

Theorem

\forall monoidal category $B \quad \forall b \in B$

$\exists!$ morphism $W \rightarrow B$ of monoidal categories with $(-) \mapsto b$

proof desired morphism $w \mapsto w_b$

(substitute b in all the blanks of the word w)

$$(e_0)_b = e$$

$$(-)_b = b$$

$$(V \square W)_b = V_b \square W_b$$

For words of fixed length n

we construct a certain "basic" graph

$$G_n = G_{n,b}$$

vertices : all words w of length n which do not involve e_0

edges : certain arrows $V_b \rightarrow W_b$ in B called basic arrows

$$d = V_b \square (V_b \square W_b) \xrightarrow{\text{associativity}} (V_b \square V_b) \square W_b \text{ of } \begin{cases} \text{basic} \\ \text{length } n \end{cases}$$

$\frac{d}{2^n}$

$(\square d) \square (\square I) = \text{one instance of boxed with identities}$

directed (involving d)

anti directed (involving d^{-1})

G_n path from U to W
composable sequences of basic arrows
from U_b to W_b

$$U_b \rightarrow W_b \text{ in } B$$

(証明の要点)

Any two paths from U to W yield, by composition, the same arrow $U_b \rightarrow W_b$ in B

(the ~~graph~~ G_n is a commutative diagram in B)

$(W^{(n)})$: the unique word of length n which has all pairs of parentheses starting in front

$$(()) \dots)$$

\exists directed path in G_n

from any word w to $W^{(n)}$
in a canonical way, successively moving outermost parentheses to the front
by instances of d .

$\forall V, W = \text{words of length } n$

$$V \rightarrow W^{(n)} \rightarrow W$$

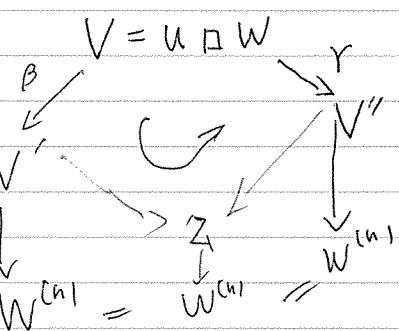
associativity
 \Downarrow
general associativity law
 $a(bc) = (ab)c$

rank ρ of a word w by recursion

$$\rho(e_0) = 0 \quad \rho(-) = 0$$

$$\rho(V \square W) = \rho(V) + \rho(W) + \text{length}(w) - 1$$

$\rho(w) = 0 \Rightarrow$ all pairs of parentheses in w start at the front



β, ρ decrease the rank

by a case subdivision

$$\beta = \gamma \quad Z = V' = V$$

$$\beta \neq \gamma \quad V = U \square W$$

ρ is

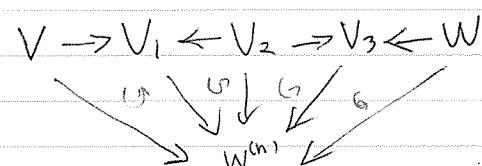
同一

$\beta = \beta' \square \beta'' : \beta$ acts inside the first factor U

$$\beta = \beta' \square \beta'' : \beta$$

$\beta = du, s, t$ where

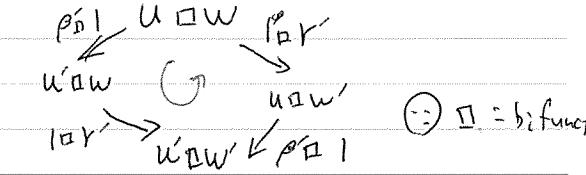
$$U = U \square W = U \square (s \square t)$$



by induction

suppose true for all V of smaller rank consider

two different directed paths starting at V with directed basic arrows $\rho \square 1$



If both act inside the same factor we can use induction on the length n

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the case when one of β or γ ,

Say β is $\beta = d = d_{U \cap S, U}$, as in the third

case above $\gamma \neq \beta$ V must act inside U or
inside W

If γ acts inside U , we use a diagram

from $U \cap S$ to $(U' \cap S) \cap t$

which commutes $\Rightarrow (f \circ g)^* \in \mathcal{L}_{\text{natural}}$