Equilibrium Existence and Fixed Point Theorems: Equivalence Results

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Abstract We show that some of the major existence theorems in mathematical economics and game theory are equivalent to Brouwer's fixed point theorem. We establish new equivalences à la Uzawa for exchange economies and production economies. Social Equilibrium Existence Theorem is also equivalent to Brouwer's fixed point theorem. Moreover, we show that most of the existence theorems in game theory are included in the equivalence. Namely, Nash equilibrium existence, the core existence and NTU value existence. Our results tell us that the use of fixed point theorem in economics and game theory was a logical necessity rather than a coincidence.

1 Introduction

Since Arrow and Debreu (1954), McKenzie (1954), Nikaido (1956) and Gale (1955), Brouwer's fixed point theorem has played a decisive role in general equilibrium analysis. Uzawa (1962) further proved that Brouwer's fixed point theorem is equivalent to Walras' existence theorem which says that for any continuous function (interpreted as an excess demand function) mapping the price simplex into the commodity space and satisfying Walras' law, there exists an equilibrium price vector. This result is known as Uzawa's equivalence theorem.

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The importance of Uzawa's equivalence theorem comes from the interpretation that proving the existence of Walrasian equilibria is just as hard as proving Brouwer's fixed point theorem, one of the deepest results in topology. See Uzawa (1962), Nikaido (1968, §16) and Debreu (1982). Nonetheless, we are a bit reluctant to accept this authoritative view on the equivalence theorem in its present form when we take a closer look at Walras' existence theorem, following Arrow and Hahn (1971, p.29): The continuous excess demand function on the price simplex implies that the demand for free goods is bounded. If the function is generated by an exchange economy, these preferences must exhibit satiation in every commodity. Unless the satiation point lies above the endowment point for every consumer, Walras' law (in equality form) might fail at some prices. What is worse, a contradiction may be unavoidable if non-satiation is required. Thus, we find it very hard to regard the excess demand function as derived from primitive data such as preference relations and initial endowments. This observation leads us to reconsidering Uzawa's equivalence theorem. To wit, our objective is to establish equivalence between Brouwer's fixed point theorem and alternative versions of Walras' existence theorem stating that there exists an equilibrium for any excess demand function satisfying economically reasonable restrictions.

In exchange economies, it is relatively easy to identify the restrictions: Continuity on the relative interior of the price simplex, Walras' law, boundedness below and a boundary condition are derived from optimization behavior on the part of consumers. See Arrow and Hahn (1971, Ch.4, theorem 8). On the other hand, Debreu's (1974) theorem on the characterization of excess demand function tells us that a given excess demand function is approximated off the boundary of the domain by another one which can be generated by an exchange economy. Thus, a proper version of Walras' existence theorem for exchange economies seems to be that any excess demand function satisfying the four restrictions has at least one equilibrium. Our first equivalence theorem states that this version of Walras'existence theorem is equivalent to Brouwer's fixed point theorem. This equivalence theorem has a corollary with a far-reaching implication: Brouwer's fixed point theorem is equivalent to the statement that Walrasian equilibria exist for a class of exchange economies comprised of a finite number of consumers with positive initial endowments and continuous, monotone, strictly convex preferences. This equivalence result fully justifies our understanding that proving the existence of Walrasian equilibria is just as hard as proving Brouwer's fixed point theorem. Our argument draws heavily on a powerful theorem of Mas-Colell (1977).

In production economies, we need to take boundary behavior of supply into account. See Artzner and Neuefeind (1978) and Neuefeind (1980). In general, an excess demand function may be unbounded from below. Thus our arguments for lower bounded excess demand functions are no longer in force. To get around the difficulty, we need to develop a novel approach. Our equivalence theorem for production economies says that Brouwer's fixed point theorem is equivalent to yet another version of Walras' existence theorem stating that an equilibrium exists for any excess demand function satisfying continuity, Walras' law and the boundary condition used by Arrow and Hahn (1971, Ch. 2). This seems like a fairly satisfactory result, considering the situation that there is no analog of Debreu's (1974) theorem for characterizing excess demand functions for production economies.

After obtaining our major results, we learned that Nishimura (1978a, Remark 2), along with a reference to Nishimura (1978b, Theorem 1), suggested our first equivalence theorem. But he did not state what was exactly his version of equivalence theorem so that it is hard to tell what kind of argument he had in mind. More important, what makes our work beyond Nishimura's suggestion is to realize that there is a link between Uzawa's equivalence theorem and the literature on the characterization of excess demand functions. Furthermore, Nishimura's suggestion is concerned with only excess demand functions with lower bounds.

Nash (1950) proved the existence of non-cooperative equilibrium for a normal form game by Kakutani's fixed point theorem. Debreu (1952) introduced abstract economies and proved the existence of social equilibrium by Eilenberg-Montgomery fixed point theorem, a generalization of Kakutani's fixed point theorem. The social equilibrium existence theorem served as the basis for the existence theorems of Arrow and Debreu (1954). It is well-known that Scarf's (1967) core existence theorem can be proved by Brouwer-type fixed point theorems. See Zhou (1994), for example. Debreu and Scarf (1963) tells us that the core has close ties with Walrasian equilibria. Furthermore, Shapley's (1969) NTU (Non-transferable Utility) value existence theorem is implied by Kakutani's fixed point theorem. Thanks to Champsaur's (1975) limit theorem, the NTU value is also closely related to Walrasian equilibria. Thus, the second objective in this paper is concerned with how to assess the technical overlap between general equilibrium theory and game theory. We do this by establishing several equivalence theorems. Our equivalence theorem for exchange economies turns out to be instrumental. To begin with, we show that social equilibrium existence theorem is equivalent to Brouwer's fixed point theorem. We use this result to demonstrate that the existence theorems of Dasgupta and Maskin (1986) for Nash equilibria are equivalent to Brouwer's fixed point theorem. This is remarkable because their theorems permit discontinuous payoff functions. Finally, we establish that both Scarf's core existence theorem and Shapley's NTU value existence theorem are equivalent to Brouwer's fixed point theorem.

The paper is organized as follows. In Section 2, we present the equivalence theorem for exchange economies and its corollary. In Section 3, we present the equivalence theorem for production economies. In Sections 4 and 5, we collect equivalence theorems exhibiting close relationships between general equilibrium theory and game theory. The last section contains a few concluding remarks.

2 Equivalence Theorems for Exchange Economies

For our purpose, it is convenient to use notation and terminology in Mas-Colell (1977) as much as possible. The commodity space is the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, whose generic element is denoted by $x = (x_1, \cdots, x_\ell)$. The consumption set of all consumers is $\mathbb{R}^\ell_+$, the non-negative orthant of $\mathbb{R}^\ell$. The interior of $\mathbb{R}^\ell_+$ is denoted by $\mathbb{R}^\ell_{++}$. The (normalized) price space is $P = \{p \in \mathbb{R}^\ell_{++} \mid p \cdot e = 1\}$, where

$e = (1, \ldots, 1) \in \mathbb{R}^\ell$ and $p \cdot e$ denotes the inner product of p and e. The closure and relative boundary of P is denoted by $\bar{P}$ and by ∂P, respectively. We sometimes identify the price space P with $S = \{p \in \mathbb{R}^\ell_{++} \mid ||p|| = 1\}$, where $||p||$ is the Euclidean norm of p. This is no harm since $p \to \frac{p}{||p||}$ maps P homeomorphically onto S. This identification is useful for application of Mas-Colell's theorem. For any $\varepsilon > 0$, let $S_\varepsilon = \{p \in S \mid p_i \geqq \varepsilon \text{ for all } i = 1, \ldots, \ell\}$.

We say that a map $f : P \to \mathbb{R}^\ell$ is an excess demand function if it satisfies the following conditions;

(C) $f : P \to \mathbb{R}^\ell$ is continuous,
(W) $p \cdot f(p) = 0$ for every $p \in P$,
(BB) There is $k \in \mathbb{R}$ such that for every $p \in P$, $f(p) \geq ke$,
(BC1) If $p_n \to p \in \partial P$ and $p_n \in P$, then $\lim_{n\to\infty} ||f(p_n)|| = +\infty$.

For every excess demand function f, let $E_f = \{p \in P | f(p) = 0\}$. This is the equilibrium price set for f. Let $\mathscr{P}$ be the set of all continuous, monotone, strictly convex, complete preorderings on $\mathbb{R}^\ell_+$. The initial endowment space is $\mathbb{R}^\ell_{++}$. Hence the space of possible consumer characteristics is $\mathscr{P} \times \mathbb{R}^\ell_{++}$. An exchange economy in this paper is a finite collection of elements in $\mathscr{P} \times \mathbb{R}^\ell_{++}$, denoted by $\mathscr{E} = \{(\succeq_j, \omega_j)\}_{j=1}^m$. An exchange economy $\mathscr{E} = \{(\succeq_j, \omega_j)\}_{j=1}^m$ generates a given excess demand function f if for all $p \in P$,

$$f(p) = \sum_{j=1}^m \varphi(p, \succeq_j, \omega_j) - \sum_{j=1}^m \omega_j, \tag{1}$$

where

$$\varphi(p, \succeq_j, \omega_j) = \{x \in \mathbb{R}^\ell_+ \mid p \cdot x \leq p \cdot \omega_j \text{ and } x \succeq_j y \text{ for all } y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_j\}.$$

The function defined by the right hand side of equation (1) is indeed an excess demand function. See Arrow and Hahn (1971, Ch.4, Theorem 8). Hence this definition is consistent with that of an excess demand function. We are concerned with excess demand functions which can be generated by some exchange economies. Now, let us review what Uzawa (1962) called Walras' Existence Theorem.

Theorem (Walras' Existence Theorem A) *Let a function $\bar{f} : \bar{P} \to \mathbb{R}^\ell$ satisfy the following conditions:*

($\bar{\mathrm{C}}$) $\bar{f} : \bar{P} \to \mathbb{R}^\ell$ *is continuous,*
($\bar{\mathrm{W}}$) $p \cdot \bar{f}(p) = 0$ *for all* $p \in \bar{P}$.

Then, there exists $p^* \in \bar{P}$ such that $\bar{f}(p^*) \leq 0$.

Uzawa (1962) calls the function $\bar{f}$ an excess demand function. But, observe that ($\bar{\mathrm{C}}$) implies that $\bar{f}$ (restricted to P) cannot be an excess demand function in our sense since it violates (BC1). Hence $\bar{f}$ cannot be generated by any exchange economy. Thus Uzawa's Equivalence Theorem, establishing equivalence between

Walras' Existence Theorem A and Brouwer's fixed point theorem, cannot fully support the interpretation that proving the existence of Walrasian equilibria is just as hard as proving Brouwer's fixed point theorem. To justify this interpretation, we need to reformulate Uzawa's Equivalence Theorem. Toward this end, we state an alternative existence theorem.

Theorem (Walras' Existence Theorem B) *If $f : P \to \mathbb{R}^\ell$ is an excess demand function, then we have $E_f \neq \emptyset$.*

Our first equivalence theorem is the following.

Theorem 1 *Walras' Existence Theorem B and Brouwer's fixed point theorem are equivalent.*

Proof It is well-known that Brouwer's fixed point theorem implies Walras' Existence Theorem B. To prove the converse, we show that Walras' Existence Theorem B implies Walras' Existence Theorem A, which is equivalent to Brouwer's fixed point theorem by Uzawa (1962). Let $\bar{f} : \bar{P} \to \mathbb{R}^\ell$ be any function satisfying ($\bar{\text{C}}$) and ($\bar{\text{W}}$). We wish to show that there exists $p^* \in \bar{P}$ such that $\bar{f}(p^*) \leq 0$. Pick some excess demand function $g : P \to \mathbb{R}^\ell$, say $g(p) = (\frac{1}{\ell p_1} - 1, \ldots, \frac{1}{\ell p_\ell} - 1)$. For each $n \in N$, define

$$h^n(p) = \bar{f}(p) + \frac{1}{n} g(p) \text{ for all } p \in P.$$

Then, it is easy to see that h^n is an excess demand function. By Walras' Existence Theorem B, there exists $p^n \in P$ such that

$$0 = h^n(p^n) = \bar{f}(p^n) + \frac{1}{n} g(p^n).$$

By compactness of $\bar{P}$, we may assume that $p^n \to p^* \in \bar{P}$. Since g is bounded from below by $-e$ and $\bar{f}$ is continuous on $\bar{P}$, we get

$$\bar{f}(p^*) = \lim_{n\to\infty} \bar{f}(p^n) = \lim_{n\to\infty} -\frac{1}{n} g(p^n) \leq \lim_{n\to\infty} \frac{1}{n} e = 0.$$

This completes the proof. □

Before asking further questions, let us formulate yet another existence theorem.

Theorem (Walras' Existence Theorem C) *For any excess demand function f generated by some exchange economy, we have $E_f \neq \emptyset$.*

At this stage, we need to recall that whether every excess demand function can be generated by some exchange economy is an open question. This suggests that investigating the equivalence between Brouwer's fixed point theorem and Walras' Existence Theorem C is a non-trivial task. We give an affirmative answer by invoking Mas-Colell's theorem (1977) and demonstrate that Brouwer's fixed point theorem is necessary and sufficient to lay a solid foundation for general equilibrium analysis.

Corollary 1 *Walras' Existence Theorem C is equivalent to Brouwer's fixed point theorem.*

Proof It is sufficient to prove that Walras' Existence Theorem C implies Walras' Existence Theorem B. Let $f : P \to \mathbb{R}^\ell$ be an excess demand function. We may identify the domain P with S via the homeomorphism $p \to \frac{p}{\|p\|}$. Pick any $\varepsilon > 0$. By Mas-Colell's theorem (1977), there exist $\mu \in (0, \varepsilon)$ and an economy whose excess demand function f^* satisfies

$$f^* \equiv f \text{ over } S_\mu \text{ and } E_{f^*} = E_f \subset S_\mu.$$

By Walras' Existence Theorem C, we have $E_{f^*} \neq \emptyset$ and hence $E_f \neq \emptyset$. □

3 Equivalence Theorem for Production Economies.

In production economies, excess demand functions may be unbounded below when the underlying production sets are unbounded above. Our argument in the previous section is no longer available in this context since it crucially depends on the lower boundedness property of excess demand functions. We need to employ novel arguments for equivalence theorem à la Uzawa for production economies. Since there is no production economy counterpart of Mas-Colell's theorem (1977), the equivalence theorem is stated in terms of excess demand functions. We follow Neuefeind (1980) for the formulation of production economies. We say that a function $f : D \to \mathbb{R}^\ell$ is an excess demand function for a production economy if it satisfies the following conditions;

(D) D is a non-empty, convex, open subset of P,
(C) $f : D \to \mathbb{R}^\ell$ is continuous,
(W) $p \cdot f(p) = 0$ for every $p \in D$,
(UBB) f is unbounded below, i.e. there exist $p \in \partial D$, a sequence $\{p^n\}$ in D with $p^n \to p$, and a commodity i such that $f_i(p^n) \to -\infty$,
(BC2) There exists $\bar{p} \in D$ such that $\bar{p} \cdot f(p^n) \to +\infty$ for any sequence $\{p^n\}$ in D with $p^n \to p \in \partial D$.

To motivate the conditions, let Y be a production set satisfying the following conditions;

Y is a non-empty, closed and strictly convex subset of $\mathbb{R}^\ell$ containing 0 and allowing free disposal, i.e., $-\mathcal{R}^\ell_+ + Y \subset Y$.

For any $p \in \mathbb{R}^\ell$, define the supply set,

$$s(p) = \{y \in Y \mid p \cdot y = \sup p \cdot Y\}$$

and let

$$D = \{p \in \mathbb{R}^\ell \mid \sum_{i=1}^{\ell} p_i = 1 \text{ and } s(p) \neq \emptyset\}.$$

It follows from Neuefeid (1980, Proposition) that condition (D) holds and that if $\{p^n\}$ is a sequence in D with $p^n \to p \in \partial D$, then the supply function obeys the following boundary behavior:

There exists a commodity i such that $s_i(p^n) \to +\infty$ and for each $\bar{p} \in D$, there exists $\varepsilon > 0$ such that $\bar{p} \cdot s(p^n) < -\varepsilon \|s(p)\|$ for sufficiently large n.

To complete the description of the economy, let $(X_j, \succeq_j, \omega_j)$ be the characteristics of consumer $j = 1, \cdots, m$ where,

(i) $X_j \subset \mathbb{R}^\ell$ is closed, convex, bounded below, and satisfies $X_j + \mathbb{R}^\ell_+ \subset X_j$,
(ii) $\succeq_j$ is a continuous, strictly convex, monotone, complete preordering on X_j,
(iii) $\omega_j \in \text{int}X_j$.

For any $(p, w) \in \mathbb{R}^\ell \times \mathbb{R}$, define the demand set of consumer j by,

$$\varphi_j(p, w) = \{x \in X_j \mid p \cdot x = w \text{ and } x \succeq_j y \text{ for all } y \in X_j \text{ with } p \cdot y \leq w\}.$$

Let θ_j be the profit share of j: $\theta_j \geq 0$, $\sum_{j=1}^m \theta_j = 1$. Finally, let

$$f(p) = \sum_{j=1}^m \varphi_j(p, p \cdot \omega_j + \theta_j \pi(p)) - s(p) - \sum_{j=1}^m \omega_j,$$

where $\pi(p) = \sup p \cdot Y$ and $p \in D$.

Then, it follows from the standard arguments that (W) and (C) hold. Condition (BC2) follows from lower boundedness of demand and the boundary behavior of supply mentioned above. There is a possibility that f is bounded below because boundary behavior of demand may completely offset that of supply. To exclude this unusual phenomena, suppose there exists $p \in \partial D \cap P$ with $\pi(p) < +\infty$. Then, condition (UBB) holds. Indeed, let $q \in D$ and let $p^n = \frac{1}{n}q + (1 - \frac{1}{n})p$ for $n \in N$. By convexity of the support function π, $\pi(p^n) \leq \frac{1}{n}\pi(q) + (1 - \frac{1}{n})\pi(p)$ for all n. Hence the sequence $\{(p^n, \ p^n \cdot \omega_j + \theta_j \pi(p^n))\}$ is contained in a compact subset of $\{(p, w) \in P \times \mathbb{R} \mid w > \inf p \cdot X_j\}$. Since d_j is continuous there, $\{d_j(p_n, p_n \cdot \omega_j + \theta_j \pi(p_n))\}$ is bounded. On the other hand, the boundary behavior of supply implies that there exists a commodity i such that $s_i(p_n) \to +\infty$. Hence, $f_i(p_n) \to -\infty$ and condition (UBB) holds.

We first formulate Walras' Existence Theorem in production economies.

Theorem (Walras' Existence Theorem D) *If f is an excess demand function for a production economy, $E_f \neq \emptyset$.*

Our second equivalence theorem is the following.

Theorem 2 *Walras' Existence Theorem D is equivalent to Brouwer's fixed point theorem.*

To prove theorem 2, we need a lemma due to Neuefeind (1980).

Lemma 1 *Let f be an excess demand function for a production economy such that there exists $\bar{p} \in D$ such that the set $\{p \in D \mid \bar{p} \cdot f(p) \leq 0\}$ has positive distance from ∂D. Then, $E_f \neq \emptyset$.*

Proof See Neuefeind (1980, Lemma 1). Note that this result is derived from Brouwer's fixed point theorem. □

Proof of Theorem 2 First, we prove that Brouwer's fixed point theorem implies Walras' Existence Theorem D. Let f be an excess demand function for a production economy and let $\bar{p}$ be the price given in (BC2). By (BC2), the set $\{p \in D \mid \bar{p} \cdot f(p) \leq 0\}$ has positive distance from ∂D. By Lemma 1, $E_f \neq \emptyset$.

To prove the converse, let $\psi : \bar{P} \to \bar{P}$ be a continuous function. We shall show that ψ has a fixed point by means of Walras' Existence Theorem D. Pick $\eta > 0$ with $\frac{1}{\ell(\ell-1)} - \eta > 0$. For each $p \in \partial P$, we have

$$p \cdot p \geq \frac{1}{(\ell - 1)} > \frac{1}{\ell} = p \cdot c,$$

where $c = (\frac{1}{\ell}, \cdots, \frac{1}{\ell})$. This gives

$$p \cdot p - p \cdot c > \frac{1}{\ell(\ell - 1)} - \eta.$$

Therefore, for each $p \in \partial P$, there exist an open neighborhood N_p of p and an open neighborhood $N_c(p)$ of c such that for all $p' \in N_p$ and for all $z \in N_c(p)$,

$$p' \cdot p' - p' \cdot z > \frac{1}{\ell(l - 1)} - \eta > 0.$$

It follows from compactness of ∂P that there exists a finite subset $\{p^1, \cdots, p^K\}$ of ∂P such that $\partial P \subset \cup_{j=1}^{K} N_{p^j} \equiv \mathcal{N}(\partial P)$. Let $N_c \equiv \cap_{j=1}^{K} N_c(p^j)$. Then, by definition, for all $p' \in \mathcal{N}(\partial P)$ and for all $z \in N_c$,

$$p' \cdot p' - p' \cdot z > \frac{1}{\ell(\ell - 1)} - \eta > 0 \tag{2}$$

For $\varepsilon \in (0, \frac{1}{\ell})$, define

$$P_\varepsilon = \{p \in P \mid p_i \geq \varepsilon \text{ for all } i = 1, \cdots, \ell\}.$$

We can choose ε sufficiently near $\frac{1}{\ell}$ so that we have $P_\varepsilon \subset N_c$. Define $g : P_\varepsilon \to P$ by

$$g(p) = \frac{1}{1 - \ell\varepsilon}(p_1 - \varepsilon, \cdots, p_\ell - \varepsilon).$$

Then, g gives a homeomorphism between P_ε and $\bar{P}$. Let $\tilde{\psi} = g^{-1} \circ \psi \circ g$. Then $\tilde{\psi}$ maps P_ε continuously into itself. It is clear that $\tilde{\psi}$ has a fixed point if and only if ψ has a fixed point. Next, for each $p \in P$, let $\rho(p)$ be the unique point in P_ε which minimizes the Euclidean distance from p. Then, ρ is a well-defined continuous function with the property that if $p \in P_\varepsilon$, then $\rho(p) = p$. Now, define $h \equiv \tilde{\psi} \circ \rho$. Since h maps P into P_ε, a fixed point of h gives that of $\tilde{\psi}$, hence that of ψ. Let $\beta(p) = \max\{\frac{1}{p_1}, \cdots, \frac{1}{p_\ell}\}$ and let $D = P$. Define a continuous function $f : D \to \mathbb{R}^\ell$ by

$$f(p) = \beta(p)\{h(p) - (\frac{p \cdot h(p)}{p \cdot p})p\}.$$

Then, we have

$$\sum_{i=1}^{\ell} f_i(p) = \beta(p)\{1 - \frac{p \cdot h(p)}{p \cdot p}\} = \beta(p)\{\frac{p \cdot p - p \cdot h(p)}{p \cdot p}\}.$$

Let $p^n \to p \in \partial P$, $p^n \in P$. Then $p^n \in N(\partial P)$ for sufficiently large n. Since $h(p^n) \in P_\varepsilon$, it follows from equation (2) that $p^n \cdot p^n - p^n \cdot h(p^n) > \frac{1}{\ell(\ell-1)} - \eta$. Then, for sufficiently large n,

$$\beta(p^n)\{\frac{p^n \cdot p^n - p^n \cdot h(p^n)}{p^n \cdot p^n}\} > \frac{\beta(p^n)}{\ell(\ell-1)}.$$

Thus, we obtain $\sum_{i=1}^{\ell} f_i(p^n) \to +\infty$ and (BC2) holds for $\bar{p} = e$. Since $\beta(p) \to +\infty$ as $p \to \partial P$, f is unbounded below by Walras' Law. Therefore, f is an excess demand function for a production economy. By Walras' Existence Theorem D, there exists $p^* \in P$ such that $f(p^*) =$. Since $\beta(p^*) > 0$, we have

$$h_i(p^*) = \{\frac{p^* \cdot h(p^*)}{p^* \cdot p^*}\}p_i^*, \text{ for each } i = 1, \cdots, \ell.$$

Summing over i gives $\frac{p^* \cdot h(p^*)}{p^* \cdot p^*} = 1$. Thus, $h(p^*) = p^*$. Therefore, p^* is a fixed point of $\tilde{\psi}$. □

4 Equivalence Theorems for Nash Equilibria

Let us turn to game theoretical equilibrium concept: Nash equilibrium. Nash (1950) established an existence theorem for Nash equilibria by means of Kakutani's fixed point theorem. Dasgupta and Maskin (1986) further generalized the result by allowing discontinuous payoff functions. Our findings about Nash equilibria are new and somewhat striking: The existence theorems of Dasgupta and Maskin are in fact equivalent to Brouwer's fixed point theorem. It has been long recognized that general equilibrium theory and non-cooperative game theory share the same grounds via fixed point theorems. Yet, our equivalence theorems clearly say more than that: the technical overlap was not a mere coincidence but a logical necessity.

An abstract economy is a list $\xi = (I, S, \{u_i\}_{i\in I}, \{F_i\}_{i\in I})$, consisting of a finite player set I, $S = \Pi_{i\in I} S_i$, where S_i is a non-empty, compact, convex subset of some finite dimensional Euclidean space, representing the strategy set of player i, $u_i : S \to \mathbb{R}$, the payoff function of player i and a correspondence F_i from S_{-i} into S_i, where $S_{-i} = \Pi_{j\neq i} S_j$. F_i is the feasibility correspondence of player i. When none of F_i implies any restrictions, i.e., $F_i(s_{-i}) = S_i$ for all $s_{-i} \in S_{-i}$ and for all $i \in I$, ξ can be identified with a normal form game $\Gamma = (I, S, \{u_i\}_{i\in I})$. We say that $s^* \in S$ is a social equilibrium for an abstract economy ξ if s_i^* maximize $u_i(s_i, s_{-i}^*)$ subject to $s_i \in F_i(s_{-i}^*)$ for each $i \in I$. If, in addition, ξ is a normal form game, then we say that s^* is a Nash equilibrium.

Debreu (1952) first studied abstract economies and proved the following.

Theorem (S.E.E.Theorem) *There exists a social equilibrium for any abstract economy* $\xi = (I, S, \{u_i\}_{i \in I}, \{F_i\}_{i \in I})$ *satisfying the following conditions. For each* $i \in I$, u_i *is continuous and* $u_i(s_i, s_{-i})$ *is quasi-concave in* s_i *for each fixed* s_{-i}. *Furthermore,* F_i *is a compact, convex valued, continuous correspondence.*

The original version of the theorem does not require quasi-concavity of payoff functions but assumes that the best response correspondences are contractible valued. We took the above version from Debreu (1982).

The following theorem is a consequence of well-known results combined with Corollary 1.

Theorem 3 *S.E.E. Theorem is equivalent to Brouwer's fixed point theorem.*

Proof Debreu (1952) tells us that Kakutani's fixed point theorem implies S.E.E. Theorem. On the other hand, Arrow and Debreu (1954) shows that S.E.E. Theorem implies Walras' Existence Theorem C. It is well-known that Brouwer's fixed point theorem is equivalent to Kakutani's fixed point theorem (see Ichiishi (1983)). Since Walras' Existence Theorem C is equivalent to Brouwer's fixed point theorem by Corollary 1, the proof is complete. □

Let $\Gamma = (I, S, \{u_i\}_{i \in I})$ be a normal form game. The payoff function $u_i : S \to \mathbb{R}$ is graph-continuous if for each $\bar{s} \in S$, there exists a function $f_i : S_{-i} \to S_i$ such that $f_i(\bar{s}_{-i}) = \bar{s}_i$ and $u_i(f_i(s_{-i}), s_{-i})$ is continuous at $\bar{s}_{-i}$. Dasgupta and Maskin (1986) proved the following two theorems using Kakutani's fixed point theorem.

Theorem (D-M Theorem 1) *There exists a Nash equilibrium for any normal form game* $\Gamma = (I, S, \{u_i\}_{i \in I})$ *satisfying the following conditions. Each* u_i *is upper semi-continuous, graph-continuous and quasi-concave in* s_i.

Theorem (D-M Theorem 2) *There exists a Nash equilibrium for any normal form game* $\Gamma = (I, S, \{u_i\}_{i \in I})$ *satisfying the following conditions. Each* u_i *is upper semi-continuous and quasi-concave in* s_i *and the function* v_i *defined by*

$$v_i(s_{-i}) = \max\{u(s_i, s_{-i}) \mid s_i \in S_i\},$$

is lower semi-continuous.

These two would be the most general existence theorems for Nash equilibria (in pure strategies). Since graph continuity of u_i implies lower semi-continuity of v_i, D-M Theorem 2 implies D-M Theorem 1. The following theorem shows not only the converse holds but also D-M Theorem 1 implies S.E.E. Theorem.

Theorem 4 *Both D-M Theorems 1 and 2 are equivalent to Brouwer's fixed point theorem.*

Proof In view of Theorem 3 and the remark in the previous paragraph, it is sufficient to show that D-M Theorem 1 implies S.S.E. Theorem. Let us be given an abstract economy $\xi = (I, S, \{u_i\}_{i \in I}, \{F_i\}_{i \in I})$. Then, we can choose a real number λ

such that $\min_{i\in I}\min\{u_i(s) \mid s \in S\} > \lambda$. For each player i, define a modified payoff function $g_i : S \to \mathbb{R}$ by

$$g_i(s_i, s_{-i}) = \begin{cases} u_i(s_i, s_{-i}) & \text{if } s_i \in F_i(s_{-i}), \\ \lambda & \text{otherwise.} \end{cases}$$

We show that the resulting normal form game $\Gamma = (I, S, \{g_i\}_{i\in I})$ satisfies the conditions of D-M Theorem 1.

Let $s_i, s_i' \in S_i$, $0 < t < 1$, and $s_{-i} \in S_{-i}$. If $ts_i + (1-t)s_i' \in F(s_{-i})$, then we have

$$\begin{aligned} g_i(ts_i + (1-t)s_i', s_{-i}) &= u_i(ts_i + (1-t)s_i', s_{-i}) \\ &> \min\{u_i(s_i, s_{-i}), u_i(s_i', s_{-i})\} \\ &> \min\{g_i(s_i, s_{-i}), g_i(s_i', s_{-i})\}. \end{aligned}$$

If $ts_i + (1-t)s_i' \notin F_i(s_{-i})$, then we have either $s_i \notin F_i(s_{-i})$ or $s_i' \notin F_i(s_{-i})$ by convexity of $F_i(s_{-i})$. Hence,

$$g_i(ts_i + (1-t)s_i', s_{-i}) = \min\{g_i(s_i, s_{-i}), g_i(s_i', s_{-i})\} = \lambda.$$

Therefore, $g_i(s_i, s_{-i})$ is quasi-concave in s_i for each fixed s_{-i}.

Let $s^n \in S$ and $s^n \to s \in S$. If $s_i \in F_i(s_{-i})$, then $g_i(s) = u_i(s) = \limsup u_i(s^n) \geq \limsup g_i(s^n)$. If $s_i \notin F_i(s_{-i})$, then $s_i^n \notin F_i(s_{-i}^n)$ for sufficiently large n since F_i is compact-valued and upper semi-continuous. Hence, we have $\limsup g_i(s^n) = g_i(s) = \lambda$. Therefore, g_i is upper semi-continuous.

Now, we show that g_i is also graph-continuous. Let $\bar{s} \in S$. First, we consider the case $\bar{s}_i \in F_i(\bar{s}_{-i})$. Let $\tilde{F}_i(s_{-i}) = \{\bar{s}_i\}$ if $s_{-i} = \bar{s}_{-i}$ and let $\tilde{F}_i(s_{-i}) = F_i(s_{-i})$ otherwise. Since $\tilde{F}_i$ is closed and convex valued and lower semi-continuous, it admits a continuous selection $f_i : S_{-i} \to S_i$. See, for example, Hildenbrand and Kirman (1988, Mathematical Appendix IV). Then, $g_i(f_i(s_{-i}), s_{-i}) = u_i(f_i(s_{-i}), s_{-i})$ is clearly continuous in s_{-i} and $f_i(\bar{s}_{-i}) = \bar{s}_i$. If $\bar{s}_i \notin F_i(\bar{s}_{-i})$, then let $f_i(s_{-i}) = \bar{s}_i$ for all s_{-i}. Let $s_i^n \to \bar{s}_i$. Since F_i is compact-valued and upper semi-continuous, $\bar{s}_i \notin F_i(s_{-i}^n)$ for sufficiently large n. Hence, $g_i(f_i(s_{-i}^n), s_{-i}^n) = g_i(\bar{s}_i, s_{-i}^n) = \lambda = g_i(f_i(\bar{s}_{-i}), s_{-i})$ for sufficiently large n. Therefore, g_i is graph-continuous.

By D-M Theorem 1, there exists a Nash equilibrium $s^* \in S$ for Γ. Suppose $s^* \notin F_i(s_{-i}^*)$. Pick $s \in F_i(s_{-i}^*)$. Then $g_i(s_i^*, s_{-i}^*) = \lambda < u_i(s_i, s_{-i}^*) = g_i(s_i, s_{-i}^*)$, a contradiction. Hence, s^* is a social equilibrium for ξ. This completes the proof. □

5 Equivalence Theorems for the Core and Non-transferable Utility Value.

In this section, we discuss two major solution concepts for NTU (Non transferable Utility) games, i.e. the core and the NTU value. An NTU game is a triple (N, F, V) satisfying the following conditions:

(1) N is a finite set of players.
(2) F is a non-empty closed subset of $\mathbb{R}^N$.
(3) For each $S \in \mathcal{N} \equiv 2^N \setminus \{\emptyset\}$, $V(S)$ is a closed subset of $\mathbb{R}^N$.

(4) For each $S \in \mathcal{N}$, $V(S)$ is comprehensive, i.e. if $x \in V(S)$ and $y \leq x$, then $y \in V(S)$.
(5) For each $x \in V(N)$, there exists $y \in F$ with $x \leq y$.
(6) For each $S \in \mathcal{N}$, if $x \in V(S)$ and $x_i = y_i$ for every $i \in S$, then $y \in V(S)$.
(7) There exists $M \in \mathbb{R}$ such that for every $S \in \mathcal{N}$, $[x \in V(S)$ and $x > b]$ implies $[x_i < M$ for every $i \in S]$, where b is defined by $b_i = \sup\{y_i | y \in V(\{i\})\}$ for each $i \in N$.

The core of an NTU game (N, F, V) is the set $C(N, F, V)$ of $x \in \mathbb{R}^N$ such that (i) $x \in F$ and (ii) there do not exist $S \in \mathcal{N}$ and $y \in V(S)$ such that $x_i < y_i$ for all $i \in S$.

A subfamily $\mathcal{B}$ of $\mathcal{N}$ is balanced if there exist non-negative numbers λ_S for $S \in \mathcal{B}$ such that $\sum_{S \in \mathcal{B}, S \ni i} \lambda_S = 1$ for every $i \in S$. An NTU game (N, F, V) is balanced if for every balanced subfamily $\mathcal{B}$ of $\mathcal{N}$ we have $\bigcap_{S \in \mathcal{B}} V(S) \subset V(N)$. Scarf (1967) proved the fundamental existence theorem for the core of an NTU game.

Theorem (The Core Existence Theorem) *The core of a balanced NTU game is not empty.*

It is known that the Core Existence Theorem can be proved by fixed point theorems. For example, Zhou (1994) used Brouwer's fixed point theorem to prove The Core Existence Theorem. With the limit theorem of Debreu and Scarf (1963), we can show the converse.

Theorem 5 *The Core Existence Theorem is equivalent to Brouwer's fixed point theorem.*

Proof It is sufficient to prove that the Core Existence Theorem implies Walras' Existence Theorem C. Thus, the arguments reduce to those in Hildenbrand and Kirman (1988, Proposition 5.2, p.169 and Theorem 5.2, p.176). We include them for completeness. Let $\mathscr{E} = \{(\succeq_j, \omega_j)\}_{j=1}^m$ be an exchange economy. For each $q \in N$, let $\mathscr{E}^q = \{(\succeq_{jk}, \omega_{jk})\}_{j=1}^m{}_{k=1}^q$ be the q-fold replica economy of $\mathscr{E}$, i.e., $(\succeq_{jk}, \omega_{jk}) = (\succeq_j, \omega_j)$ for all $j = 1, \cdots, m$ and $k = 1, \cdots, q$. Pick a utility representation u_j of $\succeq_j$ for each j and consider the NTU game (N^q, F^q, V^q) corresponding to $\mathscr{E}^q$, i.e.,

$$N^q = \{(i, j) \mid i = 1, \cdots, m, j = 1, \cdots, q\},$$

$$F^q = \{v \in \mathbb{R}^{N^q} \mid \text{There exists } (x_{jk}) \in \mathbb{R}^{N^q}$$
$$\text{such that } \sum_{j=1,k=1}^{q,m} x_{jk} = \sum_{j=1,k=1}^{m,q} \omega_{jk} \text{ and for all } (j,k) \in N^q,\ v_{jk} = u_j(x_{jk})\},$$

and for each $S \subset N^q$,

$$V^q(S) = \{v \in \mathbb{R}^{N^q} \mid \text{There exists } (x_{jk}) \in \mathbb{R}^{N^q}$$
$$\text{such that } \sum_{(j,k) \in S} x_{jk} = \sum_{(j,k) \in S} \omega_{jk} \text{ and for all } (i, j) \in S,\ v_{jk} \leq u_j(x_{jk})\}.$$

It follows from a well-known argument that (N^q, F^q, V^q) is balanced. By the Core Existence Theorem, there exists $v^q \in C(N^q, F^q, V^q)$. Then, there exists $(x^q_{jk})_{(j,k)\in N^q}$ such that $u_j(x^q_{jk}) = v^q_{jk}$ for all j, k and q. By the equal treatment property (Debreu and Scarf 1963), for each q and for each j, there exists x^q_j such that $x^q_j = x^q_{jk}$ for all k. Since $\{x^q_i\}$ is a bounded sequence for each j, we may assume that $x^q_i \to x^*_j$ as $q \to \infty$. Then, for each q, $x^* = (x^*_j)^m_{j=1}$ cannot be improved upon by any coalition in $\mathscr{E}^q$. By Debreu and Scarf's (1963) limit theorem, x^* is a competitive equilibrium for $\mathscr{E}$. This completes the proof. □

Now, we turn to the NTU value. First, we need to recall TU (transferable utility) games and the Shapley value. A TU game is a pair (N, v), where N is a nonempty finite set of players and v is a real-valued function on $\mathcal{N}$. The Shapley value φ is defined by for each $i \in N$,

$$\varphi_i(N, v) = \sum_{S \subset N\setminus\{i\}} [\frac{|S|!|N - S - 1|!}{|N|!}] \cdot [v(S \cup \{i\}) - v(S)],$$

where $|S|$ denotes the cardinality of $S \in \mathcal{N}$. Let (N, F, V) be an NTU game and let $\Delta = \{\lambda \in \mathbb{R}^N_+ \mid \sum_{i\in N} \lambda_i = 1\}$. We say that a payoff allocation $u^* \in \mathbb{R}^N$ is an NTU value of (N, F, V) if,

(1) $u^* \in F$,
(2) There exists $\lambda^* \in \Delta$ such that $\varphi_i(N, v_{\lambda^*}) = \lambda^*_i u^*_i$ for all $i \in N$, where v_{λ^*} is a TU game defined by $v_{\lambda^*}(S) = \sup\{\sum_{i\in S} \lambda^*_i u_i \mid u \in V(S)\}$ for each $S \in \mathcal{N}$.

Using Kakutani's fixed point theorem, Shapley (1969) proved the following.

Theorem (The NTU Value Existence Theorem) *There exists an NTU value for any NTU game (N, F, V) satisfying the following conditions: F is compact convex and for any $S, T \in \mathcal{N}$ with $S \cap T = \emptyset$ and for any $u \in V(S)$ and $v \in V(T)$, we have $w \in V(S \cup T)$, where $w_i = u_i$ for $i \in S$, $w_i = v_i$ for $i \in T$ and w_i is arbitrary for any other $i \in N$.*

We show that the NTU Value Existence Theorem implies Kakntani's fixed point theorem. To do this, we use Champsaur's (1975) limit theorem of Value Allocations. Since the limit theorem requires strictly concave utility functions, our argument necessarily involves approximation of preferences by strictly concavifiable ones (see Kannai (1977)).

Theorem 6 *The NTU Value Existence Theorem is equivalent to Brouwer's fixed point theorem.*

Proof It is sufficient to show that the NTU Value Existence Theorem implies Walras' Existence Theorem C. Let $\mathscr{E} = \{(\succsim_j, \omega_j)\}^m_{j=1}$ be an exchange economy. For each j, there exists a sequence $\{\succsim^r_j\}$ in $\mathscr{P}$ converging to $\succsim_j$ in the topology of closed convergence such that each $\succsim^r_j$ has a strictly concave utility representation u^r_j (see Kannai (1977)). Fix r and consider the "strictly concavified economy" $\mathscr{E}(r) = \{(u^r_j, \omega_j)\}^m_{j=1}$. Then, replicate $\mathscr{E}(r)$ q times to obtain $\mathscr{E}^q(r) = \{(u^r_{jk}, \omega_{jk})\}^{m}_{j=1}{}^{q}_{k=1}$, where $u^r_{jk} = u^r_j$ and $\omega_{jk} = \omega_j$. Now, consider the NTU game (N^q, F^q, V^q) corresponding to $\mathscr{E}(r)^q$. Let $\Lambda^q = \{(\lambda_{jk})^{m}_{j=1}{}^{q}_{k=1} \mid \lambda_{jk} = \lambda_j$ for all j, k and $\sum^m_{j=1} \lambda_j = \frac{1}{q}\}$. Applying the NTU Value Existence Theorem to (N^q, F^q, V^q), we have a symmetric value allocation (x^{qr}), i.e., $x^{qr}_{jk} = x^{qr}_j$ for all $(j,k) \in N^q$, $\sum^m_{j=1} x^{qr}_j \leq \sum^m_{j=1} \omega_j$ and $\lambda^{qr}_{jk} u^r_j(x^{qr}_{jk}) = \varphi_{jk}(N^q, v_{\lambda^{qr}})$ for all $(j,k) \in N^q$, where $\lambda^{qr} \in \Lambda^q$ with strictly positive components. (see Theorems 6.3.1 and 6.4.1 in Ichiishi (1983) for explicit arguments). By Champsaur's (1975) limit theorem, a limit point x^r of the sequence $\{x^{qr}\}^\infty_{q=1}$ is a competitive allocation for $\mathscr{E}(r)$ (see also Ichiishi (1983, Theorem 6.5.2, p.130)). Finally, consider the sequence $\{x^r\}^\infty_{r=1}$. We may assume that it has a limit point x^*. Since the sequence $\{\succeq^r_j\}$ converge to $\succeq_j$ in the topology of closed convergence for each $j = 1, \cdots, m$, x^* is a competitive allocation for $\mathscr{E}$ (see Hildenbrand (1974, Ch. 1)). □

6 Concluding Remarks

We have displayed an array of equivalence theorems that encompasses game theory and mathematical economics. The technical overlap between these two fields turned out to be a logical necessity rather than a mere coincidence. This is the main lesson we have learned from our exercises.

It is possible to develop variants of our equivalence theorems in several ways. For example, some of our equivalence theorems can be put in terms of excess demand correspondences and Kakutani's Fixed Point Theorem without any technical difficulties. Using a well-known fact that convex preferences are approximated by strongly convex preferences, we could provide another variant of equivalence theorem for exchange economies.

Issues pertinent to infinite dimensional spaces would also pose yet another question to us for future research.

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