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Theorem

$\forall$ monoidal category B $\quad \forall b \in B$

$\exists !$ morphism $W \to B$ of monoidal categories with $(-) \mapsto b$

proof desired morphism $\boxed{w \mapsto w_b}$

(substitute b in all the blanks of the word w)

$(e_0)_b = e$

$(\_)_b = b$

$(v \square w)_b = v_b \square w_b$

For words of fixed length n

we construct a certain "basic" graph

$G_n = G_{n,b}$

vertices : all words w of length n which do not involve e_0

edges : certain arrows $v_b \to w_b$ in B called basic arrows

$\alpha : u_b \square (v_b \square w_b) \to (u_b \square v_b) \square w_b$ of associativity } basic

α^{-1}

$(1 \square \alpha) \square (1 \square 1)$: one instance of α boxed with identities

{ directed (involving α)

| antidirected (involving α^{-1})

G_n path from u to w

composable sequences of basic arrows from u_b to w_b

$u_b \to w_b$ in B

(証明の要点)

Any two paths from u to w yield, by composition, the same arrow $u_b \to w_b$ in B

(the ~~great~~ graph G_n is a commutative diagram in B)

$w^{(n)}$: the unique word of length n which has all pairs of parentheses starting in front

$(((\quad) \quad) \quad)$

$\exists$ directed path in G_n from any word w to $w^{(n)}$ in a canonical way, successively moving outer most parentheses to the front by instances of α

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$\forall v, w$: words of length n

$v \to w^{(n)} \to w$

associativity

⇓

general associativity law

$a(bc) = (ab)c$

rank ρ of a word w by recursion

$\rho e_0 = 0 \qquad \rho(\_) = 0$

$\rho(v \square w) = \rho(v) + \rho(w) + \mathrm{length}(w) - 1$

$\rho w = 0 \Rightarrow$ all pairs of parentheses in w start at the front

G_n が commute することを示す.

any path $v \to w$

join each vertex to the "bottom" vertex $w^{(n)}$ by the canonical path

$$\begin{tikzcd}
v \arrow[r] \arrow[drr] & v_1 \arrow[dr] & v_2 \arrow[l] \arrow[r] \arrow[d] & v_3 \arrow[dl] & w \arrow[l] \arrow[dll] \\
& & w^{(n)} & &
\end{tikzcd}$$

by Induction

suppose true for all v of smaller rank consider

two different directed paths starting at v with directed basic arrows β and γ

$$\begin{tikzcd}
& v = u \square w \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\
v' \arrow[d] \arrow[r] & z \arrow[d] & v'' \arrow[l] \arrow[d] \\
w^{(n)} \arrow[r, equal] & w^{(n)} \arrow[r, equal] & w^{(n)}
\end{tikzcd}$$

β, γ decrease the rank

by a case subdivision

$\beta = \gamma \qquad z = v' = v'$

$\beta \neq \gamma \qquad v = u \square w$

β is　　γについても同様

$\beta = \beta' \square 1_w$: β acts inside the first factor u

$\beta = 1_u \square \beta''$: β 〃 w

$\beta = \alpha_{u,s,t}$ where $v = u \square w = u \square (s \square t)$

If both act inside the same factor u we can use inductor on the length n

$$\begin{tikzcd}
& u \square w \arrow[dl, "\beta' \square 1"'] \arrow[dr, "1 \square \gamma'"] & \\
u' \square w \arrow[dr, "1 \square \gamma'"'] & & u \square w' \arrow[dl, "\beta' \square 1"] \\
& u' \square w' &
\end{tikzcd}$$

$\square$: bifunctor

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the case when one of β or γ,

Say β is $\beta = \alpha = \alpha_{u,s,t}$ as in the third case above $\gamma \neq \beta$ γ must act inside u or inside w

If γ acts inside u, we use a diamond from $u \square (s \square t)$ to $(u' \square s) \square t$

which commute ☺ (何故なら α' is natural))

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