X}_0, \mathcal{V}_0)=\{(X_a^0, \mathcal{V}_a^0), p_{a',a}^0, A\}$ is an approximative **ANR**-resolution. By (I. 3.2) $\boldsymbol{p}^0: f^{-1}(Y_0)\to\mathcal{X}_0$ is an **ANR**-resolution.

By the assumption for each $s\in S$ there exists $a_s>a$ satisfying EP$(a, \mathcal{V}, \mathcal{K})$ for Y_s. We put $Z=\bigoplus\{X_{a_s}^s\times\{s\}: s\in S\}$ (topological sum). We define maps $q: f^{-1}(Y_0)\to Z$ and $q': Z\to X_a^0$ as follows: $q(x)=(p_{a_s}(x), s)$ for $x\in f^{-1}(Y_s)$ and $q'(z, s)=p_{a_s,a}(z)$ for $z\in X_{a_s}^s\times\{s\}$. Since $\{Y_s: s\in S\}$ is discrete in Y, $\{f^{-1}(Y_s): s\in S\}$ is discrete in X. Using Theorem 5.1.17 of Engelking [**9**, p. 379] we can easily show that $f^{-1}(Y_0)=\cup\{f^{-1}(Y_s): s\in S): s\in S\}=\bigoplus\{f^{-1}(Y_s): s\in S\}$ (topological sum). Then q is well-defined and continuous. Also q' is well-defined and continuous, and $q'q=p_a|f^{-1}(Y_0)$.

Since X_a^0 is an ANR, there exists $\mathcal{V}_1\in\mathit{Cov}(X_a^0)$ satisfying $(*)$ in (I. 5.7) for $\mathcal{V}_a^0$. There exists $\mathcal{V}_2\in\mathit{Cov}(X_a^0)$ satisfying (R2) for $\boldsymbol{p}^0$ and $\mathcal{V}_1$. Since Z is a topological sum, there exist $\mathcal{W}^s\in\mathit{Cov}(X_{a_s}^s)$, $s\in S$, such that $q'^{-1}\mathcal{V}_2>\mathcal{W}=\cup\{\mathcal{W}^s\times\{s\}: s\in S\}$ and $st\mathcal{W}^s<\mathcal{V}_{a_s}|X_{a_s}^s$ for $s\in S$. By (R1) for $\boldsymbol{p}^0$ there exist $a_3>a_2$ and a map $g: X^0_{a_3} \to Z$ such that

(5) $\qquad (q, gp^0_{a_3}) < \mathcal{W}$.

By the choice of $\mathcal{W}$, $(q'q, q'gp^0_{a_3}) < \mathcal{V}_2$ and then by the choices of q and q', $(p_{a_3, a}p^0_{a_3}, q'gp^0_{a_3}) < \mathcal{V}_2$. By the choice of $\mathcal{V}_2$ there exists $a_4 > a_3$ such that

(6) $\qquad (p_{a_3, a}p^0_{a_4, a_3}, q'gp^0_{a_4, a_3}) < \mathcal{V}_1$.

Take any $s \in S$ and any point $x \in f^{-1}(Y_s)$. Since $q(x) \in X^s_{a_s} \times \{s\}$, by (5) there exists $W_1 \in \mathcal{W}^s$ such that $q(x)$, $gp_{a_3}(x) \in W_1 \times \{s\} \subset X^s_{a_s} \times \{s\}$. Then $gp_{a_3}f^{-1}(Y_s) \subset X^s_{a_s} \times \{s\}$ and hence $\overline{p_{a_3}f^{-1}(Y_s)} \subset R_s = X^s_{a_3} \cap g^{-1}(X^s_{a_s} \times \{s\})$, because $X^s_{a_s} \times \{s\}$ is closed in Z. Since $a_3 > a_2$, by (3),

(7) $\qquad X^s_{a_3} \cap X^{s'}_{a_3} = \emptyset$ for $s \neq s'$.

Since $X^s_{a_s} \times \{s\}$ is open in Z, by (7)

(8) $\qquad R_s$ is open in X_{a_3}, $\overline{p_{a_3}f^{-1}(Y_s)} \subset R_s$ and $R_s \cap R_{s'} = \emptyset$
for $s, s' \in S$ with $s \neq s'$.

We will show the following:

(9) $\qquad \mathcal{T} = \{\overline{p_{a_3}f^{-1}(Y_s)} : s \in S\}$ is a discrete collection in X_{a_3}.

Indeed, let u be any point of X_{a_3}. We will find an open neighborhood N of u in X_{a_3} such that N meets at most one member of $\mathcal{T}$.

If $u \in X_{a_3} - X^0_{a_3}$, we choose for N any member of $\mathcal{V}_{a_3}$ which contains u. Then N misses all members of $\mathcal{T}$. To see this assume that for a given $s \in S$ we have $N \cap \overline{p_{a_3}f^{-1}(Y_s)} \neq \emptyset$. Then also $N \cap p_{a_3}f^{-1}(Y_s) \neq \emptyset$ and this implies $u \in N \subset st(p_{a_3}f^{-1}(Y_s), \mathcal{V}_{a_3}) = X^s_{a_3} \subset X^0_{a_3}$, which is a contradiction.

Now assume that $u \in X^0_{a_3}$. Then $g(u) \in Z = \bigoplus X^s_{a_s} \times \{s\}$. Therefore, there exists an $s' \in S$ such that $g(u) \in X^{s'}_{a_{s'}} \times \{s'\}$. We put $N = g^{-1}(X^{s'}_{a_{s'}} \times \{s'\})$. This set is open in $X^0_{a_3}$ and therefore open in X_{a_3}. Moreover, $u \in N$. Finally, by (8), for all $s \neq s'$ we have $\overline{p_{a_3}f^{-1}(Y_s)} \subset R_s \subset g^{-1}(X^s_{a_s} \times \{s\})$, which is disjoint from $N = g^{-1}(X^{s'}_{a_{s'}} \times \{s'\})$. Consequently, only $\overline{p_{a_3}f^{-1}(Y_{s'})}$ can meet N. Hence we have (9).

By (9) and the definition of collectionwise normality (see Engelking [**9**, p. 379]), there exists a discrete family of open sets $\{R'_s : s \in S\}$ such that $\overline{p_{a_3}f^{-1}(Y_s)} \subset R'_s$, $s \in S$. We may assume that $\overline{R'_s} \subset R_s$, $s \in S$. Then $\overline{p_{a_3}f^{-1}(Y_0)} = \cup \{\overline{p_{a_3}f^{-1}(Y_s)} : s \in S\} \subset \cup \{R'_s : s \in S\} \subset X^0_{a_3}$. By (B3) for $\boldsymbol{p}^0$, there exists $a_5 > a_4$ such that $p_{a_5, a_3}(X^0_{a_5}) \subset \cup \{R'_s : s \in S\}$. Since $p_{a_5, a_3}(X^s_{a_5}) \subset X^s_{a_3}$, we see that $p_{a_5, a_3}(X^s_{a_5}) \cap$

$R'_{s'}\subset X^s_{a_3}\cap R_{s'}\subset X^s_{a_3}\cap X^{s'}_{a_3}=\emptyset$ for $s\neq s'$. Consequently, $p_{a_5,a_3}(X^s_{a_5})\subset p_{a_5,a_3}(X^0_{a_5})\subset\cup\{R'_s: s\in S\}$ implies $p_{a_5,a_3}(X^s_{a_5})\subset R'_s$ and then $p_{a_5,a_3}(\overline{X^0_{a_5}})\subset\overline{p_{a_5,a_3}(X^0_{a_5})}\subset\overline{\cup p_{a_5,a_3}(X^s_{a_5})}\subset\overline{\cup R'_s}=\cup\overline{R'_s}\subset\cup R_s\subset\cup X^s_{a_3}=X^0_{a_3}$. Also note that $gp_{a_5,a_3}(X^s_{a_5})\subset g(R'_s)\subset g(R_s)\subset X^s_{a_s}\times\{s\}$ for $s\in S$. Now it is clear that $\mathcal{W}_1=\cup\{(gp_{a_5,a_3})^{-1}(\mathcal{W}^s\times\{s\}): s\in S\}\cup\{X_{a_5}-\overline{X^0_{a_5}}\}\in \mathcal{C}ov(X_{a_5})$. Then there exists $a_6>a_5$ such that $p^{-1}_{a_6,a_5}\mathcal{W}_1>\mathcal{U}_{a_6}$.

We will show that a_6 has the required property. Take any point $y\in Y_0$ and any map $h: K_0\to st(p_{a_6}(D_y), \mathcal{U}_{a_6})$. Then there exists $s_0\in S$ such that $y\in Y_{s_0}$, and then $gp_{a_6,a_3}h(K_0)\subset X^{s_0}_{a_{s_0}}\times\{s_0\}$. Take any $t\in K_0$. Then there exist $x\in D_y$ and $U_1\in\mathcal{U}_{a_6}$ such that $h(t)$, $p_{a_6}(x)\in U_1$. By the choice of a_6 there exists $W_2\in\mathcal{W}^{s_0}$ such that $gp_{a_6,a_3}h(t)$, $gp_{a_3}(x)\in W_2\times\{s_0\}$. By (5) $q(x)$, $gp_{a_3}(x)\in W_3\times\{s_0\}$ for some $W_3\in\mathcal{W}^{s_0}$. Then by the choice of $\mathcal{W}^{s_0}$ there exists $U_2\in\mathcal{U}_{a_{s_0}}$ such that $q(x)$, $gp_{a_6,a_3}h(t)\in U_2\times\{s_0\}$. Thus, $rgp_{a_6,a_3}h(K_0)\subset st(p_{a_{s_0}}(D_y), \mathcal{U}_{a_{s_0}})$ where r: $X^{s_0}_{a_{s_0}}\times\{s_0\}\to X^{s_0}_{a_{s_0}}$ is the projection. By the choice of a_{s_0} there exist $y'\in Y_{s_0}$ and a map $g': K\to st(p_a(D_{y'}), \mathcal{U})$ such that

$$g'|K_0=p_{a_{s_0},a}rgp_{a_6,a_3}h=q'gp_{a_6,a_3}h\,. \tag{10}$$

By (6), $(p_{a_6,a}h, q'gp_{a_6,a_3}h)<\mathcal{V}_1$. By the choice of $\mathcal{V}_1$ there exists a $\mathcal{U}$-homotopy $u: K_0\times I\to X^0_a$ such that $u_0=q'gp_{a_6,a_3}h$ and $u_1=p_{a_6,a}h$. We define a map $g'': K\times\{0\}\cup K_0\times I\to X^0_a$ as follows: $g''(t, 0)=g'(t)$ for $(t, 0)\in K\times\{0\}$ and $g''(t, t')=u(t, t')$ for $(t, t')\in K_0\times I$. By (10), g'' is well-defined and continuous. Since $g'(K)\subset st(p_a(D_{y'}), \mathcal{U})$ and u is a $\mathcal{U}$-homotopy, $\operatorname{Im} g''\subset st(p_a(D_{y'}), st\mathcal{U})$. Since (K, K_0) is a polyhedral pair, by the homotopy extension theorem (see Th. 3 of MS [**23**, p. 291]) there exists an extension $G'': K\times I\to st(p_a(D_{y'}), st\mathcal{U})$ of g''. Then we put $G: K\to st(p_a(D_{y'}), st\mathcal{U})$ as follows; $G(x)=G''(x, 1)$ for $x\in K$. Thus G is an extension of $p_{a_6,a}h$. Hence Y_0 has the required property. ■

(5.11) LEMMA. *If D_y has AEP with respect to $\mathcal{K}$, then for each $a\in A$ there exists an open neighborhood V_y of y in Y such that $\overline{V}_y$ has EP$(a, \mathcal{U}_a, \mathcal{K})$.*

PROOF. By (I. 3.2) and (2.1) $\boldsymbol{p}^0=\{p^0_a: a\in A\}: D_y\to\{D(y, a), p^0_{a',a}, A\}$ is an **ANR**-resolution of D_y. Here p^0_a and $p^0_{a',a}$ are induced by p_a and $p_{a',a}$, respectively. Since $\mathcal{K}$ is a collection of polyhedral pairs, by (4.4) $\boldsymbol{p}^0$ has EP with respect to $\mathcal{K}$. Thus for each $a\in A$ there exists $a_1>a$ satisfying (EP) with respect to $\mathcal{K}$ for a. Since $\overline{p_{a_1}(D_y)}\subset st(p_{a_1}(D_y), \mathcal{U}_{a_1})$, there exists an open subset W in X_{a_1} such that $\overline{p_{a_1}(D_y)}\subset W\subset\overline{W}\subset st(p_{a_1}(D_y), \mathcal{U}_{a_1})=D(y, a_1)$. Since $p^{-1}_{a_1}(W)$ is an open neighborhood of D_y in X and $f: X\to Y$ is a closed map, $Y-f(X-p^{-1}_{a_1}(W))$ is an open neighborhood of y in Y. Since Y is paracompact, there exists an open neighborhood V_y of y in Y such that $y\in V_y\subset \bar{V}_y\subset Y-f(X-p_{a_1}^{-1}(W))$, and then

(1) $$\overline{p_{a_1}f^{-1}(\bar{V}_y)}\subset \bar{W}\subset st(p_{a_1}(D_y),\ \mathcal{U}_{a_1})\subset st(p_{a_1}f^{-1}(\bar{V}_y),\ \mathcal{U}_{a_1}).$$

By (I. 3.2) and (2.1) $\boldsymbol{p}''=\{p''_a : a\in A\} : f^{-1}(\bar{V}_y)\to\{st(p_a f^{-1}(\bar{V}_y),\ \mathcal{U}_a),\ p''_{a',a},\ A\}$ is an **ANR**-resolution. Here p''_a and $p''_{a',a}$ are induced by p_a and $p_{a',a}$, respectively. By (1) and (B3) for $\boldsymbol{p}''$ there exists $a_2>a_1$ such that

(2) $$p_{a_2,a_1}(st(p_{a_2}f^{-1}(\bar{V}_y),\ \mathcal{U}_{a_2}))\subset st(p_{a_1}(D_y),\ \mathcal{U}_{a_1}).$$

We show that V_y and a_2 have required property. Take any $y'\in \bar{V}_y$ and any map $h : K_0\to st(p_{a_2}(D_{y'}),\ \mathcal{U}_{a_2})$. Since $p_{a_2,a_1}h : K_0\to st(p_{a_1}(D_y),\ \mathcal{U}_{a_1})$ by (2), by the choice of a_1 there exists a map $G : K\to st(p_a(D_y),\ \mathcal{U}_a)$ such that $G|K_0=p_{a_2,a}h$. Hence we have the required property. ■

(5.12) Proposition. *Let $\mathcal{K}$ be a collection of potyhedral pairs. If $D_y=f^{-1}(y)$ has AEP with respect to $\mathcal{K}$ for any $y\in Y$, then any approximative* **ANR**-*resolution $\boldsymbol{p} : X\to(\mathcal{X},\ \mathcal{U})$ has the following property:*

(*) *For each $a\in A$ there exists $a_1>a$ such that for any $y\in Y$, any $(K,\ K_0)\in\mathcal{K}$ and any map $h : K_0\to st(p_{a_1}(D_y),\ \mathcal{U}_{a_1})$, there exist $y'\in Y$ and a map $H : K\to st(p_a(D_{y'}),\ \mathcal{U}_a)$ satisfying $H|K_0=p_{a_1,a}h$.*

To prove (5.12) we need the Michael method as follows: Let Z be a space and $\mathcal{V}$ a collection of subsets of Z. We introduce the following notations:

$\mu^*(\mathcal{V})=\{A\subset Z :\ A$ is closed in Z and A is a subset of some element of $\mathcal{V}\}$.

$\sigma^*(\mathcal{V})=\{A\subset Z :\ A$ is the union of a finite collection of closed subsets of Z, whose interiors with respect to A cover A, and which are elements of $\mathcal{V}\}$.

$\Sigma^*(\mathcal{V})=\{A\subset Z :\ A$ is the union of a discrete collection of closed subsets of Z which are elements of $\mathcal{V}\}$.

(5.13) Lemma (Michael [**24**]). *If Z is a paracompact space and $\mathcal{V}$ covers Z, then $Z\in\sigma^*\Sigma^*\sigma^*\Sigma^*\mu^*(\mathcal{V})$.*

Proof of (5.12). From any approximative **ANR**-resolution $\boldsymbol{p} : X\to(\mathcal{X},\ \mathcal{U})$, by (2.1) we have an approximative **ANR**-resolution $\boldsymbol{p}'=\{p'_a : a\in A\} : X\to(\mathcal{X},\ \mathcal{U})'=\{(X'_a,\ \mathcal{U}'_a),\ p'_{a',a},\ A\}$. Here $X'_a=st(p_a(X),\ \mathcal{U}_a)$ and maps p'_a, $p'_{a',a}$ are induced by p_a, $p_{a',a}$ for $a'>a$. Trivially if $\boldsymbol{p}'$ has the property (*), then so does $\boldsymbol{p}$. Thus without loss of generality we may assume that $\boldsymbol{p}$ satisfies $st(p_a(X),\ \mathcal{U}_a)=X_a$ for all $a\in A$.

Take any $a\in A$. By (5.7) there exist $a_2>a_1>a$ such that $p_{a_1,a}^{-1}\mathcal{D}_a>st\mathcal{D}_{a_1}$ and $p_{a_2,a_1}^{-1}\mathcal{D}_{a_1}>st^3\mathcal{D}_{a_2}$. By the assumption and (5.11) there exist open neighborhoods V_y of y such that $\bar{V}_y$ has EP(a_2, $\mathcal{U}_{a_2}$, $\mathcal{K}$) for any $y\in Y$. Since Y is paracompact, $\mathcal{V}=\{V_y: y\in Y\}$ covers Y and then by (5.13) $Y\in\sigma^*\Sigma^*\sigma^*\Sigma^*\mu^*(\mathcal{V})$. By (5.8)-(5.10) any element of $\sigma^*\Sigma^*\sigma^*\Sigma^*\mu^*(\mathcal{V})$ has EP(a_2, $st^3\mathcal{D}_{a_2}$, $\mathcal{K}$) and hence so does Y. Then there exists $a_3>a_2$ satisfying EP(a_2, $st^3\mathcal{D}_{a_2}$, $\mathcal{K}$).

We show that a_3 is the required index. Take any $y\in Y$, any $(K, K_0)\in\mathcal{K}$ and any map $h: K_0\to st(p_{a_3}(D_y), \mathcal{U}_{a_3})$. By the choice of a_3 there exist $y'\in Y$ and a map $H: K\to st(p_{a_2}(D_{y'}), st^3\mathcal{D}_{a_2})$ such that $H|K_0=p_{a_3,a_2}h$. By the choices of a_1 and a_2 it is easy to show that there exists $y''\in Y$ such that $p_{a_2,a}H(K)\subset st(p_a(D_{y''}), \mathcal{U}_a)$. Hence we have the required property. ■

(5.14) THEOREM. *Let Y be a paracompact space and $f: X\to Y$ a closed onto map with (#). Let $n\geqq 0$ be an integer. If $f^{-1}(y)$ is approximatively n-connected for each $y\in Y$, then f has FEP with respect to* ${}^{n+1}_{0}\mathbf{SC}_{\mathrm{pairs}}$.

PROOF. Let $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$ be an approximative **ANR**-resolution such that $st(p_a(X), \mathcal{U}_a)=X_a$ for $a\in A$. By (4.14) and D_y has AEP with respect to ${}^{n+1}\mathbf{POL}_{\mathrm{pairs}}$.

We show (5.14) by induction on n. First we show this for $n=0$. Take any $a\in A$ and then there exists $a_0>a$ satisfying (*) in (5.12) for ${}^{1}\mathbf{POL}_{\mathrm{pairs}}$ and a. We show that a_0 is the required index. Take any $(K, L)\in{}^{1}_{0}\mathbf{SC}_{\mathrm{pairs}}$ and any partial realization $g: L\to X_{a_0}$ of (K, L) relative to $\mathcal{D}_{a_0}$. Take any 1-simplex $s=[v_0, v_1]\in K-L$. Then $g(\partial s)=g(s\cap L)\subset st(p_{a_0}(D_y), U_{a_0})$ for some $y\in Y$. By the choice of a_0 there exists a map $g_s: s\to X_a$ such that $g_s|\partial s=p_{a_0,a}g|\partial s$ and $g_s(s)\subset st(p_a(D_{y'}), \mathcal{U}_a)$ for some $y'\in Y$. We define a map $G: K\to X_a$ as follows: $G(x)=p_{a_0,a}g(x)$ for $x\in L$ and $G(x)=g_s(x)$ for $x\in s\in K-L$. Obviously G is a full realization of (K, L) relative to $\mathcal{D}_a$ and $G|L=p_{a_0,a}g$.

We assume that (5.14) holds for n and show it for $n+1$. Take any $a\in A$ and then there exists $a_1>a$ satisfying (*) in (5.12) for ${}^{n+2}\mathbf{POL}_{\mathrm{pairs}}$ and a. By (5.7) there exists $a_2>a_1$ such that $p_{a_2,a_1}^{-1}\mathcal{D}_{a_1}>st\mathcal{D}_{a_2}$. By the inductive assumption there exists $a_3>a_2$ satisfying (FEP) for a_2 and ${}^{n+1}_{0}\mathbf{SC}_{\mathrm{pairs}}$.

We show that a_3 is the required index. Take any $(K, L)\in{}^{n+2}_{0}\mathbf{SC}_{\mathrm{pairs}}$ and any partial realization $g: L\to X_{a_3}$ of (K, L) relative to $\mathcal{D}_{a_3}$. By the choice of a_3 there exists a full realization $g': K^{n+1}\to X_{a_2}$ of $(K^{n+1}\cup L, L)$ relative to $\mathcal{D}_{a_2}$ such that $g'|L=p_{a_3,a_2}g$. Take any $(n+2)$-simplex s of $K-L$. Since $g'(\partial s)\subset st(st(p_{a_2}(D_y), \mathcal{U}_{a_2}), \mathcal{D}_{a_2})$ for some $y\in Y$, by the choice of a_2, $p_{a_2,a_1}g'(\partial s)\subset st(p_{a_1}(D_{y'}), \mathcal{U}_{a_1})$ for some $y'\in Y$. Then by the choice of a_1 there exist $y''\in Y$

and a map $g'_s: s \to st(p_a(D_{y''}), \mathcal{U}_a)$ such that $g'_s|\partial s = p_{a_2, a}g'|\partial s$. Now we define a map $G: K \to X_a$ as follows: $G(x) = p_{a_2, a}g'(x)$ for $x \in K^{n+1} \cup L$ and $G(x) = g'_s(x)$ for $x \in s \in K - (K^{n+1} \cup L)$. Obviously $G|L = p_{a_2, a}g$ and for each simplex s of K, $G(s) \subset st(p_a(D_y), \mathcal{U}_a)$ for some $y \in Y$. Hence f has FEP with respect to ${}^{n+2}_{0}\mathbf{SC}_{\text{pairs}}$. ■

§ 6. The Vietoris-Smale theorem in shape.

In this section we shall give a characterization of approximatively n-connected maps and the Vietoris-Smale theorem in shape theory.

Let $f: X \to Y$ be a map. Let $\boldsymbol{p} = \{p_a : a \in A\} : X \to (\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $\boldsymbol{q} = \{q_b : b \in B\} : Y \to (\mathcal{Y}, \mathcal{V}) = \{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be approximative **ANR**-resolutions. Let $\boldsymbol{f} = \{f, f_b : b \in B\} : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.

(6.1) PROPOSITION. *Let $f: X \to Y$ be a map and Y a normal space. Then f is approximatively (-1)-connected iff $\overline{f(X)} = Y$.*

PROOF. First we assume that f is approximatively (-1)-connected. Take any $y \in Y$ and any open neighborhood V of y in Y. Since Y is normal, $\mathcal{V} = \{V, Y - \{y\}\} \in \mathcal{C}ov(Y)$. By (AI3) there exists $b \in B$ such that $\mathcal{V} > q_b^{-1} st^2 \mathcal{V}_b$. Since $\boldsymbol{f}$ is an approximative resolution of f,

(1) $\qquad (q_b f, f_b p_{f(b)}) < \mathcal{V}_b.$

By (AR2) there exists $a > f(b)$ such that $st(p_{f(b)}(X), \mathcal{U}_{f(b)}) \supset p_{a, f(b)}(X_a)$. Since $\boldsymbol{f}$ is approximatively (-1)-connected, there exists an admissible pair $(a_1, b_1) > (a, b)$ satisfying (ALP) with respect to $\{(\varDelta^0, \phi)\}$ for (a, b).

Take a map $g: \varDelta^0 \to Y_{b_1}$ such that $g(\varDelta^0) = q_{b_1}(y)$. By the choice of (a_1, b_1) there exists a map $H: \varDelta^0 \to X_a$ such that

(2) $\qquad (f_b p_{a, f(b)} H, q_{b_1, b} g) < st \mathcal{V}_b.$

By the choice of a, $p_{a, f(b)} H(\varDelta^0) \in st(p_{f(b)}(X), \mathcal{U}_{f(b)})$ and then there exist $x \in X$ and $U \in \mathcal{U}_{f(b)}$ such that $p_{f(b)}(x)$, $p_{a, f(b)} H(\varDelta^0) \in U$. By (AM1) there exists $V_1 \in \mathcal{V}_b$ such that

(3) $\qquad f_b p_{f(b)}(x), f_b p_{a, f(b)} H(\varDelta^0) \in V_1.$

By (1) and (2) there exist $V_2, V_3 \in \mathcal{V}_b$ such that

(4) $\qquad q_b f(x), f_b p_{f(b)}(x) \in V_2,$ and

(5) $\qquad f_b p_{a, f(b)} H(\varDelta^0), q_{b_1, b} g(\varDelta^0) \in st(V_3, \mathcal{V}_b).$

Since $q_{b_1,b}g(\Delta^0)=q_b(y)$, by (3)-(5), $q_bf(x)$, $q_b(y)\in st(st(V_3, \mathcal{V}_b), \mathcal{V}_b)$. By the choice of b and the definition of V, $f(x)$, $y\in V$ and then $V\cap f(X)\neq\emptyset$. Hence $f(X)$ is dense in Y.

Next we assume that $f(X)$ is dense in Y. Take any admissible pair (a, b) of $\boldsymbol{f}$. By (B4) there exists $b_1>b$ such that $st(q_b(Y), \mathcal{V}_b)\supset q_{b_1,b}(Y_{b_1})$. Take any $a_1>a$, $f(b_1)$. We will show that the admissible pair (a_1, b_1) has the required property. Take any map $g:\Delta^0\to Y_{b_1}$. By the choice of b_1 there exist $y\in Y$ and $V_4\in\mathcal{V}_b$ such that

(6) $$q_b(y),\ q_{b_1,b}g(\Delta^0)\in V_4.$$

Since $f(X)$ is dense in Y, there exists $x\in X$ such that

(7) $$f(x)\in q_b^{-1}V_4.$$

We define a map $H:\Delta^0\to X_a$ by $H(\Delta^0)=p_a(x)$. By (1) there exists $V_5\in\mathcal{V}_b$ such that

(8) $$q_bf(x),\ f_bp_{f(b)}(x)\in V_5.$$

Since $f_bp_{f(b)}(x)=f_bp_{a,f(b)}H(\Delta^0)$, by (6)-(8) $f_bp_{a,f(b)}H(\Delta^0)$, $q_{b_1,b}g(\Delta^0)\in st(V_4, \mathcal{V}_b)$. Thus $(f_bp_{a,f(b)}H, q_{b_1,b}g)<st\mathcal{V}_b$ and hence f is approximatively (-1)-connected. ■

(6.2) COROLLARY. *Let $f:X\to Y$ be a closed map and Y a normal space. Then f is approximatively (-1)-connected iff $f(X)=Y$.*

(6.3) THEOREM. *Let Y be a paracompact space and $f:X\to Y$ a closed map with (♯):*

(♯) *$f^{-1}(Y_0)$ is P-embedded in X for any closed subset Y_0 of Y.*

For each integer $n\geq -1$ the following statements are equivalent:

(i) *f is onto and $f^{-1}(y)$ is approximatively n-connected for any $y\in Y$.*

(ii) *f is onto and the decomposition $\boldsymbol{D}=\{f^{-1}(y):y\in Y\}$ has AFEP with respect to ${}^{n+1}_{0}\mathbf{SC}_{\text{pairs}}$.*

(iii) *f is approximatively n-connected.*

PROOF. When $n=-1$ (6.3) follows from (6.2). We assume $n\geq 0$. We show (iii)→(i). By (6.2) f is onto. By (3.3) f has ALP with respect to ${}^{n}\mathbf{POL}_{\text{pairs}}$ and then by (2.7) and (♯) so does $f|f^{-1}(y):f^{-1}(y)\to\{y\}$ for each $y\in Y$. By (3.4) pro-$\pi_k(f|f^{-1}(y))$: pro-$\pi_k(f^{-1}(y), *)\to$pro-$\pi_k(\{y\}, *)=0$ is an isomorphism for $0\leq k\leq n$. Then pro-$\pi_k(f^{-1}(y), *)=0$ for $0\leq k\leq n$. By (4.14) $f^{-1}(y)$ is approximatively n-connected.

(5.14) means that (i)→(ii). We show (ii)→(iii). By (I. 4.9) there exist approximative **ANR**-resolutions $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}: Y\to(\mathcal{Y}, \mathcal{V})$ and an approximative resolution $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying (RM1) and (RM2). By (RM2) and (2.1) we may assume that $st(p_a(X), \mathcal{U}_a)=X_a$ and $st(q_b(Y), \mathcal{V}_b)=Y_b$ for all $a\in A$ and all $b\in B$.

Claim 1. $\boldsymbol{f}$ has (ALP**) with respect to $(D, S)_n$.

Take any admissible pair (a, b) of $\boldsymbol{f}$ and then there exists $b_1>b$ such that $q_{b_1,b}^{-1}\mathcal{V}_b>st\mathcal{V}_{b_1}$. By (RM1) for $\boldsymbol{f}$ there exists $a_1>a$, $f(b_1)$ such that $f_b p_{a_1,f(b)}=q_{b_1,b}f_{b_1}p_{a_1,f(b_1)}$. By the assumption and (5.2) there exists $a_2>a_1$ satisfying (FEP) with respect to ${}^{n+1}_{0}\mathbf{SC}$ for a_1. By (5.7) there exists $a_3>a_2$ such that $p_{a_3,a_2}^{-1}\mathcal{D}_{a_2}>st^2\mathcal{D}_{a_3}$. Here $D(y, a)=st(p_a(D_y), \mathcal{U}_a)$ and $\mathcal{D}_a=\{D(y, a): y\in Y\}$ for $a\in A$ and $y\in Y$. Since f is closed, there exists an open neighborhood V_y of y in Y such that

(1) $$D_y\subset f^{-1}(V_y)\subset p_{a_3}^{-1}(st(p_{a_3}(D_y), \mathcal{U}_{a_3})) \text{ for each } y\in Y.$$

Since Y is paracompact, $\mathcal{V}=\{V_y: y\in Y\}\in \mathcal{C}ov(Y)$ and then there exists $b_2>b_1$ such that $\mathcal{V}>q_{b_2}^{-1}\mathcal{V}_{b_2}$.

Let $\mathcal{V}_{b_2}=\{V_m: m\in M\}$ and $M'=\{m\in M: V_m\cap q_{b_2}(Y)\neq\emptyset\}$. By Theorem 2 of Kuratowski [**17**, p. 226] there exist open sets $G(V_m)$ in Y_{b_2} for $m\in M'$ such that

(2) $$V_m\cap\overline{q_{b_2}(Y)}=G(V_m)\cap\overline{q_{b_2}(Y)} \text{ and } G(V_m)\subset V_m \text{ for } m\in M',$$

(3) for each finite subset $\{m_1, m_2, \cdots, m_s\}$ of M', $V_{m_1}\cap V_{m_2}\cap\cdots\cap V_{m_s}\cap\overline{q_{b_2}(Y)}=\emptyset$ implies $G(V_{m_1})\cap G(V_{m_2})\cap\cdots\cap G(V_{m_s})=\emptyset$.

Since $\overline{q_{b_2}(Y)}\subset\cup\{G(V_m): m\in M'\}$ by (2), by (B3) there exists $b_3>b_2$ such that $q_{b_3,b_2}(Y_{b_3})\subset\cup\{G(V_m): m\in M'\}$. Since $\{q_{b_3,b_2}^{-1}G(V_m): m\in M'\}\in\mathcal{C}ov(Y_{b_3})$, by (AI3) there exists $b_4>b_3$ such that $\{q_{b_4,b_3}^{-1}G(V_m): m\in M'\}>\mathcal{V}_{b_4}$. By (RM1) for $\boldsymbol{f}$ there exists $a_4>a_3$, $f(b_4)$ such that $f_{b_1}p_{a_4,f(b_1)}=q_{b_4,b_1}f_{b_4}p_{a_4,f(b_4)}$. Let $\mathcal{U}_{a_4}=\{U_e: e\in E\}$ and $E'=\{e\in E: U_e\cap p_{a_4}(X)\neq\emptyset\}$. Since $\overline{p_{a_4}(X)}\subset\cup\{U_e: e\in E'\}=st(p_{a_4}(X), \mathcal{U}_{a_4})$, by (B3) there exists $a_5>a_4$ such that $p_{a_5,a_4}(X_{a_5})\subset\cup\{U_e: e\in E'\}$, and then there exists $a_2>a_5$ such that $\{p_{a_6,a_5}^{-1}U_e: e\in E'\}>\mathcal{U}_{a_6}$.

We will show that the admissible pair (a_6, b_4) has the required property. To do so we take any $(\Delta^{k+1}, \partial\Delta^{k+1})\in(D, S)_n$ and any maps $g: \Delta^{k+1}\to Y_{b_4}$, $h: \partial\Delta^{k+1}\to X_{a_6}$ such that $g|\partial\Delta^{k+1}=f_{b_4}p_{a_6,f(b_4)}h$. Take $\mathcal{W}\in\mathcal{C}ov(\Delta^{k+1})$ such that $\mathcal{W}<g^{-1}(\mathcal{V}_{b_4})$ and $\mathcal{W}|\partial\Delta^{k+1}<h^{-1}\mathcal{U}_{a_6}$. There exists a simplicial complex K and a subcomplex L of K such that $|K|=\Delta^{k+1}$, $|L|=\partial\Delta^{k+1}$, L is a full subcomplex of K and for each simplex s of K there exists $W_s\in\mathcal{W}$ such that $|s|\subset W_s$. Take any simplex s of K. By the choices of b_4 and W_s there exist $V_s\in\mathcal{V}_{b_4}$ and

$m(s)\in M'$ such that $g(W_s)\subset V_s$ and $q_{b_4,b_2}(V_s)\subset G(V_{m(s)})$. Thus

(4) $\qquad q_{b_4,b_2}g(|s|)\subset G(V_{m(s)})$ for each simplex s of K.

Since $V_{m(s)}\cap q_{b_2}(Y)\neq\emptyset$, by the choice of b_2 there exists $y(s)\in Y$ such that $V_{y(s)}\supset q_{b_2}^{-1}V_{m(s)}$, and then there exists $x(s)\in X$ such that $x(s)\in f^{-1}q_{b_2}^{-1}V_{m(s)}\subset f^{-1}V_{y(s)}$. Then by (1)

(5) $\qquad p_{a_3}(x(s))\in st(p_{a_3}(D_{y(s)}), \mathcal{V}_{a_3})$ for each simplex s of K.

Now we define a map $g^0: K^0\cup L\to X_{a_2}$ as follows: $g^0(z)=p_{a_6,a_2}h(z)$ for $z\in L$ and $g^0(z)=p_{a_2}(x(v))$ for $z=v\in K^0-L$. It is easy to show that g^0 is well-defined and continuous.

Claim 2. For each simplex s of K there exists $y\in Y$ such that $g^0(s\cap(K^0\cup L))\subset st(p_{a_2}(D_y), \mathcal{V}_{a_2})$.

To prove Claim 2 we take any simplex $s=[v_0, v_1, \cdots, v_p]$ of K. By (4) for each i, $0\leqq i\leqq p$, $q_{b_4,b_2}g(v_i)\in G(V_{m(s)})\cap G(V_{m(v_i)})\neq\emptyset$ and then by (2) $V_{m(s)}\cap V_{m(v_i)}\cap q_{b_2}(Y)\neq\emptyset$. Since $V_{m(s)}\cap V_{m(v_i)}\cap q_{b_2}(Y)\neq\emptyset$, $V_{y(s)}\cap V_{y(v_i)}\supset q_{b_2}^{-1}(V_{m(s)}\cap V_{m(v_i)})\neq\emptyset$ and then by (1) $st(p_{a_3}(D_{y(s)}), \mathcal{V}_{a_3})\cap st(p_{a_3}(D_{y(v_i)}), \mathcal{V}_{a_3})\supset p_{a_3}(f^{-1}(V_{y(s)}\cap V_{y(v_i)}))\neq\emptyset$, that is,

(6) $\qquad st(p_{a_3}(D_{y(s)}), \mathcal{V}_{a_3})\cap st(p_{a_3}(D_{y(v_i)}), \mathcal{V}_{a_3})\neq\emptyset$ for $0\leqq i\leqq p$.

We consider the following cases: (i) All vertexes of s are in K^0-L. (ii) All vertexes of s are in L. (iii) Some vertexes of s are in K^0-L and some vertexes of s are in L.

We consider the case (i). In this case $s\cap(K^0\cup L)=\{v_0, v_1, \cdots, v_p\}$ and then $g^0(s\cap(K^0\cup L))=\{p_{a_2}(x(v_0)), p_{a_2}(x(v_1)), \cdots, p_{a_2}(x(v_p))\}$. Since $p_{a_3}(x(v_i))\in st(p_{a_3}(D_{y(v_i)}), \mathcal{V}_{a_3})$ for $i=0, 1, \cdots, p$ by (5), then by (6) $\{p_{a_3}(x(v_i)): i=0, 1, \cdots, p\}\subset st\{st(p_{a_3}(D_{y(s)}), \mathcal{V}_{a_3}), \mathcal{D}_{a_3}\}$. By the choice of a_3 there exists $y''(s)\in Y$ such that $p_{a_3,a_2}(st\{st(p_{a_3}(D_{y(s)}), \mathcal{V}_{a_3}), \mathcal{D}_{a_3}\})\subset st(p_{a_2}(D_{y''(s)}), \mathcal{V}_{a_2})$. Hence $g^0(s\cap(K^0\cup L))\subset st(p_{a_2}(D_{y''(s)}), \mathcal{V}_{a_2})$.

Next we consider the case (ii). Since L is a full subcomplex of K, $s\in L$ and then $g^0(s\cap(K^0\cup L))=p_{a_6,a_2}h(s)$. By the choice of W_s there exists $U_s\in\mathcal{V}_{a_6}$ such that $h(s)\subset h(W_s\cap L)\subset U_s$. By the choice of a_6 there exists $e(s)\in E'$ such that $p_{a_6,a_4}(U_s)\subset U_{e(s)}$. Since $U_{e(s)}\cap p_{a_4}(X)\neq\emptyset$, there exists $x'(s)\in X$ such that $p_{a_4}(x'(s))\in U_{e(s)}$. Put $y'(s)=fx'(s)\in Y$ and then $x'(s)\in D_{y'(s)}$. Then we have

(7) $\qquad p_{a_6,a_4}h(s)\subset st(p_{a_4}(x'(s)), \mathcal{V}_{a_4})\subset st(p_{a_4}(D_{y'(s)}), \mathcal{V}_{a_4})$

and then

$$g^0(s\cap(K^0\cup L))=p_{a_6,a_2}h(s)\subset p_{a_4,a_2}st(p_{a_4}(D_{y'(s)}), \mathcal{V}_{a4})\subset st(p_{a_2}(D_{y'(s)}), \mathcal{V}_{a_2}).$$

Finally we consider the case (iii). In this case we may assume that $v_0, v_1, \cdots, v_u$ are vertexes in K^0-L and $v_{u+1}, v_{u+2}, \cdots, v_p$ are vertexes in L for some u, $0\leqq u\leqq p$. Since L is full, $t=[v_{u+1}, v_{u+2}, \cdots, v_p]\in L$ and then $s\cap(K^0\cup L)=\{v_0, v_1, \cdots, v_u\}\cup t$. Since $t\in L$, by the case (ii) $p_{a_6,a_4}h(t)\subset U_{e(t)}$ and $p_{a_4}x'(t)\in U_{e(t)}$. Since $(f_{b_4}p_{a_4,f(b)})^{-1}\mathcal{V}_{b_4}>\mathcal{U}_{a_4}$, there exists $V'_t\in\mathcal{V}_{b_4}$ such that $f_{b_4}p_{a_4,f(b_4)}(U_{e(t)})\subset V'_t$. Since $g|L=f_{b_4}p_{a_6,f(b_4)}h|L$, $f_{b_4}p_{f(b_4)}x'(t)\in V'_t$ and $g(t)=f_{b_4}p_{a_6,f(b_4)}h(t)\subset V'_t$, Since $g(t)\subset g(s)\subset g(W_s)\subset V_s$, $V_s\cap V'_t\supset g(t)\neq\emptyset$. By the choice of b_4 there exists $m_0(t)\in M'$ such that $q_{b_4,b_2}V'_t\subset G(V_{m_0(t)})$. Since $q_{b_4,b_2}V_s\subset G(V_{m(s)})$, $G(V_{m_0(t)})\cap G(V_{m(s)})\supset q_{b_4,b_2}(V_s\cap V'_t)\neq\emptyset$. By (3), $V_{m(s)}\cap V_{m_0(t)}\cap\overline{q_{b_2}(Y)}\neq\emptyset$ and then $V_{m(s)}\cap V_{m_0(t)}\cap q_{b_2}(Y)\neq\emptyset$. By the choice of b_2 there exists $y_0(t)\in Y$ such that $V_{y_0(t)}\supset q_{b_2}^{-1}V_{m_0(t)}$. Since $V_{y(s)}\cap V_{y_0(t)}\supset q_{b_2}^{-1}(V_{m(s)}\cap V_{m_0(t)})\neq\emptyset$, by (1) $st(p_{a_3}(D_{y(s)}, \mathcal{U}_{a_3})\cap st(p_{a_3}(D_{y_0(t)}), \mathcal{U}_{a_3})\supset p_{a_3}f^{-1}(V_{y(s)}\cap V_{y_0(t)})\neq\emptyset$, that is,

(8) $$st(p_{a_3}(D_{y(s)}), \mathcal{U}_{a_3})\cap st(p_{a_3}(D_{y_0(t)}), \mathcal{U}_{a_3})\neq\emptyset\,.$$

Since f satisfies (RM2), $q_{b_4}f=f_{b_4}p_{f(b_4)}$ and then by (8) $q_{b_4}fx'(t)\in V'_t$. By the choice of $m_0(t)$, $q_{b_2}fx'(t)\in G(V_{m_0(t)})$ and then by (2) $q_{b_2}fx'(t)\in G(V_{m_0(t)})\cap\overline{q_{b_2}(Y)}=V_{m_0(t)}\cap\overline{q_{b_2}(Y)}$. Thus $y'(t)=fx'(t)\in q_{b_2}^{-1}V_{m_0(t)}\subset V_{y_0(t)}$ and then by (1) $p_{a_3}(D_{y'(t)})\subset p_{a_3}f^{-1}V_{y_0(t)}\subset st(p_{a_3}(D_{y_0(t)}), \mathcal{U}_{a_3})$. This means that

(9) $$st(p_{a_3}(D_{y_0(t)}), \mathcal{U}_{a_3})\cap st(p_{a_3}(D_{y'(t)}), \mathcal{U}_{a_3})\neq\emptyset\,.$$

Since $t\in L$, by (5)-(9) $\{p_{a_3}x(v_i): i=u+1, \cdots, p\}\cup p_{a_6,a_3}h(t)\subset\cup\{st(p_{a_3}(D_{y(v_i)}), \mathcal{U}_{a_3}): i=u+1, \cdots, p\}\cup st(p_{a_3}(D_{y'(t)}), \mathcal{U}_{a_3})\subset st(st(p_{a_3}(D_{y(s)}), \mathcal{U}_{a_3}), st\mathcal{D}_{a_3})$. By the choice of a_3 there exists $y\in Y$ such that $p_{a_3,a_2}st(st(p_{a_3}(D_{y(s)}), \mathcal{U}_{a_3}), st\mathcal{D}_{a_3})\subset st(p_{a_2}(D_y), \mathcal{U}_{a_2})$. Thus $g^0(s\cap(K^0\cup L))=\{p_{a_2}x(v_i): i=0, 1, \cdots, u\}\cup p_{a_6,a_2}h(t)\subset st(p_{a_2}(D_y), \mathcal{U}_{a_2})$. Thus we have completed the proof of Claim 2.

Since $(K, K^0\cup L)\in{}^{n+1}\mathbf{POL}_{\text{pairs}}$, by Claim 2 and the choice of a_2 there exists a map $g^1: K\to X_{a_1}$ satisfying the following conditions:

(10) $g^1|K^0\cup L=p_{a_2,a_1}g^0$ and

(11) for each simplex s of K there exists $y\in Y$ such that $g^1(s)\subset st(p_{a_1}(D_y), \mathcal{U}_{a_1})$.

Claim 3. $(f_{b_1}p_{a_1,f(b_1)}g^1, q_{b_4,b_1}g)<st^2\mathcal{V}_{b_1}$.

Take any $z\in K$ and then there exists a simplex s of K with $z\in s$. First we assume that $s\in L$. In this case by the choices of a_4, g and (10), $f_{b_1}p_{a_1,f(b_1)}g^1|s=f_{b_1}p_{a_1,f(b_1)}p_{a_2,a_1}g^0|s=f_{b_1}p_{a_6,f(b_1)}h|s=q_{b_4,b_1}f_{b_4}p_{a_6,f(b_4)}h|s=q_{b_4,b_1}g|s$. Thus Claim 3 holds in this case.

Next we assume that $s\notin L$. We put $s=[v_0, v_1, \cdots, v_p]$. Since L is full, we may assume that $v_0 \in K^0 - L$. By (11) there exists $y \in Y$ such that $g^1(s) \subset st(p_{a_1}(D_y), \mathcal{U}_{a_1})$. By (RM2) for $\boldsymbol{f}$

(12) $$q_{b_1} f = f_{b_1} p_{f(b_1)}.$$

Since $f_{b_1} p_{a_1, f(b_1)} p_{a_1}(D_y) = q_{b_1}(y)$ by (12) and $(f_{b_1} p_{a_1, f(b_1)})^{-1} \mathcal{V}_{b_1} > \mathcal{U}_{a_1}$, $f_{b_1} p_{a_1, f(b_1)} g^1(s) \subset f_{b_1} p_{a_1, f(b_1)}(st(p_{a_1}(D_y), \mathcal{U}_{a_1})) \subset st(q_{b_1}(y), \mathcal{V}_{b_1})$. Since z, $v_0 \in s$,

(13) $$f_{b_1} p_{a_1, f(b_1)} g^1(z),\ f_{b_1} p_{a_1, f(b_1)} g^1(v_0) \in st(q_{b_1}(y), \mathcal{V}_{b_1}).$$

By (10) and (12), $f_{b_1} p_{a_1, f(b_1)} g^1(v_0) = q_{b_1} f(x(v_0))$. Since $V_{m(s)}$, $V_{m(v_0)} \in \mathcal{V}_{b_2}$, there exist $V_1, V_2 \in \mathcal{V}_{b_1}$ such that $q_{b_2, b_1}(V_{m(s)}) \subset V_1$ and $q_{b_2, b_1}(V_{m(v_0)}) \subset V_2$. By the choice of $x(v_0)$, $q_{b_2} f(x(v_0)) \in V_{m(v_0)}$ and then

(14) $$f_{b_1} p_{a_1, f(b_1)} g^1(v_0) \in V_2.$$

By (2), (4) and the choice of V_1, $q_{b_2, b_1} q_{b_4, b_2} g(s) \subset q_{b_2, b_1} G(V_{m(s)}) \subset q_{b_2, b_1} V_{m(s)} \subset V_1$ and then

(15) $$q_{b_4, b_1} g(z),\ q_{b_4, b_1} g(v_0) \in V_1.$$

By (2), (4) and the choice of V_2, $q_{b_4, b_1} g(v_0) \in q_{b_2, b_1} G(V_{m(v_0)}) \subset q_{b_2, b_1}(V_{m(v_0)}) \subset V_2$, that is,

(16) $$q_{b_4, b_1} g(v_0) \in V_2.$$

From (13)-(16), $f_{b_1} p_{a_1, f(b_1)} g^1(z)$, $q_{b_4, b_1} g(z) \in st(st(V_2, \mathcal{V}_{b_1}), st\mathcal{V}_{b_1})$. Thus in this case we have the required condition and hence we have Claim 3.

We put $G = p_{a_1, a} g^1 : \Delta^{k+1} \to X_a$. By the choices of a_1, b_1, (10) and Claim 3, $(f_b p_{a, f(b)} G, q_{b_4, b} g) < st\mathcal{V}_b$ and $G | \partial \Delta^{k+1} = p_{a_6, a} h$. Thus f satisfies (ALP**) with respect to $(D, S)_n$ and hence we have Claim 1.

By Claim 1 and (3.1) $\boldsymbol{f}$ has ALP with respect to $(D, S)_n$. Thus f is approximatively n-connected. Hence we have completed the proof. ■

(6.4) COROLLARY. *Let Y be a paracompact space and $f : X \to Y$ a closed onto map with (♯). If $f^{-1}(y)$ is approximatively n-connected for each $y \in Y$, then we have the following:*

(i) *pro-$\pi_k(f)$: pro-$\pi_k(X, x) \to$ pro-$\pi_k(Y, f(x))$ is an isomorphism for $0 \leqq k \leqq n$ and epimorphism for $k = n+1$ in pro-groups for each $x \in X$.*

(ii) *$\check{\pi}_k(f) : \check{\pi}_k(X, x) \to \check{\pi}(Y, f(x))$ is an isomorphism for $0 \leqq k \leqq n$ and for each $x \in X$.*

(iii) *If $n+1 = Max(\mathrm{sd}\, X + 1, \mathrm{sd}\, Y) < \infty$, then $f : (X, x) \to (Y, f(x))$ induces a shape equivalence for each $x \in X$.*

(6.5) COROLLARY. *Let $f: X \to Y$ be a closed onto map from a paracompact space X. If $f^{-1}(y)$ is approximatively n-connected for each $y \in Y$, then* (i)-(iii) *in* (6.4) *hold.*

(6.4) follows from (3.4), (3.6), (3.7) and (6.3). (6.5) follows from (6.5) and Michael's Theorem (see Engelking [9, p. 385]). ■

We say that $f: X \to Y$ is a cell-like map, in notation CE-map, provided that $f^{-1}(y)$ has trivial shape for any $y \in Y$.

(6.6) COROLLARY. *Let $f: X \to Y$ be a closed map from a paracompact space X. If f is a CE-map, then we have the following:*

(i) *$pro\text{-}\pi_k(f): pro\text{-}\pi_k(X, x) \to pro\text{-}\pi_k(Y, f(x))$ is an isomorphism in pro-groups for each k and each $x \in X$.*

(ii) *$\check{\pi}_k(f): \check{\pi}_k(X, x) \to \check{\pi}_k(Y, f(x))$ is an isomorphism for each k and each $x \in X$.*

(iii) *If* $\mathrm{sd}\, X, \mathrm{sd}\, Y < \infty$, *$f: (X, x) \to (Y, f(x))$ induces a shape equivalence for each $x \in X$.*

(6.6) follows from (6.5). ■

(6.7) REMARK. Usually approximatively n-connected maps are called UV^n-maps (see Lacher [18]). Smale [30] and Kozlowski [13] studied these maps and showed special cases of theorems (6.4) and (6.3). Various Vietoris-Smale theorems in shape theory were studied by many authors, Bogatyi [2, 3], Dydak [4-7], Kodama [11, 12], Kuperberg [16], Kozlowski-Segal [15], Morita [27, 28]. Our results are the most general.

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