A CONDITION FOR ALGEBRAS ASSOCIATED WITH A CYCLIC QUIVER TO BE SYMMETRIC

By

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Abstract. Let \( K \) be a field, \( f(x) \) a monic polynomial in \( K[x] \) and \( K\Gamma \) the path algebra of a cyclic quiver \( \Gamma \) with \( s \) vertices and \( s \) arrows. In this paper, we give a necessary and sufficient condition for the algebra \( K\Gamma/(f(X)) \) to be a symmetric algebra, where \( X \) is the sum of all arrows in \( K\Gamma \).

1. Introduction

Let \( K \) be a field and \( \Gamma \) the cyclic quiver with \( \{e_1, \ldots, e_s\} \) as the set of vertices and \( \{a_1, \ldots, a_s\} \) as the set of arrows (\( s \geq 2 \)) such that the start point and the end point of \( a_i \) are \( e_i \) and \( e_{i+1} \), respectively. Let \( K\Gamma \) be the path algebra of \( \Gamma \). We denote the sum of all arrows by \( X : X = a_1 + \cdots + a_s \). It is known by Erdmann and Holm [EH] that \( K\Gamma/(X^p) \) is a symmetric algebra if and only if \( p \equiv 1 \pmod{s} \). In this paper, we consider the \( K \)-algebra \( A := K\Gamma/(f(X)) \) where \( f(x) \) is a monic polynomial over \( K \). Our purpose is to give a necessary and sufficient condition for \( A \) to be a symmetric algebra.

We describe the brief way to get the main theorem. First we will show that the equation \((f(X)) = (X^c h(X)) \) holds where \( h(x) \) is a monic polynomial in \( K[x^n] \) and \( c \) is an integer such that \( 0 \leq c \leq s - 1 \). Second we construct a left \( A \)-isomorphism \( \text{Hom}_K(A, K) \to A \) and also a right one (Propositions 2.3, 2.5). So we see that \( A \) is a Frobenius algebra. If \( c = 0 \) and the constant term of \( h(x) \) is nonzero, then we have a certain left \( A \)-isomorphism \( A \to A \) and also a right one (Lemma 3.3). By the above propositions and lemma, we have an isomorphism \( \text{Hom}_K(A, K) \to A \) of \( A \)-bimodules. Also if \( c = 1 \), then \( A \) is a symmetric algebra; if \( 2 \leq c \leq s - 1 \), then \( A \) is a nonsymmetric algebra (Proposition 3.5). Summarizing these statements we get the following main result; \( A \) is a

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symmetric algebra if and only if either \( c = 0 \) and the constant term of \( h(x) \) is nonzero or \( c = 1 \) holds (Theorem 3.6). Moreover, by means of the decomposition of algebras, we can compute the Hochschild cohomology ring of \( A \) in principle (Remark 3.8).

2. \( A \) is a Frobenius Algebra

Let \( s \) be a positive integer \((s \geq 2)\). By \( \Gamma \) we denote the cyclic quiver with \( \{e_1, \ldots, e_s\} \) as the set of vertices and \( \{a_1, \ldots, a_s\} \) as the set of arrows such that the start point and the end point of \( a_t \) are \( e_t \) and \( e_{t+1} \), respectively. Let \( K \) be a field and \( K\Gamma \) the path algebra of \( \Gamma \). Here we regard the index \( t \) of \( e_t \) modulo \( s \). Hence \( a_t = e_{t+1}a_te_t \) holds for \( 1 \leq t \leq s \) in \( K\Gamma \). We denote the sum of all arrows by \( X : X = a_1 + \cdots + a_s \). Then \( X^j \) is a sum of all paths of length \( j \) for \( j \geq 0 \).

Let \( f(x) \) be a monic polynomial of degree \( m \) \((m \geq 1)\) over \( K : f(x) = \alpha_0 + \alpha_1x + \cdots + \alpha_{m-1}x^{m-1} + x^m \). We consider the \( K \)-algebra \( A = K\Gamma/(f(X)) \).

For each \( i \) \((0 \leq i \leq s - 1)\), we set

\[
fi(x) = \alpha_i x^i + \alpha_{s+i}x^{s+i} + \alpha_{2s+i}x^{2s+i} + \cdots,
\]

which is the sum of the all terms of \( f(x) \) whose degree is congruent to \( i \) modulo \( s \). Then we have \( f(x) = f_0(x) + f_1(x) + \cdots + f_{s-1}(x) \). Let \( g_i(x) \) be the polynomials whose constant term is nonzero such that \( f_i(x) = x^{n_i}g_i(x) \) \((n_i \geq 0)\) if \( f_i(x) \neq 0 \), and we set \( g_i(x) = 0 \) if \( f_i(x) = 0 \). Then \( g(x) := \gcd(g_0(x), g_1(x), \ldots, g_{s-1}(x)) \) is in \( K[x^s] \) since \( g_i(x) \in K[x^s] \). If we set \( d = \min\{n_i + i \mid 0 \leq i \leq s - 1, \quad f_i(x) \neq 0\} \), then there exist an integer \( c \) \((0 \leq c \leq s - 1)\) and a monic polynomial \( h(x) \in K[x^s] \) such that \( \gcd(f_0(x), f_1(x), \ldots, f_{s-1}(x)) = x^d g(x) = x^c h(x) \). Note that \( c \) and \( h(x) \) are uniquely determined by \( f(x) \). Since \( e_{t+i}f(X)e_t = e_{t+i}f_i(X)e_t \) \((1 \leq t \leq s, 0 \leq i \leq s - 1)\), we have the following equation of ideals in \( K\Gamma \)

\[
(f(X)) = (f_0(X)) + (f_1(X)) + \cdots + (f_{s-1}(X)) = (X^c h(X)).
\]

Thus we have the following lemma.

**Lemma 2.1.** For the algebra \( A \), there exist an integer \( c \) \((0 \leq c \leq s - 1)\) and a monic polynomial \( h(x) \in K[x^s] \) such that

\[
A = K\Gamma/(X^c h(X)).
\]

**Example 2.2.** Let \( K \) be the field of rationals \( \mathbb{Q} \).

(i) Case \( s = 2 \). If \( f(x) = x - 2x^2 + x^3 \), then
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\[ f_0(x) = x^2g_0(x) = x^2 \cdot (-2), \]
\[ f_1(x) = xg_1(x) = x(1 + x^2). \]

Since \( \gcd(f_0(x), f_1(x)) = x \), we have
\[ \mathbb{Q}\Gamma/(f(X)) = \mathbb{Q}\Gamma/(X). \]

(ii) Case \( s = 3 \). If \( f(x) = x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} \), then
\[ f_0(x) = x^3g_0(x) = x^3(1 + x^3 + x^6), \]
\[ f_1(x) = x^4g_1(x) = x^4(1 + x^3 + x^6), \]
\[ f_2(x) = x^3g_2(x) = x^3(1 + x^3 + x^6). \]

Therefore \( \gcd(f_0(x), f_1(x), f_2(x)) = x^2(1 + x^3 + x^6) \), so we have
\[ \mathbb{Q}\Gamma/(f(X)) = \mathbb{Q}\Gamma/(X^2(1 + X^3 + X^6)). \]

(iii) Case \( s = 4 \). If \( f(x) = x^5 - x^6 + x^7 + 2x^9 + 2x^{11} + 2x^{13} + x^{14} + 2x^{15} + x^{17} + 2x^{18} + 2x^{19} + x^{22} + x^{23} + x^{27} \), then we write \( f(x) \) as follows:
\[ f(x) = \frac{x^5 + 2x^9 + 2x^{13} + x^{17} + (-x^6 + x^{14} + 2x^{18} + x^{22})}{f_1(x)} \]
\[ + \frac{x^7 + 2x^{11} + 2x^{15} + 2x^{19} + x^{23} + x^{27}}{f_1(x)} \]

Each of the above polynomials \( f_i(x) \) factors as follows:
\[ f_1(x) = x^5g_1(x) = x^5(1 + x^4)(1 + x^4 + x^8), \]
\[ f_2(x) = x^6g_2(x) = x^6(-1 + x^4 + x^8)(1 + x^4 + x^8), \]
\[ f_3(x) = x^7g_3(x) = x^7(1 + x^4 + x^{12})(1 + x^4 + x^8). \]

Therefore \( \gcd(f_1(x), f_2(x), f_3(x)) = x^5(1 + x^4 + x^8) \), so we have
\[ \mathbb{Q}\Gamma/(f(X)) = \mathbb{Q}\Gamma/(X^5(1 + X^4 + X^8)) \]
\[ = \mathbb{Q}\Gamma/(X(X^4 + X^8 + X^{12})). \]

Using the above notations, we set \( h(x) = k_0 + k_1x^s + \cdots + k_{n-1}x^{(n-1)s} + x^{ns} \in K[x^s] \). We will show that the \( K \)-algebra \( A = K\Gamma/(X^eh(X)) \) is a Frobenius algebra. In the rest of this paper, we use a representative elements instead of
their residue classes. We take the set \( \{X^i e_i \mid 1 \leq i \leq s, 0 \leq j \leq ns + c - 1 \} \) as a \( K \)-basis of \( A \) and also the dual basis \( \{ (X^i e_i)^* \in \text{Hom}_K(A, K) \mid 1 \leq i \leq s, 0 \leq j \leq ns + c - 1 \} \) (cf. [FS]). Then we obtain the following proposition. On that occasion we set \( k_n = 1 \) in the following.

**Proposition 2.3.** We have a left \( A \)-isomorphism \( \varphi : \text{Hom}_K(A, K) \to A \) defined by

\[
\varphi((X^i e_i)^*) = \sum_{t = m+1}^n k_t X^{i/s + c - j - 1} \quad \text{for} \quad 1 \leq i \leq s, 0 \leq j \leq ns + c - 1,
\]

where \( m \) is the integer \((-1 \leq m \leq n - 1)\) such that \( j = ms + c + r \) \((0 \leq r \leq s - 1)\). So \( A \) is a Frobenius algebra.

We prepare the following lemma for the proof of the proposition.

**Lemma 2.4.** Let \( i, j, t, t', u \) be integers with \( 1 \leq i, u \leq s, 0 \leq j \leq ns + c - 1, 1 \leq t \leq n - 1 \) and \( 0 \leq t' \leq n - 1 \). Then for \( (X^i e_i)^* \in \text{Hom}_K(A, K) \), we have

\[
X(X^i e_i)^* = \begin{cases} 
-k_0(X^{ns-1} e_{i+1})^* & \text{if } j = 0, c = 0, \\
0 & \text{if } j = 0, c \neq 0,
\end{cases}
\]

\[
\begin{cases} 
(X^{ts-1} e_{i+1})^* - k_t(X^{ns-1} e_{i+1})^* & \text{if } j = ts, c = 0, \\
(X^{ts+c-1} e_{i+1})^* - k_t(X^{ns+c-1} e_{i+1})^* & \text{if } j = t's + c, c \neq 0, \\
(X^{j-1} e_{i+1})^* & \text{otherwise},
\end{cases}
\]

\[
e_u(X^i e_i)^* = \begin{cases} 
(X^i e_i)^* & \text{if } u = i, \\
0 & \text{if } u \neq i.
\end{cases}
\]

**Proof.** Case \( c = 0 \); If \( j = 0 \), then for \( 0 \leq p \leq ns - 1 \) and \( 1 \leq q \leq s \),

\[(\dagger) \quad (X(e_i)^*)(X^p e_q) = (e_i)^*(X^p e_q X) = (e_i)^*(X^{p+1} e_{q-1}) .
\]

Here in case of \( p + 1 = ns \) and \( q - 1 \equiv i \) \((\text{mod } s)\), since \( X^{ns} = -k_0 - k_1 X^s - \cdots - k_{n-1} X^{(n-1)s} \) in \( A \), we have \( (e_i)^*(X^{ns} e_i) = (e_i)^*(-k_0 - \cdots - k_{n-1} X^{(n-1)s} e_i) = -k_0 \). Therefore

\[
\text{(equation } \dagger) = \begin{cases} 
-k_0 & \text{if } p + 1 = ns \text{ and } q - 1 \equiv i \pmod{s}, \\
0 & \text{otherwise}.
\end{cases}
\]

On the other hand \(-k_0(X^{ns-1} e_{i+1})^*(X^p e_q) = -k_0 \) if \( p = ns - 1 \) and \( q \equiv i + 1 \) \((\text{mod } s)\), 0 otherwise. Thus we have \( X(e_i)^* = -k_0(X^{ns-1} e_{i+1})^* \). If \( j = ts \), then for
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0 \leq p \leq ns - 1 \text{ and } 1 \leq q \leq s, \quad (X(X^i e_i)) (X^p e_q) = (X^{ts} e_i) (X^{p+1} e_{q-1}) = 1 \text{ if } p + 1 = ts \text{ and } q - 1 \equiv i \pmod{s}, \quad -k_i \text{ if } p + 1 = ns \text{ and } q - 1 \equiv i \pmod{s}, \quad 0 \text{ otherwise. Also } (X^{ts-1} e_{i+1})^* - k_l (X^{ts-1} e_{i+1})^* (X^p e_q) = 1 \text{ if } p = ts - 1 \text{ and } q \equiv i + 1 \pmod{s}, \quad -k_i \text{ if } p = ns - 1 \text{ and } q \equiv i + 1 \pmod{s}, \quad 0 \text{ otherwise. Thus we have } X(e_i)^* = (X^{ts-1} e_{i+1})^* - k_i (X^{ns-1} e_{i+1})^*.

Case \( c \neq 0 \): If \( j = 0 \), then for \( 0 \leq p \leq ns + c - 1 \) and \( 1 \leq q \leq s \), \((X(e_i)^* (X^p e_q) = (e_i)^* (X^p e_q X) = (e_i)^* (X^{p+1} e_{q-1})\). Here in case of \( p + 1 = ns + c \) and \( q - 1 \equiv i \mod{s} \), since \( X^{ns+c} = -k_0 X^c - k_1 X^{s+c} - \ldots - k_{n-1} X^{(n-1)s+c} \) in \( A \), we have \((e_i)^* (X^{ns+c} e_i) = (e_i)^* ((-k_0 X^c - k_1 X^{s+c} - \ldots - k_{n-1} X^{(n-1)s+c}) e_i) = 0 \). Therefore \( X(e_i)^* = 0 \). The remaining cases are clear. Hence we have the equation for \( X((X^i e_i)^*) \).

Next we will show that the second equation of lemma holds. If \( u = i \), then for \( 0 \leq p \leq ns + c - 1 \) and \( 1 \leq q \leq s \), \((e_u (X^i e_i)^*) (X^p e_q) = (X^{i} e_i)^* (X^p e_q e_i) = (X^{i} e_i)^* (X^p e_q) \) if \( q = i \), 0 if \( q \neq i \). Also we have \((X^{i} e_i)^* (X^p e_q) = (X^{i} e_i)^* (X^p e_i) \) if \( q = i \), 0 if \( q \neq i \). Hence we have \( e_u (X^i e_i)^* = (X^i e_i)^* \). If \( u \neq i \), then for \( 0 \leq p \leq ns + c - 1 \) and \( 1 \leq q \leq s \), \((e_u (X^i e_i)^*) (X^p e_q) = (X^{i} e_i)^* (X^p e_q) \) if \( q = u \), 0 if \( q \neq u \). If \( q = u \), then we have \( q = i \) because \( u \neq i \). Hence \( e_u (X^i e_i)^*) (X^p e_q) = 0 \) for \( 0 \leq p \leq ns + c - 1 \), \( 1 \leq q \leq s \). Therefore the proof of lemma is completed.

By this lemma, we will prove the Proposition 2.3.

**Proof of Proposition 2.3.** It is clear to see that \( \phi \) is an isomorphism of \( K \)-spaces. So it suffices to show that \( \phi \) is a homomorphism of left \( A \)-modules.

Hence we prove that

\[
\phi(X(X^i e_i)^*) = X\phi((X^i e_i)^*), \quad \phi(e_u (X^i e_i)^*) = e_u \phi((X^i e_i)^*),
\]

for \( 1 \leq i, u \leq s \) and \( 0 \leq j \leq ns + c - 1 \). First we will show that \( \phi(X(X^i e_i)^*) = X\phi((X^i e_i)^*) \).

We consider the case \( c = 0 \). If \( j = 0 \), then we have \( X\phi((e_i)^*) = X \sum_{t=1}^n k_t e_t X^{(t-1)s-1} = \sum_{t=1}^n k_t e_{t+1} X^{(t-1)s} = -e_{t+1} \sum_{t=0}^{n-1} k_t X^{(t)s} + \sum_{t=1}^{n-1} k_t e_{t+1} X^{(t-1)s} = -k_0 e_{t+1}, \) and \( \phi((X^i e_i)^*) = \phi((-k_0 (X^{ns-1} e_{i+1})^*) = -k_0 e_{t+1} \).

If \( j = ts \) (\( 1 \leq t \leq s - 1 \)), then we have \( X\phi((X^i e_i)^*) = X \sum_{t=1}^n k_t e_t X^{(t-1)s-1} = \sum_{t=1}^n k_t e_{t+1} X^{(t-1)s} = \sum_{t=1}^n k_t e_{t+1} X^{(t-1)s}, \) and \( \phi((X^i e_i)^*) = \phi((-k_0 (X^{ns-1} e_{i+1})^*) = -k_0 e_{t+1} \).

We consider the case \( c \neq 0 \). If \( j = 0 \), then we have \( X\phi((e_i)^*) = X \sum_{t=0}^n k_t e_t X^{(t+c-1)} = \sum_{t=0}^n k_t e_{t+1} X^{(t+c)} = 0, \) and \( \phi((X^i e_i)^*) = \phi(0) = 0 \). The remaining cases are clear. Therefore we have \( \phi(X(X^i e_i)^*) = X\phi((X^i e_i)^*) \).

Second we will show that \( \phi(e_u (X^i e_i)^*) = e_u \phi((X^i e_i)^*) \). If \( u = i \), then we have \( \phi(e_u (X^i e_i)^*) = \sum_{t=m+1}^n k_t e_t X^{(t+c-1)} = \phi((X^i e_i)^*) = \phi(e_u (X^i e_i)^*). \) If \( u \neq i \), then
we have $e_u \varphi((X^j e_i)^*) = 0$ since $e_u \neq e_i$. Also $\varphi(e_u (X^j e_i)^*) = \varphi(0) = 0$. Hence $\varphi$ is an isomorphism of left $A$-modules. This completes the proof of the proposition.

Similarly, considering the operation of $A$ onto $\text{Hom}_K(A, K)$ from the right, we get the following proposition.

**Proposition 2.5.** We have a right $A$-isomorphism $\psi : \text{Hom}_K(A, K) \to A$ defined by

$$\psi((X^j e_i)^*) = \sum_{\ell=m+1}^{n} k_{\ell} e_{i+\ell-1} X^{s_{\ell}+c-j-1}$$

for $1 \leq i \leq s$, $0 \leq j \leq ns + c - 1$,

where $m$ is the integer ($-1 \leq m \leq n - 1$) such that $j = ms + c + r$ ($0 \leq r \leq s - 1$).

**3. Main Theorem**

In this section we give a necessary and sufficient condition for the algebra $A = K\Gamma/(X^c h(X))$ to be a symmetric algebra, where $c$ is the integer such that $0 \leq c \leq s - 1$ and $h(x) = k_0 + k_1 x^3 + \cdots + x^m$. We prepare some lemmas for the proof of the main theorem.

The following fact is described in [EH].

**Lemma 3.1.** $K\Gamma/(X^p)$ ($p \geq 1$) is a symmetric algebra if and only if $p \equiv 1 \pmod{s}$.

**Proof.** We denote $K\Gamma/(X^p)$ by $B$. We set $p = ns + c$ ($0 \leq c \leq s - 1$) and $h(x) = x^{ns}$. Then the above $A$ coincides with $B$. If $p \equiv 1 \pmod{s}$, that is, $c = 1$, then $\varphi$ of Proposition 2.3 coincides with $\psi$ of Proposition 2.5. Hence $B$ is a symmetric algebra. Conversely we assume that $B$ is a symmetric algebra. We will use an indirect proof by assuming that $p \not\equiv 1 \pmod{s}$. Let $\xi$ be an isomorphism of $B$-bimodules $\text{Hom}_K(B, K) \to B$. Fix an $i$ with $1 \leq i \leq s$. Let $\xi((e_i)^*) = \sum_{j=0}^{p-1} \sum_{\ell=1}^{s} k_{j,\ell} X^j e_{\ell}$ for $k_{j,\ell} \in K$. Since $\xi$ is an isomorphism of $B$-bimodules, the equation $\xi((e_i)^*) e_u = \xi((e_i)^*) e_u$ holds for any $1 \leq u \leq s$. The left hand side equals $\sum_{j=0}^{p-1} \sum_{\ell=1}^{s} k_{j,\ell} X^j e_{\ell} e_u = \sum_{j=0}^{p-1} k_{j,u} X^j e_u$ and the right hand side equals $\sum_{j=0}^{p-1} \sum_{\ell=1}^{s} k_{j,\ell} X^j e_{\ell}$ if $i = u$, $0$ if $i \neq u$. This implies that $k_{j,\ell} = 0$ for $1 \leq \ell \leq s$ such that $\ell \neq i$ and any $0 \leq j \leq p - 1$. So we have $\xi((e_i)^*) = \sum_{j=0}^{p-1} k_{j,i} X^j e_i$. Furthermore, the equation $e_u \xi((e_i)^*) = \xi(e_u (e_i)^*)$ holds for $1 \leq u \leq s$. The left hand side equals $\sum_{j=0}^{p-1} k_{j,i} X^j e_{u-j} e_i = \sum_{j=0}^{p-1} k_{j,i} X^j e_i$ if
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\[ u \equiv i + j \imod{s}, \ 0 \text{ if } u \neq i + j \imod{s}. \] The right hand side equals 
\[ \sum_{j=0}^{p-1} k_{j,i}^i X^j e_i \text{ if } i = u, \ 0 \text{ if } i \neq u. \] This implies that \( k_{j,i}^i = 0 \) for \( 0 \leq j \leq p - 1 \) such that \( j \neq 0 \) (mod \( s \)). So we have \( \xi((e_i)^*) = \sum_{j=0}^{n} k_{j,i}^i X^j e_i \). Moreover, since \( X^i \xi((e_i)^*) = \xi(X(e_i)^*) \) and \( X(e_i)^* = 0 \) (by Lemma 2.4), it follows that \( \sum_{j=0}^{n} k_{j,i}^i X^j e_i = 0 \) in \( B \). Since the set \( \{X^j e_i|1 \leq i \leq s, 0 \leq j \leq p - 1\} \) is a \( K \)-basis of \( B \), we have \( \xi((e_i)^*) = 0 \) if \( p \neq 1 \) (mod \( s \)), a contradiction. Thus we have \( p \equiv 1 \) (mod \( s \)) if \( B \) is symmetric. This completes the proof of the lemma.

It is known by Furuya and Sanada [FS] that \( Z(K \Gamma) \) equals to \( K[X^s] \), where \( Z(K \Gamma) \) is the center of \( K \Gamma \). And an algebra isomorphism \( K \Gamma/(p_1(X) \cdots p_m(X)) \approx K \Gamma/(p_1(X)) \oplus \cdots \oplus K \Gamma/(p_m(X)) \) where each \( p_i(x) \in K[x^s] \) and \( \gcd(p_i(x), p_j(x)) = 1 \) for all \( 1 \leq i, j \leq m \) such that \( i \neq j \) is given by [FS]. By the similar way, we have the following lemma.

**Lemma 3.2.** If \( p(x) \in K[x^s] \) and \( p(x) \) is not divided by \( x \), then we have the following decomposition of algebras for the algebra \( K \Gamma/(X'^r p(X)) \) (r \( \geq 1 \)):

\[ K \Gamma/(X'^r p(X)) \approx K \Gamma/(X'^r) \oplus K \Gamma/(p(X)). \]

**Proof.** Since \( x^r \) and \( p(x) \) are relatively prime, we have \( X'^r u_1(X) + p(X) u_2(X) = 1 \) in \( K \Gamma \) for some \( u_1(x), u_2(x) \in K[x] \). Let \( z \in (X'^r) \cap (p(X)). \) If \( p(X) \in K[X^s] = Z(K \Gamma) \), then there exist \( v_1, v_2 \in K \Gamma \) such that \( z = X'^r v_1 = p(X) v_2 \). So we have \( z = z(X'^r u_1(X) + p(X) u_2(X)) = v_2 X'^r p(X) u_1(X) + X'^r p(X) v_1 u_2(X) \in (X'^r p(X)) \). Thus we have \( (X'^r) \cap (p(X)) \subset (X'^r p(X)) \). The converse inclusion is clear. By Chinese remainder theorem, we have the decomposition of algebras

\[ K \Gamma/(X'^r p(X)) = K \Gamma/((X'^r) \cap (p(X))) \approx K \Gamma/(X'^r) \oplus K \Gamma/(p(X)). \]

**Lemma 3.3.** Let \( c = 0 \). If \( k_0 \neq 0 \), then we have a left \( A \)-isomorphism \( \phi' : A \rightarrow A \) defined by \( \phi'(e_i X^j) = e_i X^{j+1} \) and a right \( A \)-isomorphism \( \psi' : A \rightarrow A \) defined by \( \psi'(e_i X^j) = e_{i+1} X^{j+1} \) for \( 1 \leq i \leq s, \ 0 \leq j \leq ns - 1 \).

**Proof.** Since \( k_0 \neq 0 \), each \( K \)-linear maps is an isomorphism of \( K \)-spaces. It is easy to show that these maps are homomorphisms of \( A \)-modules.

**Proposition 3.4.** Let \( c = 0 \). Then \( A \) is a symmetric algebra if and only if \( k_0 \neq 0 \).
Proof. If $k_0 \neq 0$, then by Propositions 2.3, 2.5 and Lemma 3.3, we have the left $A$-isomorphism $\varphi' \circ \varphi : \text{Hom}_K(A, K) \to A((X/e_i)^*) \mapsto \sum_{\ell=m+1}^n k_{\ell}e_{\ell}X^{(\ell-j)}$ and also we have the right $A$-isomorphism $\psi' \circ \psi : \text{Hom}_K(A, K) \to A((X/e_i)^*) \mapsto \sum_{\ell=m+1}^n k_{\ell}e_{\ell}X^{(\ell-j)}$. Thus $\varphi' \circ \varphi$ coincides with $\psi' \circ \psi$, so this is the isomorphism of $A$-bimodules. This means that $A$ is a symmetric algebra. Conversely we assume that $k_0 = 0$. Then there exists an integer $t$ ($1 \leq t \leq n$) such that $h(x) = x^{ts}h_0(x)$ where the constant term of $h_0(x) (e K[x^s])$ is nonzero. By Lemma 3.2, we have the following decomposition of $A$:

$$A \approx K \Gamma/(X^{ts}) \oplus K \Gamma/(h_0(X)).$$

For the decomposition, $K \Gamma/(X^{ts})$ is a nonsymmetric algebra by Lemma 3.1. Hence $A$ is a nonsymmetric algebra too ([EN, Proposition 1]). This completes the proof of the lemma.

Proposition 3.5. If $c = 1$, then $A$ is a symmetric algebra, and if $2 \leq c \leq s - 1$, then $A$ is a nonsymmetric algebra.

Proof. For the algebra $A$, there exists the integer $t$ ($0 \leq t \leq n$) such that $(X^{t}h(X)) = (X^{ts}+c)h_0(X)$ where the constant term of $h_0(x) \in K[x^s]$ is nonzero. Then, by Lemma 3.2, we have the following decomposition:

$$A = K \Gamma/(X^{ts+c}h_0(X)) \approx K \Gamma/(X^{ts+c}) \oplus K \Gamma/(h_0(X)).$$

By Proposition 3.4, $K \Gamma/(h_0(X))$ is a symmetric algebra. By Lemma 3.1, if $c = 1$, then $K \Gamma/(X^{ts+1})$ is a symmetric algebra, and if $2 \leq c \leq s - 1$, then $K \Gamma/(X^{ts+c})$ is a nonsymmetric algebra.

We summarize the above results as follows.

Theorem 3.6. $A$ is a symmetric algebra if and only if either $c = 0$ and $k_0 \neq 0$ hold or $c = 1$ holds.

Example 3.7. In Example 2.2, the algebras of the cases (i), (iii) are symmetric algebras, but one of the case (ii) is a nonsymmetric algebra.

Remark 3.8. We saw that there is a decomposition $A = K \Gamma/(X^{ts+c}h_0(X)) \approx K \Gamma/(X^{ts+c}) \oplus K \Gamma/(h_0(X))$ where the constant term of $h_0(x) \in K[x^s]$ is nonzero and $0 \leq c \leq s - 1$. For the decomposition of $A$, the Hochschild cohomology ring
of the first term is given by [EH], and also one of the second term is given by [FS]. Therefore the Hochschild cohomology ring of $A$ is obtained by these facts.

For example, we denote the $QG/(X^2(1 + X^3 + X^6))$ (isomorphic to $QG/(X^2) \oplus QG/(1 + X^3 + X^6)$) in Example 2.2 (ii) by $C$. We will compute the even Hochschild cohomology ring $HH^{ev}(C) = \bigoplus_{i \geq 0} HH^{2i}(C)$. By [EH, Section 4.8], the even Hochschild cohomology ring $HH^{ev}(QG/(X^2))$ is isomorphic to $Q[y_2, y_6]/(y_2^2, y_2y_6)$ where $\deg y_2 = 2$ and $\deg y_6 = 6$. Also, by [FS, Propositions 3.2, 3.7], the even Hochschild cohomology ring $HH^{ev}(QG/(1 + X^3 + X^6))$ is isomorphic to $Q[z_0]/(1 + z_0 + z_0^2)$ where $\deg z_0 = 0$. Thus we have

$$HH^{ev}(C) \simeq Q[y_2, y_6]/(y_2^2, y_2y_6) \oplus Q[z_0]/(1 + z_0 + z_0^2).$$

**References**


