

Variational Problems on Compact
Riemannian Homogeneous Spaces

Osamu IKAWA

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Osamu IKAWA

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Contents

Introduction	3
1 Equivariant minimal immersions between compact Riemannian homogeneous spaces	9
1 Preliminaries	9
2 A construction of equivariant minimal immersions of compact Riemannian homogeneous spaces into Grassmann manifolds	11
3 Another construction of equivariant minimal immersions of compact Riemannian homogeneous spaces into Grassmann manifolds	15
4 A construction of totally geodesic immersions of compact irreducible Riemannian symmetric spaces into Grassmann manifolds	20
5 The Jacobi differential operators of equivariant minimal immersions	24
6 An example	31
2 Invariant connections on compact simple Lie groups	37
1 Preliminaries	37
2 The structure of the space of invariant connections in the case where $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g}	39
3 The case where $\rho(\mathfrak{l})$ contains a regular element of \mathfrak{g}	46
4 An example	51
Bibliography	53

Introduction

This thesis is composed of four papers [9], [10], [11] and [12], in which we dealt with minimal immersions and Yang-Mills connections.

In differential geometry, there are many problems which arise from variational problems in physics. Harmonic mappings, minimal immersions and Yang-Mills connections are the typical examples. These are the critical points for variational problems on energy, volume and the square norm of the curvature forms, respectively. We shall deal with these problems on compact Riemannian homogeneous spaces including spheres and Grassmann manifolds. Compact Riemannian homogeneous spaces play a central role in the theory of Riemannian geometry of positive curvature. We can apply group-theoretic methods to these spaces.

In Chapter 1, we shall construct harmonic mappings and minimal immersions from compact Riemannian homogeneous spaces into Grassmann manifolds in two ways (see Theorem A and B).

Let M and N be two compact connected Riemannian manifolds. A smooth mapping $F : M \rightarrow N$ is called *harmonic* if it is an extremal of the energy. Moreover, if harmonic mapping $F : M \rightarrow N$ is an isometric immersion, then F is a *minimal* immersion. An isometric immersion $F : M \rightarrow N$ is called *totally geodesic* if F carries every geodesic of M to a geodesic of N . A totally geodesic immersion is especially minimal. The existence and construction of minimal immersions and harmonic mappings are interesting and important problems in various situations. There are many studies on minimal immersions into a sphere whose starting point is theorem of T. Takahashi (see [3],[4],[29]).

Let G be a compact connected Lie group and K be a closed subgroup of G . Then $M = G/K$ is a compact Riemannian homogeneous space with a G -invariant Riemannian metric \langle, \rangle . For a field $E = \mathbf{R}, \mathbf{C}$ or \mathbf{H} , put

$$U(n, E) = \begin{cases} O(n) & (E = \mathbf{R}), \\ U(n) & (E = \mathbf{C}), \\ Sp(n) & (E = \mathbf{H}). \end{cases}$$

Put $G_{n,m}(E) = U(n+m, E)/U(n, E) \times U(m, E)$, which we call the E -Grassmann manifold consisting of all n -dimensional E -subspaces in V . Let $F : M = G/K \rightarrow G_{n,m}(E)$ be an equivariant mapping. Then there exists a Lie homomorphism $\rho : G \rightarrow U(n+m, E)$ with

$\rho(K) \subset U(n, E) \times U(m, E)$ such that $F(gK) = \rho(g)U(n, E) \times U(m, E)$ for each $g \in G$. Put $V = E^{n+m}$, $V_1 = E^n$, $V_2 = E^m$. Then $V = V_1 + V_2$ (direct sum). Put

$$\text{Hom}_K(V_1, V_2) = \{A \in \text{Hom}(V_1, V_2); \rho(k)A = A\rho(k) \text{ for each } k \in K\}.$$

Theorem A *If $\text{Hom}_K(V_1, V_2) = \{0\}$ and $V_i (i = 1, 2)$ is not G -invariant, then F is a nonconstant harmonic mapping. Furthermore, if G acts irreducibly on V , then F is E -full (see §1.2 for definition). Moreover, if K acts irreducibly on $T_o(M)$, then F is a minimal immersion with respect to a multiple of the G -invariant Riemannian metric \langle, \rangle on M .*

Take a nontrivial \mathbf{R} -spherical representation (ρ, V) of (G, K) . Then there exists a nonzero vector $v_0 \in V$ such that

$$\rho(k)v_0 = v_0 \quad \text{for each } k \in K.$$

Take a G -invariant inner product \langle, \rangle on V . Put

$$\begin{aligned} V_0 &= \mathbf{R}v_0, \\ V_1 &= \rho(\mathfrak{m})v_0, \\ V_2 &= \text{the orthogonal projection of } \text{span}\{\rho(X)\rho(Y)v_0; X, Y \in \mathfrak{m}\} \\ &\quad \text{to } (V_0 + V_1)^\perp, \\ &\quad \dots \\ V_k &= \text{the orthogonal projection of } \text{span}\{\rho(X_1) \cdots \rho(X_k)v_0; X_1, \dots, X_k \in \mathfrak{m}\} \\ &\quad \text{to } (V_0 + \cdots + V_{k-1})^\perp, \\ &\quad \dots, \end{aligned}$$

where we denote the differential representation of ρ of G by the same symbol ρ . There exists an integer m such that

$$\begin{aligned} V &= \sum_{i=0}^m V_i \text{ (the orthogonal direct sum of } K\text{-invariant subspaces),} \\ V_i &\neq \{0\} \quad \text{for } 0 \leq i \leq m. \end{aligned}$$

Put $S_m = \{0, \dots, m\}$. For subsets $P(\neq \emptyset), Q(\neq \emptyset)$ with $S_m = P \cup Q$ (disjoint union), put $V_P = \sum_{p \in P} V_p, V_Q = \sum_{q \in Q} V_q, a = \dim V_P, b = \dim V_Q$. Then $V = V_P + V_Q$ (orthogonal direct sum of K -invariant subspaces). Put

$$\begin{aligned} F : M = G/K &\rightarrow G_{a,b}(\mathbf{R}) = SO(a+b)/S(O(a) \times O(b)); \\ gK &\mapsto \rho(g)V_P = \rho(g)S(O(a) \times O(b)). \end{aligned}$$

Theorem B *F is a nonconstant \mathbf{R} -full equivariant harmonic mapping. If the linear isotropy action of K is irreducible, then F is a minimal immersion. In particular, if we put $P = \{0\}, Q = \{1, \dots, m\}$, then F is a minimal immersion of M into a projective space.*

Theorem C If (G, K) is a compact irreducible symmetric pair and put $P = \{\text{even}\}$, $Q = \{\text{odd}\}$, then F is a totally geodesic immersion.

A minimal immersion is said to be *stable* if the second variation of the volume is non-negative. There does not exist a stable minimal immersion into a sphere. A minimal immersion is stable if and only if the minimum eigenvalue of the Jacobi operator is non-negative. In §1.5, we shall calculate the eigenvalues of the Jacobi operator by using group theoretic methods.

In Chapter 2, we shall study Yang-Mills connections on compact simple Lie groups.

Maxwell's electromagnetic theory provides the simplest example of a gauge theory, with the field equation being given by Maxwell's equations (see [5],[20],[25]). We therefore begin with a brief review of Maxwell's equations. We denote by $\mathbf{R}_3^4 = (\mathbf{R}^4, dt^2 - \sum_{i=1}^3 dx_i^2)$ the four dimensional Minkowski space-time. We denote the magnetic field, the electric field, the charge density and the current density by $\mathbf{B} = (B_1, B_2, B_3)$ ($B_i : \mathbf{R}_3^4 \rightarrow \mathbf{R}$), $\mathbf{E} = (E_1, E_2, E_3)$ ($E_j : \mathbf{R}_3^4 \rightarrow \mathbf{R}$), $\rho : \mathbf{R}_3^4 \rightarrow \mathbf{R}$ and $\mathbf{J} : \mathbf{R}_3^4 \rightarrow \mathbf{R}$, respectively. We define 2-form F on \mathbf{R}_3^4 by

$$F = E_1 dx_1 \wedge dt + E_2 dx_2 \wedge dt + E_3 dx_3 \wedge dt + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

Then we have

$$dF = (\text{div}\mathbf{B})dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^3 \left(\frac{\partial B_i}{\partial t} + (\text{rot}\mathbf{E})_i \right) dt \wedge dx_i \wedge dx_{i+1},$$

and

$$\delta F = (\text{div}\mathbf{E})dt + \left(-\frac{\partial \mathbf{E}}{\partial t} + \text{rot}\mathbf{B} \right) \cdot d\mathbf{r}, \quad \text{where we put } d\mathbf{r} = (dx_1, dx_2, dx_3).$$

Maxwell's equations are given by:

$$dF = 0, \quad \delta F = j, \quad \text{where we put } j = \rho dt + \mathbf{J}d\mathbf{r}.$$

The first equation, which means Faraday's law of induction and the non-existence of the magnetic monopoles, holds regardless of the charge density or the current density. The second means Gauss' law and Ampere's law. We rewrite the above equations by using a scalar potential φ and a vector potential \mathbf{A} : The equation $dF = 0$ implies that the electromagnetic field F is derivative from a 1-form $A = A_0 dt + \sum_{i=1}^3 A_i dx_i$, i.e. $F = F_A = dA$. We call A the gauge potential. Remark that $F_{A+d\chi} = F_A$ for $\chi \in C^\infty(\mathbf{R}_3^4)$. If we put $\varphi = -A_0$ and $\mathbf{A} = (A_1, A_2, A_3)$, then

$$\mathbf{B} = \text{rot}\mathbf{A}, \quad \mathbf{E} = -\text{grad}\varphi - \frac{\partial \mathbf{A}}{\partial t}.$$

The equation $\delta F = j$ implies that

$$-\Delta\varphi - \frac{\partial}{\partial t}\operatorname{div}\mathbf{A} = \rho, \quad \frac{\partial}{\partial t}\operatorname{grad}\varphi + \frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} + \operatorname{grad}\operatorname{div}\mathbf{A} = \mathbf{J}.$$

The source-free field equations are obtained by setting $j = 0$ and can be written as

$$dF = 0 \text{ (Bianchi's identity),} \quad \delta F = 0 \text{ (Yang-Mills equation).}$$

The above formulation admits immediate generalization to the case when the Minkowski space-time and gauge potential \mathbf{A} is replaced by a semi-Euclidean space $\mathbf{R}_p^n = (\mathbf{R}^n, ds^2 = \sum_{i=1}^{n-p} dx_i^2 - \sum_{j=n-p+1}^n dx_j^2)$ and a \mathfrak{g} -valued 1-form $A = \sum_{i=1}^n A_i dx_i$ ($A_i \in C^\infty(\mathbf{R}_p^n, \mathfrak{g})$) on \mathbf{R}_p^n , where \mathfrak{g} is a Lie algebra of a compact Lie group G : Fix a bi-invariant Riemannian metric on G . For a \mathfrak{g} -valued 1-form A we define a \mathfrak{g} -valued 2-form F_A by

$$F_A = dA + \frac{1}{2}[A \wedge A] = \frac{1}{2} \sum F_{ij} dx_i \wedge dx_j, \text{ where } F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j].$$

For a \mathfrak{g} -valued k -form $\theta = \frac{1}{k!} \sum \theta_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$ we define a \mathfrak{g} -valued $(k+1)$ -form $d_A \theta$ by

$$d_A \theta = \frac{1}{k!} \sum (\nabla_j \theta_{i_1 \dots i_k}) dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}, \text{ where } \nabla_j \theta_{i_1 \dots i_k} = \frac{\partial \theta_{i_1 \dots i_k}}{\partial x_j} + [A_j, \theta_{i_1 \dots i_k}].$$

The Bianchi's identity $d_A F_A = 0$ holds. A gauge potential A is called a *Yang-Mills connection* if A satisfies the following:

$$d_A * F_A = 0 \quad (\text{Yang-Mills equation}).$$

We denote the set of all gauge potentials and the set of all \mathfrak{g} -valued k -forms by \mathcal{A} and $\Omega^k(\mathfrak{g})$, respectively. The gauge transformation group $\mathcal{G} = C^\infty(\mathbf{R}_3^4, G)$ acts on $\Omega^k(\mathfrak{g})$ and \mathcal{A} as follows:

- (1) $f^* \xi = \operatorname{Ad}(f^{-1})\xi$ for $\xi \in \Omega^k(\mathfrak{g}), f \in \mathcal{G}$,
- (2) $f^* A = f^{-1} df + \operatorname{Ad}(f^{-1})A$ for $A \in \mathcal{A}, f \in \mathcal{G}$.

Here $f^{-1} df$ is a pull-back of the Maurer-Cartan form on G by f . We have:

- (1) $F_{f^* A} = f^* F_A, \|F_{f^* A}\| = \|F_A\|$,
- (2) If A is Yang-Mills, then $f^* A$ is also Yang-Mills.

If we set $G = U(1)$, then the Yang-Mills equation is equivalent to Maxwell's equations. $G = SU(2) \times U(1)$ was used in Weinberg-Salam theory combining the weak interaction and electromagnetic theory, and $G = SU(3)$ was used in QCD(=quantum chromodynamics). $G = SU(5)$, $SO(10)$, E_6 and Poincare group were used in order to unify the electromagnetic interaction, the weak interaction, the strong interaction and the gravity into one geometry.

The above formulation admits immediate generalization to the case when the semi-Euclidean space and the gauge potential are replaced by a Riemannian manifold M and a connection on a principal G -bundle P over M : Let Ω_Λ denote the curvature form of a connection Λ on P . A critical point of the *Yang-Mills functional*

$$\Lambda \mapsto \frac{1}{2} \int_M \|\Omega_\Lambda\|^2$$

is called a *Yang-Mills connection*. A Yang-Mills connection Λ is said to be *stable* if the second variation of the Yang-Mills functional is non-negative. A flat connection is a stable Yang-Mills connection. H. T. Laquer dealt with Yang-Mills connection on compact symmetric spaces (see [17],[18],[19]) and he [16] proved that (0)-connection on a compact Lie group is an unstable Yang-Mills connection. A compact Riemannian manifold M is said to be *Yang-Mills instable* if, for every choice of G and every principal G -bundle P over M , stable Yang-Mills connection is always flat. S. Kobayashi, Y. Ohnita and M. Takeuchi [15] classified the compact simply connected irreducible symmetric spaces of type I which are Yang-Mills instable. In their paper, they gave a following question:

Is every simply connected compact simple Lie group Yang-Mills instable?

We consider an equivariant G -bundle P over a compact connected simple Lie group L and invariant connections on P . Every equivariant G -bundle P is obtained by a Lie homomorphism $\rho : L \rightarrow G$. The space of invariant connections on the principal G -bundle $P = K \times_\rho G$ over L is identified with

$$\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \{\Lambda \in \text{Hom}(\mathfrak{l}, \mathfrak{g}); [\rho(X), \Lambda(Y)] = \Lambda([X, Y]) \text{ for } X, Y \in \mathfrak{l}\},$$

where $\text{Hom}(\mathfrak{l}, \mathfrak{g})$ is the space of linear mappings from the vector space \mathfrak{l} to the vector space \mathfrak{g} .

We determine the structure of the space of invariant connections when $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} (see §2.2 for definition).

Theorem D *Assume $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} and that G is simple.*

(1) *If $\text{rank}(L) \geq 2$, then $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \mathbf{R}\rho$.*

(2) If $\text{rank}(L) = 1$, then there exist $\Gamma_1, \dots, \Gamma_{2s} \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that

$$\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \mathbf{R}\rho + \sum_{i=1}^{2s} \mathbf{R}\Gamma_i;$$

and the set of flat invariant connections is given by

$$\left\{ \pm \frac{1}{2}\rho \right\} \cup \left\{ \sum_{i=1}^{2s} a_i \Gamma_i; \sum_{i=1}^{2s} a_i^2 = \frac{1}{8} \right\};$$

and the set of Yang-Mills invariant connections except flat connections is given by

$$\{0\} \cup \left\{ \pm \frac{1}{4}\rho + \frac{1}{2} \sum_{i=1}^{2s} a_i \Gamma_i; \sum_{i=1}^{2s} a_i^2 = \frac{1}{8} \right\}.$$

Applying this result to Yang-Mills invariant connection, we get that any non-flat Yang-Mills invariant connection is instable when $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} (Corollary 2.2.1).

Theorem E *Assume $\rho(\mathfrak{l})$ contains a regular element of \mathfrak{g} . Then any non-flat Yang-Mills homogeneous connection is instable.*

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Chapter 1

Equivariant minimal immersions between compact Riemannian homogeneous spaces

1.1 Preliminaries

Let G (resp. U) be a compact connected Lie group with Lie algebra \mathfrak{g} (resp. \mathfrak{u}) and K (resp. L) be a closed subgroup of G (resp. U) with Lie algebra \mathfrak{k} (resp. \mathfrak{l}). Then $M = G/K$ (resp. $N = U/L$) is a compact Riemannian homogeneous space with a G (resp. U)-invariant Riemannian metric. Since K (resp. L) is compact, M (resp. N) is reductive, that is, there exists an $\text{Ad}(K)$ (resp. $\text{Ad}(L)$)-invariant subspace \mathfrak{m} (resp. \mathfrak{p}) such that

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} \text{ (direct sum)} \quad (\text{resp. } \mathfrak{u} = \mathfrak{l} + \mathfrak{p}).$$

We call \mathfrak{m} (resp. \mathfrak{p}) a Lie subspace of M (resp. N). We identify the tangent space $T_o(M)$ (resp. $T_o(N)$) at $o = \pi(e)$ with \mathfrak{m} (resp. \mathfrak{p}) in a natural manner, where π is the natural projection of G (resp. U) onto M (resp. N). The differential mapping k_* ($k \in K$) (resp. l_* ($l \in L$)) acting on $T_o(M)$ (resp. $T_o(N)$) corresponds to $\text{Ad}(k)$ (resp. $\text{Ad}(l)$) on \mathfrak{m} (resp. \mathfrak{p}), that is,

$$k_* \pi_* X = \pi_* \text{Ad}(k) X \quad \text{for each } X \in \mathfrak{m}.$$

Hence we have

$$\frac{d}{dt} (\exp tY)_* \pi_* X|_{t=0} = \pi_* [Y, X] \quad Y \in \mathfrak{k}, X \in \mathfrak{m}. \quad (1.1.1)$$

Let $F : M \rightarrow N$ be an equivariant mapping, that is, there exists a Lie homomorphism $\rho : G \rightarrow U$ with $\rho(K) \subset L$ such that $F(gK) = \rho(g)L$ for each $g \in G$. We get

$$F_* X = (\rho_* X)_{\mathfrak{p}} \quad \text{for each } X \in \mathfrak{m}. \quad (1.1.2)$$

We denote by ∇ and R the covariant derivative and the Riemannian curvature tensor of M , respectively. We denote by $\bar{\nabla}$ and \bar{R} for N in the same way. For each $X \in \mathfrak{g}$ we define a Killing vector field $X^* \in \mathfrak{X}(M)$ by

$$X_x^* = \frac{d}{dt} \exp tX \cdot x|_{t=0} \in T_x(M).$$

We have by the Koszul formula (see [7, p. 48, (2)])

$$(\nabla_{X^*} Y^*)_o = -[X_m, Y_{\mathfrak{k}}] - \frac{1}{2}[X_m, Y_m]_m \quad \text{for } X, Y \in \mathfrak{g}. \quad (1.1.3)$$

From the above equation, we have

$$\begin{aligned} \nabla_{g_*v} X &= \frac{d}{dt} (\exp(-t\text{Ad}(g)v))_* X_{g \exp tv \cdot K}|_{t=0} + \frac{1}{2} g_* [v, g_*^{-1} X_{gK}]_m \\ &\text{for } v \in \mathfrak{m}, g \in G, X \in \mathfrak{X}(M), \end{aligned} \quad (1.1.4)$$

$$\begin{aligned} R(X, Y)Z &= -\frac{1}{2}[[X, Y]_m, Z]_m - \frac{1}{4}[[Y, Z]_m, X]_m + \frac{1}{4}[[X, Z]_m, Y]_m - [[X, Y]_{\mathfrak{k}}, Z] \\ &\text{for } X, Y, Z \in \mathfrak{m}. \end{aligned} \quad (1.1.5)$$

Let $F: M \rightarrow N$ be an equivariant isometric immersion. Then there exists a Lie homomorphism $\rho: G \rightarrow U$ with $\rho(K) \subset L$ such that $F(gK) = \rho(g)L$ for each $g \in G$. Let A and B denote the shape operator and the second fundamental form of F , respectively. Take an orthonormal basis $\{X_i\}_{1 \leq i \leq p}$ of \mathfrak{g} with $\{X_i\}_{1 \leq i \leq m} \subset \mathfrak{m}$ and $\{X_j\}_{m+1 \leq j \leq p} \subset \mathfrak{k}$.

Proposition 1.1.1 (1)

$$B(X, Y) = -[(\rho_* X)_{\mathfrak{p}}, (\rho_* Y)_{\mathfrak{l}}] - \frac{1}{2}[(\rho_* X)_{\mathfrak{p}}, (\rho_* Y)_{\mathfrak{p}}]_{\mathfrak{p}} + \frac{1}{2}(\rho_*([X, Y]_m))_{\mathfrak{p}}$$

for $X, Y \in \mathfrak{m}$.

$$(2) F \text{ is minimal if and only if } \sum_{i=1}^m [(\rho_* X_i)_{\mathfrak{l}}, (\rho_* X_i)_{\mathfrak{p}}] = 0.$$

Proof: (1) is obtained from (1.1.2) and (1.1.3). (2) is clear from (1). ■

We review some elementary results on representation theory of compact connected Lie groups without proof.

Lemma 1.1.2 Let (ρ, V) be a real irreducible representation of G . $(\rho^{\mathbb{C}}, V^{\mathbb{C}})$ is not a complex irreducible representation of G if and only if there exists a complex irreducible representation (τ, W) of G such that $(\rho, V) = (\tau_{\mathbb{R}}, W_{\mathbb{R}})$, where we denote by $(\rho^{\mathbb{C}}, V^{\mathbb{C}})$ (resp. $(\tau_{\mathbb{R}}, W_{\mathbb{R}})$) the complex (resp. real) representation of G obtained by extension (resp. restriction) of the coefficient field of (ρ, V) (resp. (τ, W)) to \mathbb{C} (resp. \mathbb{R}).

Lemma 1.1.3 Let (ρ, V) be a complex irreducible representation of G . $(\rho_{\mathbf{R}}, V_{\mathbf{R}})$ is not a real irreducible representation of G if and only if there exists a real irreducible representation (τ, W) of G such that $(\rho, V) = (\tau^{\mathbf{C}}, W^{\mathbf{C}})$.

Lemma 1.1.4 Let (ρ, V) be a complex irreducible representation of G . $(\rho^{\mathbf{H}}, V^{\mathbf{H}})$ is not a quaternion irreducible representation of G if and only if there exists a quaternion irreducible representation (τ, W) of G such that $(\rho, V) = (\tau_{\mathbf{C}}, W_{\mathbf{C}})$, where we denote by $(\rho^{\mathbf{H}}, V^{\mathbf{H}})$ (resp. $(\tau_{\mathbf{C}}, W_{\mathbf{C}})$) the quaternion (resp. complex) representation of G obtained by extension (resp. restriction) of the coefficient field of (ρ, V) (resp. (τ, W)) to \mathbf{H} (resp. \mathbf{C}).

Lemma 1.1.5 Let (ρ, V) be a quaternion irreducible representation of G . $(\rho_{\mathbf{C}}, V_{\mathbf{C}})$ is not a complex irreducible representation of G if and only if there exists a complex irreducible representation (τ, W) of G such that $(\rho, V) = (\tau^{\mathbf{H}}, W^{\mathbf{H}})$.

1.2 A construction of equivariant minimal immersions of compact Riemannian homogeneous spaces into Grassmann manifolds

Let G be a compact connected Lie group and K be a closed subgroup of G . Then $M = G/K$ is a compact Riemannian homogeneous space with a G -invariant Riemannian metric $\langle \cdot, \cdot \rangle$. For a field $E = \mathbf{R}, \mathbf{C}$ or \mathbf{H} , put

$$U(n, E) = \begin{cases} O(n) & (E = \mathbf{R}), \\ U(n) & (E = \mathbf{C}), \\ Sp(n) & (E = \mathbf{H}). \end{cases}$$

Put $G_{n,m}(E) = U(n+m, E)/U(n, E) \times U(m, E)$, which we call the E -Grassmann manifold consisting of all n -dimensional E -subspaces in V . Let $F : M = G/K \rightarrow G_{n,m}(E)$ ($n \geq 1, m \geq 1$) be an equivariant mapping. Then there exists a Lie homomorphism $\rho : G \rightarrow U(n+m, E)$ with $\rho(K) \subset U(n, E) \times U(m, E)$ such that $F(gK) = \rho(g)U(n, E) \times U(m, E)$ for each $g \in G$. Put $V = E^{n+m}$, $V_1 = E^n$, $V_2 = E^m$. Then $V = V_1 + V_2$ (direct sum) and the Lie algebra \mathfrak{u} of $U(n+m, E)$ acts on V , naturally. Put $\mathfrak{l} = \text{Lie}(U(n) \times U(m))$ and

$$\mathfrak{p} = \{A \in \mathfrak{u} ; AV_1 \subset V_2, AV_2 \subset V_1\}.$$

Then $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ is the canonical decomposition of \mathfrak{u} . Put

$$\text{Hom}_K(V_1, V_2) = \{A \in \text{Hom}(V_1, V_2); \rho(k)A = A\rho(k) \text{ for each } k \in K\}.$$

We will define that F is E -full. Let V'_1 and V'_2 be subspaces of V_1 and V_2 , respectively. Put $n' = \dim_E V'_1$ and $m' = \dim_E V'_2$. Then $U(n' + m', E)$ is considered as a closed subgroup

of $U(n+m, E)$ in a natural manner. So $G_{n',m'}(E)$ is a totally geodesic submanifold of $G_{n,m}(E)$. The mapping F is said to be E -full when the image $F(M)$ is not contained in these totally geodesic submanifolds $G_{n',m'}(E)$ with $n' + m' < n + m$.

Theorem A *If $\text{Hom}_K(V_1, V_2) = \{0\}$ and $V_i (i = 1, 2)$ is not G -invariant, then F is a nonconstant harmonic mapping. Furthermore, if G acts irreducibly on V , then F is E -full. Moreover, if K acts irreducibly on $T_o(M)$, then F is a minimal immersion with respect to a multiple of the G -invariant Riemannian metric \langle, \rangle on M .*

Proof: Let $H \in \mathfrak{p}$ denote the tension field of F at o (see [6, Chap. I, §2] for the definition of tension field). Then by homogeneity F is harmonic if and only if $H = 0$. Since $H\rho(k) = \rho(k)H$ for each $k \in K$, we have $H = 0$. So F is a nonconstant harmonic mapping.

We assume that K acts irreducibly on $T_o(M)$. We define a symmetric linear transformation A of $T_o(M)$ by

$$\langle X, AY \rangle = (F_*X, F_*Y) \quad \text{for } X, Y \in T_o(M),$$

where $(,)$ denote a $U(n+m, E)$ -invariant Riemannian metric on $G_{n,m}(E)$. Since A is a K -homomorphism, A is a scalar operator by the irreducibility of the action of K . The scalar is clearly nonnegative. So if F were not an isometric (more precisely, homothetic) immersion, then $F_* = 0$. Thus V_i is G -invariant from (1.1.2) and the connectedness of G . So F is an isometric minimal immersion. If G acts irreducibly on V , then F is clearly E -full. ■

Example (Equivariant minimal immersions of S^2 into Grassmann manifolds)

Let (ρ, V) be any $SU(2)$ - E -irreducible representation. Put $K = S(U(1) \times U(1))$. Let $V = \sum_i W_i$ be a K - E -irreducible decomposition of V . We have

$$W_i \cong W_j \text{ (} K\text{-isomorphic)} \iff i = j \text{ (see §1.6, Lemma 1.6.1)}. \quad (1.2.6)$$

Let $V_i \neq \{0\} (i = 1, 2)$ be a K - E -invariant subspace of V such that $V = V_1 + V_2$ (direct sum). Put $n = \dim_E V_1, m = \dim_E V_2$. If we put $F : S^2 = SU(2)/K \rightarrow G_{n,m}(E); gT \mapsto \rho(g)U(n, E) \times U(m, E)$ for $g \in G$, then F is a full minimal immersion from (1.2.6) and Theorem A. ■

We will apply Theorem A. Let $M (\neq \{\text{a single point}\})$ be a compact Riemannian homogeneous space. The identity component G of the group of all isometries of M is compact. The action of G on M is effective and transitive. The subgroup $K = \{g \in G; g \cdot o = o\}$ of G is closed and called isotropy group of M at o .

A G - E -irreducible representation (ρ, V) is called an E -spherical representation of the pair (G, K) , if $V_K = \{v \in V; \rho(k)v = v \text{ for each } k \in K\} \neq \{0\} (E = \mathbf{R}, \mathbf{C}, \mathbf{H})$. The dimension of V and V_K is called the *degree* and the *multiplicity* of (ρ, V) , respectively.

Lemma 1.2.1 *If $K \neq \{e\}$, then there exists an E -spherical representation (ρ, V) such that $V_K \neq V$.*

Proof: We may assume that $E = \mathbf{C}$ by Lemma 1.1.3 and Lemma 1.1.4. Let $L^2(G/K)$ denote the space of complex valued functions f on G/K with

$$\int_{G/K} |f(x)|^2 dx < \infty.$$

Put

$$L^2(G, K) = \{f \in L^2(G/K); f(kx) = f(x) \text{ for each } k \in K, x \in M\}.$$

Since $K \neq \{e\}$, we get $L^2(G/K) \neq L^2(G, K)$. If $V = V_K$ for each \mathbf{C} -spherical representation (ρ, V) , then we have $L^2(G/K) = L^2(G, K)$ from Peter-Weyl theorem (see [30, p.20]). This is a contradiction. ■

The manifold M is said to have *irreducible linear isotropy group*, if K acts irreducibly on $T_o(M)$.

Lemma 1.2.2 *We assume that M has irreducible linear isotropy group.*

- (1) *The degree of any nontrivial \mathbf{R} -spherical representation of (G, K) is greater than or equal to $\dim M + 1$.*
- (2) *If $\dim M \geq 2$, then the degree of any nontrivial \mathbf{C} -spherical representation is greater than or equal to 2.*

Proof: (1) is obtained from Theorem of T.Takahashi [29]. But, for the sake of completeness, we give a proof. For each nontrivial \mathbf{R} -spherical representation (ρ, V) of (G, K) , put

$$F : M = G/K \rightarrow V; gK \mapsto \rho(g)v,$$

where $v \in V_K$ and $\|v\| = 1$ with respect to a G -invariant inner product on V . Then we can prove that F is an immersion in the same way as the proof of Theorem A. Clearly the image $F(M)$ is contained in the unit hypersphere in V . So we get the conclusion. (2) is obtained from (1) and Lemma 1.1.3. (Furthermore, F is minimal (see [31, Proposition 8.1, p. 21])) ■

Proposition 1.2.3 (1) *A compact Riemannian homogeneous space of dimension ≥ 2 with irreducible linear isotropy group admits an equivariant minimal immersion into an E -Grassmann manifold.*

- (2) *There exists a nonconstant equivariant harmonic mapping from a compact Riemannian homogeneous space with non trivial isotropy group into an E -Grassmann manifold.*

Proof: (1) Take (ρ, V) as in Lemma 1.2. 1 . Put $V_1 = V_K$ and $V_2 = V_K^\perp$ with respect to a G -invariant inner product on V . Then the assertion follows from Theorem A.

(2) It is obtained from Theorem A and Lemma 1.2. 1 in the same way as the proof of (1). ■

If M is a compact Riemannian symmetric space, then (G, K) is a compact symmetric pair (see §1.4 for definition).

Lemma 1.2. 4 *Let (G, K) be a compact symmetric pair.*

(1) *The multiplicity of any \mathbf{C} -spherical representation of (G, K) equals to 1.*

(2) *The multiplicity of any \mathbf{R} -spherical representation of (G, K) equals to 1 or 2.*

(3) *Any \mathbf{H} -spherical representation of (G, K) is obtained from the extension of coefficient field of a G - \mathbf{C} -irreducible representation to \mathbf{H} and the multiplicity is equal to 1.*

Proof: (1) We refer to [30, p. 104, Theorem 5.5]. (2) is obtained from (1) and Lemma 1.1. 2. (3) is obtained from (1) and Lemma 1.1. 5 . ■

Lemma 1.2. 5 *Let M be a compact irreducible Riemannian symmetric space of dimension ≥ 2 . Then the degree of any nontrivial \mathbf{H} -spherical representation of (G, K) is greater than or equal to 2.*

Proof: It is obtained from Lemma 1.2. 2 ,(2) and Lemma 1.2. 4 ,(3). ■

Proposition 1.2. 6 (1) *Let M be a compact irreducible Riemannian symmetric space of dimension ≥ 2 .*

(i) *M admits an equivariant minimal immersion into a real projective space or $G_{2,n}(\mathbf{R})$.*

(ii) *M admits an equivariant minimal immersion into a complex projective space.*

(iii) *M admits an equivariant minimal immersion into a quaternion projective space.*

(2) *Let M be a compact Riemannian symmetric space with non trivial isotropy group.*

(i) *There exists a nonconstant equivariant harmonic mapping from M into a real projective space or $G_{2,n}(\mathbf{R})$.*

(ii) *There exists a nonconstant equivariant harmonic mapping from M into a complex projective space.*

(iii) *There exists a nonconstant equivariant harmonic mapping from M into a quaternion projective space.*

Proof: (1) It is obtained from Theorem A, Lemma 1.2. 2 , Lemma 1.2. 4 and Lemma 1.2. 5 in the same way as the proof of Proposition 1.2. 3 .

(2) It is obtained from Theorem A and Lemma 1.2. 4 in the same way as the proof of Proposition 1.2. 3 . ■

1.3 Another construction of equivariant minimal immersions of compact Riemannian homogeneous spaces into Grassmann manifolds

Let G be a compact connected Lie group with Lie algebra \mathfrak{g} and K be a closed subgroup of G with Lie algebra \mathfrak{k} . Take a bi-invariant Riemannian metric \langle, \rangle on G and denote also by \langle, \rangle the induced $\text{Ad}(G)$ -invariant inner product on $\mathfrak{m} = \mathfrak{k}^\perp$. Thus $M = (M^n, \langle, \rangle) = G/K$ is a compact Riemannian homogeneous space. The subspace \mathfrak{m} of \mathfrak{g} is naturally identified with the tangent space $T_o(M)$ of M at $o = \pi(e)$, where $\pi : G \rightarrow M$ is a natural projection.

Take a nontrivial \mathbf{R} -spherical representation (ρ, V) of (G, K) . Then there exists a nonzero vector $v_0 \in V$ such that

$$\rho(k)v_0 = v_0 \quad \text{for each } k \in K.$$

Take a G -invariant inner product \langle, \rangle on V . Put

$$\begin{aligned} V_0 &= \mathbf{R}v_0, \\ V_1 &= \rho(\mathfrak{m})v_0, \\ V_2 &= \text{the orthogonal projection of } \text{span}\{\rho(X)\rho(Y)v_0; X, Y \in \mathfrak{m}\} \\ &\quad \text{to } (V_0 + V_1)^\perp, \\ &\dots \\ V_k &= \text{the orthogonal projection of } \text{span}\{\rho(X_1) \cdots \rho(X_k)v_0; X_1, \dots, X_k \in \mathfrak{m}\} \\ &\quad \text{to } (V_0 + \cdots + V_{k-1})^\perp, \\ &\dots \end{aligned} \tag{1.3.7}$$

where we denote the differential representation of ρ of G by the same symbol ρ . Since ρ is irreducible, there exists an integer m such that

$$\begin{aligned} V &= \sum_{i=0}^m V_i \text{ (the orthogonal direct sum of } K\text{-invariant subspaces),} \\ V_i &\neq \{0\} \quad \text{for } 0 \leq i \leq m. \end{aligned}$$

Since ρ is nontrivial, we get $m \geq 1$. Put $S_m = \{0, \dots, m\}$. For subsets $P(\neq \emptyset), Q(\neq \emptyset)$ with $S_m = P \cup Q$ (disjoint union), put $V_P = \sum_{p \in P} V_p, V_Q = \sum_{q \in Q} V_q, a = \dim V_P, b = \dim V_Q$. Then $V = V_P + V_Q$ (orthogonal direct sum of K -invariant subspaces). Put

$$\begin{aligned} F : M = G/K &\rightarrow G_{a,b}(\mathbf{R}) = SO(a+b)/S(O(a) \times O(b)); \\ gK &\mapsto \rho(g)V_P = \rho(g)S(O(a) \times O(b)). \end{aligned} \tag{1.3.8}$$

We prove the following theorem.

Theorem B *F is a nonconstant \mathbf{R} -full equivariant harmonic mapping. If the linear isotropy action of K is irreducible, then F is a minimal immersion. In particular, if we put $P = \{0\}, Q = \{1, \dots, m\}$, then F is a minimal immersion of M into a projective space.*

In order to prove this, we prove lemmas. First, we note that $\rho(\mathfrak{m})V_k \subset V_0 + \cdots + V_{k+1}$ for $k = 0, \dots, m$, where we put $V_{m+1} = \{0\}$.

Lemma 1.3.1

$\rho(\mathfrak{m})V_k \subset V_{k-1} + V_k + V_{k+1}$ for $k = 0, 1, \dots, m$, where we put $V_{-1} = \{0\}$.

Proof: We prove this by induction on k . It is clear when $k = 0$. We assume that this lemma holds until k . For $0 \leq i \leq k-1$, by the hypothesis of induction, we get $\langle \rho(\mathfrak{m})V_{k+1}, V_i \rangle = \langle V_{k+1}, \rho(\mathfrak{m})V_i \rangle = \{0\}$. ■

We denote an orthonormal basis of \mathfrak{m} and \mathfrak{k} by $\{E_i\}_{1 \leq i \leq n}$ and $\{E_{n+j}\}_{1 \leq j \leq l}$, respectively. We remark that the Casimir operator

$$C = \sum_{i=1}^{n+l} \rho(E_i)^2$$

of ρ is a scalar operator because C is a G -invariant symmetric transformation and ρ is irreducible. For $v \in V$, we denote the V_k -component of v by v_{V_k} .

Lemma 1.3.2 $\sum_{i=1}^n \rho(E_i)(\rho(E_i)v_k)_{V_{k+1}} \in V_k + V_{k+1}$ for each $v_k \in V_k$

Proof: We have

$$V_k \ni Cv_k = \sum_{i=1}^{n+l} \rho(E_i)(\rho(E_i)v_k)_{V_k} + \sum_{i=1}^{n+l} \rho(E_i)(\rho(E_i)v_k)_{V_{k-1}} + \sum_{i=1}^{n+l} \rho(E_i)(\rho(E_i)v_k)_{V_{k+1}}.$$

Hence we have by Lemma 1.3.1

$$\sum_{i=1}^{n+l} \rho(E_i)(\rho(E_i)v_k)_{V_{k+1}} \in V_k + V_{k+1}.$$

Since V_k is K -invariant, we get the conclusion. ■

The Lie algebra \mathfrak{u} of $SO(a+b)$ acts on V , naturally. Put $\mathfrak{l} = \text{Lie}(S(O(a) \times O(b)))$ and $\mathfrak{p} = \{T \in \mathfrak{u}; TV_P \subset V_Q, TV_Q \subset V_P\}$. Then $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ is the canonical decomposition of \mathfrak{u} . For $T \in \mathfrak{u}$, we denote the \mathfrak{p} (resp. \mathfrak{l})-component of T by $T_{\mathfrak{p}}$ (resp. $T_{\mathfrak{l}}$).

Lemma 1.3.3 $\sum_{i=1}^n (\rho(E_i)_{\mathfrak{l}} \rho(E_i)_{\mathfrak{p}} + \rho(E_i)_{\mathfrak{p}} \rho(E_i)_{\mathfrak{l}}) = 0$.

Proof: For each $v_k \in V_k$, we have

$$\sum_{i=1}^n \rho(E_i)^2 v_k = C v_k - \sum_{i=1}^l \rho(E_{n+j})^2 v_k \in V_k.$$

Since

$$\sum_{i=1}^n \rho(E_i)^2 = \sum_{i=1}^n \left(\rho(E_i)_l \rho(E_i)_p + \rho(E_i)_p \rho(E_i)_l \right) + \sum_{i=1}^n \left(\rho(E_i)_l \rho(E_i)_l + \rho(E_i)_p \rho(E_i)_p \right),$$

we get the conclusion. ■

Proof of Theorem B: Let $H \in \mathfrak{p}$ denote the tension of F at o . By (1.1.3), we have $H = \sum_{i=1}^n [\rho(E_i)_l, \rho(E_i)_p]$. From Lemma 1.3. 3, we have

$$H = 2 \sum_{i=1}^n \rho(E_i)_l \rho(E_i)_p = -2 \sum_{i=1}^n \rho(E_i)_p \rho(E_i)_l.$$

If $0, 1 \in P$ or $0, 1 \in Q$, then we have $Hv_0 = 0$ by $\rho(E_i)_p v_0 = 0$. If $0 \in P, 1 \in Q$ or $0 \in Q, 1 \in P$, then we have $Hv_0 = 0$ by $\rho(E_i)_l v_0 = 0$. Hence we have $H|_{V_0} = 0$. We assume that $H|_{(V_0 + \dots + V_j)} = 0$. We will prove $H|_{V_{j+1}} = 0$. Clearly, we have $HV_{j+1} \subset \sum_{i=0}^{j+3} V_i$. By the hypothesis, we have

$$0 = \langle H \sum_{i=0}^j V_i, V_{j+1} \rangle = - \langle \sum_{i=0}^j V_i, HV_{j+1} \rangle.$$

Hence we have $HV_{j+1} \subset V_{j+1} + V_{j+2} + V_{j+3}$. Let χ_P and χ_Q denote the characteristic functions of P and Q , respectively:

$$\chi_P(k) = \begin{cases} 1 & (k \in P) \\ 0 & (k \in Q), \end{cases} \quad \chi_Q(k) = \begin{cases} 1 & (k \in Q) \\ 0 & (k \in P). \end{cases}$$

For each $v_{j+1} \in V_{j+1}$, we have by Lemma 1.3. 2 and the hypothesis of induction

$$\begin{aligned}
& \left(\sum_{i=1}^n \rho(E_i) \lrcorner \rho(E_i) \right) \mathfrak{p} v_{j+1} \\
= & \chi_P(j+1) \sum_{k=j}^{j+2} \chi_Q(k) \sum_{i=1}^n \sum_{l=k-1}^{k+1} \chi_Q(l) (\rho(E_i) (\rho(E_i) v_{j+1}) v_k) v_l \\
& + \chi_Q(j+1) \sum_{k=j}^{j+2} \chi_P(k) \sum_{i=1}^n \sum_{l=k-1}^{k+1} \chi_P(l) (\rho(E_i) (\rho(E_i) v_{j+1}) v_k) v_l \\
= & \chi_P(j+1) \chi_Q(j+2) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+2} \\
& + \chi_P(j+1) \chi_Q(j+2) \chi_Q(j+3) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+3} \\
& + \chi_Q(j+1) \chi_P(j+2) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+2} \\
& + \chi_Q(j+1) \chi_P(j+2) \chi_P(j+3) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+3}, \\
& \left(\sum_{i=1}^n \rho(E_i) \mathfrak{p} \rho(E_i) \right) \lrcorner v_{j+1} \\
= & \chi_P(j+1) \sum_{k=j}^{j+2} \chi_P(k) \sum_{i=1}^n \sum_{l=k-1}^{k+1} \chi_Q(l) (\rho(E_i) (\rho(E_i) v_{j+1}) v_k) v_l \\
& + \chi_Q(j+1) \sum_{k=j}^{j+2} \chi_Q(k) \sum_{i=1}^n \sum_{l=k-1}^{k+1} \chi_P(l) (\rho(E_i) (\rho(E_i) v_{j+1}) v_k) v_l \\
= & \chi_P(j+1) \chi_Q(j+2) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+1}) v_{j+2} \\
& + \chi_P(j+1) \chi_P(j+2) \chi_Q(j+3) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+3} \\
& + \chi_Q(j+1) \chi_P(j+2) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+1}) v_{j+2} \\
& + \chi_Q(j+1) \chi_Q(j+2) \chi_P(j+3) \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+3}.
\end{aligned}$$

Hence, if $j+1, j+2 \in P$ or $j+1, j+2 \in Q$, then we have $(\sum_{i=1}^n \rho(E_i) \lrcorner \rho(E_i) \mathfrak{p}) v_{j+1} = 0$. If $j+1 \in P, j+2 \in Q$ or $j+1 \in Q, j+2 \in P$, then we have by Lemma 1.3. 3

$$\begin{aligned}
\left(\sum_{i=1}^n \rho(E_i) \mathfrak{p} \rho(E_i) \right) \lrcorner v_{j+1} &= \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+1}) v_{j+2} \in V_{j+2}, \\
\left(\sum_{i=1}^n \rho(E_i) \lrcorner \rho(E_i) \mathfrak{p} \right) v_{j+1} &= \sum_{i=1}^n (\rho(E_i) (\rho(E_i) v_{j+1}) v_{j+2}) v_{j+2} = 0.
\end{aligned}$$

Hence we have $H|_{V_{j+1}} = 0$. Therefore F is a nonconstant harmonic mapping.

If the isotropy action of K is irreducible, F is an isometric immersion. ■

Let (ρ, V) be a complex (resp. quaternion) spherical representation of (G, K) . Put

$$V_K = \{v \in V; \rho(k)v = v \text{ for each } k \in K\} (\neq \{0\}).$$

If there exists a nonzero vector $v_0 \in V_K$ such that

$$\langle \rho(g)v_0, v_0 \rangle \in \mathbf{R} \text{ for each } g \in G, \quad (1.3.9)$$

then we can construct a harmonic mapping from M into a complex (resp. quaternion) Grassmann manifold in the same way as the proof of Theorem A. Condition (1.3.9) means:

Proposition 1.3.4 *A complex (resp. quaternion)-spherical representation (ρ, V) satisfies (1.3.9) if and only if there exists a real spherical representation (τ, W) of (G, K) such that*

$$(\rho, V) = (\tau, W)^{\mathbf{C}} \quad (\text{resp. } (\tau, W)^{\mathbf{H}}), \quad (1.3.10)$$

where $(\tau, W)^{\mathbf{C}}$ (resp. $(\tau, W)^{\mathbf{H}}$) denote the complex (resp. quaternion) representation of G obtained by extension of the coefficient field of (τ, W) to \mathbf{C} (resp. \mathbf{H}).

Proof: Clearly (1.3.10) implies (1.3.9). Conversely we assume (1.3.9). If we put

$$W = \mathbf{R}\text{-linear span of } \{\rho(g)v_0; g \in G\},$$

then (1.3.10) is concluded. ■

If (G, K) is a compact symmetric pair of rank one, then every \mathbf{C} (or \mathbf{H})-spherical representation (ρ, V) of (G, K) satisfies (1.3.9) (see [31, p. 25, Cor. 8.2]).

We prove lemmas needed later (§1.4).

Lemma 1.3.5 $\rho(X_1) \cdots \rho(X_k)v_0 \equiv \rho(X_{\tau(1)}) \cdots \rho(X_{\tau(k)})v_0 \pmod{V_0 + \cdots + V_{k-1}}$ for $X_1, \dots, X_k \in \mathfrak{m}, \tau \in \mathfrak{S}_k$, where we denote by \mathfrak{S}_k the symmetric group of degree k .

Proof: We have

$$\begin{aligned} & \rho(X_1) \cdots \rho(X_i)\rho(X_{i+1}) \cdots \rho(X_k)v_0 \\ &= \rho(X_1) \cdots \rho(X_{i-1})\rho(X_{i+1})\rho(X_i)\rho(X_{i+2}) \cdots \rho(X_k)v_0 \\ & \quad + \rho(X_1) \cdots \rho(X_{i-1})\rho([X_i, X_{i+1}])\rho(X_{i+2}) \cdots \rho(X_k)v_0. \end{aligned}$$

Hence we get the conclusion. ■

Lemma 1.3.6

$V_k =$ the orthogonal projection of $\text{span}\{\rho(X)^k v_0; X \in \mathfrak{m}\}$ to $(V_0 + \cdots + V_{k-1})^\perp$.

Proof: We prove this by induction on k . It is clear when $k = 0$. We assume that this lemma holds for k . From this, we get

$$V_{k+1} = \begin{array}{l} \text{the orthogonal projection of } \text{span}\{\rho(X)\rho(Y)^k v_0; X, Y \in \mathfrak{m}\} \\ \text{to } (V_0 + \cdots + V_k)^\perp. \end{array}$$

From Lemma 1.3.5, we have

$$\rho(X + lY)^{k+1} v_0 \equiv \sum_{s=0}^{k+1} \binom{k+1}{s} l^s \rho(X)^{k+1-s} \rho(Y)^s v_0 \pmod{V_0 + \cdots + V_k}$$

for $l = 1, 2, \dots, k+2$. By the formula of Van der Monde, we have

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2^0 & 2^1 & \cdots & 2^{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ (k+2)^0 & (k+2)^1 & \cdots & (k+2)^{k+1} \end{pmatrix} = \prod_{1 \leq i < j \leq k+2} (j - i) \neq 0.$$

Hence the vector $\rho(X)\rho(Y)^k v_0$ is a linear combination of $\rho(X + Y)^{k+1} v_0, \dots, \rho(X + (k+2)Y)^{k+1} v_0 \pmod{V_0 + \cdots + V_k}$. ■

1.4 A construction of totally geodesic immersions of compact irreducible Riemannian symmetric spaces into Grassmann manifolds

Let G be a compact connected Lie group with Lie algebra \mathfrak{g} , K a closed subgroup of G with Lie algebra \mathfrak{k} . The pair (G, K) is called a *compact symmetric pair* if there exists an involutive automorphism θ of G such that K lies between $K_\theta = \{g \in G; \theta(g) = g\}$ and the identity component $(K_\theta)_0$ of K_θ . A compact symmetric pair (G, K) is said to be *irreducible* if the adjoint action of K on \mathfrak{m} is irreducible.

Let (G, K) be a compact irreducible symmetric pair. An $\text{Ad}(G)$ and θ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} naturally induces a G -invariant Riemannian metric on $M = G/K$. M is a compact Riemannian symmetric space with respect to the G -invariant Riemannian metric. Since θ is an involutive automorphism, we have a canonical orthogonal decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

Put P as in (1.3.8), §1.3 with $P = \{\text{even}\}$, $Q = \{\text{odd}\}$, then we have the following theorem.

Theorem C F is a totally geodesic immersion.

In order to prove this, we prove the following lemma.

Lemma 1.4. 1

$\rho(\mathfrak{m})V_k \subset V_{k-1} + V_{k+1}$ for $k = 0, 1, \dots, m$, where we put $V_{-1} = V_{m+1} = \{0\}$.

Proof: We prove this by induction on k . It is clear when $k = 0$. We assume that this lemma holds until k . From Lemma 1.3. 1, it is sufficient to prove $\langle \rho(\mathfrak{m})V_{k+1}, V_{k+1} \rangle = \{0\}$. When k is even, put $k = 2l$. For $X \in \mathfrak{m}$, by the hypothesis of induction, we get

$$\begin{aligned} \rho(X)v_0 &\in V_1, \\ \rho(X)^2v_0 &\in V_0 + V_2, \\ &\dots \\ \rho(X)^{2l}v_0 &\in V_0 + V_2 + \dots + V_{2l}, \\ \rho(X)^{2l+1}v_0 &\in V_1 + V_3 + \dots + V_{2l+1}. \end{aligned}$$

For $Y \in \mathfrak{m}$, we get

$$\rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}} = \rho(Y)\rho(X)^{2l+1}v_0 - \sum_{s=0}^{l-1} \rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2s+1}}.$$

By the hypothesis of induction, we get

$$\sum_{s=0}^{l-1} \rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2s+1}} \in V_0 + V_2 + \dots + V_{2l}.$$

For each $Z \in \mathfrak{m}$, we have

$$\begin{aligned} &\langle \rho(Y)\rho(X)^{2l+1}v_0, \rho(Z)^{2l+1}v_0 \rangle \\ &= \langle \rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}} + \sum_{s=0}^{l-1} \rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2s+1}}, \sum_{t=0}^l (\rho(Z)^{2l+1}v_0)_{V_{2t+1}} \rangle \\ &= \langle \rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}}, \sum_{t=0}^l (\rho(Z)^{2l+1}v_0)_{V_{2t+1}} \rangle \\ &= \langle \rho(Y)(\rho(X)^{2l+1}v_0)_{V_{2l+1}}, (\rho(Z)^{2l+1}v_0)_{V_{2l+1}} \rangle. \end{aligned}$$

From Lemma 1.3. 6, it is sufficient to prove

$$\langle \rho(Y)\rho(X)^{2l+1}v_0, \rho(Z)^{2l+1}v_0 \rangle = 0 \text{ for each } X, Y, Z \in \mathfrak{m}.$$

For $X_1, \dots, X_{2l+2}, Y_1, \dots, Y_{2l+1} \in \mathfrak{m}, \sigma \in \mathfrak{S}_{2l+2}$, by the hypothesis of induction and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, we have

$$\begin{aligned} &\langle \rho(X_1) \cdots \rho(X_{2l+2})v_0, \rho(Y_1) \cdots \rho(Y_{2l+1})v_0 \rangle \\ &= \langle \rho(X_{\sigma(1)}) \cdots \rho(X_{\sigma(2l+2)})v_0, \rho(Y_1) \cdots \rho(Y_{2l+1})v_0 \rangle. \end{aligned}$$

Hence we have

$$\begin{aligned}
\langle \rho(W)^{2l+2}v_0, \rho(Z)^{2l+1}v_0 \rangle &= \langle \rho(W)^{2l+1}\rho(Z)v_0, \rho(Z)^{2l}\rho(W)v_0 \rangle \\
&= \langle \rho(W)^{2l}\rho(Z)^2v_0, \rho(Z)^{2l-1}\rho(W)^2v_0 \rangle \\
&\dots \\
&= \langle \rho(W)^{l+1}\rho(Z)^{l+1}v_0, \rho(Z)^l\rho(W)^{l+1}v_0 \rangle \\
&= \langle \rho(Z)^{l+1}\rho(W)^{l+1}v_0, \rho(Z)^l\rho(W)^{l+1}v_0 \rangle \\
&= 0
\end{aligned}$$

for each $W, Z \in \mathfrak{m}$.

Hence we have

$$\begin{aligned}
0 &= \langle \rho(Y + mX)^{2l+2}v_0, \rho(Z)^{2l+1}v_0 \rangle \\
&= \sum_{i=0}^{2l+2} \binom{2l+2}{i} m^i \langle \rho(Y)^{2l+2-i}\rho(X)^i v_0, \rho(Z)^{2l+1}v_0 \rangle
\end{aligned}$$

for $X, Y, Z \in \mathfrak{m}, m = 1, \dots, 2l+3$. By the formula of Van der Monde, we have

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ 2^0 & 2^1 & \dots & 2^{2l+2} \\ \vdots & \vdots & \ddots & \vdots \\ (2l+3)^0 & (2l+3)^1 & \dots & (2l+3)^{2l+2} \end{pmatrix} = \prod_{1 \leq i < j \leq 2l+3} (j-i) \neq 0.$$

Hence we have $\langle \rho(Y)\rho(X)^{2l+1}v_0, \rho(Z)^{2l+1}v_0 \rangle = 0$. When k is odd, we can prove this in the same way. ■

Proof of Theorem C: We have $\rho(\mathfrak{m}) \subset \mathfrak{p}$ by Lemma 1.4.1. Hence F is a totally geodesic immersion. ■

Remark 1.4.2 Let (ρ, V) be a complex (resp. quaternion) spherical representation which satisfy (1.3.9). Then we can construct a totally geodesic immersion of M into a complex (resp. quaternion) Grassmann manifold in the same way as the proof of Theorem C.

The next example is not contained in Theorem A.

Example $(G, K) = (SU(n), SO(n))(n \geq 3)$.

Since G acts on \mathbf{C}^n naturally, G acts on a complex vector space $W = (\sigma, W) = S^2(\mathbf{C}^n) = \text{span}\{u \cdot v = \frac{1}{2}(u \otimes v + v \otimes u); u, v \in \mathbf{C}^n\}$. Let $\{e_i\}_{1 \leq i \leq n}$ denote the canonical basis of \mathbf{C}^n . Put $v_0 = \sum_{i=1}^n e_i^2 \in W$. Then we have $\sigma(k)v_0 = v_0$ for each $k \in K$. Put $(\rho, V) = (\sigma, W)_{\mathbb{R}}$. Then (ρ, V) is a nontrivial real spherical representation of (G, K) (see Lemma 3.5). The

canonical inner product on $\mathbf{C}^n = \mathbf{R}^{2n}$ naturally induces a G -invariant inner product on V . We define K -invariant subspaces V_k as in (1.3.7). Then we have

$$\begin{aligned} V_0 &= \mathbf{R}v_0, \\ V_1 &= \sum_{1 \leq i < j \leq n} \mathbf{R}\sqrt{-1}e_i \cdot e_j + \left\{ \sum_{i=1}^n x_i \sqrt{-1}e_i^2; x_i \in \mathbf{R}(1 \leq i \leq n), \sum_{i=1}^n x_i = 0 \right\}, \\ V_2 &= \sum_{1 \leq i < j \leq n} \mathbf{R}e_i \cdot e_j + \left\{ \sum_{i=1}^n x_i e_i^2; x_i \in \mathbf{R}(1 \leq i \leq n), \sum_{i=1}^n x_i = 0 \right\}, \\ V_3 &= \mathbf{R}\sqrt{-1}v_0, \\ V &= \sum_{i=0}^3 V_i. \end{aligned}$$

Put F as in (1.3.8), then F is a minimal immersion. Since $V_0 \cong V_3$, $V_1 \cong V_2$ (K -isomorphic), this example is not contained in Theorem A.

In order to prove the irreducibility of (ρ, V) , we prove a few lemmas.

Lemma 1.4.3 *Let (σ, W) be a complex irreducible representation of a compact connected Lie group G . If there exists a weight λ of (σ, W) such that $-\lambda$ is not a weight of (σ, W) , then $(\sigma, W)_{\mathbf{R}}$ is a real irreducible representation of G .*

Proof: If $(\sigma, W)_{\mathbf{R}}$ were not irreducible, then there exists a real representation (ρ, V) of G such that $(\sigma, W) = (\rho, V)^{\mathbf{C}}$ by Lemma 1.1.3. Let J denote the conjugation of W with respect to V . Then J is a conjugate G -linear mapping with $J^2 = 1$. Let T be a maximal torus of G with Lie algebra \mathfrak{t} . Let v_{λ} be a nonzero weight vector of λ , that is,

$$\rho(H)v_{\lambda} = \sqrt{-1}\lambda(H)v_{\lambda} \quad \text{for each } H \in \mathfrak{t}.$$

Since J is conjugate G -linear, we have

$$\rho(H)Jv_{\lambda} = -\sqrt{-1}\lambda(H)Jv_{\lambda} \quad \text{for each } H \in \mathfrak{t}.$$

Since $-\lambda$ is not a weight, we have $Jv_{\lambda} = 0$. Since $J^2 = 1$, this is a contradiction. ■

Lemma 1.4.4 *(σ, W) is a complex irreducible representation of G .*

Proof: We first let E_{ij} denote the matrix whose i -th row and j -th column are given by $\delta_{ir}\delta_{js}$.

It is sufficient to prove that the complexification $\mathfrak{sl}(n, \mathbf{C})$ of $\mathfrak{su}(n)$ acts irreducibly on W . Suppose $W_0 (\neq \{0\})$ is an $\mathfrak{sl}(n, \mathbf{C})$ -invariant subspace of W . In order to prove $W_0 = W$, first we show $v_0 \in W_0$. Let $v = \sum_{1 \leq k < l \leq n} a_{kl} e_k \cdot e_l \in W_0$. Put $i = \min\{k; a_{kl} \neq 0\}$, $j = \min\{l; a_{kl} \neq 0\}$. We may assume $(i, j) \neq (n, n)$. If $i = j (< n)$, then we have $W_0 \ni \sigma(E_{ni})^2 v = 2a_{ii} e_n^2$. If $i < j$, then we have $W_0 \ni \sigma(E_{ni})v = \sum_{j \leq l \leq n} a_{il} e_l \cdot e_n$. Hence, if $i < j = n$, then we have $W_0 \ni a_{in} e_n^2$. If $i < j < n$, then we have $W_0 \ni \sigma(E_{nj})\sigma(E_{ni})v = a_{ij} e_n^2$. Hence we get $e_n^2 \in W_0$. For $1 \leq i \leq n-1$, we have $W_0 \ni \sigma(E_{in})^2 e_n^2 = 2e_i^2$. Hence we have $v_0 \in W_0$. Since $W = \text{span}\{\rho(G)v_0\}$, we have $W_0 = W$. ■

Lemma 1.4.5 (ρ, V) is a real irreducible representation of G .

Proof: Put

$$T = S(\underbrace{U(1) \times \cdots \times U(1)}_{n\text{-times}})$$

and

$$\mathfrak{t} = \{\sqrt{-1}\text{diag}\{x_1, \dots, x_n\}; x_i \in \mathbf{R}(1 \leq i \leq n), \sum_{i=1}^n x_i = 0\}.$$

Then T is a maximal torus of G with Lie algebra \mathfrak{t} . For $H = \sqrt{-1}\text{diag}\{x_1, \dots, x_n\} \in \mathfrak{t}$, we have $\sigma(H)(e_i \cdot e_j) = \sqrt{-1}(x_i + x_j)e_i \cdot e_j$. Since $n \geq 3$, this shows that (ρ, V) is irreducible by Lemma 1.2.4 and Lemma 1.2.5. \blacksquare

1.5 The Jacobi differential operators of equivariant minimal immersions

In this section, we succeed the notation in §1.1 and we assume that F is minimal. We define symmetric linear transformatoins \bar{R}_x and \tilde{A}_x on the normal space $N_x(M)$ at x as follows:

$$\bar{R}_x(v) = \sum_{i=1}^m (\bar{R}(e_i, v)e_i)^\perp, \quad \tilde{A}_x(v) = \sum_{i=1}^m B(e_i, A^v e_i)$$

for $v \in N_x(M)$, where $\{e_i\}_{1 \leq i \leq m}$ is an orthonormal basis of $T_x(M)$. Clearly, we get $\tilde{A} = 0$, if F is totally geodesic. Let $N(M)$ denote the normal bundle of M and $\Gamma(N(M))$ denote the vector space of all C^∞ -sections of $N(M)$. Let Δ denote the negative of the rough Laplacian of the normal connection ∇^\perp of $N(M)$, that is,

$$-\Delta V = \sum_{1 \leq i, j \leq m} g^{ij} \nabla_{E_i}^\perp \nabla_{E_j}^\perp V - \sum_{1 \leq i, j \leq m} g^{ij} \nabla_{\nabla_{E_i} E_j}^\perp V \quad \text{for } V \in \Gamma(N(M)),$$

where $\{E_i\}_{1 \leq i \leq m}$ is a local frame field of M , $g_{ij} = \langle E_i, E_j \rangle$ and $(g^{ij})_{1 \leq i, j \leq m} = (g_{ij})_{1 \leq i, j \leq m}^{-1}$. Since the Jacobi differential operator

$$J = \Delta + \bar{R} - \tilde{A}$$

is a strongly elliptic linear differential operator, it has discrete eigenvalues:

$$\text{Spec}(J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \rightarrow \infty\}.$$

The minimal immersion F is said to be *stable* if the second variation of the volume of F is nonnegative for every variation. The minimal immersion F is stable if and only if $\lambda_1 \geq 0$

(see [28], pp.73-74). Since the orthogonal complement \mathfrak{m}^\perp in \mathfrak{p} is identified with $N_o(M)$ in a natural manner, we may consider \bar{R} and \bar{A} as symmetric linear transformations on \mathfrak{m}^\perp .

Let (U, L) be a compact symmetric pair. An $\text{Ad}(U)$ and θ -invariant inner product \langle, \rangle on \mathfrak{u} naturally induces a U -invariant Riemannian metric on N . Since $\mathfrak{p} = \{X \in \mathfrak{u}; \theta(X) = -X\}$, we have $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$.

Lemma 1.5.1 (1)

$$\bar{R}(v) = \sum_{i=1}^m [F_* X_i, [F_* X_i, v]_{\mathfrak{l}}]_{\mathfrak{m}^\perp} + \frac{1}{4} \sum_{i=1}^m [F_* X_i, [F_* X_i, v]_{\mathfrak{p}}]_{\mathfrak{m}^\perp}$$

for $v \in \mathfrak{m}^\perp$.

If (U, L) is a compact symmetric pair, then

$$\bar{R}(v) = \sum_{i=1}^m [F_* X_i, [F_* X_i, v]]_{\mathfrak{m}^\perp} \quad \text{for } v \in \mathfrak{m}^\perp.$$

$$(2) \quad \tilde{A}(v) = - \sum_{i=1}^m [(\rho_* X_i)_{\mathfrak{l}} + \frac{1}{2}(\rho_* X_i)_{\mathfrak{p}}, [(\rho_* X_i)_{\mathfrak{l}} + \frac{1}{2}(\rho_* X_i)_{\mathfrak{p}}, v]_{F_* \mathfrak{m}}]_{\mathfrak{m}^\perp} \quad \text{for } v \in \mathfrak{m}^\perp.$$

If (U, L) is a compact symmetric pair, then

$$\tilde{A}(v) = - \sum_{i=1}^m [\rho_* X_i, [\rho_* X_i, v]_{F_* \mathfrak{m}}]_{\mathfrak{m}^\perp} \quad \text{for } v \in \mathfrak{m}^\perp.$$

Proof: (1) This follows from (1.1.5).

(2) By using (1.1.3), we have

$$\begin{aligned} \langle A^v X_i, X_j \rangle &= \langle v, (\bar{\nabla}_{(\rho_* X_i)^*} (\rho_* X_j)^*)_o \rangle \\ &= - \langle v, [(\rho_* X_i)_{\mathfrak{p}}, (\rho_* X_j)_{\mathfrak{l}}] + \frac{1}{2} [(\rho_* X_i)_{\mathfrak{p}}, (\rho_* X_j)_{\mathfrak{p}}]_{\mathfrak{p}} \rangle. \end{aligned}$$

Hence we have by Lemma 1.5.1 (1)

$$\langle \tilde{A}(v), w \rangle = - \sum_{i=1}^m \langle [(\rho_* X_i)_{\mathfrak{l}} + \frac{1}{2}(\rho_* X_i)_{\mathfrak{p}}, [(\rho_* X_i)_{\mathfrak{l}} + \frac{1}{2}(\rho_* X_i)_{\mathfrak{p}}, v]_{F_* \mathfrak{m}}]_{\mathfrak{m}^\perp}, w \rangle$$

for $v, w \in \mathfrak{m}^\perp$. ■

The group K acts on \mathfrak{m}^\perp by $\text{Ad}(\rho(k))(k \in K)$. We denote by $(\text{Ad} \circ \rho)^\perp$ this action of K on \mathfrak{m}^\perp . We identify $\Gamma(N(M))$ with

$$C^\infty(G; \mathfrak{m}^\perp)_K = \{\varphi \in C^\infty(G; \mathfrak{m}^\perp); \varphi(gk) = (\text{Ad} \circ \rho)^\perp(k^{-1})\varphi(g) \text{ for } g \in G, k \in K\}$$

by the following correspondence:

$$C^\infty(G; \mathfrak{m}^\perp)_K \ni \varphi \mapsto \tilde{\varphi} \in \Gamma(N(M)); \tilde{\varphi}(gK) = \rho(g)_* \varphi(g) \text{ for } g \in G.$$

We define an action L_x (resp. $\rho(x)_*$) of G on $C^\infty(G; \mathfrak{m}^\perp)_K$ (resp. $\Gamma(N(M))$) as follows:

$$\begin{aligned} (L_x \varphi)(g) &= \varphi(x^{-1}g) \text{ for } \varphi \in C^\infty(G; \mathfrak{m}^\perp)_K, x, g \in G, \\ (\rho(x)_* V)_{gK} &= \rho(x)_* V_{x^{-1}gK} \text{ for } V \in \Gamma(N(M)), x, g \in G. \end{aligned}$$

The action L_x ($x \in G$) on $C^\infty(G; \mathfrak{m}^\perp)_K$ corresponds to $\rho(x)_*$ on $\Gamma(N(M))$. We also denote by J, Δ and \tilde{A} the operators on $C^\infty(G; \mathfrak{m}^\perp)_K$ corresponding to the operators J, Δ and \tilde{A} on $\Gamma(N(M))$, respectively. Let $C = -\sum_{i=1}^p X_i^2$ denote the negative of the Casimir differential operator of G .

Lemma 1.5.2

$$\begin{aligned} \Delta \varphi &= C\varphi + \sum_{i=m+1}^p [\rho_* X_i, [\rho_* X_i, \varphi]] - 2 \sum_{i=1}^m [(\rho_* X_i)_\Gamma, X_i \varphi]_{\mathfrak{m}^\perp} \\ &\quad - \sum_{i=1}^m [(\rho_* X_i)_\Gamma, [(\rho_* X_i)_\Gamma, \varphi]]_{\mathfrak{m}^\perp} - \tilde{A}(\varphi) - \sum_{i=1}^m [(\rho_* X_i)_\mathfrak{p}, X_i \varphi]_{\mathfrak{m}^\perp} \\ &\quad - \frac{1}{2} \sum_{i=1}^m [(\rho_* X_i)_\Gamma, [(\rho_* X_i)_\mathfrak{p}, \varphi]_{\mathfrak{p}}]_{\mathfrak{m}^\perp} - \frac{1}{2} \sum_{i=1}^m [(\rho_* X_i)_\mathfrak{p}, [(\rho_* X_i)_\Gamma, \varphi]]_{\mathfrak{m}^\perp} \\ &\quad - \frac{1}{4} \sum_{i=1}^m [(\rho_* X_i)_\mathfrak{p}, [(\rho_* X_i)_\mathfrak{p}, \varphi]_{\mathfrak{p}}]_{\mathfrak{m}^\perp} \text{ for } \varphi \in C^\infty(G; \mathfrak{m}^\perp)_K. \end{aligned}$$

If (U, L) is a compact symmetric pair, then

$$\begin{aligned} \Delta \varphi &= C\varphi + \sum_{i=m+1}^p [\rho_* X_i, [\rho_* X_i, \varphi]] - 2 \sum_{i=1}^m [(\rho_* X_i)_\Gamma, X_i \varphi]_{\mathfrak{m}^\perp} \\ &\quad - \sum_{i=1}^m [(\rho_* X_i)_\Gamma, [(\rho_* X_i)_\Gamma, \varphi]]_{\mathfrak{m}^\perp} - \tilde{A}(\varphi) \text{ for } \varphi \in C^\infty(G; \mathfrak{m}^\perp)_K. \end{aligned}$$

Proof: For $V = \tilde{\varphi} \in \Gamma(N(M))$, we have

$$-(\Delta V)(o) = \left(\sum_{i=1}^m \bar{\nabla}_{(\rho_* X_i)_*} \bar{\nabla}_{(\rho_* X_i)_*} V \right)^\perp(o) + \tilde{A}(V_o).$$

Put $W_i = \bar{\nabla}_{(\rho_* X_i)_*} V$ ($1 \leq i \leq m$). Then we have by (1.1.1) and (1.1.4)

$$\begin{aligned} W_{i, \exp t \rho_* X_i L} &= \frac{d}{ds} (\exp(-s \rho_* X_i))_* V_{\exp(t+s) \rho_* X_i L} |_{s=0} \\ &\quad + (\exp t \rho_* X_i)_* [(\rho_* X_i)_\Gamma, (\exp(-t \rho_* X_i))_* V_{\exp t \rho_* X_i L}] \\ &\quad + \frac{1}{2} (\exp t \rho_* X_i)_* [(\rho_* X_i)_\mathfrak{p}, (\exp(-t \rho_* X_i))_* V_{\exp t \rho_* X_i L}]_{\mathfrak{p}}. \end{aligned}$$

In particular

$$W_{i,o} = \frac{d}{ds}(\exp(-s\rho_* X_i))_* V_{\exp s\rho_* X_i L|s=0} + [(\rho_* X_i)_l, V_o] + \frac{1}{2}[(\rho_* X_i)_p, V_o]_p.$$

Thus we have by (1.1.1) and (1.1.4)

$$\begin{aligned} & (\bar{\nabla}_{(\rho_* X_i)}^* \bar{\nabla}_{(\rho_* X_i)}^* V)(o) \\ &= (\bar{\nabla}_{(\rho_* X_i)}^* W_i)_o \\ &= \frac{d}{dt}(\exp(-t\rho_* X_i))_* W_{i,\exp t\rho_* X_i L|t=0} + [(\rho_* X_i)_l, W_{i,o}] + \frac{1}{2}[(\rho_* X_i)_p, W_{i,o}]_p \\ &= \frac{\partial^2}{\partial t \partial s}(\exp(-(t+s)\rho_* X_i))_* V_{\exp(t+s)\rho_* X_i L|t=s=0} \\ &\quad + 2[(\rho_* X_i)_l, \frac{d}{dt}(\exp(-t\rho_* X_i))_* V_{\exp t\rho_* X_i L|t=0}] + [(\rho_* X_i)_l, [(\rho_* X_i)_l, V_o]] \\ &\quad + [(\rho_* X_i)_p, \frac{d}{dt}(\exp(-t\rho_* X_i))_* V_{\exp t\rho_* X_i L|t=0}]_p + \frac{1}{2}[(\rho_* X_i)_l, [(\rho_* X_i)_p, V_o]_p] \\ &\quad + \frac{1}{2}[(\rho_* X_i)_p, [(\rho_* X_i)_l, V_o]]_p + \frac{1}{4}[(\rho_* X_i)_p, [(\rho_* X_i)_p, V_o]_p]_p. \end{aligned}$$

So we get

$$\begin{aligned} -(\Delta\varphi)(e) &= (\sum_{i=1}^m X_i^2 \varphi)(e) + 2 \sum_{i=1}^m [(\rho_* X_i)_l, (X_i \varphi)(e)]_{\mathfrak{m}^\perp} \\ &\quad + \sum_{i=1}^m [(\rho_* X_i)_l, [(\rho_* X_i)_l, \varphi(e)]]_{\mathfrak{m}^\perp} + \tilde{A}(\varphi(e)) + \sum_{i=1}^m [(\rho_* X_i)_p, (X_i \varphi)(e)]_{\mathfrak{m}^\perp} \\ &\quad + \frac{1}{2} \sum_{i=1}^m [(\rho_* X_i)_l, [(\rho_* X_i)_p, \varphi(e)]]_{\mathfrak{p} \mathfrak{m}^\perp} + \frac{1}{2} \sum_{i=1}^m [(\rho_* X_i)_p, [(\rho_* X_i)_l, \varphi(e)]]_{\mathfrak{m}^\perp} \\ &\quad + \frac{1}{4} \sum_{i=1}^m [(\rho_* X_i)_p, [(\rho_* X_i)_p, \varphi(e)]]_{\mathfrak{p} \mathfrak{m}^\perp}. \end{aligned}$$

Since $\Delta L_x = L_x \Delta$ for $x \in G$ by the equivariance of F , we get the conclusion. \blacksquare

We define operators $J_i : C^\infty(G; \mathfrak{m}^\perp)_K \rightarrow C^\infty(G; \mathfrak{m}^\perp)_K (i = 1, 2)$ by

$$\begin{aligned} J_1 \varphi &= \sum_{i=1}^p [\rho_* X_i, X_i \varphi]_{\mathfrak{m}^\perp} + \sum_{i=1}^p [\rho_* X_i, [\rho_* X_i, \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}, \\ J_2 \varphi &= \sum_{i=1}^p [(\rho_* X_i)_p, X_i \varphi]_{\mathfrak{m}^\perp} + \sum_{i=1}^p [(\rho_* X_i)_p, [\rho_* X_i, \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp} \end{aligned}$$

for $\varphi \in C^\infty(G; \mathfrak{m}^\perp)_K$.

Remark 1.5.3 It follows that $\sum_{i=1}^p [\rho_* X_i, X_i \varphi]_{\mathfrak{m}^\perp}$, $\sum_{i=1}^p [\rho_* X_i, [\rho_* X_i, \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}$, $\sum_{i=1}^p [(\rho_* X_i)_p, X_i \varphi]_{\mathfrak{m}^\perp}$ and $\sum_{i=1}^p [(\rho_* X_i)_p, [\rho_* X_i, \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp} \in C^\infty(G; \mathfrak{m}^\perp)_K$ for $\varphi \in C^\infty(G; \mathfrak{m}^\perp)_K$. Moreover each of the above four operators is commutative with L_x for $x \in G$. If (U, L) is a compact symmetric pair, then $J_2 = 0$. \blacksquare

Theorem 1.5.4 (1)

$$\begin{aligned} J\varphi &= C\varphi - 2J_1\varphi + [\sum_{i=1}^p (\text{ad } \rho_* X_i)^2 \varphi]_{\mathfrak{m}^\perp} + J_2\varphi - \sum_{i=1}^p [\rho_* X_i, [(\rho_* X_i)_p, \varphi]_{F, \mathfrak{m}}]_{\mathfrak{m}^\perp} \\ &\quad + \frac{1}{2} \sum_{i=1}^p [(\rho_* X_i)_p, [(\rho_* X_i)_p, \varphi]_{F, \mathfrak{m}}]_{\mathfrak{m}^\perp} \end{aligned}$$

for $\varphi \in C^\infty(G; \mathfrak{m}^\perp)_K$.

(2) $J_1 = 0$ if and only if $[\rho_* X, v]_{\mathfrak{m}^\perp} = 0$ for $X \in \mathfrak{m}, v \in \mathfrak{m}^\perp$.

(3) $J_2 = 0$ if and only if $[(\rho_* X)_\mathfrak{p}, v]_{\mathfrak{m}^\perp} = 0$ for $X \in \mathfrak{m}, v \in \mathfrak{m}^\perp$.

(4) If (U, L) is a compact symmetric pair, then

$$J\varphi = C\varphi - 2J_1\varphi + \left[\sum_{i=1}^p (\text{ad } \rho_* X_i)^2 \varphi \right]_{\mathfrak{m}^\perp} \text{ for } \varphi \in C^\infty(G; \mathfrak{m}^\perp)_K.$$

Proof: (1) follows from Lemma 1.5.1, Lemma 1.5.2 and minimal condition (Proposition 1.1.1, 4.1(2)).

(2) and (3) are obtained in the same way as [26, I, p. 138, Proposition 4.2.2]. (4) follows from (1). ■

Remark 1.5.5 It follows that $C\varphi, [\sum_{i=1}^p (\text{ad } \rho_* X_i)^2 \varphi]_{\mathfrak{m}^\perp}, \sum_{i=1}^p [\rho_* X_i, [(\rho_* X_i)_\mathfrak{p}, \varphi]_{F, \mathfrak{m}}]_{\mathfrak{m}^\perp}$ and $\sum_{i=1}^p [(\rho_* X_i)_\mathfrak{p}, [(\rho_* X_i)_\mathfrak{p}, \varphi]_{F, \mathfrak{m}}]_{\mathfrak{m}^\perp} \in C^\infty(G; \mathfrak{m}^\perp)_K$ for $\varphi \in C^\infty(G; \mathfrak{m}^\perp)_K$. Moreover each of the above four operators is commutative with L_x for $x \in G$. ■

Let $D(G)$ be the set of equivalence classes of finite dimensional \mathbf{C} -irreducible representations of G . Let $c_\sigma (\geq 0)$ be the eigenvalue of the negative of the Casimir operator of $(\sigma, W) \in D(G)$. (By the formula of Freudenthal, we can determine c_σ (see [30], p.205).) For $(\sigma, W) \in D(G)$, put

$$(W^* \otimes (\mathfrak{m}^\perp)^{\mathbf{C}})_0 = \{\alpha \in W^* \otimes (\mathfrak{m}^\perp)^{\mathbf{C}}; (\sigma^* \otimes (\text{Ad} \circ \rho)^\perp)(k)\alpha = \alpha \text{ for } k \in K\}.$$

Put

$$D(G; K, (\text{Ad} \circ \rho)^\perp) = \{(\sigma, W) \in D(G); (W^* \otimes (\mathfrak{m}^\perp)^{\mathbf{C}})_0 \neq \{0\}\}.$$

For $(\sigma, W) \in D(G; K, (\text{Ad} \circ \rho)^\perp)$, we define a symmetric linear mapping $J_\sigma \in \text{End}((W^* \otimes (\mathfrak{m}^\perp)^{\mathbf{C}})_0)$ as follows:

$$\begin{aligned} J_\sigma = & c_\sigma 1 - 2 \left\{ \sum_{i=1}^p \sigma^*(X_i) \otimes [\rho_* X_i, *]_{\mathfrak{m}^\perp} + 1_{W^*} \otimes \sum_{i=1}^p [\rho_* X_i, [\rho_* X_i, *]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp} \right\} \\ & + 1_{W^*} \otimes [\sum_{i=1}^p (\text{ad } \rho_* X_i)^2 *]_{\mathfrak{m}^\perp} \\ & + \left\{ \sum_{i=1}^p \sigma^*(X_i) \otimes [(\rho_* X_i)_\mathfrak{p}, *]_{\mathfrak{m}^\perp} + 1_{W^*} \otimes \sum_{i=1}^p [(\rho_* X_i)_\mathfrak{p}, [\rho_* X_i, *]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp} \right\} \\ & + 1_{W^*} \otimes \left\{ \frac{1}{2} \sum_{i=1}^p [(\rho_* X_i)_\mathfrak{p}, [(\rho_* X_i)_\mathfrak{p}, *]_{F, \mathfrak{m}}]_{\mathfrak{m}^\perp} - \sum_{i=1}^p [\rho_* X_i, [(\rho_* X_i)_\mathfrak{p}, *]_{F, \mathfrak{m}}]_{\mathfrak{m}^\perp} \right\}. \end{aligned}$$

Clearly, if (U, L) is a compact symmetric pair, then

$$\begin{aligned} J_\sigma = & c_\sigma 1 - 2 \left\{ \sum_{i=1}^p \sigma^*(X_i) \otimes [\rho_* X_i, *]_{\mathfrak{m}^\perp} + 1_{W^*} \otimes \sum_{i=1}^p [\rho_* X_i, [\rho_* X_i, *]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp} \right\} \\ & + 1_{W^*} \otimes [\sum_{i=1}^p (\text{ad } \rho_* X_i)^2 *]_{\mathfrak{m}^\perp}. \end{aligned}$$

By virtue of the Peter-Weyl theorem for homogeneous vector bundles, the problem of computing the spectra of J is reduced to the eigenvalue problem of the linear mapping J_σ with $(\sigma, W) \in D(G; K, (\text{Ad} \circ \rho)^\perp)$ (see [26, I, §5]).

Theorem 1.5.6 For $(\sigma, W) \in D(G; K, (\text{Ad} \circ \rho)^\perp)$, let $\{\lambda_{\sigma;1}, \dots, \lambda_{\sigma;m_\sigma}\}$ be the eigenvalues of J_σ , where $m_\sigma = \dim(W^* \otimes (\mathfrak{m}^\perp)^{\mathbb{C}})_0$. Then

$$\text{Spec}(J) = \bigcup_{(\sigma, W) \in D(G; K, (\text{Ad} \circ \rho)^\perp)} \underbrace{\{\lambda_{\sigma;1}, \dots, \lambda_{\sigma;1}\}}_{d_\sigma}, \dots, \underbrace{\{\lambda_{\sigma;m_\sigma}, \dots, \lambda_{\sigma;m_\sigma}\}}_{d_\sigma},$$

where $d_\sigma = \dim W$.

Remark 1.5.7 If (U, L) is a compact symmetric pair and F is a totally geodesic immersion, then

$$J_\sigma = c_\sigma 1 + 1_{W^*} \otimes \sum_{i=1}^p (\text{ad} \rho_* X_i)^2 *.$$

K. Mashimo, Y. Ohnita and H. Tasaki studied the stability of various totally geodesic submanifolds in compact symmetric spaces using this formula. ([21], [22], [23] and [27]) ■

Remark 1.5.8 We state K. Mashimo's study [24] related to Theorem 1.5.6. Let G be a compact connected simple Lie group and σ an automorphism on G of order 3. Take an $\text{Ad}(G)$ and $d\sigma$ -invariant inner product \langle, \rangle on \mathfrak{g} . Put $K = \{g \in G; \sigma(g) = g\}$. We consider the Cartan embedding

$$\Psi_\sigma : G/K \rightarrow G; gK \mapsto g\sigma(g^{-1}).$$

The induced Riemannian metric on M by the Cartan embedding Ψ_σ is a normal homogeneous metric: For $X, Y \in \mathfrak{m}$,

$$\langle \Psi_{\sigma*}(X), \Psi_{\sigma*}(Y) \rangle = 3\langle X, Y \rangle$$

By Proposition 1.1.1 (1), the second fundamental form B of the Cartan embedding Ψ_σ is given by

$$B(X, Y) = \frac{1}{2}[\sigma(X), Y] - \frac{1}{2}[X, \sigma(Y)] \in \mathfrak{k} \quad \text{for } X, Y \in \mathfrak{m}.$$

Put

$$\begin{aligned} U &= G \times G, \\ L &= \{(g, g); g \in G\}, \\ \mathfrak{l} &= \{(X, X); X \in \mathfrak{g}\}, \\ \mathfrak{p} &= \{(X, -X); X \in \mathfrak{g}\} \\ \rho &: G \rightarrow G \times G; g \mapsto (g, \sigma(g)). \end{aligned}$$

When we identify G with U/L by $G \cong U/L; ab^{-1} \mapsto (a, b)L$, we get

$$\Psi_\sigma : G \rightarrow U/L; gK \mapsto \rho(g)L,$$

$$\mathfrak{m}^\perp = \{(Y, -Y); Y \in \mathfrak{k}\} \cong \mathfrak{k}.$$

We assume that Ψ_σ is minimal. For $X \in \mathfrak{m}, Y \in \mathfrak{k}$,

$$\begin{aligned} [\rho(X), (Y, -Y)]_{\mathfrak{m}^\perp} &= (\tfrac{1}{2}[X + \sigma(X), Y], -\tfrac{1}{2}[X + \sigma(X), Y]) \\ &= 0, \end{aligned}$$

which implies that $J_1 = 0$ by Theorem 1.5.4, (2). Hence we get

$$J_\sigma = c_\sigma 1 + 1_{W^*} \otimes \left[\sum_{i=1}^p (\text{ad}_{\rho_*} E_i)^2 * \right]_{\mathfrak{m}^\perp}.$$

K. Mashimo studied the stability of Cartan embeddings of compact Riemannian 3-symmetric spaces by using this formula. ■

1.6 An example

In this section, we denote by G (resp. K) the special unitary group $SU(2)$ of degree 2 (resp. the closed subgroup $S(U(1) \times U(1))$ of $SU(2)$). Basic notations are same in §4. We define an $\text{Ad}(G)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} by

$$\langle X, Y \rangle = -2\text{Tr}(XY) \quad \text{for } X, Y \in \mathfrak{g}.$$

We also denote by $\langle \cdot, \cdot \rangle$ the induced G -invariant Riemannian metric on $S^2 = G/K$. Then the Riemannian manifold $(S^2, \langle \cdot, \cdot \rangle)$ is of constant curvature 1. We choose an orthonormal basis $\{e_i\}_{1 \leq i \leq 3}$ of \mathfrak{g} as follows:

$$e_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, e_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 = \frac{1}{2} \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

First we write down all G - E -irreducible representations:

Lemma 1.6.1 G - E -irreducible representation (ρ, V) is one of the following:

(1) The case of $E = \mathbf{C}$:

There exists an orthonormal basis $\{f_k\}_{0 \leq k \leq n}$ of V such that

$$\begin{aligned} \rho(e_3)f_k &= \frac{\sqrt{-1}}{2}(-n+2k)f_k, \\ \rho(e_1)f_k &= \frac{\sqrt{-1}}{2} \left\{ \sqrt{(n-k)(k+1)}f_{k+1} + \sqrt{k(n-k+1)}f_{k-1} \right\}, \\ \rho(e_2)f_k &= \frac{1}{2} \left\{ -\sqrt{(n-k)(k+1)}f_{k+1} + \sqrt{k(n-k+1)}f_{k-1} \right\} \end{aligned}$$

for $0 \leq k \leq n$.

(2) The case of $E = \mathbf{R}$:

(2-a) There exists an orthonormal basis $\{h_0\} \cup \{f_k, g_k\}_{1 \leq k \leq n}$ of V such that

$$\begin{aligned} \rho(e_3)h_0 &= 0, \\ \rho(e_3)f_k &= kg_k, \\ \rho(e_3)g_k &= -kf_k, \\ \rho(e_1)h_0 &= \sqrt{\frac{n(n+1)}{2}}g_1, \\ \rho(e_1)f_k &= \frac{1}{2}\sqrt{(n-k)(n+k+1)}g_{k+1} + \frac{1}{2}\sqrt{(n+k)(n-k+1)}g_{k-1}, \\ \rho(e_1)g_k &= -\frac{1}{2}\sqrt{(n-k)(n+k+1)}f_{k+1} - \frac{1}{2}\sqrt{(n+k)(n-k+1)}f_{k-1}, \\ \rho(e_2)h_0 &= -\sqrt{\frac{n(n+1)}{2}}f_1, \\ \rho(e_2)f_k &= -\frac{1}{2}\sqrt{(n-k)(n+k+1)}f_{k+1} + \frac{1}{2}\sqrt{(n+k)(n-k+1)}f_{k-1}, \\ \rho(e_2)g_k &= -\frac{1}{2}\sqrt{(n-k)(n+k+1)}g_{k+1} + \frac{1}{2}\sqrt{(n+k)(n-k+1)}g_{k-1} \end{aligned}$$

for $1 \leq k \leq n$, where we put $f_0 = \sqrt{2}h_0, g_0 = 0$.

(2-b) There exists an orthonormal basis $\{f_k, g_k\}_{0 \leq k \leq 2n-1}$ of V such that

$$\begin{aligned}\rho(e_3)f_k &= \frac{1}{2}(-2n+1+2k)g_k, \\ \rho(e_3)g_k &= -\frac{1}{2}(-2n+1+2k)f_k, \\ \rho(e_1)f_k &= \frac{1}{2}\sqrt{(2n-1-k)(k+1)}g_{k+1} + \frac{1}{2}\sqrt{k(2n-k)}g_{k-1}, \\ \rho(e_1)g_k &= -\frac{1}{2}\sqrt{(2n-1-k)(k+1)}f_{k+1} - \frac{1}{2}\sqrt{k(2n-k)}f_{k-1}, \\ \rho(e_2)f_k &= -\frac{1}{2}\sqrt{(2n-1-k)(k+1)}f_{k+1} + \frac{1}{2}\sqrt{k(2n-k)}f_{k-1}, \\ \rho(e_2)g_k &= -\frac{1}{2}\sqrt{(2n-1-k)(k+1)}g_{k+1} + \frac{1}{2}\sqrt{k(2n-k)}g_{k-1}\end{aligned}$$

for $0 \leq k \leq 2n-1$.

(3) The case of $E = \mathbf{H}$:

(3-a) There exists an orthonormal basis $\{f_k\}_{0 \leq k \leq n-1}$ of V such that

$$\begin{aligned}\rho(e_3)f_k &= \frac{i}{2}(-2n+1+2k)f_k, \\ \rho(e_1)f_k &= \frac{i}{2}(1-\delta_{k,n-1})\left\{\sqrt{(2n-1-k)(k+1)}f_{k+1} + \sqrt{k(2n-k)}f_{k-1}\right\} \\ &\quad + \frac{\delta_{k,n-1}}{2}\left\{n(-1)^{n+1}\mathbf{k}f_{n-1} + i\sqrt{(n-1)(n+1)}f_{n-2}\right\}, \\ \rho(e_2)f_k &= \frac{1}{2}(1-\delta_{k,n-1})\left\{-\sqrt{(2n-1-k)(k+1)}f_{k+1} + \sqrt{k(2n-k)}f_{k-1}\right\} \\ &\quad + \frac{\delta_{k,n-1}}{2}\left\{-n(-1)^{n+1}\mathbf{j}f_{n-1} + \sqrt{(n-1)(n+1)}f_{n-2}\right\}\end{aligned}$$

for $0 \leq k \leq n-1$.

(3-b) There exists an orthonormal basis $\{f_k\}_{0 \leq k \leq 2n}$ of V such that

$$\begin{aligned}\rho(e_3)f_k &= i(-n+k)f_k, \\ \rho(e_1)f_k &= \frac{i}{2}\left\{\sqrt{(2n-k)(k+1)}f_{k+1} + \sqrt{k(2n-k+1)}f_{k-1}\right\}, \\ \rho(e_2)f_k &= \frac{1}{2}\left\{-\sqrt{(2n-k)(k+1)}f_{k+1} + \sqrt{k(2n-k+1)}f_{k-1}\right\}\end{aligned}$$

for $0 \leq k \leq 2n$, where we denote the differential representation of the representation ρ of G by the same symbol ρ .

Proof: (1) is obtained from [8, Theorem 1.3, p. 599].

(2) is obtained from (1), Lemma 1.1.2 and Lemma 1.1.3:

(2-a) When $n = 2m$ in (1), put

$$\begin{aligned}p_k &= \begin{cases} f_{k+m} - f_{-k+m} & (k = \text{odd}), \\ f_{k+m} + f_{-k+m} & (k = \text{even}), \end{cases} \\ q_k &= \begin{cases} \sqrt{-1}(f_{k+m} + f_{-k+m}) & (k = \text{odd}), \\ \sqrt{-1}(f_{k+m} - f_{-k+m}) & (k = \text{even}), \end{cases} \quad h_0 = \sqrt{2}f_m = \frac{1}{\sqrt{2}}p_0.\end{aligned}$$

and

$$W = \mathbf{R}h_0 + \sum_{k=1}^m (\mathbf{R}p_k + \mathbf{R}q_k).$$

Then W is a G - \mathbf{R} -irreducible representation. We rewrite $m = n, p_k = f_k, q_k = g_k$ and $W = V$. We get (2-a).

(2-b) When $n = 2m - 1$ in (1), $((\rho_{2m-1})_{\mathbf{R}}, (V_{2m-1})_{\mathbf{R}})$ is a G - \mathbf{R} -irreducible representation. Put $g_k = \sqrt{-1}f_k$. We rewrite $m = n$. We get (2-b).

(3) is obtained from (1), Lemma 1.1.4 and Lemma 1.1.5:

(3-a) When $n = 2m - 1$ in (1), we define a conjugate G -linear mapping \mathfrak{J} by

$$\mathfrak{J}f_k = (-1)^k f_{2m-1-k}.$$

Since $\mathfrak{J}^2 = -1$, (ρ_{2m-1}, V_{2m-1}) is considered as a G - \mathbf{H} -representation (σ_m, W_m) . $\{f_k\}_{0 \leq k \leq m-1}$ is an orthonormal basis of (σ_m, W_m) and $f_{2m-1-k} = (-1)^k \mathfrak{J}f_k$. We rewrite $m = n$ and $(\sigma_n, W_n) = (\rho_n, V_n)$. We get (3-a).

(3-b) When $n = 2m$ in (1), $((\rho_{2m})^{\mathbf{H}}, (V_{2m})^{\mathbf{H}})$ is a G - \mathbf{H} -irreducible representation. We rewrite $m = n$. We get (3-b). ■

For G - E -irreducible representation (ρ, V) , put

$$X = \begin{cases} \{0, 1, \dots, n\} & \text{(if } (\rho, V) \text{ is type(1))}, \\ \{1, \dots, n\} & \text{(if } (\rho, V) \text{ is type(2-a))}, \\ \{0, 1, \dots, 2n - 1\} & \text{(if } (\rho, V) \text{ is type(2-b))}, \\ \{0, 1, \dots, n - 1\} & \text{(if } (\rho, V) \text{ is type(3-a))}, \\ \{0, 1, \dots, 2n\} & \text{(if } (\rho, V) \text{ is type(3-b))}. \end{cases}$$

For subsets $P (\neq \emptyset), Q (\neq \emptyset)$ of X with $X = P \cup Q$ (disjoint union), put

$$V_P = \begin{cases} \sum_{p \in P} \mathbf{C}f_p & \text{(if } (\rho, V) \text{ is type(1))}, \\ \mathbf{R}h_0 + \sum_{p \in P} (\mathbf{R}f_p + \mathbf{R}g_p) & \text{(if } (\rho, V) \text{ is type(2-a))}, \\ \sum_{p \in P} (\mathbf{R}f_p + \mathbf{R}g_p) & \text{(if } (\rho, V) \text{ is type(2-b))}, \\ \sum_{p \in P} \mathbf{H}f_p & \text{(if } (\rho, V) \text{ is type(3-a) or (3-b))}, \end{cases}$$

$$V_Q = \begin{cases} \sum_{q \in Q} \mathbf{C}f_q & \text{(if } (\rho, V) \text{ is type(1))}, \\ \sum_{q \in Q} (\mathbf{R}f_q + \mathbf{R}g_q) & \text{(if } (\rho, V) \text{ is type(2-a))}, \\ \sum_{q \in Q} (\mathbf{R}f_q + \mathbf{R}g_q) & \text{(if } (\rho, V) \text{ is type(2-b))}, \\ \sum_{q \in Q} \mathbf{H}f_q & \text{(if } (\rho, V) \text{ is type(3-a) or (3-b))}, \end{cases}$$

$$a = \dim_E V_P, \quad b = \dim_E V_Q.$$

Then $V_P(\neq \{0\})$ and $V_Q(\neq \{0\})$ are K - E -invariant subspaces of V and

$$V = V_P + V_Q(\text{direct sum}), \quad \text{Hom}_K(V_P, V_Q) = \{0\}.$$

Thus

$$F : S^2 = G/K \rightarrow G_{a,b}(E) = U(a+b)/U(a) \times U(b); gK \mapsto \rho(g)U(a) \times U(b) \text{ for } g \in G$$

is a full minimal immersion by Theorem A. Note that $S^2, G_{n,2}(\mathbf{R}), G_{m,n}(\mathbf{C})$ are Hermitian symmetric spaces.

Proposition 1.6.2 (1) *The case where (ρ, V) is type(1):*

F is totally geodesic if and only if $P = \{\text{even}\}, Q = \{\text{odd}\}$ (or $P = \{\text{odd}\}, Q = \{\text{even}\}$),

F is a Kähler immersion if and only if $P = \{0, 1, \dots, k\}, Q = \{k+1, \dots, n\}$ (or $Q = \{0, 1, \dots, k\}, P = \{k+1, \dots, n\}$).

(2) (2-a) *The case where (ρ, V) is type(2-a):*

F is totally geodesic if and only if $P = \{\text{even}\}, Q = \{\text{odd}\}$,

F is a Kähler immersion if and only if $Q = \{n\}$,

(2-b) *The case where (ρ, V) is type(2-b):*

F is totally geodesic if and only if $P = \{\text{even}\}, Q = \{\text{odd}\}$ (or $P = \{\text{odd}\}, Q = \{\text{even}\}$),

F is a Kähler immersion if and only if $P = \{0\}, \{2n-1\}$ (or $Q = \{0\}, \{2n-1\}$).

(3) (3-a) *The case where (ρ, V) is type(3-a): F is not totally geodesic,*

(3-b) *The case where (ρ, V) is type(3-b):*

F is totally geodesic if and only if $P = \{\text{even}\}, Q = \{\text{odd}\}$ (or $P = \{\text{odd}\}, Q = \{\text{even}\}$).

Proof: It follows from Proposition 4.1(1). ■

For example, when (ρ, V) is type (2-a) and $P = \{1\}, Q = \{2\}$, we calculate $\text{Spec}(J)$ by using Theorem 1.5.6. In this case, since F is a Kähler immersion, F is stable (see [28], p.76, Theorem 3.5.1).

Theorem 1.6.3 *The spectra of J is given as follows:*

$$\text{Spec}(J) = \{\lambda_n^\pm = (n+3)(n-2) \pm \sqrt{6(n+3)(n-2)} \quad (m(\lambda_n^\pm) = 2(2n+1)); n = 2, 3, \dots\},$$

where we denote the multiplicity of λ_n^\pm by $m(\lambda_n^\pm)$.

Proof: In this case,

$$V_P = \mathbf{R}h_0 + \mathbf{R}f_1 + \mathbf{R}g_1, \quad V_Q = \mathbf{R}f_2 + \mathbf{R}g_2.$$

The expression matrix of $\rho(e_i)$ ($1 \leq i \leq 3$) with respect to an orthonormal basis $\{h_0, f_1, g_1, f_2, g_2\}$ of V is as follows:

$$\begin{aligned} \rho(e_1) &= \begin{pmatrix} 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \\ \rho(e_3) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}. \end{aligned}$$

Hence we have

$$\mathfrak{m}^\perp = \left\{ v(x, y, z, w) = \begin{pmatrix} 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & w \\ 0 & 0 & 0 & w & -z \\ -x & -z & -w & 0 & 0 \\ -y & -w & z & 0 & 0 \end{pmatrix}; x, y, z, w \in \mathbf{R} \right\}$$

and

$$[\rho(e_3), v(x, y, z, w)] = v(-2y, 2x, -3w, 3z).$$

Put

$$\mathfrak{m}_1^\perp = \{v(x, y, 0, 0); x, y \in \mathbf{R}\}, \quad \mathfrak{m}_2^\perp = \{v(0, 0, z, w); z, w \in \mathbf{R}\}.$$

Then we have

$$\mathfrak{m} = \mathfrak{m}_1^\perp + \mathfrak{m}_2^\perp \quad (K\text{-irreducible decomposition})$$

and

$$\sum_{i=1}^3 [\rho(e_i), [\rho(e_i), *]_{\mathfrak{m}_1^\perp}]_{\mathfrak{m}_1^\perp} = \begin{cases} -7id & \text{on } \mathfrak{m}_1^\perp, \\ -12id & \text{on } \mathfrak{m}_2^\perp. \end{cases}$$

Put

$$v_\pm = \sqrt{2}v(1, \pm\sqrt{-1}, 0, 0) \in (\mathfrak{m}_1^\perp)^C, \quad w_\pm = \sqrt{2}v(0, 0, 1, \pm\sqrt{-1}) \in (\mathfrak{m}_2^\perp)^C.$$

Then

$$[\rho(e_3), v_\pm] = \mp 2\sqrt{-1}v_\pm, \quad [\rho(e_3), w_\pm] = \mp 3\sqrt{-1}w_\pm.$$

Hence we have

$$D(G, K, (\text{Ad} \circ \rho)^\perp) = \{(\rho_{2n}, V_{2n}); n = 2, 3, \dots\},$$

where we denote by (ρ_{2n}, V_{2n}) the complex irreducible representation of G of degree $2n + 1$. The expression matrix of $\sum_{i=1}^3 \rho_{2n}^*(e_i) \otimes [\rho(e_i), *]_{\mathfrak{m}^\perp}$ with respect to an orthogonal basis $\{f_{n+2}^* \otimes v_-, f_{n+3}^* \otimes w_-, f_{n-2}^* \otimes v_+, f_{n-3}^* \otimes w_-\}$ of $(V_{2n}^* \otimes (\mathfrak{m}^\perp)^{\mathbb{C}})_0$ is as follows:

$$\sum_{i=1}^3 \rho_{2n}^*(e_i) \otimes [\rho(e_i), *]_{\mathfrak{m}^\perp} = \begin{pmatrix} 4 & \alpha & 0 & 0 \\ \alpha & 9 & 0 & 0 \\ 0 & 0 & 4 & -\alpha \\ 0 & 0 & -\alpha & 9 \end{pmatrix},$$

where we put $\alpha = \frac{1}{2}\sqrt{6(n+3)(n-2)}$. Since the eigenvalue of the Casimir operator of (ρ_{2n}, V_{2n}) is $n(n+1)$, we have

$$J_{\rho_{2n}} = \{n(n+1) - 12\}id - 2 \begin{pmatrix} -3 & \alpha & 0 & 0 \\ \alpha & -3 & 0 & 0 \\ 0 & 0 & -3 & -\alpha \\ 0 & 0 & -\alpha & -3 \end{pmatrix}.$$

Hence we get the conclusion by Theorem 1.5. 6 . ■

Chapter 2

Invariant connections on compact simple Lie groups

2.1 Preliminaries

Let L be a compact connected simple Lie group with Lie algebra \mathfrak{l} . Take an $\text{Ad}(L)$ -invariant inner product \langle, \rangle on \mathfrak{l} . Let G be another compact connected Lie group with Lie algebra \mathfrak{g} . Take an $\text{Ad}(G)$ -invariant inner product \langle, \rangle on \mathfrak{g} . Let $\rho : L \rightarrow G$ be a Lie homomorphism. We denote the differential Lie homomorphism of ρ by the same symbol ρ . Put

$$K = L \times L \supset H = \{(l, l); l \in L\} \cong L \quad ((l, l) \leftrightarrow l) \quad \text{and} \quad M = K/H.$$

We define an inner product \langle, \rangle on \mathfrak{k} by

$$\langle (X, Y), (Z, W) \rangle = 2(\langle X, Z \rangle + \langle Y, W \rangle) \quad \text{for} \quad X, Y, Z, W \in \mathfrak{l}.$$

We define an $\text{Ad}(H)$ -invariant subspace \mathfrak{m} of \mathfrak{k} by

$$\mathfrak{m} = \{(X, -X); X \in \mathfrak{l}\}.$$

Then we have:

$$\mathfrak{k} = \mathfrak{h} + \mathfrak{m} \quad (\text{direct sum}).$$

The induced $\text{Ad}(H)$ -invariant inner product on \mathfrak{m} naturally induces a K -invariant Riemannian metric on M . The mapping

$$\mathfrak{m} \rightarrow \mathfrak{l}; \quad \left(\frac{1}{2}X, -\frac{1}{2}X \right) \mapsto X$$

is a linear isomerty from \mathfrak{m} onto \mathfrak{l} . In this correspondence, we have

$$(\text{Ad}(H), \mathfrak{m}) \cong (\text{Ad}(L), \mathfrak{l}).$$

The mapping

$$(a, b)H \mapsto ab^{-1}$$

is an isometry from M onto L .

We define a Lie homomorphism $\bar{\rho}$ from H into G by

$$\bar{\rho} : H \rightarrow G; (l, l) \mapsto \rho(l).$$

Every Lie homomorphism from H into G is obtained in this way. The space of invariant connections on the principal G -bundle $P = K \times_{\bar{\rho}} G$ over M is identified with

$$\begin{aligned} \text{Hom}_{\rho}(\mathfrak{l}, \mathfrak{g}) &= \{\Lambda \in \text{Hom}(\mathfrak{l}, \mathfrak{g}); \text{Ad}(\rho(a))\Lambda(Y) = \Lambda(\text{Ad}(a)Y), a \in L, Y \in \mathfrak{l}\} \\ &= \{\Lambda \in \text{Hom}(\mathfrak{l}, \mathfrak{g}); [\rho(X), \Lambda(Y)] = \Lambda([X, Y]) \text{ for } X, Y \in \mathfrak{l}\} \end{aligned}$$

by Wang's theorem ([14, pp.106-107, theorem 11.5]), where $\text{Hom}(\mathfrak{l}, \mathfrak{g})$ is the space of linear mappings from the vector space \mathfrak{l} to the vector space \mathfrak{g} . Remark that $\mathbf{R}\rho$ is contained in $\text{Hom}_{\rho}(\mathfrak{l}, \mathfrak{g})$. The curvature form Ω of an invariant connection $\Lambda \in \text{Hom}_L(\mathfrak{l}, \mathfrak{g})$ is an alternative linear mapping from $\mathfrak{l} \times \mathfrak{l}$ to \mathfrak{g} which is given by

$$2\Omega(X, Y) = -\frac{1}{4}\rho([X, Y]) + [\Lambda(X), \Lambda(Y)].$$

In particular, the curvature form Ω_t of $t\rho \in \mathbf{R}\rho$ is

$$2\Omega_t(X, Y) = \left(t^2 - \frac{1}{4}\right)\rho([X, Y]).$$

Hence $\Lambda = \pm\frac{1}{2}\rho$ are flat connections, which are called (\pm) -connection, respectively. A critical point of the Yang-Mills functional $\Lambda \mapsto \|\Omega\|^2$ is called a Yang-Mills connection. An invariant connection $\Lambda \in \text{Hom}_{\rho}(\mathfrak{l}, \mathfrak{g})$ is Yang-Mills if and only if for each $X \in \mathfrak{l}$

$$\sum_{i=1}^n [\Lambda(E_i), \Omega(E_i, X)] = 0 \quad (\text{Yang-Mills equation}),$$

where $\{E_1, \dots, E_n\}$ is an orthonormal basis of \mathfrak{l} . This equation is independent of the choice of an orthonormal basis $\{E_i\}_{i=1}^n$. In particular, $\Lambda = 0$ is Yang-Mills connection, which is called (0) -connection.

Take a maximal torus T of G and denote by \mathfrak{t} the Lie algebra of T . The complexification $\mathfrak{t}^{\mathbb{C}}$ of \mathfrak{t} is a Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} . For each $\alpha \in \mathfrak{t}$, put

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}}; [H, X] = \sqrt{-1}\langle \alpha, H \rangle X \text{ for all } H \in \mathfrak{t}\}.$$

An element $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ if $\mathfrak{g}_{\alpha} \neq \{0\}$. Let Σ denote the set of all nonzero roots of $\mathfrak{g}^{\mathbb{C}}$. Fix a lexicographic ordering on \mathfrak{t} . Let Σ^+ and $\Pi = \{\alpha_i\}_{i=1}^r$

denote the set of all positive roots and the set of all simple roots, respectively. Then we obtain the root space decompositions of \mathfrak{g}^G and \mathfrak{g} :

$$\begin{aligned}\mathfrak{g} &= \mathfrak{t} + \sum_{\alpha \in \Sigma^+} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha) \quad (\text{orthogonal direct sum}), \\ \mathfrak{g}^G &= \mathfrak{t}^G + \sum_{\alpha \in \Sigma} \mathbf{C}E_\alpha \quad (\text{direct sum});\end{aligned}$$

(A1) $\|F_\alpha\| = \|G_\alpha\| = 1,$

(A2) $E_{-\alpha} = -\eta E_\alpha,$ where we denote η the conjugation of \mathfrak{g}^G with respect to $\mathfrak{g},$

(A3) $E_\alpha \in \mathfrak{g}_\alpha$ i.e., $[H, E_\alpha] = \sqrt{-1}\langle \alpha, H \rangle E_\alpha$ for $H \in \mathfrak{t}, \alpha \in \Sigma,$

(A4) $F_\alpha = E_\alpha - E_{-\alpha}, G_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$ for $\alpha \in \Sigma^+.$

2.2 The structure of the space of invariant connections in the case where $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g}

In this section we use the same notation in §2.1.

We say that $\rho(\mathfrak{l})$ is a *regular subalgebra* of \mathfrak{g} if there exist a subspace \mathfrak{h} of \mathfrak{t} and a subset Δ^+ of Σ^+ such that

$$\rho(\mathfrak{l}) = \mathfrak{h} + \sum_{\alpha \in \Delta^+} (\mathbf{R}F_\alpha + \mathbf{R}G_\alpha).$$

Put $\Delta = \Delta^+ \cup (-\Delta^+).$ Then

(B1) If α, β are in $\Delta,$ and $\alpha + \beta \in \Sigma,$ then $\alpha + \beta \in \Delta,$

(B2) \mathfrak{h} is a linear closure of $\Delta.$

We decompose $\mathfrak{g} = \sum_i \mathfrak{g}_i + \mathfrak{z}$ where \mathfrak{g}_i is a simple ideal of \mathfrak{g} and \mathfrak{z} is the center of \mathfrak{g} (see [7, p.132, Cor. 6.3; Prop. 6.6]). Denote ρ_i the \mathfrak{g}_i -component of $\rho.$ Then $\rho = \sum \rho_i$ and $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \sum_i \text{Hom}_{\rho_i}(\mathfrak{l}, \mathfrak{g}_i).$ Hence we may assume that G is simple in order to determine the structure of $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}).$

Let φ be an inner automorphism of G and denote ρ' the Lie homomorphism of L into G defined by $\varphi \circ \rho.$ Then the mapping $\varphi : \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) \rightarrow \text{Hom}_{\rho'}(\mathfrak{l}, \mathfrak{g}); \Lambda \mapsto \varphi \circ \Lambda$ is a linear isomorphism and for $\Lambda \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}), \|\Omega_\Lambda\| = \|\Omega_{\varphi \circ \Lambda}\|.$ Hence we may identify ρ' with $\rho.$

Theorem D ([12]) *Assume $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} and that G is simple.*

(1) *If $\text{rank}(L) \geq 2,$ then $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \mathbf{R}\rho.$*

(2) If $\text{rank}(L) = 1$, then there exist $\Gamma_1, \dots, \Gamma_{2s} \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that

$$\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \mathbf{R}\rho + \sum_{i=1}^{2s} \mathbf{R}\Gamma_i;$$

and the set of flat invariant connections is given by

$$\left\{ \pm \frac{1}{2}\rho \right\} \cup \left\{ \sum_{i=1}^{2s} a_i \Gamma_i; \sum_{i=1}^{2s} a_i^2 = \frac{1}{8} \right\};$$

and the set of Yang-Mills invariant connections except flat connections is given by

$$\{0\} \cup \left\{ \pm \frac{1}{4}\rho + \frac{1}{2} \sum_{i=1}^{2s} a_i \Gamma_i; \sum_{i=1}^{2s} a_i^2 = \frac{1}{8} \right\}.$$

Corollary 2.2.1 Assume $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} . Then any non-flat Yang-Mills invariant connection is instable.

Definition We say that ρ is *indecomposable* if whenever $\rho = \rho_1 + \rho_2$ and $\rho_i : \mathfrak{l} \rightarrow \mathfrak{g}$ is a Lie homomorphism such that $[\rho_1(\mathfrak{l}), \rho_2(\mathfrak{l})] = 0$ then $\rho_1 = 0, \rho_2 = \rho$ or $\rho_2 = 0, \rho_1 = \rho$.

Corollary 2.2.2 Consider the following three conditions (C1),(C2) and (C3):

(C1) ρ is indecomposable,

(C2) Flat invariant connections are only (\pm) -connections,

(C3) (0)-connection is a unique non-flat Yang-Mills invariant connection.

Then (C1) and (C2) are equivalent. The condition (C3) implies (C1). Moreover if $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} , then (C1) implies (C3).

Remark 2.2.3 In general, (C1) does not imply (C3). In fact we will give an example satisfying the following three conditions (see §2.4):

(D1) ρ is indecomposable,

(D2) There exists a non-flat Yang-Mills invariant connection Λ such that

$$\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) = \mathbf{R}\rho + \mathbf{R}\Lambda \quad (\text{orthogonal direct sum}),$$

(D3) Λ attains a local minimum on the space of invariant connections $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$. ■

Proposition 2.2.4 ([12]) *Assume $\rho(\mathfrak{l})$ is a regular subalgebra of \mathfrak{g} and that G is simple. If $\dim \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) \geq 2$, then there exists $\alpha \in \Delta^+$ such that*

$$\rho(\mathfrak{l}) = \mathbf{R}\alpha + \mathbf{R}F_\alpha + \mathbf{R}G_\alpha \cong \mathfrak{su}(2).$$

If we set

$$\Sigma(\alpha) = \{\beta \in \Sigma; \langle \alpha, \beta \rangle = \|\alpha\|^2\},$$

then there exist $\beta_i > 0, \gamma_i < 0 (1 \leq i \leq s)$ such that

$$\Sigma(\alpha) = \{\alpha\} \cup \{\beta_i, \gamma_i\}_{1 \leq i \leq s}$$

and that

$$\pm \alpha + \beta_i - \beta_j, \pm \alpha + \beta_i - \gamma_j \in \Sigma (i \neq j), \quad \beta_i + \gamma_i = 2\alpha.$$

Such pairs (\mathfrak{g}, α) can be classified and essentially (i.e., as in the sense we mentioned in this section) they must be enlisted in the table below.

\mathfrak{g}	α	$\dim \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$
$\mathfrak{so}(2n+1)$	α_n	$2n-1$
$\mathfrak{sp}(n)$	α_1	3
\mathfrak{f}_4	α_3	7

Here we adopt the same notations and numberings of the simple roots given in the Bourbaki's table [1].

Table 1

Canonical numbering of the simple roots

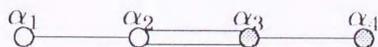
$\mathfrak{so}(2n+1)$



$\mathfrak{sp}(n)$



\mathfrak{f}_4



The black circles denote the shorter roots.

Proof of Proposition 2.2. 4 :

Let $p : \mathfrak{t} \rightarrow \mathfrak{h}$ denote the orthogonal projection. If $\dim \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g}) \geq 2$, then we have

$$p(\Sigma - \Delta) \supset \Delta. \quad (2.2.1)$$

In fact there exists $\Lambda \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that $\Lambda \notin \mathbf{R}\rho$. Since $\Lambda(\mathfrak{l}) \cap \rho(\mathfrak{l})$ is an ideal of $\rho(\mathfrak{l})$, we get $\Lambda(\mathfrak{l}) \cap \rho(\mathfrak{l}) = \{0\}$.

We use the classification of compact simple Lie algebras ([7, Ch. X, §6]).

Case 1. $\mathfrak{g} = \mathfrak{su}(n), \mathfrak{so}(n), e_6, e_7, e_8$: The condition (2.2.1) does not hold since the length of roots of $\mathfrak{g}^{\mathbb{C}}$ are the same.

Case 2. $\mathfrak{g} = \mathfrak{g}_2$: The condition (2.2.1) does not hold.

Case 3. $\mathfrak{g} = \mathfrak{so}(2n+1)$: Since the Weyl group $W(G)$ of G operates simply transitively on the set of systems of simple roots of Σ (see [2, p. 205]), we may assume that Δ contains the shortest simple root α_n . Suppose there existed $1 \leq i \leq n-1$ such that $\epsilon_i \in \Delta$. Then $\alpha_n + \epsilon_i \in \Delta$, which is impossible. Hence we get $\Delta = \{\pm\alpha_n\}$ and

$$\Sigma(\alpha_n) = \{\alpha_n\} \cup \{\beta_i = \epsilon_n + \epsilon_i, \gamma_i = \epsilon_n - \epsilon_i; 1 \leq i \leq n-1\}.$$

Case 4. $\mathfrak{g} = \mathfrak{sp}(n)$: Since the shortest simple roots $\alpha_1, \dots, \alpha_{n-1}$ are mutually transformed by the action of $W(G)$, we may assume $\alpha_1 \in \Delta$. The candidates for $x \in \Sigma$ which satisfies $px = \alpha_1$ are $x = 2\epsilon_1$ and $x = -2\epsilon_2$. In both cases, $\epsilon_1 + \epsilon_2 = \pm(x - px) \in \mathfrak{h}^\perp$. Since the set of the shortest roots which are orthogonal to $\epsilon_1 + \epsilon_2$ is

$$\{\pm\alpha_1\} \cup \{\pm\epsilon_i \pm \epsilon_j; 3 \leq i < j\},$$

we get $\Delta = \{\pm\alpha_1\} \cup (\{\pm\epsilon_i \pm \epsilon_j; 3 \leq i < j\} \cap \Delta)$. Here $\{\pm\alpha_1\}$ and $\{\pm\epsilon_i \pm \epsilon_j; 3 \leq i < j\}$ are orthogonal to each other, which implies that $\Delta = \{\pm\alpha_1\}$.

Case 5 $\mathfrak{g} = \mathfrak{f}_4$: We may assume $\alpha_3 \in \Delta$. Suppose there existed $1 \leq i \leq 3$ such that $\epsilon_i \in \Delta$. Then $\alpha_3 - \epsilon_i \in \Delta$, which is impossible. Hence $\epsilon_i \notin \Delta (1 \leq i \leq 3)$. Remark that

$$\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \mp \epsilon_4 = \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \mp \epsilon_4) \in \Sigma.$$

Suppose that $\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$ and $\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$ were both in Δ . Then $\epsilon_1 + \epsilon_4 \in \Delta$. This is a contradiction. Hence these do not take place in the same time. Using the same argument and considering a reflection with respect to $\epsilon_i (2 \leq i \leq 4)$, we may assume that the candidates for Δ are as follows:

$$\Delta = \{\pm\alpha_3, \left\{ \pm\alpha_3, \pm\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4), \pm\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) \right\}\}.$$

Suppose the latter took place. Then $px = \alpha_3$ implies $x = \alpha_3$. This is a contradiction. Hence we get $\Delta = \{\pm\alpha_3\}$ and

$$\Delta(\alpha_3) = \{\alpha_3\} \cup \{\beta_i = \epsilon_4 + \epsilon_i, \gamma_i = \epsilon_4 - \epsilon_i\}_{1 \leq i \leq 3}.$$

In order to prove Theorem D, we prove lemmas. ■

Lemma 2.2.5 *Take $t \in \mathbf{R}$ such that $\cos(t\|\alpha\|) = -1$. Then*

- (1) $\text{Ad}(\exp tF_\alpha)E_{\beta_i} = \text{Ad}(\exp(-tF_\alpha))E_{\beta_i}$,
- (2) $\text{Ad}(\exp tF_\alpha)E_{-\beta_i} = \text{Ad}(\exp(-tF_\alpha))E_{-\beta_i}$,
- (3) $\text{Ad}(\exp tF_\alpha)E_{\gamma_i} = \text{Ad}(\exp(-tF_\alpha))E_{\gamma_i}$,
- (4) $\text{Ad}(\exp tF_\alpha)E_{-\gamma_i} = \text{Ad}(\exp(-tF_\alpha))E_{-\gamma_i}$.

Proof of Lemma 2.2.5 :

(1) First we remark

$$\text{Ad}(\exp 2t \frac{\alpha}{\|\alpha\|})G_\alpha = G_\alpha, \quad \text{Ad}(\exp 2t \frac{\alpha}{\|\alpha\|})E_{\beta_i} = E_{\beta_i}.$$

Take $s \in \mathbf{R}$ such that $\sin(s\|\alpha\|) = -1$. Then we have

$$\text{Ad}(\exp sG_\alpha)F_\alpha = \frac{\alpha}{\|\alpha\|},$$

which implies that

$$\begin{aligned} & \text{Ad}(\exp sG_\alpha)\text{Ad}(\exp 2tF_\alpha)E_{\beta_i} \\ &= \text{Ad}(\exp 2t\text{Ad}(\exp sG_\alpha)F_\alpha)\text{Ad}(\exp sG_\alpha)E_{\beta_i} \\ &= \text{Ad}(\exp 2t \frac{\alpha}{\|\alpha\|})\text{Ad}(\exp sG_\alpha)E_{\beta_i} \\ &= \text{Ad}(\exp s\text{Ad}(\exp 2t \frac{\alpha}{\|\alpha\|})G_\alpha)\text{Ad}(\exp 2t \frac{\alpha}{\|\alpha\|})E_{\beta_i} \\ &= \text{Ad}(\exp sG_\alpha)E_{\beta_i}. \end{aligned}$$

- (2) This is the conjugation of the equation (1).
- (3) This is obtained in the same way as the proof (1).
- (4) This is the conjugation of the equation (3). ■

Lemma 2.2.6 *An invariant connection Λ in $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ is flat if and only if*

$$[\Lambda(E_\alpha), \Lambda(E_{-\alpha})] = -\frac{\sqrt{-1}}{8}\rho(\alpha).$$

Proof of Lemma 2.2.6: Assume $\Lambda \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ satisfies the above equation, which is equivalent to

$$[\Lambda(F_\alpha), \Lambda(G_\alpha)] = \frac{1}{4}\rho(\alpha).$$

Take $t \in \mathbf{R}$ such that $\sin(t\|\alpha\|) = 1$, then we have

$$\text{Ad}(\exp tF_\alpha)\alpha = -\|\alpha\|G_\alpha, \quad \text{Ad}(\exp tF_\alpha)G_\alpha = \frac{\alpha}{\|\alpha\|},$$

which implies that

$$[\Lambda(F_\alpha), \Lambda(\alpha)] = -\frac{1}{4}\|\alpha\|^2\rho(G_\alpha).$$

Take $u \in \mathbf{R}$ such that $\sin(u\|\alpha\|^2) = 1$, then we have

$$\text{Ad}(\exp u\alpha)F_\alpha = G_\alpha, \quad \text{Ad}(\exp u\alpha)G_\alpha = -F_\alpha,$$

which implies that

$$[\Lambda(G_\alpha), \Lambda(\alpha)] = \frac{1}{4}\|\alpha\|^2\rho(F_\alpha).$$

Hence Λ is a flat connection. The converse is trivial. ■

Lemma 2.2.7 (1) $[E_{\beta_i}, E_{-\beta_j}] + [E_{\gamma_j}, E_{-\gamma_i}] = 0 \quad (i \neq j),$

(2) $[E_{\beta_i}, E_{-\gamma_j}] + [E_{\beta_j}, E_{-\gamma_i}] = 0 \quad (i < j),$

(3) $[E_{\gamma_i}, E_{-\beta_j}] + [E_{\gamma_j}, E_{-\beta_i}] = 0 \quad (i < j).$

Proof of Lemma 2.2.7:

(1) Take $t \in \mathbf{R}$ such that $\cos(t\|\alpha\|) = -1$. From Lemma 2.2.5, we have

$$[E_{\beta_i}, E_{-\beta_j}] + [E_{\gamma_j}, E_{-\gamma_i}] = [E_{\beta_i}, E_{-\beta_j}] - \text{Ad}(\exp tF_\alpha)[E_{\beta_i}, E_{-\beta_j}].$$

Since $\pm\alpha + \beta_i - \beta_j \notin \Sigma$, we get the assertion.

(2) This is obtained in the same way as the proof of (1).

(3) This is the conjugation of the equation (2). ■

Proof of Theorem E: (1) This is clear from Proposition 2.2.4. We show (2). Take $t \in \mathbf{R}$ such that $\cos(t|\alpha|) = -1$ and put

$$\begin{aligned} E'_{\beta_i} &= \text{Ad}(\exp tF_\alpha)E_{-\gamma_i}, & E'_{-\beta_i} &= -\eta E'_{\beta_i} = \text{Ad}(\exp tF_\alpha)E_{\gamma_i}, \\ F'_{\beta_i} &= E'_{\beta_i} - E'_{-\beta_i}, & G'_{\beta_i} &= \sqrt{-1}(E'_{\beta_i} + E'_{-\beta_i}). \end{aligned}$$

Since these vectors $E'_{\pm\beta_i}, F'_{\beta_i}$ and G'_{β_i} fill the same role of $E_{\pm\beta_i}, F_{\beta_i}$ and G_{β_i} (i.e., these vectors satisfy the conditions (A1)~(A4)), we rewrite $E_{\pm\beta_i} = E'_{\pm\beta_i}, F_{\beta_i} = F'_{\beta_i}, G_{\beta_i} = G'_{\beta_i}$. By Lemma 2.2.5, these vectors satisfy the following condition: If $\cos(t|\alpha|) = -1$, then $E_{\pm\beta_i} = \text{Ad}(\exp tF_\alpha)E_{\mp\gamma_i}$. For $1 \leq i \leq s$, we can define $\Lambda_i, \Lambda_{i+s} \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})^C$ by

$$\Lambda_i(E_\alpha) = E_{\beta_i}, \quad \Lambda_{i+s}(E_\alpha) = E_{\gamma_i}.$$

We get

$$\Lambda_i(E_{-\alpha}) = -E_{\gamma_i}, \quad \Lambda_{i+s}(E_{-\alpha}) = -E_{-\beta_i}$$

and that $\{\rho\} \cup \{\Lambda_i, \Lambda_{i+s}\}_{1 \leq i \leq s}$ forms a basis of $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})^C$. From Lemma 2.2.6 and Lemma 2.2.7, we get that $\Lambda \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ is flat if and only if

$$\Lambda = \pm \frac{1}{2}\rho \quad \text{or} \quad \Lambda = \sum_{i=1}^s (b_i \Lambda_i + c_i \Lambda_{i+s}), \quad \sum_{i=1}^s b_i c_i = -\frac{1}{8}.$$

The set

$$\{\Gamma_i = \Lambda_i - \Lambda_{i+s}, \Gamma_{i+s} = \sqrt{-1}(\Lambda_i + \Lambda_{i+s})\}_{i=1}^s$$

forms a basis of $\text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$. Thus we get the set of flat invariant connections. For every $\Gamma \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$, there exist $x, y \in \mathbf{R}$ and flat invariant connection $\Gamma_0 = \sum_{i=1}^{2s} a_i \Gamma_i$ ($\sum_{i=1}^{2s} a_i^2 = \frac{1}{8}$) such that $\Gamma = x\rho + y\Gamma_0$. From Yang-Mills equation, Γ is Yang-Mills if and only if

$$\Gamma = 0, \quad \pm \frac{1}{2}\rho \quad (\text{flat}), \quad \pm \Gamma_0 \quad (\text{flat}) \quad \text{or} \quad (x, y) = \left(\pm \frac{1}{4}, \pm \frac{1}{2}\right).$$

Proof of Corollary 2.2.1: For example, we show the instability of a Yang-Mills invariant connection of the form:

$$\Gamma = \frac{1}{4}\rho + \frac{1}{2}\Gamma_0,$$

which appears in the proof of Theorem D. Let Ω_t be the curvature form of

$$\Gamma_t = \left(\frac{1}{4} - \frac{t}{2}\right)\rho + \left(\frac{1}{2} + t\right)\Gamma_0.$$

Then we get

$$2\Omega_t(X, Y) = \left(-\frac{1}{8} + \frac{1}{2}t^2\right)\rho([X, Y]) + \left(\frac{1}{4} - t^2\right)\Gamma_0([X, Y]),$$

which implies that there exists a positive constant c such that

$$\frac{d^2}{dt^2} \|\Omega_t\|_{t=0}^2 = -c \|\Gamma_0 + \frac{1}{2}\rho\|^2 < 0.$$

Proof of Corollary 2.2. 2 : If $\rho = \rho_1 + \rho_2$ is a non-trivial decomposition of ρ , then $\frac{1}{2}(\rho_1 - \rho_2)$ is a flat invariant connection except the (\pm) -connection and $\frac{1}{2}\rho_1$ is a non-flat Yang-Mills connection except the (0) -connection. Hence (C2) implies (C1), and (C3) implies (C1). We show (C1) implies (C2). Let Λ be any flat invariant connection. Put

$$\rho_1 = \frac{1}{2}\rho + \Lambda, \quad \rho_2 = \frac{1}{2}\rho - \Lambda.$$

Then $\rho = \rho_1 + \rho_2$ is a decomposition of ρ . Since ρ is indecomposable, $\rho_1 = 0$ or $\rho_1 = \rho$. Hence $\Lambda = \pm \frac{1}{2}\rho$.

We remark that in the case where we deal in Proposition 2.2. 4, ρ is not indecomposable. Hence we have the last half part.

2.3 The case where the subalgebra $\rho(\mathfrak{l})$ contains a regular element of \mathfrak{g}

We may assume that ρ is injective.

Theorem 2.3. 1 ([11]) *If $\rho(\mathfrak{l})$ contains a regular element of \mathfrak{g} , then (C1) implies (C3).*

Theorem E ([11]) *Assume $\rho(\mathfrak{l})$ contains a regular element of \mathfrak{g} . Then any non-flat Yang-Mills invariant connection is instable.*

Proof of Theorem 2.3. 1 and Theorem E:

It is sufficient to prove that for each non-flat Yang-Mills connection $\Lambda \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$, there exists $\alpha (= \alpha_\Lambda) \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that

(E1) $\alpha = 0$ implies $\Lambda = 0$,

(E2) $\rho = \alpha + (\rho - \alpha)$ is a decomposition of ρ , and $\rho - \alpha \neq 0$,

(E3) $\frac{d^2}{dt^2} \|\Omega_t\|_{t=0} < 0$, where Ω_t is the curvature form of $\Lambda + t(\rho - \alpha)$.

Applying Whitehead's vanishing theorem of cohomology group (see [13, p.95, Theorem 13]) for the representation $(\text{ad} \circ \rho, \mathfrak{g})$ of \mathfrak{l} , we have following:

If $\Lambda_1, \Lambda_2 \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ satisfy

$$(F1) \quad [\Lambda_1(X), \Lambda_2(Y)] = -[\Lambda_1(Y), \Lambda_2(X)],$$

$$(F2) \quad \mathfrak{S}_{X,Y,Z}[\rho(X), [\Lambda_1(Y), \Lambda_2(Z)]] = 0, \text{ where } \mathfrak{S}_{X,Y,Z} \text{ is the sum over the cyclic permutations of } X, Y, Z,$$

then there exists $\Lambda_3 \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that

$$[\Lambda_1(X), \Lambda_2(Y)] = \Lambda_3([X, Y]).$$

Remark that under the condition (F1), the condition (F2) is equivalent to

$$\mathfrak{S}_{X,Y,Z}[\Lambda_1([X, Y]), \Lambda_2(Z)] = 0.$$

Since $\rho(\mathfrak{l})$ contains a regular element of \mathfrak{g} , $[\Lambda_1, \Lambda_2]$ is skew-symmetric automatically. In fact, take Cartan subalgebras \mathfrak{t} and \mathfrak{h} of \mathfrak{l} and \mathfrak{g} respectively such that $\rho(\mathfrak{t}) \subset \mathfrak{h}$. Then

$$[\rho(\mathfrak{t}), \Lambda_i(\mathfrak{t})] = \Lambda_i([\mathfrak{t}, \mathfrak{t}]) = 0.$$

This implies $\Lambda_i(\mathfrak{t}) \subset \mathfrak{h}$ by assumption. In particular, $[\Lambda_1(\mathfrak{t}), \Lambda_2(\mathfrak{t})] = 0$ and $[\Lambda_1(H), \Lambda_2(H)] = 0$ for $H \in \mathfrak{t}$. Since $\mathfrak{l} = \text{Ad}(L)\mathfrak{t}$ (see [7, p.248, Theorem 6.4]), we get $[\Lambda_1(X), \Lambda_2(X)] = 0$.

Let $\Lambda \in \text{Hom}_L(\mathfrak{l}, \mathfrak{g})$ be any non-flat Yang-Mills invariant connection. First we prove $\mathfrak{S}_{X,Y,Z}[\rho(X), [\Lambda(Y), \Lambda(Z)]] = 0$ using the classification of compact simple Lie algebras. The vector space

$$V = \mathfrak{l} \wedge \mathfrak{l} = \text{span}\{X \wedge Y; X, Y \in \mathfrak{l}\}$$

is an \mathfrak{l} -module by the \mathfrak{l} -action:

$$(\text{ad}Z)(X \wedge Y) = [Z, X] \wedge Y + X \wedge [Z, Y].$$

The space

$$W = \text{span}\{[\Lambda(X), \Lambda(Y)]; X, Y \in \mathfrak{l}\}$$

is an $\text{ad}(\rho(\mathfrak{l}))$ -invariant subspace of \mathfrak{g} . We consider the \mathfrak{l} -homomorphism Φ from V onto W which is defined by

$$\Phi : V = \mathfrak{l} \wedge \mathfrak{l} \rightarrow W; X \wedge Y \mapsto [\Lambda(X), \Lambda(Y)].$$

Since Φ is surjective, $V/V_0 \cong W$ as \mathfrak{l} -modules, where $V_0 = \text{Ker}\Phi$. On the other hand, we consider the \mathfrak{l} -homomorphism Ψ from V into \mathfrak{l} which is defined by

$$\Psi : V = \mathfrak{l} \wedge \mathfrak{l} \rightarrow \mathfrak{l}; X \wedge Y \mapsto [X, Y].$$

Since $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}$, Ψ is surjective. We show that the irreducibility of $V_1 = \text{Ker}\Psi$. We denote by $\mathfrak{l}^C, \mathfrak{t}^C$ and ρ^C the complexifications of $\mathfrak{l}, \mathfrak{t}$ and ρ respectively. The complex Lie algebra

\mathfrak{l}^C is simple. We denote by Δ the set of nonzero roots of \mathfrak{l}^C with respect to \mathfrak{t}^C . For $\alpha \in \Delta$, there exists a non-zero vector $E_\alpha \in \mathfrak{l}^C$ such that

$$[H, E_\alpha] = \alpha(H)E_\alpha \quad \text{for all } H \in \mathfrak{t}^C.$$

We have a direct-sum decomposition:

$$\mathfrak{l}^C = \mathfrak{t}^C + \sum_{\alpha \in \Delta} \mathbb{C}E_\alpha.$$

Fix a lexicographic ordering on \mathfrak{t} . We denote by δ_0 the highest root of Δ and by $\{\alpha_1, \dots, \alpha_r\}$ the set of simple roots of Δ . The set

$$\{\delta_0 - \alpha_i \in \Delta\} \neq \emptyset$$

is a single-point set $\{\delta_1\}$ or two-point set $\{\delta_1, \delta_2\}$, and the set consists of two points if and only if $\mathfrak{l} = \mathfrak{su}(m)$.

In the case where $\{\delta_0 - \alpha_i \in \Delta\} = \{\delta_1\}$, we define an \mathfrak{l} -invariant subspace $V_1(\delta_0 + \delta_1)$ of V_1^C by

$$V_1(\delta_0 + \delta_1) = \text{ad}(U(\mathfrak{l}^C))(E_{\delta_0} \wedge E_{\delta_1}),$$

where $U(\mathfrak{l}^C)$ is the universal enveloping algebra of \mathfrak{l}^C . The highest weight of $V_1(\delta_0 + \delta_1)$ is $\delta_0 + \delta_1$ and the multiplicity of $\delta_0 + \delta_1$ is equal to 1. Hence $V_1(\delta_0 + \delta_1)$ is irreducible. By virtue of Weyl's degree formula (see [13, p.257]), we get

$$\dim V_1(\delta_0 + \delta_1) = \frac{\dim \mathfrak{l}(\dim \mathfrak{l} - 3)}{2} = \dim V_1.$$

Hence $V_1^C = V_1(\delta_0 + \delta_1)$. In particular, V_1^C is irreducible so is V_1 .

In the case where $\{\delta_0 - \alpha_i \in \Delta\} = \{\delta_1, \delta_2\}$, we define \mathfrak{l} -invariant subspaces $V_1(\delta_0 + \delta_1)$ and $V_1(\delta_0 + \delta_2)$ of V_1^C by

$$\begin{aligned} V_1(\delta_0 + \delta_1) &= \text{ad}(U(\mathfrak{l}^C))(E_{\delta_0} \wedge E_{\delta_1}), \\ V_1(\delta_0 + \delta_2) &= \text{ad}(U(\mathfrak{l}^C))(E_{\delta_0} \wedge E_{\delta_2}). \end{aligned}$$

For $i = 1, 2$, the highest weight of $V_1(\delta_0 + \delta_i)$ is $\delta_0 + \delta_i$ and the multiplicity of $\delta_0 + \delta_i$ is equal to 1. Hence $V_1(\delta_0 + \delta_i)$ ($i = 1, 2$) is irreducible. By virtue of Weyl's dimensionality formula, we get

$$\dim V_1(\delta_0 + \delta_1) = \dim V_1(\delta_0 + \delta_2) = \frac{1}{2} \dim V_1.$$

Hence we have

$$V_1^C = V_1(\delta_0 + \delta_1) + V_1(\delta_0 + \delta_2) \quad (\text{direct sum}).$$

We denote by $W(L)$ the Weyl group of L . Clearly, there exist $\sigma_1, \sigma_2 \in W(L)$ such that

$$\sigma_1(\delta_0 + \delta_1) = -(\delta_0 + \delta_2), \quad \sigma_2(\delta_0 + \delta_2) = -(\delta_0 + \delta_1).$$

Hence V_1 is real irreducible, whethere $\{\delta_0 - \alpha_i \in \Delta\}$ is a single point set or two points set. So we get

$$V_1 = \text{ad}(U(\mathfrak{l}))(\mathfrak{t} \wedge \mathfrak{t}) \subset V_0.$$

Hence Φ naturally induces \mathfrak{l} -homomorphism φ from V/V_1 onto W defined by

$$\varphi: V/V_1 \rightarrow W; \overline{X \wedge Y} \mapsto [\Lambda(X), \Lambda(Y)],$$

where $\overline{X \wedge Y}$ is the equivalence class of $X \wedge Y$. From Jacobi's identity, we have

$$\begin{aligned} \mathfrak{S}_{X,Y,Z} \text{ad}(Z) \overline{X \wedge Y} &= \mathfrak{S}_{X,Y,Z} (\overline{[Z, X] \wedge Y + X \wedge [Z, Y]}) \\ &= 2\mathfrak{S}_{X,Y,Z} \overline{[Z, X] \wedge Y} \\ &= 0. \end{aligned}$$

Hence we have

$$0 = \varphi(\mathfrak{S}_{X,Y,Z} \text{ad}(Z) \overline{X \wedge Y}) = \mathfrak{S}_{X,Y,Z} [\rho(Z), [\Lambda(X), \Lambda(Y)]].$$

By Whitehead's vanishing theorem of cohomology group, there exists $\alpha \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that

$$\alpha([X, Y]) = 4[\Lambda(X), \Lambda(Y)].$$

By Jacobi's identity, we have

$$\mathfrak{S}_{X,Y,Z} [\alpha([X, Y]), \Lambda(Z)] = \frac{1}{4} \mathfrak{S}_{X,Y,Z} [[\Lambda(X), \Lambda(Y)], \Lambda(Z)] = 0.$$

By Whitehead's vanishing theorem of cohomology group, there exists $\Gamma \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ such that

$$[\alpha(X), \Lambda(Y)] = \Gamma([X, Y]).$$

Since Λ is Yang-Mills, we have

$$\begin{aligned} -\frac{c}{4} \Gamma(X) &= \frac{1}{4} \sum_{i=1}^n [\Lambda(E_i), \alpha([E_i, X])] \\ &= \sum_{i=1}^n [\Lambda(E_i), [\Lambda(E_i), \Lambda(X)]] \\ &= -\frac{c}{4} \Lambda(X), \end{aligned}$$

where c is the eigenvalue of the negative of the Casimir operator of $(\text{ad}, \mathfrak{l})$. Hence $\Gamma = \Lambda$, that is,

$$[\alpha(X), \Lambda(Y)] = \Lambda([X, Y]).$$

Hence we get (E1). We show α is a Lie homomorphism. From Jacobi's identity, we have

$$\begin{aligned} \frac{1}{4}[\alpha(X), \alpha([Z, W])] &= [\alpha(X), [\Lambda(Z), \Lambda(W)]] \\ &= [[\alpha(X), \Lambda(Z)], \Lambda(W)] + [\Lambda(Z), [\alpha(X), \Lambda(W)]] \\ &= [\Lambda([X, Z]), \Lambda(W)] + [\Lambda(Z), [\Lambda([X, W])]] \\ &= \frac{1}{4}\alpha([[X, Z], W] + [Z, [X, W]]) \\ &= \frac{1}{4}\alpha([X, [Z, W]]). \end{aligned}$$

Hence $\alpha \in \text{Hom}_\rho(\mathfrak{l}, \mathfrak{g})$ is a Lie homomorphism. So, if we put $\delta = \rho - \alpha$, then $\rho = \alpha + \delta$ is a decomposition of ρ . The curvature form Ω of Λ is given by $\Omega(X, Y) = -\frac{1}{4}\delta([X, Y])$. Since Λ is not flat, we have $\delta \neq 0$. Hence we have (E2). Since $[\delta(X), \Lambda(Y)] = 0$, the curvature form Ω_t of $\Lambda + t\delta$ is given by

$$\Omega_t(X, Y) = \frac{4t^2 - 1}{4}\delta([X, Y]).$$

Hence we have (E3). ■

2.4 An example

When $\rho(\mathfrak{l})$ does not contain any regular element of \mathfrak{g} or $\rho(\mathfrak{l})$ is not a regular subalgebra of \mathfrak{g} , the (0)-connection is not necessarily a unique non-flat Yang-Mills invariant connection, even if ρ is indecomposable. We show such an example. Put $L = SU(m)$ for $m \geq 3$. We define an $\text{Ad}(L)$ -invariant inner product \langle, \rangle on \mathfrak{l} by

$$\langle X, Y \rangle = -\text{tr}(XY) \quad \text{for } X, Y \in \mathfrak{l}.$$

The inner product \langle, \rangle naturally induces a Hermitian inner product \langle, \rangle on $\mathfrak{l}^{\mathbb{C}}$. Put $G = SU(\mathbb{C})$ and $\rho = \text{Ad} : L \rightarrow G$. In this case, $\rho(\mathfrak{l})$ does not contain any regular element of \mathfrak{g} . We define an $\text{Ad}(G)$ -invariant inner product \langle, \rangle on \mathfrak{g} by

$$\langle A, B \rangle = \sum_{i=1}^{m^2-1} \langle AE_i, BE_i \rangle \quad \text{for } A, B \in \mathfrak{g},$$

where $\{E_i\}_{1 \leq i \leq m^2-1}$ is an orthonormal basis of \mathfrak{l} . We define an invariant connection $\Lambda \in \text{Hom}_{\rho}(\mathfrak{l}, \mathfrak{g})$ by

$$(\Lambda(X))(Y) = \frac{-m}{2\sqrt{m^2+4}} \left\{ (XY + YX) - \frac{2}{m} \text{tr}(XY) 1_m \right\},$$

where 1_m is the identity matrix (cf. [18]).

Remark 2.4.1 If $m = 2$, then $\Lambda = 0$. ■

Proposition 2.4.2 *In this example, the conditions (D1), (D2) and (D3) in Remark 2.2.3 are satisfied.*

Proof: By simple calculation, we have $\text{Hom}_{\rho}(\mathfrak{l}, \mathfrak{g}) = \mathbf{R}\rho + \mathbf{R}\Lambda$ (orthogonal direct sum). This result and Corollary 2.2.2 imply (D1).

The equations

$$\sum_{i=1}^{m^2-1} [E_i, [E_i, X]] = -2mX, \quad \sum_{i=1}^{m^2-1} E_i^2 = -\frac{m^2-1}{m} 1_m$$

and

$$\begin{aligned} & [\Lambda(X), [\Lambda(Y), \Lambda(Z)]](W) \\ &= \frac{m^2}{4(m^2+4)} \Lambda([X, [Y, Z]])(W) \\ & \quad + \frac{m}{m^2+4} \left\{ \text{tr}(YW) \Lambda(X)Z - \text{tr}(ZW) \Lambda(X)Y \right. \\ & \quad \left. - \text{tr}(Y \Lambda(X)W)Z + \text{tr}(Z \Lambda(X)W)Y \right\} \end{aligned}$$

implies that Λ is a non-flat Yang-Mills invariant connection. Hence we have (D2).

Put $\Lambda(x, y) = \frac{x}{2}\rho + y\Lambda$ and $f(x, y) = 4\|\Omega(x, y)\|^2$, where $\Omega(x, y)$ is the curvature form of $\Lambda(x, y)$. The equations

$$\begin{aligned}\sum_{i,j} \|\rho([E_i, E_j])\|^2 &= 4m^2(m^2 - 1), \\ \sum_{i,j} \|\Lambda([E_i, E_j])\|^2 &= \frac{m^2(m^2 - 1)(m^2 - 4)}{m^2 + 4}, \\ \sum_{i,j} \|\Lambda(E_i), \Lambda(E_j)\|^2 &= \frac{m^2(m^2 - 1)(m^2 - 4)}{4(m^2 + 4)}\end{aligned}$$

implies that

$$\begin{aligned}f(x, y) = m^2(m^2 - 1) \left\{ \frac{1}{4}(x^2 - 1)^2 + \frac{m^2 - 4}{4(m^2 + 4)}y^4 \right. \\ \left. + \frac{m^2 - 4}{m^2 + 4}x^2y^2 + \frac{m^2 - 4}{2(m^2 + 4)}(x^2 - 1)y^2 \right\}.\end{aligned}$$

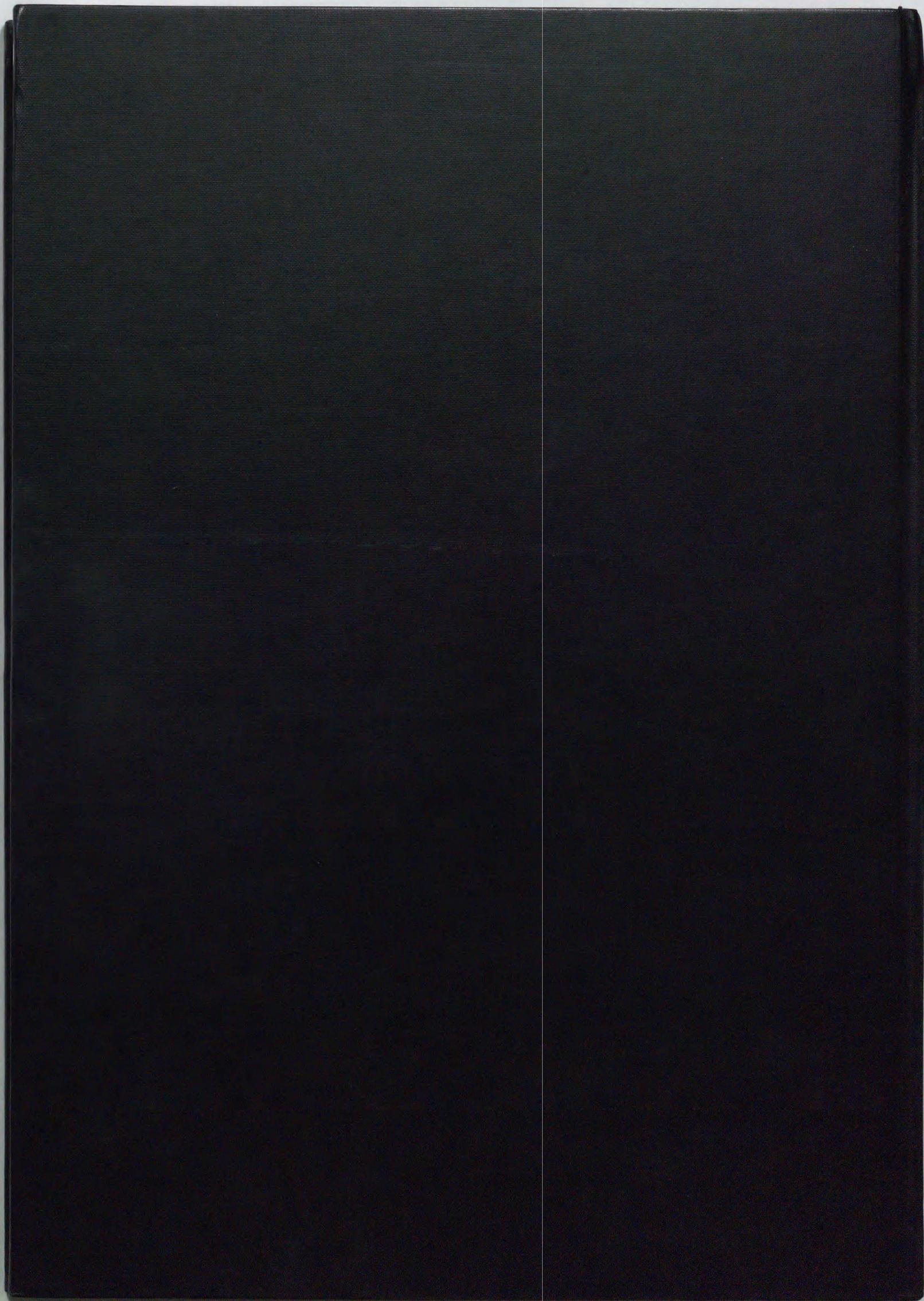
Hence f attains a local minimum at $(0, 1)$. Hence we have (D3). ■

Bibliography

- [1] N. Bourbaki, "Groupes et algebres de Lie", Hermann, Paris, 1975.
- [2] T. Brocker and T. Dieck, "Representations of Compact Lie Group", Springer-verlag, New York, 1985.
- [3] M. P. Do Carmo and N. R. Wallach, Representations of compact groups and minimal immersions into spheres, *J. Differential Geometry* 4(1970), pp. 91-104.
- [4] M. P. Do Carmo and N. R. Wallach, Minimal immersions of spheres into spheres, *Ann. Math.* 93(1971), pp. 43-62.
- [5] T. Eguchi, P. B. Gilkey and A. J. Hanson, Gravitation, gauge theories and differential geometry, *Physical Reports (Review Section of Physical Letters)* 66, no. 6(1980)pp. 213-393.
- [6] J. Eells, Jr. and J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* 86(1964), pp. 187-198.
- [7] S. Helgason, "Differential Geometry, Lie Groups and Symmetric Spaces", Academic Press, New York, San Francisco, London, 1978.
- [8] S. Helgason, "Groups and Geometric Analysis, Academic Press, New York, San Francisco, London, 1984.
- [9] O. Ikawa, Equivariant minimal immersions of compact Riemannian homogeneous spaces into compact Riemannian homogeneous spaces, *Tsukuba J. Math.*, 1993, vol. 17, pp. 169-188.
- [10] O. Ikawa, Harmonic mappings, minimal and totally geodesic immersions of compact Riemannian homogeneous spaces into Grassmann manifold, *Kodai Math. J.*, 1993, vol. 16, pp. 295-305.
- [11] O. Ikawa, Yang-Mills homogeneous connections on compact simple Lie groups, *Kodai Math. J.*, 1996, vol. 19, pp. 168-176.

- [12] O. Ikawa, Invariant connections on compact simple Lie groups, *Algebras Groups and Geometries*, 1996, vol. 13, pp. 93-104.
- [13] N. Jacobson, "Lie algebras", Dover Publications, 1962.
- [14] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry I", Interscience Publishers, 1963.
- [15] S. Kobayashi, Y. Ohnita and M. Takeuchi, On instability of Yang-Mills connections, *Math. Z.*, 193, pp. 165-189(1986).
- [16] H. T. Laquer, Stability properties of the Yang-Mills functional near the canonical connection, *Michigan Math. J.* 31(1984), pp. 139-159.
- [17] H. T. Laquer, Invariant gauge automorphisms of homogeneous principal bundles, *Geometriae Dedicata* 33(1990), pp. 27-35.
- [18] H. T. Laquer, Invariant affine connections on Lie groups, *Trans. Amer. Math. Soc.* vol 331, no. 2, (1992), pp. 541-551.
- [19] H. T. Laquer, Invariant affine connections on symmetric spaces, *Proc. Amer. Math. Soc.* vol. 115(1992), pp. 447-454.
- [20] K. B. Marathe and G. Martucci, The geometry of gauge fields, *JGP*, vol. 6(1989).
- [21] K. Mashimo and H. Tasaki, Stability of maximal tori in compact Lie groups, *Algebras, Groups and Geometries*, vol. 7, (1990), pp. 114-126.
- [22] K. Mashimo and H. Tasaki, Stability of closed Lie subgroups in compact Lie groups, *Kodai Math. J.*, vol. 13 (1990), pp. 181-203.
- [23] K. Mashimo, On the stability of the Cartan embeddings of compact symmetric spaces, *Arch. Math.*, Vol. 58 (1992), pp. 500-508.
- [24] K. Mashimo, Cartan Embeddings of Compact Riemannian 3-Symmetric Spaces, *Tokyo J. Math.*, vol. 19 (1996), pp. 353-364.
- [25] I. Mogi and M. Itoh, "Differential geometry and gauge theory", Kyoritu Shuppan, 1986(in Japanese).
- [26] T. Nagura, On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces I,II,III, *Osaka J. Math.* vol. 18(1981), pp. 115-145; vol 19 (1982), pp. 79-124; vol. 19(1982), pp. 241-281.

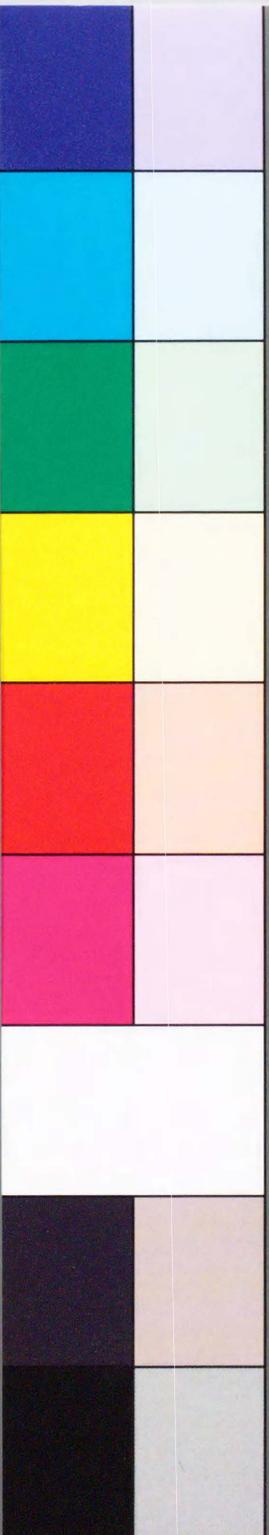
- [27] Y. Ohnita, On stability of minimal submanifolds in compact symmetric spaces, *Compositio Math.*, vol. 64(1987), pp. 157-189.
- [28] J. Simons, Minimal varieties in riemannian manifolds, *Ann. of Math.* vol 88, 1968, pp. 62-105.
- [29] T. Takahashi, Minimal immersions of Riemannian manifolds, *J. Math. Soc. Japan*, vol 18,1966, pp. 380-385.
- [30] M. Takeuchi, Modern spherical functions, *Amer. Math. Soc.*, *Translations of Mathematical Monographs*, vol. 135, 1994.
- [31] N. R. Wallach, Minimal immersions of symmetric spaces into spheres, *Symmetric Spaces*, Marcel Dekker, Inc, New York, 1972.



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