Essays on Competitive Equilibria in Markets with Indivisibilities: Theory and Applications

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ABSTRACT

This dissertation consists of three essays on markets with indivisibilities where each buyer wants at most one indivisible good and each seller provides more than one unit. Unlike the standard literature of markets with indivisibilities, quasi-linearity is not assumed for utility functions of buyers.

The first essay (Chapter 2) studies the structure of competitive equilibria. The main result shows that for each type (say t) of indivisible goods, if there are multiple competitive equilibrium prices for type t, then the competitive equilibrium quantity for type t is unique; in the same manner, if there are multiple competitive equilibrium quantities for type t, then the competitive equilibrium price for type t is unique. As a corollary of the main result, the set of competitive price vectors shrinks to a unique point when a market has a large number of sellers. It is also argued that the main result cannot be extended to a market model where each buyer may demand more than one unit of indivisible goods.

The second essay (Chapter 3) evaluates the difference between the upper and lower bounds of the set of competitive price vectors. The upper and lower bounds are calculated by certain systems of equations, respectively. The main result shows that the difference between the upper and lower bounds of competitive price vectors is bounded by the difference of incomes of two specific households. The main result implies that the difference tends to zero when the number of households is large and their incomes are distributed in a relatively continuous manner. and thus, the calculated upper (lower) bound is a good approximation for a competitive price vector.

The third essay (Chapter 4) studies the relation between income distribution and housing rents based on the market model with indivisibilities (housing is classified into finite categories by quality). In particular, it is examined that how rising income inequality affects a competitive rent vector. The main result shows that there are three cases when income inequality increased: (1) competitive rents rise in every housing category, (2) rents rise in upper-categories but fall in lower-categories, or (3) rents fall in every category. The second result shows that case (1) is a special case. It is also argued with numerical examples that it tends to show case (3) as the diminishing rate of marginal utility for housing quality gets larger; equivalently, the diminishing rate of marginal utility for composite goods gets smaller.

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Mathematical Notations

Throughout the dissertation, vectors are written by small letters a, b, x, y, etc., while sets are written by capital letters A, B, X, Y, etc. The t-th component of vector x is written by x_t. Other symbols and notations are listed below (In the list, vectors are T-dimension).

Symbol	*Meaning*
$x \geq y$	$x_t \geq y_t$ for all $t = 1, ..., T$.
$\lvert x_t \rvert$	The absolute value of x_t.
ax	The inner product of a and x ($\sum_{t=1}^{T} a_t x_t$).
$X \subseteq Y$	The set Y weakly includes X ($x \in X$ implies $x \in Y$).
$X \backslash Y$	The set $\{x : x \in X \text{ and } x \notin Y\}$.
$\lvert X \rvert$	The cardinality of the set X.

Chapter 1

Introduction

1.1 Markets with indivisibilities

In the traditional Arrow–Debreu's general equilibrium model, commodities are assumed to be perfectly divisible. The perfect divisibility enables us to apply calculus depending on continuity of utility functions to the model. The assumption of perfect divisibility is suitable for economies where the amount of consumption/production is large for every economic agent and also the amount can be freely chosen. On the other hand, markets with indivisibilities are common in the real world (e.g., housing, labor or license markets). These markets are not suitable for general equilibrium model, because in such markets, small (discrete) number of units are demanded/supplied by consumers and/or producers. Unlike general equilibrium model, in markets with indivisibilities for which the differential method is not applicable, it can not be said that an analytical method is well established. In this dissertation, we aim to develop methods for studying markets with indivisibilities. In particular, we focus on the market model where each consumer wants at most one indivisible commodity.

The seminal study of markets with indivisibilities is found in Böhm-Bawerk (1891). The author considered the horse market where the economic agents are divided into sellers and buyers, each seller owns one horse for sale under his reservation price, and a buyer wants to buy exactly one horse under his valuation price. The horses to be traded are assumed to be *homogeneous*, i.e., all the horses are non-differentiated and exchanged in the same market price. In such a model, the author studied how one (or both) side competition affects the market price formation.

Shapley and Shubik (1972) also studied two-sided market where each buyer (seller) demands (supplies) at most one unit (their model is called the *assignment game*). The authors apply the framework of cooperative game theory. Unlike Böhm-Bawerk, the authors allow commodity differentiation for indivisible goods. In this sense, Böhm-Bawerk's market game is a special case of the assignment game. This generalization allows us to study such situations that each buyer wants (chooses) exactly one unit from several types but the same kind of indivisible units.[1] The authors proved the non-emptiness of the core by using linear programming problem. The assumption of quasi-linearity (QL) for players' utility function is then crucial for the application of linear programming (nevertheless, QL is not necessary for the existence of the core).[2] The authors also proved that the core always coincides with the set of competitive allocations. This equivalence theorem shows a difference between the market with and without indivisibilities, because in general equilibrium model the equivalence between the core and competitive allocation is obtained only under a large replica economy. Another result by the authors is that the core contains two specific imputations: buyer-optimal imputation and seller-optimal imputation. Buyer-optimal imputation corresponds to the minimum competitive price vector, and seller-optimal corresponds to the maximum competitive price vector.

As with the assignment game by Shapley and Shubik, most literature of market with indivisibilities assumed QL on a utility function. QL requires linearity for utility of money, which ignores income effects on buyer's demand on indivisible commodities. Therefore, QL is inappropriate to markets where objects of trade are large relative to the expenditure such as housings. Kaneko (1982) generalized the assignment game to the model where QL is not required for buyers' utility functions and each seller may provides more than one indivisible good of the same type. This market model is called a generalized assignment market (GAM). While Shapley and Shubik applied linear programming for the proof of the existence of the core, Kaneko used the main theorem of Scarf (1967) for the existence of that.[3] The author also proved the equivalence between the core and competitive equilibria under some condition.

[1] For instance, a typical household wants one dwelling in a lifetime, but his preference for housing types may different from each other (e.g., sizes, locations, etc.). The assignment game can describes such a situation.

[2] Formally, the assignment game is defined as follows. Let $M = \{1, ..., m\}$ be the set of buyers and $N = \{1, ..., n\}$ is the set of sellers. Let $v_{ij} \geq 0$ be the buyer $i \in M$'s valuation price for indivisible object of seller $j \in N$, and $r_j \geq 0$ be the reservation price of seller j. By the assumption of quasi-linear utility function, the assignment game can be simply described by matrix form $A = (a_{ij})_{(i,j) \in M \times N}$ where $a_{ij} = \max\{v_{ij} - r_j, 0\}$ and its characteristic function is defined by $v(S) = \max[a_{i_1 j_1} + a_{i_2 j_2} + \cdots + a_{i_k j_k}]$ where $i_1, ..., i_k \in S \cap M$ and $j_1, ..., j_k \in S \cap N$.

[3] The main theorem (Theorem 1) of Scarf (1967) states that the core of a *balanced game* is non-empty. Kaneko (1982) proved a generalized assignment game is balanced game [the proof relies on the main theorem of Shapley and Scarf (1974)].

This dissertation follows the GAM model by Kaneko. As stated earlier, the existence of competitive equilibrium is guaranteed by the study of Kaneko. On the other hand, there may exist multiple equilibria, and the structure of equilibria is not clear. This makes applications of GAM in comparative statics difficult. Therefore, the main purpose of our study is to clarify the structure of competitive equilibria in the GAM model. To be precise, we study characteristics of the set of competitive equilibria. Furthermore, as an application for the GAM model and our obtained results, we study comparative statics analyses in housing markets.

1.2 Brief review of other related literature

Here, we briefly summarize other researches related to our study.

In this dissertation, we also investigate the existency of competitive equilibrium in markets without QL where each buyer may demand "more than one unit" of an indivisible good. Under the assumption QL, some researchers study an existence condition of the core/competitive equilibria in such markets. Kelso and Crawford (1982) showed that the *gross substitute* (GS) condition is sufficient for the existence of the core/competitive equilibria.[4] Gul and Stacchetti (1999) also showed GS is a necessary condition for the existence of equilibria. However, it is an open question whether their results can be extended to market models "without" QL. For an answer, we give an example where GS holds but no competitive equilibria without QL (Section 2.4 of Chapter 2).

We also state Pareto efficiency and incentive compatibility of the GAM model (Section 2.2 and Appendix A of Chapter 2). These properties are studied in a field of auction theory.[5] We show that every competitive equilibrium in GAM satisfies Pareto efficiency (Appendix A of Chapter 2). Furthermore, we discuss incentive compatibility in GAM (Section 2.2 of Chapter 2). Incentive compatibility is considered as an important property to design the trading mechanism, since in such a one-shot trade, agents have incentives to disguise with their own preferences to influence the final outcome. The most related study is Demange and Gale (1985). The authors proved that in Shapley and Shubik's assignment game without QL,

[4] GS is a condition about a buyer's individual demand correspondence. In words, a demand satisfies GS iff rise in price for some goods causes the demands for the other goods remain the same or increase.

[5] In auction theory, trades are considered to be held *one-shot.* On the other hand, the GAM model supposes an application for housing markets, and a competitive equilibrium is considered as market equilibrium after *long time trading* rather than *one-shot trading.*

an allocation rule to select a minimum competitive equilibrium satisfies strategy-proofness for buyers.[6] Recent research by Serizawa and Morimoto (2015) proved that in Demange and Gale's model without sellers, an allocation rule to select a minimum competitive equilibrium is the only rule to satisfy strategy-proofness. Since the assumption for buyers in GAM is same as Demange and Gale's model, we can directly extend their results, i.e., in the GAM model, an allocation rule to select a minimum competitive equilibrium is the only rule to satisfy strategy-proofness for buyers.

We also investigate fairness of competitive equilibrium allocation in GAM (Appendix C of Chapter 4). Svensson (1983), Alkan, Demange and Gale (1991) and Sakai (2007) studied the theory of fairness to markets with indivisibilities.[7,8] The main problem is how to "fairly" allocate indivisibles by monetary translation: the indivisibles represents not only private goods but also bads with public nature (e.g., a society tries to determine the place of garbage-disposal facilities). It is shown by Svensson (and also Alkan et al.) that there exists a equitable allocation; every equitable allocation is Pareto efficient (thus every equitable allocation is fair) and; the set of equitable allocation coincides with the set of competitive allocation with equal income. In Appendix C of Chapter 4, we briefly mention fairness of competitive equilibrium in our market model. It is a result that a competitive allocation is fair if and only if every household has the same income.

1.3 Chapter overviews

The dissertation consists of three essays (Chapter 2, 3 and 4). Chapter 2 investigates mathematical structures of the set of competitive equilibria under basic assumptions. Chapter 3 investigates the difference between the upper and lower bounds of the set of competitive price vectors under some additional assumptions. The market model of Chapter 3 aims an application for housing markets. Chapter 4 gives comparative statics analyses in the market model of Chapter 3. Details are described below.

Chapter 2 examines characteristics of the set of competitive equilibria in GAM. As men-

[6] We give definitions. An *allocation* (or *auction*) *rule* is a function from the set of agents' preference profiles to the allocation. An allocation rule is *strategy-proof* iff it is a dominant strategy for each agent to announce his true preferences.

[7] Their model also assumed that each buyer demands at most one unit of an indivisible good.

[8] The original definition of fairness is first given by Foley (1967). According to him, an allocation is called *fair* iff (1) an allocation is Pareto efficient and (2) every agent is utility maximized with his consumption compared to any other agent's consumption.

tioned earlier, there exists a competitive equilibrium in GAM (Kaneko, 1982). On the assumption of the existence of a competitive equilibrium, we show the structure of the set of competitive equilibria. Let $T(\geq 1)$ be the number of types for indivisible goods and fix arbitrarily a type t $(1 \leq t \leq T)$. The main result shows that the set of competitive equilibria has the *non-simultaneous multiplicity* structure for each type of indivisible goods: if there are multiple competitive equilibrium prices for type t, then competitive equilibrium quantity of type t is unique; equivalently, if there are multiple competitive quantities for type t, competitive price of type t is unique. This structure is well-known in the case of no commodity differentiation (Böhm-Bawerk's market game). The main result implies that even if we allow commodity differentiation, this non-simultaneous multiplicity holds *separately* for each type of goods. Based on the main result, we give the second result on the evaluation of the sizes of the sets of competitive prices and quantities. As an application of these two results, we give a shrinkage theorem on the set of competitive price vectors. We also study whether our results can be extended to a market model where each buyer may demand more than one unit of an indivisible good: we argue that our results can not be extended even if demand corresponding of each buyer satisfies GS condition.

Chapter 3 evaluates the difference between the upper and lower bounds of the set of competitive price vectors in the application model of GAM. In the analysis, the following assumptions are added: identical utility function and normality of indivisible goods. This market model assumes the rental housing market and is introduced by Kaneko (1983) and Kaneko, Ito and Osawa (2006). Under additional assumptions, the upper and lower bounds of the competitive price set are calculated by certain systems of equations, respectively. The upper (lower) bound coincides with the maximum (minimum) competitive price vector under some condition. The main result shows that the difference between the upper and lower bounds of competitive price vectors is bounded by the difference of incomes of specific two households. Since this two households are adjacent to each other with respect to income, the main result implies that the difference tends to zero when the number of households is large and their incomes are thickly distributed. Therefore, the calculated upper (lower) bound is a good approximation for a competitive price vector. Throughout Chapter 3, the main result of Chapter 2 is applied and used for the analysis.

Chapter 4 studies the relation between income distribution and housing rents based on the rental housing market model by Kaneko, Ito and Osawa (2006), where housing is classified into finite categories by quality. In particular, we investigate how rising income inequality affects a competitive rent vector. The main comparative statics result shows that when household income inequality increases, either (1) increased rent for every housing category; (2) increased rent for housing of upper-quality category but decreased rent for that of lower-quality category; (3) decreased rent for every housing category. (1) and (3) seem as counterintuitive because it is natural that rising income inequality causes a decline in rent for lower categories and a rise in rent for upper categories. Indeed, the second result shows that case (1) is a special case, while (3) may not be a special. We also argue with numerical examples that it tends to show case (3) as the diminishing rate of marginal utility for housing quality gets larger or the diminishing rate of marginal utility for composite goods gets smaller. Since the diminishing rate of marginal utility is related to the marginal rate of substitution, this observation implies that there is a certain tendency between the marginal rate of substitution and rent changes. Note that analyses of Chapter 4 rely on evaluation result on the competitive rent vector, which is shown in Chapter 3.

Chapter 2

Characteristics of Competitive Equilibria in Assignment Markets

2.1 Introduction

In this chapter, we study the structure of the set of competitive equilibria in an assignment market. This market consists of two types of economic agents; sellers and buyers. The objects of trade are several types of indivisible goods and a perfectly divisible good (money). Each seller may provide multiple units of an indivisible good, but each buyer demands at most one unit of an indivisible good. We adopt the model of the generalized assignment market (abbreviate it as GAM) from Kaneko (1982).

The GAM model is a generalization of Shapley and Shubik's (1972) assignment market model in that each seller may provide multiple units of an indivisible good and the quasi-linearity (QL) assumption on utility functions of buyers is removed. Kaneko (1982) proved the existence of a competitive equilibrium in the GAM model, while Kaneko (1983) applied the GAM model to housing markets.

The GAM/assignment model targets economic problems of indivisible objects such as houses, cars, and labor. There is a salient difference from the standard general equilibrium model with perfectly divisible goods (cf., Mas-Colell, Whinston and Green, 1995). One example is that the core of the assignment model coincides with the set of competitive allocations (Kaneko, 1982; Quinzii, 1984), while in the general equilibrium model, this coincidence can be obtained in a large replica economy (Debreu and Scarf, 1963). The structure of competitive equilibria in the GAM model also differs considerably from those in the general equilibrium

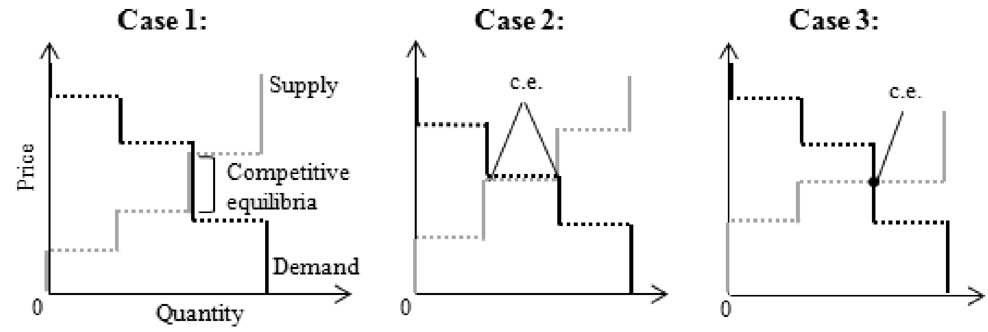

Figure 2.1: The possibilities for the structure of c.e. without commodity differentiation.

model. In this paper, we give three theorems for the structure of competitive equilibria, from which we can observe clear differences between the structures of competitive equilibria for the general equilibrium and GAM models.

The main theorem of this paper is Theorem 2.3.1 in Section 2.3. We provide another theorem, Theorem 2.3.2, on the evaluation of competitive prices/quantities. From those theorems, we obtain the shrinkage result, Theorem 2.5.2, which states that as the market size increases, the set of competitive prices shrinks to a unique price. In this introduction, we describe Theorem 2.3.1, and briefly mention the other theorems.

Let $T(\geq 1)$ be the number of types of indivisible goods, and let t be an integer with $1 \leq t \leq T$.

Theorem 2.3.1. If there are multiple competitive prices for good t, then the equilibrium quantity of t is unique; if there are multiple equilibrium quantity for good t, then the competitive price of t is unique.

Thus, Theorem 2.3.1 shows that it is not possible that the market has multiple competitive prices and equilibrium quantities for some indivisible good t.

Theorem 2.3.1 is better understood in the case where all indivisible goods are homogeneous, i.e., $T = 1$. This special case is known as the Böhm-Bawerk horse market (Böhm-Bawerk, 1891). When $T = 1$, the demand and supply schedules are expressed on two-dimensional surface, as illustrated in Fig. 2.1. Their intersection constitutes the set of competitive equilibria. As in Fig. 2.1, there are three possibilities for the structure of competitive equilibria. In Case 1, there are multiple equilibrium prices and a unique equilibrium quantity, in Case 2, there are multiple quantities and a unique price, and in Case 3, both are uniquely determined.[1] Theorem 2.3.1 shows that even if we allow commodity differentiation ($T > 1$), this structure holds separately for each type of an indivisible good.

In the literature, some extended model is also considered where each buyer may demand more than one unit of an indivisible good. It is known that the extended model has a competitive equilibrium under the gross substitutes (GS) assumption on the individual demand correspondence and under the QL assumption for the buyers (see Kelso and Crawford, 1982, Gul and Stacchetti, 1999). It may be wondered if Theorem 2.3.1 can be extended to such a model. We show that under the GS and QL assumptions, Theorem 2.3.1 can be extended to such an extended model.

However, since our model targets an economic situation where each unit of an indivisible good is non-negligible relative to a buyer's income, we would like to remove the QL assumption from our study. We provide an example, with the GS but without the QL assumption, where Theorem 2.3.1 fails. Thus, the theorem cannot be extended only under the GS assumption. In fact, we give another example satisfying GS but having no competitive equilibria.

Theorem 2.3.2 characterizes the size of the set of competitive prices for good t (equilibrium quantities, respectively) in terms of marginal costs for sellers.

Based on Theorems 2.3.1 and 2.3.2, we obtain a shrinkage result, Theorem 2.5.2 on the set of competitive prices for a large GAM. Shapley and Shubik (1972) observed, for the homogeneous case ($T = 1$), that the set of competitive prices shrinks to a unique price when a market becomes large and dense. They expected that this would also hold in the general case ($T > 1$), but also stated a difficulty caused by the increase of the dimensionality of the set of equilibria. In fact, we directly obtain their expected result from Theorems 2.3.1 and 2.3.2, while avoiding the difficulty indicated by them. Since Theorems 2.3.1 and 2.3.2 hold for each type t, we meet no difficulty in the dimensionality of the set of equilibria; a shrinkage result can be obtained separately for each type of an indivisible good.

For notational simplicity, except for Section 2.5, we assume that for each $t = 1, ..., T$, all the indivisible goods of type t are provided by only one seller. However, this assumption can be made without loss of generality when considering a competitive equilibrium. This aggregation result will be discussed in Section 2.5.1.

This paper is organized as follows. Section 2.2 presents the GAM model. Section 2.3

[1] Let $mc(y)$ ($y \in \mathbb{Z}_+$) be the seller's marginal cost of additional one unit at supply y. In Case 3, it holds that $mc(y^*) = mc(y^* + 1)$ for supply y^*, and the competitive equilibrium is uniquely determined with the price $mc(y^*)$ and supply $y^* + 1$.

presents two theorems about the structure of competitive equilibria. Section 2.4 is concerned with the extendibility of our main theorem to an extended market model. Section 2.5 shows the aggregation result of the sellers, and shows the shrinkage theorem on the competitive prices in a large GAM. Conclusions and closing remarks are presented in Section 2.6.

2.2 Generalized assignment markets

We denote the generalized assignment market model by (M, N), where $M = \{1', \ldots, m'\}$ denotes the *set of buyers* and $N = \{1, \ldots, n\}$ denotes the *set of sellers*. There are T-types of indivisible goods to be traded for a perfectly divisible good, called *money*.

The *consumption set* for a buyer is given as $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$, where for $t \neq 0$, $\mathbf{e}^t$ is the T-dimensional unit vector with t-th component 1 and $\mathbf{e}^0 = \mathbf{0}$, and $\mathbb{R}_+$ is the set of non-negative real numbers. A consumption vector $(\mathbf{e}^t, d) \in X$ with $t > 0$ means that a buyer consumes one unit of indivisible good of type-t and d amount of (perfectly divisible) money. For $t = 0$, no indivisible goods are consumed. The *initial endowment* of each buyer $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$, that is, buyer $i \in M$ initially has an income I_i and no indivisible goods. Each buyer wants to buy at most one unit of an indivisible good by paying part of I_i.

We define buyer i's *utility function* as $u_i : X \to \mathbb{R}$. We assume the following for u_i:

Assumption A1 (*Continuity and Monotonicity*). For each $x_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u_i(x_i, d)$ is a continuous and strictly monotone increasing function with respect to d.

Assumption A2 (*Boundary condition*). $u_i(\mathbf{e}^0, I_i) > u_i(\mathbf{e}^t, 0)$ for all $t = 1, ..., T$.

A1 needs no explanation. A2 means that a buyer prefers to keep his initial endowment to consuming any indivisible good by paying all his income I_i.

Each seller $j \in N$ provides indivisible goods of exactly one type, but each may provide more than one unit. We divide the set N into $N_1, \ldots, N_T$, where N_t is the set of all sellers who provide indivisible good t. Let $t = 1, ..., T$. We define the *cost function* of seller $j \in N_t$ as $c_j : \mathbb{Z}_+ \to \mathbb{R}_+$, where $\mathbb{Z}_+$ is the set of non-negative integers, and $c_j(y_j)$ represents the cost (in terms of money) of producing y_j units of indivisible goods t. For each $j \in N_t$, we define the *marginal cost* $mc_j(y_j) := c_j(y_j + 1) - c_j(y_j)$ for $y_j \in \mathbb{Z}_+$. We assume the following for c_j :

Assumption B1 (*No fixed cost*). $c_j(0) = 0$ and $c_j(0) < c_j(1)$.

Assumption B2 (*Convexity*). $mc_j(y_j) \leq mc_j(y_j + 1)$ for all $y_j \in \mathbb{Z}_+$.

The first assumption means that no fixed costs are required, but that a positive cost is required for production. Assumption B2 is a discrete version of convexity, meaning that a marginal cost increases by one additional unit. Note that this cost function is one-dimensional case of M$^\natural$-convex function (Murota, 2003).

The model given in Shapley and Shubik (1972) can be regarded as a special case of the above GAM model. They assumed that each buyer $i \in M$ wants to buy at most one unit of indivisible good with a quasi-linear (QL) utility function, i.e., $u_i(\mathbf{e}^t, d) = u_i(\mathbf{e}^t, 0) + d$ for all $(\mathbf{e}^t, d) \in X$; and each seller $j \in N$ has one unit of an indivisible good for sale with reservation price $r_j > 0$. In A1 and A2, we do not assume quasi-linearity and allow income effects in buyers' behavior. A seller in Shapley and Shubik's model is expressed in our model as a seller having the cost function $c_j(y_j)$ with $c_j(1) = r_j$ and $c_j(y_j) =$"large" for $y_j \geq 2$.

For notational simplicity, we assume that the set N_t of sellers of type t consists of one seller, i.e.,

$$N_t = \{t\} \text{ for all } t = 1, ..., T. \tag{1.1}$$

This means that the sellers of type t can be represented by one *aggregated seller*. This assumption can be made *without loss of generality*, as far as the competitive equilibrium is concerned. This will be shown in Section 2.5.1.

In the GAM model, we consider the concept of a competitive equilibrium. Let $(p, x, y) = ((p_1, \ldots, p_T),\ (x_{1'}, \ldots, x_{m'}),\ (y_1, \ldots, y_T))$ be a triple of $p \in \mathbb{R}^T_+$, $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}^{m'}$ and $y \in \mathbb{Z}^T_+$.

Definition 2.2.1 (*Competitive Equilibrium*). We say that (p, x, y) is a *competitive equilibrium* iff:

(1) Utility Maximization under the Budget Constraint: for all $i \in M$,

(i): $I_i \geq px_i$, where $px_i = \sum_{t=1}^T p_t x_{it}$;

(ii): $u_i(x_i, I_i - px_i) \geq u_i(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$ with $I_i \geq px'_i$.

(2) Profit Maximization: for all $t \in N$,

$p_t y_t - c_t(y_t) \geq p_t y'_t - c_t(y'_t)$ for all $y'_t \in \mathbb{Z}_+$.

(3) Balance of the Total Demand and Supply: $\sum_{i \in M} x_i = \sum_{t=1}^T y_t \mathbf{e}^t$.

Note that since each x_i is a T-dimensional vector and each y_t is a scalar, we need to multiply y_t by $\mathbf{e}^t$. Note also that by assumption B2, condition (2) can be rewritten as $mc_t(y_t -$

1) $\leq p_t \leq mc_t(y_t)$ for all $t \in N$.[2] We abbreviate competitive equilibrium as c.e.

Kaneko (1982) and Kaneko and Yamamoto (1986) prove the existence of a c.e. in (M, N).

Theorem 2.2.2 (*Existence*). *There exists a c.e.* (p, x, y) *in* (M, N).

We denote the set of all c.e. in (M, N) by $\mathbb{C}$. We say that a pair (x, y) is a *competitive allocation* iff $(p, x, y) \in \mathbb{C}$ for some $p \in \mathbb{R}^T_+$. Let $\mathbb{A}_C$ be the set of all competitive allocations in (M, N). We say that p is a *competitive price vector* iff $(p, x, y) \in \mathbb{C}$ for some $(x, y) \in \mathbb{A}_C$. Let $\mathbb{P}_C$ be the set of all competitive price vectors in (M, N). Note that every competitive equilibrium is Pareto efficient in (M, N) (see Appendix A).

Various authors have studied the structure of the set of c.e./core in assignment markets with/without quasi-linearity (QL) for utility functions. In Shapley-Shubik's assignment model without the assumption QL, the core coincides with the set of competitive allocations, while in GAM model, the latter is included in the core. The converse does not necessarily hold. Kaneko (1982) gave a sufficient condition for the equivalence between the core and the set of c.e., namely that for each seller $j \in N_t$, there is another seller j' who is of the same type as j (Theorem 10, p. 227). It is also known that, in the assignment model with QL, the set of competitive price vectors has a lattice structure, which guarantees the existence of the maximum and minimum competitive price vectors (cf. Shapley and Shubik, 1972; Mishra and Talman,2010). GAM model also satisfies that the set $\mathbb{P}_C$ is lattice (cf. Miyake, 1994).

We conclude this section by stating incentive compatibility of our market model. Incentive compatibility is studied in a field of auction theory.[3] The most related study is Demange and Gale (1985). The authors proved that in Shapley and Shubik's assignment game without QL, an allocation rule to select a minimum competitive equilibrium satisfies strategy-proofness for buyers.[4] Recent research by Serizawa and Morimoto (2015) proved that in Demange and Gale's model without sellers, an allocation rule to select a minimum competitive equilibrium is the only rule to satisfy strategy-proofness. Since the assumption for buyers in GAM is same as Demange and Gale's model, we can directly extend their results, i.e., in the GAM model, an allocation rule to select a minimum competitive equilibrium is the only rule to satisfy strategy-proofness for buyers.

[2] We stipulate $mc_t(-1) = 0$.

[3] In auction theory, trades are considered to be held *one-shot*. In such a one-shot trade, agents have incentives to disguise with their own preferences to influence the final outcome. Therefore, it is an important problem to study what rule which determine an allocation is compatible with such incentives.

[4] We give definitions. An *allocation* (or *auction*) *rule* is a function from the set of agents' preference profiles to the allocation. An allocation rule is *strategy-proof* iff it is a dominant strategy for each agent to announce his true preferences.

2.3 Characteristics of competitive equilibria

We present two theorems in Section 2.3.1. The first theorem states that for each indivisible good t, it separately holds that the multiplicity of competitive prices for good t implies a unique equilibrium quantity for t. The second theorem is about the sizes of the sets of competitive prices and allocations. We can separately evaluate the c.e. for each type of good using both theorems. The proofs of these theorems will be given in Section 2.3.2.

2.3.1 The structure and size of competitive equilibria

For each $t = 1, ..., T$, we denote the sizes of the sets of competitive prices and competitive allocations for indivisible good t by

$$\delta_t(\mathbb{P}_C) := \max\{\left|p_t - p'_t\right| : p, p' \in \mathbb{P}_C\};$$
$$\delta_t(\mathbb{A}_C) := \max\{\left|y_t - y'_t\right| : (x, y), (x', y') \in \mathbb{A}_C\}.$$

Since $\mathbb{P}_C$ and $\mathbb{A}_C$ are compact sets under our assumptions, we can take the maximum value for the above definition. When $\delta_t(\mathbb{P}_C) > 0$, there are at least two different competitive prices for good t, and when $\delta_t(\mathbb{P}_C) = 0$, there is a unique competitive price. The other cases are interpreted in a similar manner.

The first theorem is about possible cases of $\delta_t(\mathbb{P}_C)$ and $\delta_t(\mathbb{A}_C)$. The proof will be given in Section 2.3.2.

Theorem 2.3.1 (*Non-simultaneous Multiplicity for Competitive Equilibria*). *Let* $t = 1, ..., T$. *Then either* (1), (2) *or* (3) *holds:*

(1) $\delta_t(\mathbb{P}_C) > 0$ *and* $\delta_t(\mathbb{A}_C) = 0$.

(2) $\delta_t(\mathbb{P}_C) = 0$ *and* $\delta_t(\mathbb{A}_C) > 0$.

(3) $\delta_t(\mathbb{P}_C) = 0$ *and* $\delta_t(\mathbb{A}_C) = 0$.

The theorem is equivalent to the statement that $\delta_t(\mathbb{P}_C) > 0$ implies $\delta_t(\mathbb{A}_C) = 0$ (and $\delta_t(\mathbb{A}_C) > 0$ implies $\delta_t(\mathbb{P}_C) = 0$). Theorem 2.3.1.(1)-(3) correspond to Cases 1-3 in Fig. 2.1. As in Fig. 2.1, Theorem 2.3.1 is clear in the GAM without commodity differentiation ($T = 1$). Theorem 2.3.1 states that even if we allow commodity differentiation ($T > 1$), non-simultaneous multiplicity of competitive prices and quantities holds separately for each good. Note that assertion (3) has two subcases: Fig. 2.1, Case 3 depicts one subcase of (3), where the supply schedule has a flat part. In the other subcase, the demand schedule has a flat part.

Theorem 2.3.1 is related to Mishra and Talman (2010), Theorem 6, p.14.[5] They studied the structure of the set of c.e. in an assignment market with the QL assumption and the no-seller assumption meaning that indivisible goods are assumed to be already supplied. Their theorem states that $\mathbb{P}_C$ has an interior point if and only if there exists a unique efficient allocation. The no-seller assumption is essential for the if part: it may not hold in the presence of sellers. Fig. 2.1, Case 3 is a counterexample for this, where the price and allocation are uniquely determined. The only-if part can directly be compared to our Theorem 2.3.1. It follows from Theorem 2.3.1 that if $\delta_t(\mathbb{P}_C) > 0$ for any $t = 1, ..., T$, then the competitive allocation is uniquely determined.

Using Theorem 2.3.1, we evaluate the sizes of the sets $\mathbb{P}_C$ and $\mathbb{A}_C$ for good t. For this evaluation, when the equilibrium quantity for good t is unique, i.e., $\delta_t(\mathbb{A}_C) = 0$, we denote it by y_t^*. When $\delta_t(\mathbb{P}_C) = 0$, we denote the price by p_t^*. As stated in Theorem 2.3.1, the sufficient condition for $\delta_t(\mathbb{A}_C) = 0$ ($\delta_t(\mathbb{P}_C) = 0$, respectively) is $\delta_t(\mathbb{P}_C) > 0$ ($\delta_t(\mathbb{A}_C) > 0$). The proof of Theorem 2.3.2 is given in Section 2.3.2.

Theorem 2.3.2 (*The Size of the Set of Competitive Equilibria*). *Let* $t = 1, ..., T$. *Then,*

(1) $\delta_t(\mathbb{P}_C) > 0$ *implies* $mc_t(y_t^* - 1) \leq p_t \leq mc_t(y_t^*)$ *for all* $p \in \mathbb{P}_C$.

(2) $\delta_t(\mathbb{A}_C) > 0$ *implies* $\delta_t(\mathbb{A}_C) \leq |\{y_t \in \mathbb{Z}_+ : p_t^* = mc_t(y_t)\}|$ *and* $p_t^* = mc_t(\widehat{y}_t)$, *where* $\widehat{y}_t := \min\{y_t \in \mathbb{Z}_+ : (x, y) \in \mathbb{A}_C\}$.[6]

Assertion (1) states that if there are multiple competitive prices for good t, then all the competitive prices of good t are bounded by the marginal costs $mc_t(y_t^* - 1)$ and $mc_t(y_t^*)$. (2) states that if there are multiple equilibrium quantities for good t, then $\delta_t(\mathbb{A}_C)$ is restricted by the condition of seller t's marginal costs. Although (2) allows multiple equilibrium quantities for good t, the magnitude of multiplicity is expected to be rather small. For example, if the cost function c_t is strictly convex, then (2) implies $\delta_t(\mathbb{A}_C) \leq 1$. The additional $p_t^* = mc_t(\widehat{y}_t)$ means that the competitive price of good t is the marginal cost $mc_t(\widehat{y}_t)$. In sum, even if there are multiple competitive prices or quantities, they are not distantly located.

[5] Theorem 6 of Mishra and Talman (2010) is based on Theorem 5 of them. Theorem 5 is more specific; it states that $p \in \mathbb{P}_C$ is an interior point if and only if each good is demanded by a unique buyer and every buyer demands exactly one good. Since our model eliminates the QL assumption and takes sellers explicitly, their proof of Theorem 5 cannot be directly applied to our model. Nevertheless, we conjecture that this theorem can be extended to our model.

[6] $|X|$ is the cardinality of the set X.

2.3.2 Proofs of Theorems 2.3.1 and 2.3.2

We give proofs of Theorems 2.3.1 and 2.3.2. First we start with the following lemmas.

Lemma 2.3.3. *Let $(p, x, y), (p', x', y') \in \mathbb{C}$ and $t = 1, ..., T$. Then $p_t < p'_t$ implies $y_t \leq y'_t$.*

Proof. We have $p_t y_t - c_t(y_t) \geq p_t y'_t - c_t(y'_t)$ and $p'_t y'_t - c_t(y'_t) \geq p'_t y_t - c_t(y_t)$ by seller t's profit maximization condition. By these inequalities, we have $p_t y_t + p'_t y'_t \geq p_t y'_t + p'_t y_t$. Hence, we obtain $y_t(p_t - p'_t) \geq y'_t(p_t - p'_t)$. This inequality, together with $p_t < p'_t$ implies $y_t \leq y'_t$. ∎

Lemma 2.3.4. *Let $(p, x, y), (p', x', y') \in \mathbb{C}$, $t \neq t'$ and suppose $p_t \geq p'_t$ and $p_{t'} < p'_{t'}$. Then there is no $i \in M$ such that $x_i = \mathbf{e}^t$ and $x'_i = \mathbf{e}^{t'}$.*

Proof. Let $p_t \geq p'_t$. We suppose that $x_i = \mathbf{e}^t$ and $x'_i = \mathbf{e}^{t'}$ for some $i \in M$. It suffices to show that $p_{t'} \geq p'_{t'}$. By utility maximization for i, we have

$$u_i(\mathbf{e}^t, I_i - p_t) \geq u_i(\mathbf{e}^{t'}, I_i - p_{t'}) \text{ and } u_i(\mathbf{e}^{t'}, I_i - p'_{t'}) \geq u_i(\mathbf{e}^t, I_i - p'_t). \tag{1.2}$$

Since $p_t \geq p'_t$, we have, by Assumption A1, $u_i(\mathbf{e}^t, I_i - p'_t) \geq u_i(\mathbf{e}^t, I_i - p_t)$. This, together with the the first inequality of (1.2), implies $u_i(\mathbf{e}^t, I_i - p'_t) \geq u_i(\mathbf{e}^{t'}, I_i - p_{t'})$. Also the second inequality implies that $u_i(\mathbf{e}^{t'}, I_i - p'_{t'}) \geq u_i(\mathbf{e}^{t'}, I_i - p_{t'})$. By Assumption A1, we have $p_{t'} \geq p'_{t'}$. ∎

We get the following lemma from Lemmas 2.3.1 and 2.3.2.

Lemma 2.3.5. *Let $(p, x, y), (p', x', y') \in \mathbb{C}$ and $t = 1, ..., T$. Then $p_t \neq p'_t$ implies $y_t = y'_t$.*

Proof. Suppose $p_t < p'_t$. We show $y_t = y'_t$. Let

$$K = \{k : 1 \leq k \leq T,\ p_k < p'_k\} \text{ and } L = \{1, ..., T\} \backslash K.$$

It follows from Lemma 2.3.3 that $y_k \leq y'_k$ for all $k \in K$. Hence, $\sum_{k \in K} y_k \leq \sum_{k \in K} y'_k$. If the converse of this inequality holds, then $y_k = y'_k$ should be the case for all $k \in K$. Hence, it suffices to show that $\sum_{k \in K} y_k \geq \sum_{k \in K} y'_k$.

Now, let

$$M(K) = \{i \in M : x_i = \mathbf{e}^k \text{ for some } k \in K\};$$
$$M(L) = \{i \in M : x_i = \mathbf{e}^l \text{ for some } l \in L\};$$
$$M'(K) = \{i \in M : x'_i = \mathbf{e}^k \text{ for some } k \in K\}.$$

Then $\{M(K), M(L), \{i \in M : x_i = \mathbf{e}^0\}\}$ is a partition of M. Now let us show $M(K) \supseteq M'(K)$. By Lemma 2.3.4, for any $l \in L$ and $k \in K$, there is no $i \in M$ such that $x_i = \mathbf{e}^l$

and $x'_i = \mathbf{e}^k$, i.e., $M(L) \cap M'(K) = \emptyset$. Furthermore, by Assumption A1 and $p_k < p'_k$ for any $k \in K$, there is no $i \in M$ such that $x_i = \mathbf{e}^0$ and $x'_i = \mathbf{e}^k$, i.e., $\{i \in M : x_i = \mathbf{e}^0\} \cap M'(K) = \emptyset$. Since $M(K) \cup M(L) \cup \{i \in M : x_i = \mathbf{e}^0\} = M$ is a partition of M, we have $M(K) \supseteq M'(K)$.

By the condition of the balance of total demand and supply, we have $|M(K)| = \sum_{k \in K} y_k$ and $|M'(K)| = \sum_{k \in K} y'_k$. By the above inclusion result, we have $\sum_{k \in K} y_k = |M(K)| \geq |M'(K)| = \sum_{k \in K} y'_k$. ∎

We now prove Theorems 2.3.1 and 2.3.2.

Proof of Theorem 2.3.1. We prove the following equivalent assertion: $\delta_t(\mathbb{P}_C) > 0$ implies $\delta_t(\mathbb{A}_C) = 0$. Suppose $\delta_t(\mathbb{P}_C) > 0$, i.e., there exist $(p^1, x^1, y^1), (p^2, x^2, y^2) \in \mathbb{C}$ such that $p^1_t > p^2_t$. By Lemma 2.3.5, we have $y^1_t = y^2_t$. Let $(p, x, y) \in \mathbb{C}$. Again, by Lemma 2.3.5, $y_t = y^2_t$ if $p_t = p^1_t$; $y_t = y^1_t$ if $p_t \neq p^1_t$. Thus, the equilibrium quantity of good t is unique, i.e., $\delta_t(\mathbb{A}_C) = 0$. ∎

Proof of Theorem 2.3.2.(1). $\delta_t(\mathbb{P}_C) > 0$ implies $\delta_t(\mathbb{A}_C) = 0$ by Theorem 2.3.1. Let y^*_t be the unique equilibrium quantity for good t. Let $p \in \mathbb{P}_C$. Then we have $mc_t(y^*_t - 1) \leq p_t \leq mc_t(y^*_t)$ by seller t's profit maximization condition. ∎

Proof of (2). $\delta_t(\mathbb{A}_C) > 0$ implies $\delta_t(\mathbb{P}_C) = 0$ by Theorem 2.3.1. Let p^*_t be the unique competitive price for good t. Suppose, on the contrary, $\delta_t(\mathbb{A}_C) > |\{y_t \in \mathbb{Z}_+ : mc_t(y_t) = p^*_t\}|$. Let $n = |\{y_t \in \mathbb{Z}_+ : mc_t(y_t) = p^*_t\}|$. Let $(x, y), (x', y') \in \mathbb{A}_C$ with $y'_t + n + 1 \leq y_t$. Then we have $mc_t(y_t - 1) \leq p^*_t \leq mc_t(y'_t)$ by seller t's profit maximization condition. This and Assumption B2 imply that $p^*_t = mc_t(y'_t) = \cdots = mc_t(y'_t + n)$. Thus, we obtain $|\{y_t \in \mathbb{Z}_+ : mc_t(y_t) = p^*_t\}| \geq n + 1$, which contradicts the definition of n.

We now prove $p^*_t = mc_t(\widehat{y}_t)$. Let $\widehat{y}_t = \min\{y_t \in \mathbb{Z}_+ : (x, y) \in \mathbb{A}_C\}$. Then we have $p^*_t \leq mc_t(\widehat{y}_t)$ by the seller t's profit maximization condition. On the other hand, let $(x^1, y^1) \in \mathbb{A}_C$ with $y^1_t \geq \widehat{y}_t + 1$. Similarly, we have $mc_t(y^1_t - 1) \leq p^*_t$. This and Assumption B2 imply $mc_t(\widehat{y}_t) \leq p^*_t$. ∎

2.4 Difficulties in extending Theorem 2.3.1

In Section 2.3, we presented two theorems on the structure of the set of c.e. in GAM model. The main theorem is Theorem 2.3.1, and the other is obtained based on this theorem. The key assumption is that each buyer wants at most one unit of an indivisible good. Here, we consider whether Theorem 2.3.1 could be preserved in markets where each buyer may demand more than one unit of an indivisible good.

In the literature on markets with indivisible goods, the following conditions are typically used: Quasi-linearity (QL) and gross substitutability (GS). Under those conditions, it is known (cf., Kelso and Crawford, 1982; Gul and Stacchetti, 1999) that a market with indivisibilities has a c.e. In fact, we see that Theorem 2.3.1 can be extended under the QL condition. Without the QL condition, but with the GS condition, we give a counterexample for the extension of Theorem 2.3.1. We also give another example having no c.e. under the GS condition.

We consider a market model (M, N) that is the same as a GAM model, but with the assumption that each buyer may demand more than one unit of an indivisible good. Each buyer $i \in M$ has a utility function u_i defined on $A \times \mathbb{R}_+$, where $A \subseteq \mathbb{Z}_+^T$. We define the individual demand correspondence $D_i : \mathbb{R}_+^T \twoheadrightarrow A$ as:

$$D_i(p) := \{x : x \in B_i(p) \text{ and } u_i\left(x, I_i - px\right) \geq u_i\left(x', I_i - px'\right) \text{ for all } x' \in B_i(p)\},$$

where $B_i(p) := \{x : x \in A \text{ and } I_i \geq px\}$.Then the GS and QL conditions are defined by the following manner.

Condition QL (*Quasi-linearity*). The utility function is expressed as $u_i(x, d) = v_i(x) + d$ for all $(x, d) \in A \times \mathbb{R}_+$ for some $v_i : A \to \mathbb{R}$.

Condition GS (*Gross substitutability*)**.** For any $p, p' \in \mathbb{R}_+^T$ with $p \leq p'$ and for any $x \in D_i(p)$, there exists $x' \in D_i(p')$ such that $x_t \leq x'_t$ for all t with $p_t = p'_t$.

In QL, the function $v_i(\cdot)$ is interpreted as the valuation price for consumption of indivisible goods. GS states that when the prices for some goods increase from p, the demands for the other goods remain the same or increase.

Under Condition QL, Theorem 2.3.1 can be directly extended:

Proposition 2.4.1. Let (M, N) be an extended GAM model satisfying condition QL, and let $t = 1, ..., T$, $(p, x, y), (p', x', y') \in \mathbb{C}$. Then $p_t \neq p'_t$ implies $y_t = y'_t$.

Proof. Since (p, x, y) is a c.e., it holds that for each $i \in M$, $v_i(x_i) + (I_i - px_i) \geq v_i(x'_i) + (I_i - px'_i)$. Thus, $\sum_{i \in M}(v_i(x_i) - px_i) \geq \sum_{i \in M}(v_i(x'_i) - px'_i)$. Also, it holds that $\sum_{i \in M}(v_i(x'_i) - p'x'_i)$ $\geq \sum_{i \in M} v_i((x_i) - p'x_i)$. By adding each side of the latter two inequalities, we have $p[\sum_{i \in M} x'_i$

$-\sum_{i\in M} x_i] \geq p'[\sum_{i\in M} x'_i - \sum_{i\in M} x_i]$. Since the market is balanced (Definition 2.2.1.(3)), this inequality is equivalent to

$$p(y'-y) \geq p'(y'-y) \tag{1.3}$$

Since the seller side is the same as a GAM model in Section 3, we can apply Lemma 2.3.3. By this, we have $y_t \leq y'_t$ for all t with $p_t < p'_t$. This and Eq. (1.3) imply $y_t = y'_t$ for all t with $p_t \neq p'_t$. ∎

Proposition 2.4.1 is the same claim as Lemma 2.3.5 of Section 2.3, which implies Theorem 2.3.1. Thus, Theorem 2.3.1 holds under Condition QL, independent of the existence of a competitive equilibrium. Under QL, Condition GS is known as a sufficient condition for the existence of a c.e. Under QL, but without GS, Kelso and Crawford (1982) gave a counterexample for the existence of a c.e.

For our study of markets with indivisible goods, it would be more appropriate to eliminate Condition QL since we aim to apply our study to markets including housing and car markets; each indivisible unit is large relative to the expenditure of a household. In such markets, income effects are non-negligible, but QL ignores these (cf. Kaneko and Wooders, 2004 for further explanations on QL).

Now, we focus on the extendibility of Theorem 2.3.1 without assuming Condition QL. Here, we give a counterexample where Theorem 2.3.1 fails without QL, even in the case of one indivisible good. At the same time, this example satisfies Condition GS in the trivial sense.

Example 2.4.2 (*Failure of Theorem 2.3.1*). Consider a market with one buyer $1'$ and one seller 1. Buyer $1'$ has an initial income $I_{1'} = 5$, and may demand at most two units of the indivisible good. His utility function $u_{1'} : \{0,1,2\} \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$u_{1'}(x,d) = \begin{cases} 2x+d & \text{if } x \leq 1, d \leq 4, \\ 2+2x+\dfrac{1}{2}d & \text{if } x \leq 1, d > 4. \end{cases} \qquad u_{1'}(2,d) = \begin{cases} 3.5+d & \text{if } d \leq 2, \\ 4.5+\dfrac{1}{2}d & \text{if } d > 2. \end{cases}$$

For each x, the function $u_{1'}(x,d)$ is continuous with respect to d: for each $x = 0, 1$ (or 2), $u_{1'}(x,d)$ kinks at $d = 4$ ($d = 2$). The marginal utility from d decreases at the kink. Thus, this function is concave with respect to d. In fact, it is also concave with respect to x for each d; for example, when $d = 2$, $u_{1'}(1,2) - u_{1'}(0,2) = 2 > 1.5 = u_{1'}(2,2) - u_{1'}(1,2)$. For each $x = 0,1,2$, function $u_{1'}$ is depicted in Fig. 2.2.

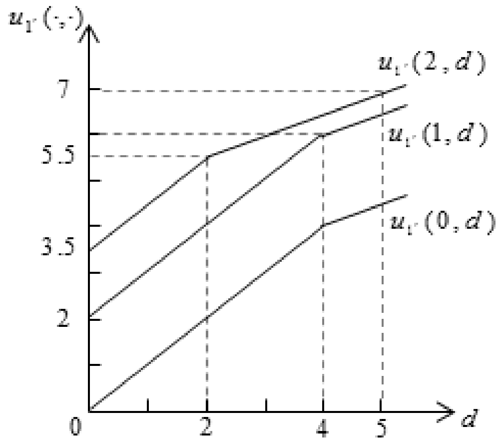

Figure 2.2: Buyer's utility function in Ex. 2.4.2.

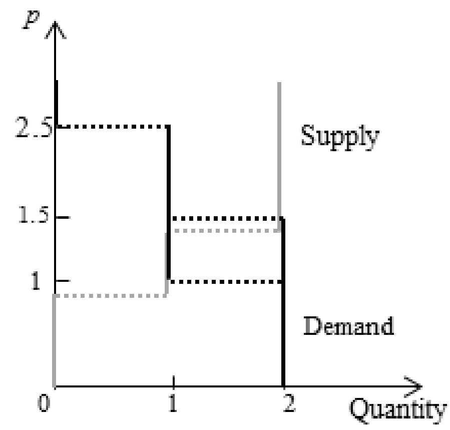

Figure 2.3: Demand and supply schedules in Ex. 2.4.2.

The seller's cost function is given by $c_1(0) = 0$, $c_1(1) = 0.9$, $c_1(2) = 2.3$, $c_1(3) =$ "large." In this example, the supply schedule is expressed as a step function, shown by the gray curve in Fig. 2.3, in the same way as in Fig. 2.1: the price for the second step is $2.3 - 0.9 = 1.4$, which is a competitive price. However, the demand schedule is not a simple step function: for each price p with $1 \leq p \leq 1.5$, the demand schedule takes two values, that is, it forms the rectangle with $1 \leq p \leq 1.5$ and $x = 1, 2$, as depicted in Fig. 2.3. We see that the set of c.e. is given by

$$\{(p, 1, 1) : 1 \leq p \leq 1.4\} \cup \{(p, 2, 2) : 1.4 \leq p \leq 1.5\}.$$

Thus, we have multiple competitive prices and multiple competitive quantities. This implies that Theorem 2.3.1 fails, since $D_{1'}(p) = \{1, 2\}$ for $1 \leq p \leq 1.5$. Indeed, for p with $1 \leq p \leq 1.5$, $u_{1'}(1, 5 - p) = u_{1'}(2, 5 - 2p) = 7 - p > 4.5 = u_{1'}(0, 5)$.

Without Condition QL, but with GS, the existence of a c.e. is not necessarily guaranteed; we show this with an example of one indivisible good that also satisfies GS in the trivial sense.

Example 2.4.3 (*Non-existence of competitive equilibria*). Consider a market with one buyer $1'$ and one seller 1. Buyer $1'$ has an initial income $I_{1'} = 4$, and may demand at most two units of the indivisible good. His utility function $u_{1'} : \{0, 1, 2\} \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$u_{1'}(0,d) = \begin{cases} d & \text{if } d \leq 4, \\ \frac{8}{3} + \frac{1}{3}d & \text{if } d > 4. \end{cases} \quad u_{1'}(1,d) = \begin{cases} 1.9 + d & \text{if } d \leq 2, \\ 2.9 + \frac{1}{2}d & \text{if } 2 < d \leq 4, \\ \frac{107}{30} + \frac{1}{3}d & \text{if } d > 4. \end{cases}$$

$$u_{1'}(2,d) = \begin{cases} 3.8 + d & \text{if } d \leq 1, \\ \frac{67}{15} + \frac{1}{3}d & \text{if } d > 1. \end{cases}$$

For each fixed x, the function $u_{1'}(x, d)$ is continuous in d, and since the marginal utility from d decreases at the kink, the function is concave in d. For a fixed d, $u_{1'}(x, d)$ is concave with respect to x. For each $x = 0, 1, 2$, Fig. 2.4 depicts function $u_{1'}$.

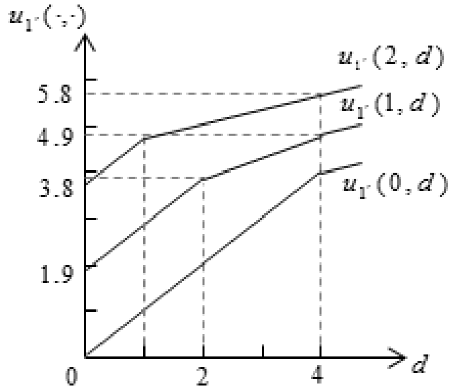

Figure 2.4: Buyer's utility function in Ex. 2.4.3.

The seller's cost function is given by $c_1(0) = 0$, $c_1(1) = 1.5$ and $c_1(2) =$"large." Fig. 2.5 shows the demand and supply schedules, which both are step functions. The quantity supplied is always 1 at price $p > 1.5$. Since $D_{1'}(p) = \{0\}$ for $p > 1.9$ and $D_{1'}(p) = \{2\}$ for $p < 1.9$, the candidate equilibrium price must be $p = 1.9$, but $1 \notin D_{1'}(1.9)$. Indeed, for $p = 1.9$, $u_{1'}(2, 4-2p) = u_{1'}(0, 4) = 4 > 3.95 = u_{1'}(1, 4-p)$, meaning that 0 or 2 to 1 is strictly preferred. The demand schedule has a hole at $(p, y) = (1.9, 1)$. Thus, the above example has no c.e.

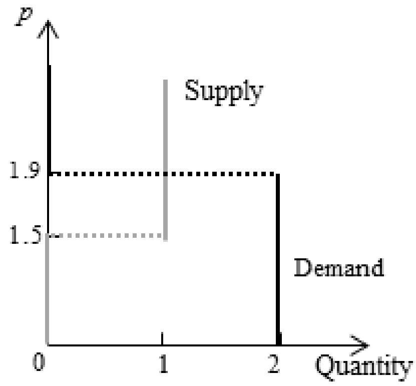

Figure 2.5: Demand and supply schedules in Ex. 2.4.3.

Let us see how the demand schedule has a hole at $(p, y) = (1.9, 1)$.

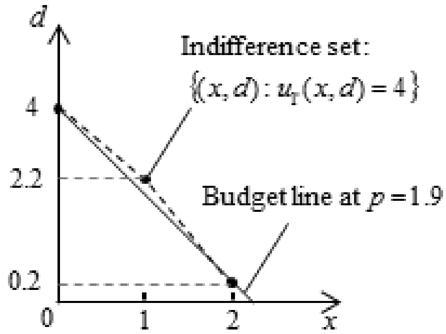

Figure 2.6: Buyer's indifference set and budget line at $p = 1.9$ in Ex. 2.4.3.

Fig. 2.6 depicts the buyer's indifference curve, $\{(x, d) \in \{0, 1, 2\} \times \mathbb{R}_+ : u_{1'}(x, d) = 4\} = \{(0, 4), (1, 2.2), (2, 0.2)\}$, and the budget line at $p = 1.9$. This indifference curve is non-convex, while function satisfies the convexity for each x and d: Bundles $(0, 4)$ and $(2, 0.2)$ are on the budget line, however, $(1, 2.2)$ is an exterior of the budget line. Thus, the demand correspondence has a hole at $(p, y) = (1.9, 1)$.

2.5 Aggregation of sellers and shrinkage of competitive prices

In the previous discussion, we assumed Eq. (1.1): $N_t = \{t\}$ for all $t = 1, ..., T$, i.e., only one seller provides indivisible goods of type t. In this section, we present that the assumption of (1.1) does not loose any generality in the consideration of c.e. This result allows us to consider markets with many sellers while preserving Theorems 2.3.1 and 2.3.2. As an application of our Theorems 2.3.1 and 2.3.2, we also show the shrinkage result on competitive prices where the market gets dense with sellers.

2.5.1 Aggregation of sellers

Here, we show how the set N_t is aggregated into one seller $\{t\}$.

Let (M, N) be the original assignment market without (1.1), i.e., $|N_t| \geq 1$ for all $t = 1, ..., T$, where each landlord $j \in N_t$ provides indivisible goods of type-t with the cost function $c_j : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying Assumptions B1 and B2. Now, let $N^o = \{1, ..., T\}$ be the *set of aggregated sellers*. Our problem is to define the cost function $\widetilde{c}_t$ of each aggregated seller $t \in N^o$ preserving the structure of c.e. in (M, N).

The following theorem explains the equivalence of c.e. between (M, N) and (M, N^o).

Theorem 2.5.1 (*Aggregation of Sellers*). *Let (M, N) be an assignment market without requiring* (1.1). *There exist cost functions $\widetilde{c}_t : \mathbb{Z}_+ \to \mathbb{R}_+$ satisfying Assumption B1 and B2 for $t \in N^o$ such that,*

(1) *if (p, x, y) is a c.e. in (M, N), then there is a $\widetilde{y} \in \mathbb{Z}^T_+$ such that $(p, x, \widetilde{y})$ is a c.e. in (M, N^o).*

(2) *if $(p, x, \widetilde{y})$ is a c.e. in (M, N^o), then there is a $y \in \mathbb{Z}^n_+$ such that (p, x, y) is a c.e. in (M, N).*

The set N_t for type t is aggregated into $\{t\}$: the essential part is to define the aggregated cost function $\widetilde{c}_t$ satisfying Assumptions B1 and B2. Once this $\widetilde{c}_t$ is appropriately defined, (1) the supplies $\{y_j\}_{j\in N_t}$ are aggregated into $\widetilde{y}_t = \sum_{j\in N_t} y_j$ and this aggregation makes a c.e. in (M, N^o); (2) the aggregated $\widetilde{y}_t$ is divided into $\{y_j\}_{j\in N_t}$ and this makes a c.e. in (M, N).

We now construct the aggregated cost function. Let us fix an arbitrary $t \in N^o$. For each $j \in N_t$, define the sequence $\{mc_j(y) : y \in \mathbb{Z}_+\}$.[7] By Assumption B2, this is an increasing sequence. We then generate the new increasing sequence $\{\widetilde{mc}_t(y) : y \in \mathbb{Z}_+\}$ by reordering all the components of the set of sequences $\{\{mc_j(y) : y \in \mathbb{Z}_+\} : j \in N_t\}$ in the ascending order: in the start, we choose the smallest one from $\{mc_j(0) : j \in N_t\}$. If we choose $mc_{j_0}(0)$ here, we next choose the smallest one from $\{\{mc_j(0) : y_j \in \mathbb{Z}_+\} : j \in N_t \backslash \{j_0\}\} \cup \{mc_{j_0}(1)\}$. The following shows an example of the construction of $\widetilde{mc}_t$ with $|N_t| = 3$.

$$
\begin{array}{ccccccccccccc}
mc_1(0) & \leq & mc_2(0) & \leq & mc_1(1) & \leq & mc_3(0) & \leq & mc_2(1) & \leq & mc_2(2) & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\widetilde{mc}_t(0) & & \widetilde{mc}_t(1) & & \widetilde{mc}_t(2) & & \widetilde{mc}_t(3) & & \widetilde{mc}_t(4) & & \widetilde{mc}_t(5) & \cdots
\end{array}
$$

[7] $mc_j(y_j) = c_j(y_j + 1) - c_j(y_j)$.

By using $\{\widetilde{mc}_t(y) : y \in \mathbb{Z}_+\}$, we define the *cost function of aggregated seller* t, $\widetilde{c}_t : \mathbb{Z}_+ \to \mathbb{R}_+$ by $\widetilde{c}_t(0) = 0$ and $\widetilde{c}_t(y) = \sum_{k \leq y-1} \widetilde{mc}_t(k)$ for $y \in \mathbb{Z}_+ \backslash \{0\}$. This $\widetilde{c}_t$ satisfies Assumptions B1 and B2. We can then show Theorem 2.5.1. In the proof, we stipulate $mc_j(-1) = \widetilde{mc}_t(-1) = 0$.

Proof of (1). Let (p, x, y) be a c.e. in (M, N) and $t \in N^o$. By the profit maximization condition, $mc_j(y_j - 1) \leq p_t \leq mc_j(y_j)$ for every $j \in N_t$.

Let $\widetilde{y}_t = \sum_{j \in N_t} y_j$. By the definition of $\{\widetilde{mc}_t(y) : y \in \mathbb{Z}_+\}$, it holds that

$$\begin{aligned} \widetilde{mc}_t(\widetilde{y}_t - 1) &= \max_{j \in N_t}[mc_j(y_j - 1)], \\ \widetilde{mc}_t(\widetilde{y}_t) &= \min_{j \in N_t}[mc_j(y_j)]. \end{aligned} \tag{1.4}$$

Eqs. (1.4) and the above profit maximization condition imply $\widetilde{mc}_t(\widetilde{y}_t - 1) \leq p_t \leq \widetilde{mc}_t(\widetilde{y}_t)$, that is, the aggregated seller t maximizes his profit with production unit $\widetilde{y}_t$. Since t is arbitrarily chosen, a triple $(p, x, (\widetilde{y}_1, ..., \widetilde{y}_T))$ is a c.e. in (M, N^o). ∎

Proof of (2). Let $(p, x, \widetilde{y})$ be a c.e. in (M, N^o) and $t \in N^o$. By the profit maximization condition, $\widetilde{mc}_t(\widetilde{y}_t - 1) \leq p_t \leq \widetilde{mc}_t(\widetilde{y}_t)$.

By the definition of $\{\widetilde{mc}_t(y) : y \in \mathbb{Z}_+\}$, there exist $\{y_j\}_{j \in N_t}$ such that

$$y_j \in \arg\min_{y \in \mathbb{Z}_+}[mc_j(y) : \widetilde{mc}_t(\widetilde{y}_t) \leq mc_j(y)] \text{ for } j \in N_t,$$

$$\textstyle\sum_{j \in N_t} y_j = \widetilde{y}_t.$$

For $\widetilde{y}_t$ and $\{y_j\}_{j \in N_t}$, the same equalities as Eqs. (1.4) holds. Eqs. (1.4) and the above profit maximization condition imply $mc_j(y_j - 1) \leq p_t \leq mc_j(y_j)$ for all $j \in N_t$, that is, each seller j maximizes his profit with production unit y_j. Since t is arbitrarily chosen, a triple $(p, x, \{\{y_j\}_{j \in N_t}\}_{t \in N^o})$ is a c.e. in (M, N). ∎

2.5.2 Shrinkage of competitive prices

The following passage is from Shapley and Shubik (1972) (pp. 127-128):

> "If the number of traders is increased on both sides of the market, in such a way that their valuations for the products brought to market become more and more diverse (but remain bounded in a suitable sense), then the core will tend to shrink in size."

In the context of this paper, the set of competitive price vectors corresponds to the core. The paper asserts that this shrinkage is easily obtained for the case of homogeneous goods. It continues:

> "In the more general model, however, the increasing dimensionality of the solution and the space in which it is defined make a precise discussion of the shrinkage phenomenon more difficult."

Here, we analyze their observation. For this, we eliminate assumption (1.1), that is, we have multiple sellers for type t. This allows us to consider the situation where *the number of traders is increased.* Nevertheless, we still use the aggregated cost function $\widetilde{c}_t$ to apply Theorems 2.3.1 and 2.3.2.

We consider a sequence of assignment markets $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$. Let $t = 1, ..., T$. We express the idea of Shapley and Shubik (1972) quoted above in terms of $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$ as follows:

Condition $\mathbf{D}_t$ (*Denseness of Marginal Costs*). There are some constants α_t and β_t ($0 < \alpha_t < \beta_t$) such that for any ν, (M^ν, N^ν) satisfies

(1) $|N_t^\nu| \to \infty$ as $\nu \to \infty$;

(2) $\alpha_t \leq \widetilde{mc}_t^\nu(y_t) \leq \beta_t$ for all $y_t \leq |M^\nu|$;

(3) $\max\limits_{1 \leq y_t \leq |M^\nu|} [\widetilde{mc}_t^\nu(y_t) - \widetilde{mc}_t^\nu(y_t - 1)] \leq (\beta_t - \alpha_t) / |N_t^\nu|$.

Condition D_t.(2) states that the aggregated marginal costs, $\widetilde{mc}_t^\nu(y_t)$, are in the same interval (bounded) for the relevant domains, though the size of (M^ν, N^ν) becomes large. (3) states that the marginal costs are densely distributed for large ν. However, (1) requires only the number of sellers of type t, $|N_t^\nu|$, to become large: the number of buyers, $|M^\nu|$, may be bounded, but it would be natural to require it to become large proportionally to $|N_t^\nu|$.

The graphical illustration of Condition D_t is given by Fig. 2.7. The marginal costs are distributed in the same interval: when the market is small ($\nu = 1$), the distribution of marginal costs is sparse; and when the market is large (large ν), the distribution becomes dense.

Let $\mathbb{P}_C^\nu$ be the set of all competitive price vectors in (M^ν, N^ν) for $\nu \geq 1$.

Theorem 2.5.2 (*Shrinkage of Competitive Prices*). *Let $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$ be a sequence of GAM satisfying Condition D_t. Then $\delta_t(\mathbb{P}_C^\nu) \to 0$ as $\nu \to \infty$.*

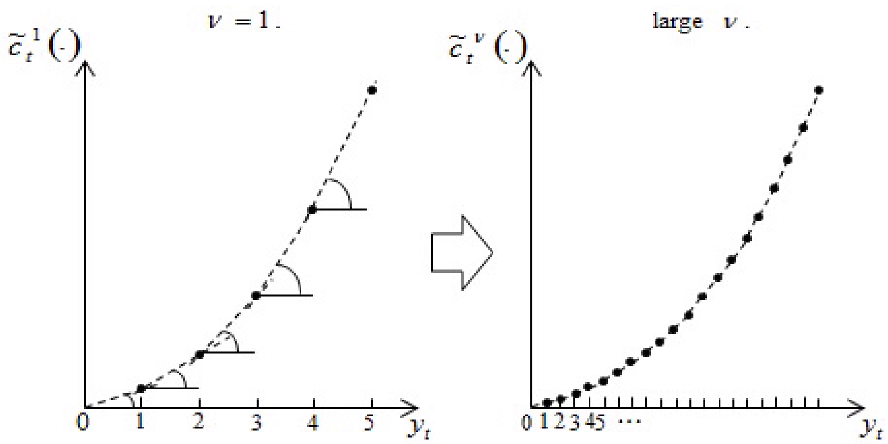

Figure 2.7: An illustration of Condition D_t.

Although the theorem claims that the size of the set of competitive prices shrinks to zero, it does not imply that the set $\mathbb{P}_C^\nu$ converges to a price vector, i.e., $\mathbb{P}_C^\nu$ may fluctuate and have multiple limit points. In this paper, we do not have a limit model of the sequence $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$. If one wants to have such a model, a candidate for it is the assignment market model with a continuum of buyers and sellers given by Gretsky, Ostroy and Zame (1999), or the f-core model by Kaneko and Wooders (1996). However, it is the point of Theorem 2.5.2 that for a large and dense market, the competitive prices are almost uniquely determined.

To prove this theorem, we restate Theorem 2.3.2.(1). Let $p_t^{\max} := \max\{p_t : p \in \mathbb{P}_C\}$ and $p_t^{\min} := \min\{p_t : p \in \mathbb{P}_C\}$.[8] By definition, we have $\delta_t(\mathbb{P}_C) = p_t^{\max} - p_t^{\min}$. Then, we restate Theorem 2.3.2.(1) as follows:

$$\text{If } \delta_t(\mathbb{P}_C) > 0, \text{ then } \delta_t(\mathbb{P}_C) = p_t^{\max} - p_t^{\min} \leq mc_t(y_t^*) - mc_t(y_t^* - 1). \tag{1.5}$$

Using (1.5), we can prove the above theorem.

Proof. Let ν be an arbitrary natural number, and let $\delta_t(\mathbb{P}_C^\nu) > 0$ and $y_t^{\nu*}$ be the unique equilibrium quantity for good t in (M^ν, N^ν). By (1.5) and $y_t^{\nu*} \leq |M^\nu|$, we have $\delta_t(\mathbb{P}_C^\nu) \leq \widetilde{mc}_t^\nu(y_t^{\nu*}) - \widetilde{mc}_t^\nu(y_t^{\nu*} - 1) \leq \max_{1 \leq y_t \leq |M^\nu|}[\widetilde{mc}_t^\nu(y_t) - \widetilde{mc}_t^\nu(y_t - 1)]$. This inequality, together with Condition D_t imply that $\delta_t(\mathbb{P}_C^\nu) \leq (\beta_t - \alpha_t)/|N_t^\nu|$, and the right hand side of the inequality tends to zero as $\nu \to \infty$. ∎

[8]Since the set of competitive price vectors $\mathbb{P}_C$ is a compact set, these maximum and minimum are well defined.

Theorem 2.5.2 states the shrinkage result for a fixed type t. When Condition D_t holds for all $t = 1, ..., T$, we get the shrinkage result for the competitive price vector set $\mathbb{P}_C^\nu$. Although they indicated a possible difficulty caused from higher dimensionality for the heterogeneous goods case, our Theorems 2.3.1 and 2.3.2 guarantee that it is sufficient to treat each indivisible good separately.

As mentioned in Section 2.2, Kaneko (1982) provided a sufficient condition for the core to coincide with the set of c.e. This sufficient condition holds naturally for a large and dense market. Thus, we obtain the shrinkage result for the core.

As stated above, Gretsky et al. (1999) gave an assignment market model with a continuum of buyers and sellers with Condition QL. They show the generic uniqueness of an equilibrium price vector. Theorem 2.5.2 may be regarded as a finite version of their theorem, without QL. Thus, we conjecture their result for the continuum assignment market without QL.

2.6 Conclusions

We have studied the structure of the set of competitive equilibria in the GAM model. The main result (Theorem 2.3.1) states that if there are multiple competitive prices for indivisible good t, the equilibrium quantity for t is uniquely determined; and that if there are multiple equilibrium quantities for t, the competitive price for t is uniquely determined. This result enables us to study the relationship between competitive prices and quantities for each indivisible good t. From this result, we obtained Theorem 2.3.2 evaluating the sizes of competitive price and quantity sets for each good t.

In Section 2.4, we discussed difficulties in extending Theorem 2.3.1 to a market model where each buyer may demand more than one unit of an indivisible good. We showed that under the quasi-linear utility assumption for buyers, Theorem 2.3.1 can be extended. Without quasi-linearity, however, we gave an example satisfying gross substitutability where Theorem 2.3.1 fails. Furthermore, we gave an example satisfying gross substitutability, but having no competitive equilibria.

In Section 2.5.1, we showed the aggregation result of the sellers, i.e., it is sufficient to consider models where, for each type t, a single seller provides units of indivisible good t. This aggregation can be made as far as competitive equilibria are concerned. We also presented in Section 2.5.2, the shrinkage theorem of competitive prices when the market becomes large and dense with sellers.

We may apply the GAM model to rental housing markets and/or second-hand automobile markets. Those markets are typically dense in the sense that the numbers of sellers and buyers are large and there are many similar sellers and buyers. Kaneko (1983) and Kaneko, Ito and Osawa (2006) adopted the GAM model for the analysis of rental housing markets, making some assumptions specific to their studies. By Theorems 2.3.2 and 2.5.2, the competitive prices are restrictive and hence their studies could be done under more general assumptions.

Chapter 3

Evaluation of Competitive Price Vectors in Markets with Indivisibilities

3.1 Introduction

In this chapter, we evaluate competitive price vectors in housing markets with indivisibilities. This market is an application model of the generalized assignment market (GAM) by Kaneko (1982). In the market, the agents are divided into buyers and sellers, the objects to be traded (houses) are treated as indivisible goods and classified into finite categories, and each buyer demands at most one unit of an indivisible good. It is known that there exists a competitive equilibrium while it may not be uniquely determined. In particular, there exist the maximum and minimum competitive price vectors.

As with the general equilibrium model, this non-uniqueness brings a problem to applications of this model in comparative statics. Fig. 3.1 depicts a supply and demand schedules in the market without commodity differentiation. The intersection of two schedules constitutes the set of competitive equilibria. Since any $p \in \left[p^{\min}, p^{\max}\right]$ is a candidate for an equilibrium price, a comparative statics result may differ depending on price p. This problem is inherited to the market with commodity differentiation.

To clear this problem, we evaluate the difference between the maximum and minimum competitive price vectors. Here, we introduce a summary of one of our evaluation results, Theorem 3.3.4 in Section 3.3.2. Theorem 3.3.4 states that the difference between the max-

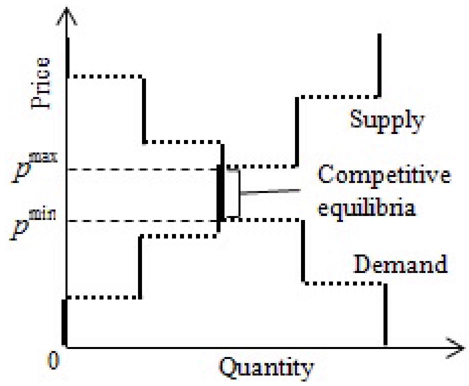

Figure 3.1: A supply and demand schedules without commodity differentiation.

imum and minimum competitive price vectors is bounded by the income difference of two specific *income-neighboring* buyers. This implies that the price difference is considered to be small, and this tendency is remarkable when the number of buyers is large and its income distribution gets dense. Thus, as far as a quasi large market is concerned, we can see that comparative statics results are approximately the same whether we use any equilibrium price vectors. To confirm our results, in Section 3.4, we give three numerical examples and shrinkage result on differential price vectors.

In our analyses, the maximum (minimum, respectively) competitive price (rent) vector is calculated as the solution of the specific system of equations. This system of equations is constructed from buyers' indifference conditions. The important assumptions for this approach are "homogeneous utility function" and "normality of the quality of indivisible objects."

Our evaluation results are related to Sai (2014) studying the structure of the set of competitive equilibria under weaker conditions. One of his result is that the difference between the maximum and minimum competitive price vectors is characterized by the difference in sellers' marginal costs. On the other hand, we characterize the difference by household incomes.

This paper is organized as follows. Section 3.2 formulates our market model and gives our definition of competitive equilibrium. Section 3.3 first introduces two systems of equations from which we can derive two representative solutions for the maximum and minimum competitive price vectors. This section then outlines the main results of our study. Section 3.4 provides some numerical examples and an application of our theorems. Section 3.5 presents concluding remarks.

3.2 The market model

This section introduces the rental housing market model of Kaneko et al. (2006). In Section 3.2.1, we give our basic assumptions and the definition of competitive equilibrium. In Section 3.2.2, we introduce additional assumptions that facilitate our study.

3.2.1 General formulation

The *rental housing market* is denoted by (M, N), where $M = \{1, \ldots, m\}$ denotes the set of households, and $N = \{1', \ldots, n'\}$ denotes the set of landlords. The apartments are classified into finite *categories* $1, 2, ..., T$.

Each household $i \in M$ initially has an income $I_i > 0$ but no dwelling. The household wants to rent at most one apartment unit paying rent from his income. Without loss of generality, we can assume that the households are ordered in their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. The *consumption set* is written by $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$, where $\mathbf{e}^k$ is the T-dimensional unit vector with $\mathbf{e}^k_k = 1$ ($\mathbf{e}^0 = \mathbf{0}$), and $\mathbb{R}_+$ is the set of nonnegative real numbers. A consumption vector $(\mathbf{e}^k, c) \in X$ with $k \neq 0$ means that household i rents one unit of the k-th category of an apartment and enjoys the consumption $c = I_i - p_k$, where p_k is the rent of the k-th apartment. For $k = 0$, no apartment is consumed. An *initial endowment* of $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$.

A *utility function* of household i is given by $u_i : X \to \mathbb{R}$. We make the following assumption.

Assumption A. For each $i \in M$ and $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u_i(x_i, c)$ is a continuous and strictly monotone function of c, and $u_i(\mathbf{e}^0, I_i) > u_i(\mathbf{e}^k, 0)$ for all $k = 1, ..., T$.

The first part of Assumption A allows a utility function to have an income effect. An inequality in the last part means the indispensability of money.

Each landlord $j \in N$ provides apartments of exactly one category (say k), but may provide more than one unit. The landlord has a cost function $C_j(y_j) : \mathbb{Z}_+ \to \mathbb{R}_+$, where $\mathbb{Z}_+$ is the set of nonnegative integers. For each $y_j \in \mathbb{Z}_+$, $C_j(y_j)$ represents the cost (in terms of money) of supplying y_j units of the k-th category. We make the following assumption for $C_j(\cdot)$.

Assumption B. For each $j \in N$, $C_j(0) = 0$ and $C_j(y_j + 1) - C_j(y_j) \leq C_j(y_j + 2) - C_j(y_j + 1)$ for all $y_j \in \mathbb{Z}_+$.

The first part of Assumption B means that no fixed cost is required for no production. The last part is a discrete version of the standard convexity assumption on a cost function, meaning that the marginal cost is increasing.

For notational simplicity, we assume that *only one* landlord k provides apartments in the k-th category. Thus, the set N becomes $\{1, \ldots, T\}$, and landlord $k \in N$ is the only landlord providing the k-th apartments. As far as the competitive equilibrium is concerned, this can be assumed *without loss of generality.*[1]

Let $(p, x, y) = ((p_1, \ldots, p_T), (x_1, \ldots, x_m), (y_1, \ldots, y_T))$ be a triple of $p \in \mathbb{R}^T_+$, $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}^m$ and $y \in \mathbb{Z}^T_+$. The competitive equilibrium is defined by the following.

Definition 3.2.1. We say that a triple (p, x, y) is a *competitive equilibrium* iff

(UM): for all $i \in M$,

(i) $I_i - px_i \geq 0$;

(ii) $u_i(x_i, I_i - px_i) \geq u_i(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$ with $I_i - px'_i \geq 0$.

(PM): for all $k = 1, ..., T$, $p_k y_k - C_k(y_k) \geq p_k y'_k - C_k(y'_k)$ for all $y'_k \in \mathbb{Z}_+$.

(BDS): $\sum_{i \in M} x_i = \sum_{k=1}^T y_k \mathbf{e}^k$.

Condition UM is utility maximization condition under the budget constraint of a household. PM is the profit maximization condition of a landlord. BDS means a balance of demand and supply. Under Assumptions A and B, we have a competitive equilibrium in (M, N).

Theorem 3.2.2 (Kaneko, 1982; Kaneko and Yamamoto, 1986)**.** *There exists a competitive equilibrium (p, x, y) in a rental housing market (M, N).*

We say that $p = (p_1, \ldots, p_T)$ is a *competitive rent vector* iff (p, x, y) is a competitive equilibrium for some $x \in \{\mathbf{0}, \mathbf{e}^1, \ldots, \mathbf{e}^T\}^m$ and $y \in \mathbb{Z}^T_+$. Note that in there exists multiple competitive equilibria. In particular, the maximum and minimum competitive rent vectors exist (denote them by $p^{\max}$ and $p^{\min}$), which play an important role in our analysis.[2]

Theorem 3.2.3. *There exist the maximum and minimum competitive rent vectors in (M, N).*

The result close to Theorem 3.2.3 is found in Miyake (1994) or Demange and Gale (1985). The complete proof of this theorem will be shown in Appendix B. We say that (p, x, y)

[1] Under this simplification, each seller is interpreted as the *aggregated landlord*. A detailed discussion is given by Chapter 2, Section 2.5.1 [or Sai (2014), Section 5.1].

[2] A competitive rent vector p is the maximum (minimum) iff $p \geq p'$ ($p \leq p'$) for every competitive rent vector p'.

is a *maximum* (*minimum*) *competitive equilibrium* iff p is the maximum (minimum). By definition, the maximum (minimum) competitive rent vector is uniquely determined, while multiple maximum (minimum) competitive equilibria may exist. Kaneko et al. (2006) and their subsequent papers (e.g. Ito (2007)) focused on a maximum competitive equilibrium and used it for comparative statics analyses.

3.2.2 Specific assumptions for (M, N)

In addition to Assumptions A and B, we assume that every household has an identical utility function:

Assumption C. $u_i(\cdot, \cdot) = u_j(\cdot, \cdot)$ for all $i, j \in M$.

From now on, we simplify the utility function u_i as u. In an urban economics context, Assumption C implies that the housing market (M, N) represents a mono-centric city, and all the households commute to an identical business district. Thus, under C, each household is characterized only by his initial income. One may think Assumption C implies an identical apartment preference for each household. However, this concern will be eliminated by the next assumption.

Assumption D. If $u(x_i, c) = u(x'_i, c')$, and $c < c'$, then $u(x_i, c + \delta) > u(x'_i, c' + \delta)$ for any $\delta > 0$.

Assumption D is the normality assumption on the quality of apartments. In D, apartment x_i is better than x'_i because a household living in x_i with a smaller consumption c is indifferent to living in x'_i with a larger consumption c'. This implies that, for each household, the demand shifts to a better apartment or remains the same if their income increases.

We also put another assumptions.

Assumption E. If $u(x_i, c) > u(x'_i, c')$, then $u(x_i, c) = u(x'_i, c' + \delta)$ for some $\delta > 0$.

Assumption F. $u(\mathbf{e}^1, 0) > u(\mathbf{e}^2, 0) > \cdots > u(\mathbf{e}^T, 0)$.[3]

Assumption E means that housing quality of any apartment is substitutable for money. Assumption F is regarding the quality of apartments. By Assumption F, the apartments are numbered according to their quality level. The first category is the best one and the T-th category is the worst one.

[3] Assumption F together with A, D and E imply $u(\mathbf{e}^1, c) > u(\mathbf{e}^2, c) > \cdots > u(\mathbf{e}^T, c)$ for all $c \in \mathbb{R}_+$.

3.3 Rent equations and the evaluation of competitive rent vectors

In Section 3.3.1, we introduce two systems of equations: the *upper and lower rent equations.* The solution of the upper (lower) rent equation is called the *upper (lower) differential rent vector*, corresponding to the maximum (minimum) competitive rent vector, under some conditions. Using both differential rent vectors, we present two theorems on the evaluation of the difference between the lower and upper differential rent vectors in Section 3.3.2. Section 3.3.3 gives proofs of two theorems.

3.3.1 Rent equations and differential rent vectors

Here, we give some lemmas and more detailed assumptions. The following lemma has an important role in the derivation of the rent equation.

Lemma 3.3.1 (Kaneko et al., 2006). *Let (p, x, y) be a competitive equilibrium. Then,*
(1): If $k' < k$ and $x_i = \mathbf{e}^k$ for some i, then $p_k < p_{k'}$.
(2): If $x_i = \mathbf{e}^k$, $x_j = \mathbf{e}^{k'}$ and $I_i > I_j$ for some i, j, then $k \leq k'$.

This lemma states that, in equilibrium, (1) the price of a better apartment is higher, and (2) a household with a higher income rents a better apartment. Note that in (1), it may be possible that no one rents an apartment in the k'-th category, while the k-th apartment is rented by someone. To eliminate such a case, we assume that there is a category f dividing the apartments into active categories and inactive categories:

Assumption G. Let (p, x, y) be a competitive equilibrium. There exists some category $f \leq T$ such that $y_k > 0$ for $k = 1, ..., f$ and $y_k = 0$ for $k = f, ..., T$.

We call this f the *marginal category.*

Recall that all the set of households $M = \{1, ..., m\}$ is ordered by their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. We next define the household with the lowest income in each active category. Let (p, x, y) be a competitive equilibrium. For each $k = 1, ..., f$, let

$$G(k) := \sum_{t=1}^{k} y_t.$$

By Lemma 3.3.1.(2), $G(k)$ is the household having the lowest income in the k-th category.

We call $G(k)$ the *boundary household* and $I_{G(k)}$ the *boundary income* of the k-th category. Note that $G(k)$ may also differ for different competitive equilibria.

We now introduce two systems of equations with unknowns $(r_1, ..., r_f)$.

Definition 3.3.2. (1) (Kaneko et al., 2006): We call the following system of equations the *upper rent equation*:

$$\left.\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1}) &= u(\mathbf{e}^{f}, I_{G(f-1)} - r_{f}), \\ u(\mathbf{e}^{f-2}, I_{G(f-2)} - r_{f-2}) &= u(\mathbf{e}^{f-1}, I_{G(f-2)} - r_{f-1}), \\ &\vdots \\ u(\mathbf{e}^{1}, I_{G(1)} - r_{1}) &= u(\mathbf{e}^{2}, I_{G(1)} - r_{2}). \end{aligned}\right\} \tag{2.1}$$

(2): We call the following system of equations the *lower rent equation*:

$$\left.\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - r_{f-1}) &= u(\mathbf{e}^{f}, I_{G(f-1)+1} - r_{f}), \\ u(\mathbf{e}^{f-2}, I_{G(f-2)+1} - r_{f-2}) &= u(\mathbf{e}^{f-1}, I_{G(f-2)+1} - r_{f-1}), \\ &\vdots \\ u(\mathbf{e}^{1}, I_{G(1)+1} - r_{1}) &= u(\mathbf{e}^{2}, I_{G(1)+1} - r_{2}). \end{aligned}\right\} \tag{2.2}$$

Definition 3.3.2.(1) was introduced by Kaneko, et al. (2006). Each system of equations has f unknowns, while this is constituted by $f-1$ equations. Eq. (2.1) states that boundary household $G(k)$ is indifferent between renting the $k+1$-th apartment at rent r_{k+1} and renting the k-th apartment at rent r_k. The difference between Eqs. (2.1) and (2.2) is the replacement of the boundary income $I_{G(k)}$ by $I_{G(k)+1}$. In Eq. (2.1) (Eq. (2.2), respectively), if r_f is given, then the unknown r_{f-1} is uniquely determined by the first equation. In the same manner, the remaining unknowns $r_{f-2}, ..., r_1$ are recursively determined. We say that a solution of Eq. (2.1) is an *upper differential rent vector* and denote it by $(\overline{r}_1, ..., \overline{r}_f)$; a solution of Eq. (2.2) is a *lower differential rent vector* and denote it by $(\underline{r}_1, ..., \underline{r}_f)$. In particular, if $\overline{r}_f$ is given with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f)$ an upper differential rent vector is uniquely determined and satisfies $\overline{r}_1 > \cdots > \overline{r}_{f-1} > \overline{r}_f$.[4]

We then have the following relations for a differential rent vector and a competitive rent vector.

[4] A lower differential rent vector is also uniequely determined under a similar condition. See Kaneko et al. (2006), Lemma 2.5.

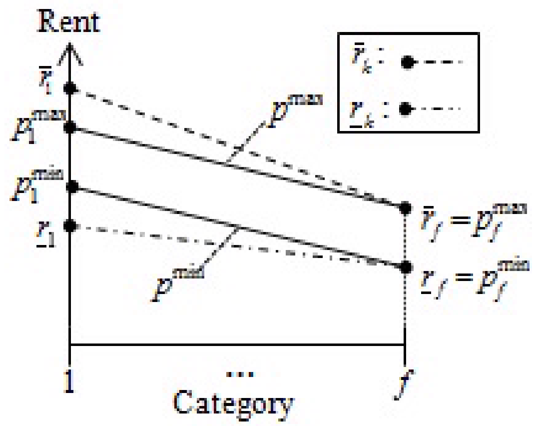

Figure 3.2: An illustration of Theorem 3.3.3.

Theorem 3.3.3.(1) *Let* (p, x, y) *be a maximum competitive equilibrium and* $(\overline{r}_1, ..., \overline{r}_f)$ *be the upper differential rent vector determined by* $\overline{r}_f = p_f$. *Then* $\overline{r}_k \geq p_k$ *for all* $k = 1, ..., f-1$.
(2) *Let* (p, x, y) *be a minimum competitive equilibrium and* $(\underline{r}_1, ..., \underline{r}_f)$ *be the lower differential rent vector determined by* $\underline{r}_f = p_f$. *Then* $\underline{r}_k \leq p_k$ *for all* $k = 1, ..., f-1$.

Proof is in Appendix B. Theorem 3.3.3 states that the upper and lower differential rent vectors correspond to an upper and lower bounds of the set of competitive rent vectors, respectively. An illustration of Theorem 3.3.3 is given by Fig. 3.2: in Fig. 3.2, the vertical axis (price) is continuous, while the horizontal axis (category) is discrete.

Kaneko et al. (2006) provided two sufficient conditions for an upper differential rent vector to coincide with the maximum competitive rent vector. We can also expect a similar condition for a lower competitive rent vector to coincide with the minimum competitive rent vector.

Theorem 3.3.4.(1) (Kaneko et al., 2006). *Let* (p, x, y) *be a maximum competitive equilibrium. If at least one of the following holds:*

(i) $I_{G(k)} = I_{G(k)+1}$ *for each* $k = 1, ..., f-1$;

(ii) $p_k < C_k(y_k + 1) - C_k(y_k)$ *for each* $k = 1, ..., f-1$,

then the upper differential rent vector $(\overline{r}_1, ..., \overline{r}_f)$ *determined by* $\overline{r}_f = p_f$ *coincides with* $(p_1, ..., p_f)$.

(2) *Let* (p, x, y) *be a minimum competitive equilibrium. If at least one of the following holds:*

(i) $I_{G(k)} = I_{G(k)+1}$ *for each* $k = 1, ..., f-1$;

(ii) $p_k > C_k(y_k) - C_k(y_k - 1)$ *for each* $k = 1, ..., f-1$,

then the lower differential rent vector $(\underline{r}_1, ..., \underline{r}_f)$ *determined by* $\underline{r}_f = p_f$ *coincides with* $(p_1, ..., p_f)$.

The proof of (1) is found in Kaneko et al. (2006) and (2) is proved in the dual manner (Appendix B). Condition (i) of each theorem is the same, stating that the boundary income of the k-th category coincides with the income of the first household in the $k+1$-th category. Each (ii) has a dual structure: both state that a profit maximization condition strictly holds for the k-th category (i.e., a competitive price p_k does not coincide with a marginal cost for the k-th category). In sum, (i) implies that when the number of households is large and the income distribution is more or less dense (Condition (i) holds approximately), then the upper and lower differential rent vectors can be regarded as approximations of the maximum and minimum competitive rent vectors, respectively.

3.3.2 The difference between the upper and lower differential rent vectors

In the previous section, we showed that the upper (lower) differential rent vector is an upper (lower) bound of a relevant part of the competitive rent set, and under a some condition, the upper (lower) rent vector coincides with the maximum (minimum) competitive rent vector. This section evaluates the difference between the upper and lower differential rent vectors. By this, the difference between the maximum and minimum competitive rent vectors is also evaluated.

Here, we assume the case that $p_k^{\min} < p_k^{\max}$ for all $k = 1, ..., T$. This and Theorem 3.1 by Sai (2014) imply that for any competitive equilibria (p, x, y) and (p', x', y'),

$$y_k = y'_k \text{ for all } k = 1, ..., T. \tag{2.3}$$

By this assumption, a marginal category f and the boundary households $G(k)$ $(1 \leq k \leq f)$ are uniquely determined in the market (M, N).

The following theorem concerns the relationship between the income difference and the rent difference. The proof is found in Section 3.3.3.

Theorem 3.3.5. *Let* $(\overline{r}_1, ..., \overline{r}_f)$ *and* $(\underline{r}_1, ..., \underline{r}_f)$ *be the upper and lower differential rent vectors determined by* $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f)$, $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)+1} - \underline{r}_f)$ *and*

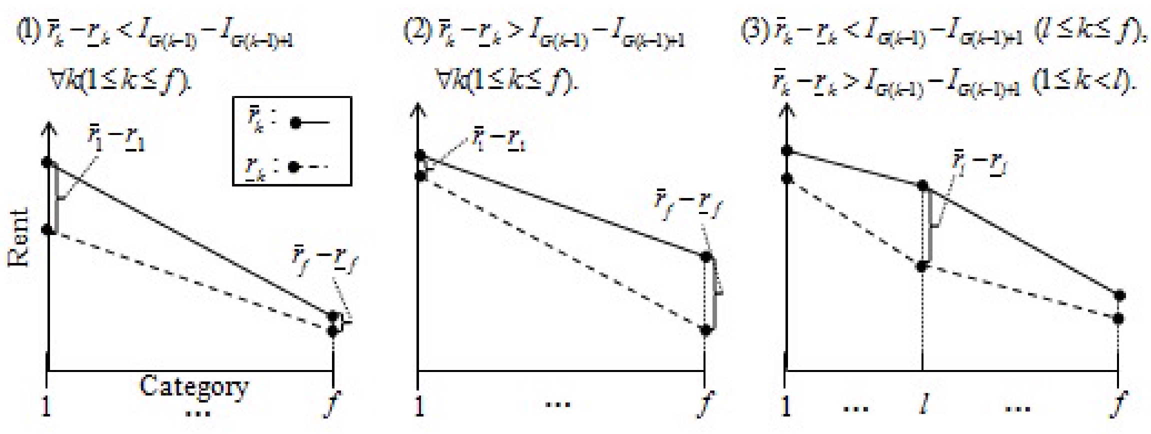

Figure 3.3: Shapes of rent differences (three cases).

$\bar{r}_f \geq \underline{r}_f$, *and* $k = 1, ..., f$. *Then,*

$$\bar{r}_k - \underline{r}_k \leq I_{G(k-1)} - I_{G(k-1)+1} \text{ if and only if } \bar{r}_k - \underline{r}_k \leq \bar{r}_{k-1} - \underline{r}_{k-1}. \tag{2.4}$$

Note that $\leq$ *of Eq. (2.4) can be replaced by* $\geq$, $>$, $<$ *or* $=$.

The form of Theorem 3.3.5 is similar to the *Basic comparative statics theorem* of Kaneko (1983) and Kaneko et al. (2006). Nevertheless, the meaning is different. Theorem 3.3.5 states that the rent difference of the k-th category is smaller than the income difference of two neighboring households numbered $G(k-1)$ and $G(k-1)+1$ if and only if the rent difference of $k-1$ is greater than the rent difference of k. This implies that we can reduce the comparison of the differences $\bar{r}_k - \underline{r}_k$ and $\bar{r}_{k-1} - \underline{r}_{k-1}$ to the comparison of the differences $\bar{r}_k - \underline{r}_k$ and $I_{G(k-1)} - I_{G(k-1)+1}$.

Fig. 3.3 depicts three examples of a shape of rent differences $\bar{r}_k - \underline{r}_k$. Fig.3.3.(1) explains the case of the statement (2.4) holds for each k. In this case, the difference $\bar{r}_k - \underline{r}_k$ gradually increases as k reaches 1. Fig.3.3.(2) explains the case where the opposite inequality of (2.4) holds for each k. In this case, the difference $\bar{r}_k - \underline{r}_k$ gradually decreases as k reaches 1. The remaining Fig.3.3.(3) explains the case where there is some category $l = 1, ..., f$ such that an inequality of (2.4) switches at l: the difference $\bar{r}_k - \underline{r}_k$ gradually increases for $k = l, ..., f$ and decreases for $k = 1, ..., l-1$. Numerical examples are given in Section 3.4.1.

The next theorem evaluates the rent difference by the income difference.

Theorem 3.3.6. *Let* $(\bar{r}_1, ..., \bar{r}_f)$ *and* $(\underline{r}_1, ..., \underline{r}_f)$ *be the upper and lower differential rent vectors determined by* $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - \bar{r}_f)$, $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)+1} - \underline{r}_f)$ *and*

$\overline{r}_f \geq \underline{r}_f$. *Suppose that* $\overline{r}_f - \underline{r}_f \leq I_{G(f-1)} - I_{G(f-1)+1}$. *Then,*

$$0 \leq \overline{r}_k - \underline{r}_k \leq \max_{k \leq l \leq f-1} \{I_{G(l)} - I_{G(l)+1}\} \textit{ for all } k = 1, ..., f-1.$$

Theorem 3.3.6 states that if the rent difference $\overline{r}_f - \underline{r}_f$ of the marginal category f is less than the income difference $I_{G(f-1)} - I_{G(f-1)+1}$ of two neighboring households, then the rent difference of the k-th ($k \leq f-1$) category is bounded by at most the largest income difference $I_{G(l)} - I_{G(l)+1}$ ($k \leq l \leq f-1$).

In our study, the rent of the marginal category f is considered to be uniquely determined.[5] Then, the upper and lower differential rent vectors are determined by the same marginal rent $p_f = \overline{r}_f = \underline{r}_f$: thus, the supposition of Theorem 3.3.6 holds. Under this situation, the theorem implies that the rent differences $\overline{r}_k - \underline{r}_k$ for each k are rather small. In particular, when we target a considerably large housing market with a dense household income distribution (i.e., the equality $I_{G(k)} = I_{G(k)+1}$ approximately holds), the difference can be approximated by zero. Consequently, the comparative statics results are not very different, whether or not we use the upper or lower differential rent vectors.

Recall $p^{\max}$ ($p^{\min}$) is the maximum (minimum) competitive rent vector in the market (M, N). Theorem 3.3.6 and Theorem 3.3.3 imply the following assertion:

$$\begin{aligned} p_k^{\max} - p_k^{\min} &\leq \max\nolimits_{k \leq l \leq f-1} \{I_{G(l)} - I_{G(l)+1}\} \textit{ for all } k = 1, ..., f-1 \\ \textit{if} \qquad \underline{r}_f &\leq p_f^{\min} \leq p_f^{\max} \leq \overline{r}_f, \end{aligned} \tag{2.5}$$

that is, the difference of the maximum and minimum competitive rents of k-th category is also bounded by the largest income difference $I_{G(l)} - I_{G(l)+1}$ with $k \leq l \leq f-1$. This implies the shrinkage result on the competitive rent vector set, which will be presented in Section 3.4.2.

3.3.3 Proofs of Theorems 3.3.5 and 3.3.6

Proof of Theorem 3.3.5. (*Only if*) By Eq. (2.2), we have $u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \underline{r}_{k-1}) = u(\mathbf{e}^k, I_{G(k-1)+1} - \underline{r}_k)$. Let $\delta = I_{G(k-1)} - \overline{r}_k - (I_{G(k-1)+1} - \underline{r}_k) \geq 0$. By Assumption D,

[5] For instance, Kaneko et al. (2006) adopted the estimated rent $\widetilde{p}_f$ from the real rent data as the differential rent r_f, and Ito (2007) adopted the (constant) marginal cost of the marginal category a_f as r_f.

$u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \underline{r}_{k-1} + \delta) \geq u(\mathbf{e}^k, I_{G(k-1)+1} - \underline{r}_k + \delta)$, that is,

$$\begin{aligned} u\left(\mathbf{e}^{k-1}, I_{G(k-1)} - \underline{r}_{k-1} - \overline{r}_k + \underline{r}_k\right) &\geq u(\mathbf{e}^k, I_{G(k-1)} - \overline{r}_k) \\ &= u(\mathbf{e}^{k-1}, I_{G(k-1)} - \overline{r}_{k-1}) \text{ by Eq. (2.1).} \end{aligned}$$

This inequality together with Assumption A imply $I_{G(k-1)} - \underline{r}_{k-1} - \overline{r}_k + \underline{r}_k \geq I_{G(k-1)} - \overline{r}_{k-1}$, that is, $\overline{r}_k - \underline{r}_k \leq \overline{r}_{k-1} - \underline{r}_{k-1}$. ∎

(*If*) We prove the contraposition of the claim. Suppose that $\overline{r}_k - \underline{r}_k > I_{G(k-1)} - I_{G(k-1)+1}$. By Eq. (2.1), we have $u(\mathbf{e}^{k-1}, I_{G(k-1)} - \overline{r}_{k-1}) = u(\mathbf{e}^k, I_{G(k-1)} - \overline{r}_k)$. Let $\delta = I_{G(k-1)+1} - \underline{r}_k - (I_{G(k-1)} - \overline{r}_k) \geq 0$. By Assumption D, $u(\mathbf{e}^{k-1}, I_{G(k-1)} - \overline{r}_{k-1} + \delta) > u(\mathbf{e}^k, I_{G(k-1)} - \overline{r}_k + \delta)$, that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \overline{r}_{k-1} - \underline{r}_k + \overline{r}_k) &> u\left(\mathbf{e}^k, I_{G(k-1)+1} - \underline{r}_k\right) \\ &= u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \underline{r}_{k-1}) \text{ by Eq. (2.2).} \end{aligned}$$

This inequality together with Assumption A imply $I_{G(k-1)+1} - \overline{r}_{k-1} - \underline{r}_k + \overline{r}_k > I_{G(k-1)+1} - \underline{r}_{k-1}$, that is, $\overline{r}_k - \underline{r}_k > \overline{r}_{k-1} - \underline{r}_{k-1}$. ∎

Proof of Theorem 3.3.6. We proof this by mathematical induction over $k = f-1,\ f-2$,..., 1. Let $k = f-1$. By the hypothesis and Assumption A, we have $u(\mathbf{e}^f, I_{G(f-1)+1} - \underline{p}_f) \leq u(\mathbf{e}^f, I_{G(f-1)} - \overline{p}_f)$. The left hand side is equal to $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1})$ by Eq. (2.2), and the right hand side is equal to $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1})$ by Eq. (**2.1**). Hence, we have $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) \leq u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1})$. This and Assumption A imply

$$\overline{r}_{f-1} - \underline{r}_{f-1} \leq I_{G(f-1)} - I_{G(f-1)+1}. \tag{2.6}$$

Let $\delta = I_{G(f-1)} - I_{G(f-1)+1} \geq 0$. Since $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1}) = u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f)$ by Eq. (2.1), we have, by Assumption D, $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1} - \delta) \leq u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f - \delta)$. This inequality is restated as

$$\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \overline{r}_{f-1}) &\leq u\left(\mathbf{e}^f, I_{G(f-1)+1} - \overline{r}_f\right) \\ &\leq u\left(\mathbf{e}^f, I_{G(f-1)+1} - \underline{r}_f\right) \\ &= u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) \text{ by Eq. (2.2).} \end{aligned}$$

This and Assumption A imply $I_{G(f-1)+1} - \overline{r}_{f-1} \le I_{G(f-1)+1} - \underline{r}_{f-1}$, that is, $\overline{r}_{f-1} \ge \underline{r}_{f-1}$. By this and Eq. (2.6), we have the relation $0 \le \overline{r}_{f-1} - \underline{r}_{f-1} \le I_{G(f-1)} - I_{G(f-1)+1}$.

Suppose that for $k = j$ with $1 < j \le f-1$,

$$0 \le \overline{r}_j - \underline{r}_j \le \max_{j \le l \le f-1}\{I_{G(l)} - I_{G(l)+1}\}. \tag{2.7}$$

Then, for $k = j-1$,

(i) Suppose $\overline{r}_j - \underline{r}_j \le I_{G(j-1)} - I_{G(j-1)+1}$. Then, $u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) \le u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$. The left hand side is equal to $u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1})$ by Eq. (2.2), and the right hand side is equal to $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1})$ by Eq. (2.1). Hence, $u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) \le u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1})$. This and Assumption A imply $I_{G(j-1)+1} - \underline{r}_{j-1} \le I_{G(j-1)} - \overline{r}_{j-1}$, that is,

$$\overline{r}_{j-1} - \underline{r}_{j-1} \le I_{G(j-1)} - I_{G(j-1)+1}. \tag{2.8}$$

Let $\delta = I_{G(j-1)} - I_{G(j-1)+1} \ge 0$. Since $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1}) = u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$ by Eq. (2.1), we have, by Assumption D, $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1} - \delta) \le u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j - \delta)$. This inequality is restated as

$$\begin{aligned} u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \overline{r}_{j-1}) &\le u(\mathbf{e}^j, I_{G(j-1)+1} - \overline{r}_j) \\ &\le u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) \text{ by Eq. (2.7)} \\ &= u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) \quad \text{by Eq. (2.2).} \end{aligned}$$

This and Assumption A imply $I_{G(j-1)+1} - \overline{r}_{j-1} \le I_{G(j-1)+1} - \underline{r}_{j-1}$, that is, $\overline{r}_{j-1} \ge \underline{r}_{j-1}$. By this and Eq. (2.8), we get

$$0 \le \overline{r}_{j-1} - \underline{r}_{j-1} \le I_{G(j-1)} - I_{G(j-1)+1}. \tag{2.9}$$

(ii) Suppose $\overline{r}_j - \underline{r}_j > I_{G(j-1)} - I_{G(j-1)+1}$. Then, $u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) > u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$. This together with Eq. (2.1) and Eq. (2.2) we have $u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) > u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1})$. This and Assumption A imply $I_{G(j-1)+1} - \underline{r}_{j-1} > I_{G(j-1)} - \overline{r}_{j-1}$, that is,

$$\overline{r}_{j-1} - \underline{r}_{j-1} > I_{G(j-1)} - I_{G(j-1)+1}. \tag{2.10}$$

Let $\delta = \overline{r}_{j-1} - \underline{r}_{j-1} - (I_{G(j-1)} - I_{G(j-1)+1}) \ge 0$. Since $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1}) = u(\mathbf{e}^j, I_{G(j-1)} -$

$\overline{r}_j$) by Eq. (2.1), we have, by Assumption D, $u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j + \delta) \leq u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1} + \delta)$. This is restated as

$$\begin{aligned} u(\mathbf{e}^j, I_{G(j-1)+1} - \overline{r}_j + \overline{r}_{j-1} - \underline{r}_{j-1}) &\leq (\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) \\ &= u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) \quad \text{by Eq. (2.2).} \end{aligned}$$

This and Assumption A imply $I_{G(j-1)+1} - \overline{r}_j + \overline{r}_{j-1} - \underline{r}_{j-1} < I_{G(j-1)+1} - \underline{r}_j$, that is, $\overline{r}_{j-1} - \underline{r}_{j-1} < \overline{r}_j - \underline{r}_j$. By this and Eq. (2.10), we get $I_{G(j-1)} - I_{G(j-1)+1} < \overline{r}_{j-1} - \underline{r}_{j-1} < \overline{r}_j - \underline{r}_j$. This together with Eq. (2.7) implies

$$0 \leq I_{G(j-1)} - I_{G(j-1)+1} < \overline{r}_{j-1} - \underline{r}_{j-1} \leq \max_{j \leq l \leq f-1}\{I_{G(l)} - I_{G(l)+1}\}.$$

This inequality together with Eq. (2.9), we have

$$0 \leq \overline{r}_{j-1} - \underline{r}_{j-1} \leq \max_{j-1 \leq k \leq f-1}\{I_{G(k)} - I_{G(k)+1}\}.$$

Hence, for all $k = 1, ..., f-1$, we have $0 \leq \overline{r}_k - \underline{r}_k \leq \max_{k \leq l \leq f-1}\{I_{G(l)} - I_{G(l)+1}\}$. ∎

3.4 Examples and the application

3.4.1 Numerical examples

Here, we will show three examples. The settings for the first and second examples are the same except for the number of apartment units w_k and households m. These examples show the rent difference in a large market is smaller than the rent difference in a sparse market. The third example is the case where the hypothesis of Theorem 3.3.6 fails.

Assume that there are six categories of apartments ($T = 6$). Let w_k ($k = 1, ..., 6$) be the number of apartment units for rent in the k-th category.[6] We assume that the same number of households come into the market, and that all the apartments are ultimately rented. That is, the marginal category is $f = 6$ and the number of households $m = \sum_{k=1}^{6} w_k$. Assume that

[6] Then, a cost function for landlord k can be expressed as $C_k = \begin{cases} a_k y_k \text{ if } y_k \leq w_k \\ \text{“large” if } y_k > w_k \end{cases}$, where a_k is a constant and “large” is greater than I_1.

k	1	2	3	4	5	6
$\overline{r}_k$	225.7	162.5	100.1	70.3	43.4	20
$\underline{r}_k$	220.4	158.3	97.2	68.2	42.3	20
$\overline{r}_k - \underline{r}_k$	5.3	4.2	2.8	2.1	1.2	0

Table 3.1: Differential rent vectors in Ex. 3.4.1

k	1	2	3	4	5	6
$\overline{r}_k$	223.4	161	99.4	69.9	43.3	20
$\underline{r}_k$	222.1	160	98.7	69.4	43	20
$\overline{r}_k - \underline{r}_k$	1.3	1.0	0.7	0.5	0.3	0

Table 3.2: Differential rent vectors in Ex. 3.4.2

each household has the following utility function:

$$u(\mathbf{e}^k, c) = h_k + \sqrt{c} \text{ for } k = 0, 1, ..., 6,$$

where $h_1 = 9$, $h_2 = 7$, $h_3 = 5$, $h_4 = 4$, $h_5 = 3$, $h_6 = 2$ and $h_0 = 0$. This utility function satisfies Assumption A, C,D, E and F. Also, we assume the income of each household is uniformly distributed over the interval $[100, 500]$.

Example 3.4.1. Let $w_k = 5$ for each $k = 1, ..., 6$. Then, $m = 5 \times 6 = 30$ and we have $I_{G(k)} - I_{G(k)+1} \simeq 13.8$ for each k. Let $\overline{r}_6 = \underline{r}_6 = 20$. Under these settings, we can calculate the upper and lower differential rent vectors by the rent Eqs. (2.1) and (2.2). Table 3.1 shows the calculation results of $\overline{r}_k$, $\underline{r}_k$ and the difference $\overline{r}_k - \underline{r}_k$.

By Table 3.1, an inequality $\overline{r}_k - \underline{r}_k \leq \max_{k \leq j \leq f-1}\{I_{G(j)} - I_{G(j)+1}\}$ of Theorem 3.3.6 holds for each k. We also observe $\overline{r}_k - \underline{r}_k < \overline{r}_{k-1} - \underline{r}_{k-1}$ for each k. This is consistent with Theorem 3.3.5 because $\overline{r}_k - \underline{r}_k < I_{G(k-1)} - I_{G(k-1)+1}$ for each k (which corresponds to Fig. 3.3.(1) in Section 3.3.2). To sum up, the difference $\overline{r}_k - \underline{r}_k$ is smaller than $I_{G(k)} - I_{G(k)+1}$ for each k; however, the difference tends to larger as a category gets better.

Example 3.4.2. Let $w_k = 20$ for each $k = 1, ..., 6$. Then, $m = 20 \times 6 = 120$ and we have $I_{G(k)} - I_{G(k)+1} \simeq 3.4$ for each k. Let $\overline{r}_6 = \underline{r}_6 = 20$. Table 3.2 shows the calculation results of $\overline{r}_k$, $\underline{r}_k$ and $\overline{r}_k - \underline{r}_k$.

As with Example 3.4.1, the difference $\overline{r}_k - \underline{r}_k$ is smaller than $I_{G(k)} - I_{G(k)+1}$ for each k; however, it tends to larger as a category gets better. Compared to Table 3.1, the difference $\overline{r}_k - \underline{r}_k$ is significantly smaller for each k.

We next give another example where the hypothesis of Theorem 3.3.6 fails. This example shows that whereas the rent difference $\overline{r}_k - \underline{r}_k$ exceeds the income difference $I_{G(k)} - I_{G(k)+1}$,

k	1	2	3	4	5	6
$\overline{r}_k$	223.4	161	99.4	69.9	43.3	20
$\underline{r}_k$	219	156.5	94.7	65.1	38.4	15
$\overline{r}_k - \underline{r}_k$	4.4	4.5	4.7	4.8	4.9	5

Table 3.3: Differential rent vectors in Ex. 3.4.3

the rent difference tends to decrease as k goes to 1.

Example 3.4.3. Let $w_k = 20$ for each k ($I_{G(k)} - I_{G(k)+1} \simeq 3.4$). Let $\overline{r}_6 = 20$ and $\underline{r}_6 = 15$. Then, we have $\overline{r}_6 - \underline{r}_6 = 5 > 3.4 \simeq I_{G(5)} - I_{G(5)+1}$; that is, the hypothesis of Theorem 3.3.6 fails. Table 3.3 shows the calculation results of $\overline{r}_k$, $\underline{r}_k$ and, $\overline{r}_k - \underline{r}_k$.

From Table 3.3, we have $\overline{r}_k - \underline{r}_k > \max_{k \leq j \leq f-1}\{I_{G(j)} - I_{G(j)+1}\}$ for each k (Theorem 3.3.6 fails). On the other hand, the difference $\overline{r}_k - \underline{r}_k$ tends to decrease as k reaches 1. This is consistent with Theorem 3.3.5 because $\overline{r}_k - \underline{r}_k > I_{G(k-1)} - I_{G(k-1)+1}$ for each k (which corresponds to Fig. 3.3.(2) in Section 3.3.2). Note that this example does not explain the necessity of the condition $\overline{r}_f - \underline{r}_f \leq I_{G(f-1)} - I_{G(f-1)+1}$ for Theorem 3.3.6. It may be possible that Theorem 3.3.6 holds but $\overline{r}_f - \underline{r}_f > I_{G(f-1)} - I_{G(f-1)+1}$.

3.4.2 Shrinkage of differential/competitive rent vectors with many households

As a consequence of our results in Section 3.3, we show a shrinkage result on a differential (or competitive) rent vector. In Chapter 2, Section 2.5.2, we showed the set of competitive price vectors shrinks to a unique point as the number of sellers become large. According to our Theorem 3.3.3 and 3.3.6, we will also obtain a similar result for markets with large number of households.

Let $\{(M^\nu, N^\nu)\}_{\nu=0}^{\infty}$ be a sequence of rental housing markets. We assume for each ν, a market (M^ν, N^ν) satisfies Assumptions A-G. We consider the situation where for a large ν, the market has many households and their income distribution gets dense. This is formalized by the following condition.

Condition 3.4.4. There is some constant $\alpha > 0$ such that for any ν, $\{(M^\nu, N^\nu)\}_{\nu=0}^{\infty}$ satisfies

(1) $|M^\nu| \to \infty$ as $\nu \to \infty$;

(2) $I_i^\nu \leq \alpha$ for all $i \in M^\nu$;

(3) $\max_{i \in M^\nu \setminus \{1\}}[I_{i-1}^\nu - I_i^\nu] \leq \alpha / |M^\nu|$.

Condition 3.4.4.(1) and (2) imply that, although the number of households becomes large, the income of each household is bounded; while (1) and (3) imply that an interval of two adjacent incomes tends to be small as the number of households becomes large. In sum, Condition 3.4.4 means that the income distribution of (M^ν, N^ν) becomes denser as ν gets larger.

For the market (M^ν, N^ν) $(\nu \geq 0)$, let $p^{\max \nu}$ $(p^{\min \nu})$ be the maximum (minimum) competitive rent vector, f^ν be a marginal category and $(\overline{r}_1^\nu, ..., \overline{r}_{f^\nu}^\nu)$, $(\underline{r}_1^\nu, ..., \underline{r}_{f^\nu}^\nu)$ be the upper and lower differential rent vectors determined by $\overline{r}_{f^\nu}^\nu = p_{f^\nu}^{\max \nu}$ and $\underline{r}_{f^\nu}^\nu = p_{f^\nu}^{\min \nu}$. Then, we have the following theorem.

Theorem 3.4.5. *Suppose that* $\{(M^\nu, N^\nu)\}_{\nu=0}^\infty$ *satisfies Condition 3.4.4 and for each* $\nu \geq 1$, $0 \leq \overline{r}_{f^\nu} - \underline{r}_{f^\nu} \leq I_{G(f^\nu-1)}^\nu - I_{G(f^\nu-1)+1}^\nu$. *Then,* $\sum_{k=1}^{f^\nu} \overline{r}_k - \underline{r}_k \to 0$ *as* $\nu \to \infty$.

Proof. By Theorem 3.3.6, we have $0 \leq \sum_{k=1}^{f^\nu-1} \overline{r}_k - \underline{r}_k \leq \sum_{k=1}^{f^\nu-1} \max_{k \leq l \leq f^\nu-1}[I_{G(l)}^\nu - I_{G(l)+1}^\nu]$. The right hand side of the inequality is not greater than $\alpha / |M^\nu|$ by Condition 3.4.3, which tends to zero as $\nu \to \infty$. ∎

Theorem 3.4.5 together with Eq. (**2.5**) in Section 3.3.2 imply that a competitive rent vector $(p_1^\nu, ..., p_{f^\nu}^\nu)$ of a relevant part also shrinks to a unique point: $\sum_{k=1}^{f^\nu} p_k^{\max \nu} - p_k^{\min \nu} \to 0$ as $\nu \to \infty$.[7]

3.5 Conclusions

We have evaluated the difference between the upper and lower differential rent vectors in a rental housing market model by Kaneko et al. (2006), where the identical utility function and the normality of the quality of housing are assumed. The upper (lower) differential rent vector is the solution of system of equations. In general, the upper (lower) differential rent vector is an upper (lower) bound of the set of competitive rent vectors. It coincides with the maximum (minimum) competitive rent vectors under some condition.

Our main result (Theorem 3.3.6) is that the rent difference of k-th category is smaller than the largest income difference between specific neighboring households numbered $G(l)$ and $G(l)+1$ $(k \leq l \leq f-1)$. This implies that the rent difference can be regarded as small and consequently, the difference between the maximum and minimum competitive rent vectors is also small. Furthermore, the difference shrinks to zero as the market becomes larger and the

[7] Indeed, the remaining categories $f+1, ..., T$ are inessential in our market model since no units in the k-th category $(f < k \leq T)$ are traded (nevertheless, a competitive rent p_k $(k > f)$ is determined with $p_k \leq C_k(1)$).

household income distribution becomes denser (Theorem 3.4.5). Another result (Theorem 3.3.5) indicates that we can reduce the comparison of two rent differences of the k-th and $k-1$-th categories into a comparison of the rent differences of the k-th category and the income differences of neighboring households numbered $G(k-1)$ and $G(k-1)+1$. Our results argue that a differential rent vector is a good approximation for a competitive rent vector; and furthermore, when we study a considerably large market, comparative statics results are similar whether we use an upper or lower differential rent vectors.

Chapter 4

Comparative Statics in Housing Markets with Indivisibilities: How Rising Income Inequality Affects on Housing Rents?

4.1 Introduction

In this chapter, we present the impact of rising household income inequality on housing rents. The market model that we adopt is the *rental housing market model* by Kaneko, Ito and Osawa (2006), which is an application of the assignment model (Shapley and Shubik, 1972; Kaneko, 1982) where the agents are divided into buyers and sellers, each buyer (household) demands at most one apartment unit, and each seller (landlord) provides some apartment units. The apartments as indivisible commodities are classified into finite categories $1, ..., T$ based on their qualities. The goods other than apartments are aggregated and consumed as composite good (money). A household utility function is assumed to be *homogeneous*, and allows *income effect* on housing qualities.

It is known that this market model guarantees the existence of a competitive equilibrium (Kaneko, 1982; Kaneko and Yamamoto, 1986). Furthermore, a competitive rent vector can be calculated by a solution for a certain system of equations (the solution is called the *differential rent vector*). In our analysis, we directly consider the differential rent vector rather than the competitive rent vector.

Here, we briefly introduce our comparative statics results. The first result is that the impact of an increase in income inequality can be divided by three cases: (1) rise in competitive rent at every category, (2) rise at higher categories and decline at lower categories or (3) decline at every category. The second result characterized (1)-(3) by the location of household who divides the households into the income-increased group and the decreased group. From this characterization, we show (1) is an extreme case, while (2) and also (3) of a counterintuitive results are possible.

We also show some tendency found between the rent change and diminishing rate of marginal utility by numerical example: the rent change tends to show (3) as *a diminishing rate of marginal utility for housing gets larger*; the rent change tends to show the case (2) *a diminishing rate of marginal utility for composite goods gets larger*. These observations implies that there is a certain low between marginal rate of substitution and a rent change.

Here, we introduce related literature. Kaneko et al. (2006) studied effects of changes in incomes of boundary households on a competitive rent vector. The boundary household is defined for each category of apartments. This household plays a crucial role in the model. The authors showed that when the boundary income difference is larger (smaller) for a better category of apartments, the rent difference forms convex (concave) shape.

Ito (2007) presented the effects of a rise in only the boundary household income of category k on competitive rents, under a more restricted assumption on a utility function. The author showed that rents are unchanged at $k+1, ..., T$, increase at $1, ..., k$ and a rent difference of each category $1, ..., k-2$ is smaller for a better category of apartments.

Määttänen and Terviö (2014) studied the effect of rising income inequality on house prices in the one-sided assignment model. One-sided means that the agents are potentially seller and buyer. The authors assume a continuum of agents and housing types (thus, an analytical method is calculus), and the homogeneity and normality on the utility functions. The authors presented a similar result to our main result. Braid (1981) also studied the effects of parameter changes on rent distributions under the two-sided version of Määttänen and Terviö's framework.

This paper is organized as follows. Section 4.2 formulates the market model and explains some notions for the study. Section 4.3 examines the impact of rising income inequality on a competitive rent distribution. Section 4.4 gives additional studies on the relation between an income distribution and a rent distribution by numerical examples. Section 4.5 presents conclusions and some remarks.

4.2 The market model

The *rental housing market model* (Kaneko et al., 2006) is denoted by (M, N), where the symbol $M = \{1, \ldots, m\}$ denotes the set of *households*, and $N = \{1, \ldots, T\}$ denotes the set of *landlords*. The objects of trade are apartments (indivisible) and money (perfectly divisible). The apartments are classified into a finite number of T categories by their housing attributes (e.g., housing size and commuting time). Each landlord $k \in N$ supplies of apartment units of the k-th category (thus k is the only landlord providing the k-th apartments).[1]

Each household $i \in M$ initially has an income $I_i > 0$ but no dwelling. The household wants to live in some apartment and use income to pay rent. Without loss of generality, we can assume that the households are ordered in their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. The *consumption set* is written by $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$, where $\mathbf{e}^k$ is the T-dimensional unit vector with k-th component is 1 ($\mathbf{e}^0 = \mathbf{0}$), and $\mathbb{R}_+$ is the set of nonnegative real numbers. A *consumption bundle* $(\mathbf{e}^k, c) \in X$ with $k \neq 0$ means that household i rents one apartment unit of category k and enjoys the consumption $c = I_i - p_k$ paying rent p_k of category k. For $k = 0$, no apartment is consumed. An initial endowment of $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$. Each household has an *identical utility function* $u : X \to \mathbb{R}$ satisfying the following assumption:

Assumption A. For each $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u(x, c)$ is a continuous and strictly monotone function of c, and $u(\mathbf{e}^0, I_i) > u(\mathbf{e}^k, 0)$ for all $k = 1, ..., T$.

The identical utility function implies that a housing market (M, N) represents a monocentric city, and every household commutes to the same business district. In Assumption A, continuity and monotonicity of money are standard; the latter inequality means the indispensability of money. We also assume the following B-D on $u(\cdot, \cdot)$.

Assumption B. If $u(x_i, c) = u(x'_i, c')$, and $c < c'$, then $u(x_i, c + \delta) > u(x'_i, c' + \delta)$ for any $\delta > 0$.

[1] The original model of Kaneko et al (2006) assume that $|N| \geq T$ and there are more than one seller providing apartments of type k $(= 1, ..., T)$. As far as competitive equilibrium is concerned, we can assume *without of generality* that only one seller provides apartments of type $k(= 1, ..., T)$ (thus the set N becomes $N = \{1, ..., T\}$). See Section 5 of Sai (2014).

Assumption C. If $u(x_i, c) > u(x'_i, c')$, then $u(x_i, c) = u(x'_i, c' + \delta)$ for some $\delta > 0$.

Assumption D. $u(\mathbf{e}^1, 0) > u(\mathbf{e}^2, 0) > \cdots > u(\mathbf{e}^T, 0)$.

Assumption B is the *normality* assumption on the quality of apartments in the following sense. In B, the k-th apartment has a better quality than k', since living in k with smaller consumption c is indifferent to living in k' with larger c'. When an income is increased by the same magnitude $\delta > 0$, the household strictly demands a better apartment. The normality implies that even if we assume an identical utility function, households having different incomes demand different qualities of apartments. Assumption C means that the housing quality of an apartment is substitutable for money. Assumption D means that the apartment qualities are strictly ordered numerically.[2]

We next define the seller side. Each landlord $k \in N = \{1, ..., T\}$ provides apartments of k-th category. The landlord has a *cost function* $C_k(y_k) : \mathbb{Z}_+ \to \mathbb{R}_+$, where $\mathbb{Z}_+$ is the set of nonnegative integers. For each $y_k \in \mathbb{Z}_+$, $C_k(y_k)$ represents the cost (in terms of money) of supplying y_k units of apartments of k-th category. In this study, we employ the following simple form of $C_k(\cdot)$.

Assumption E. For each $k \in N$, $C_k(y_k)$ is expressed as

$$C_k(y_k) = \begin{cases} a_k y_k & \text{if } y_k \leq w_k, \\ \text{“large”} & \text{if } y_k \geq w_k + 1, \end{cases}$$

In Assumption E, the constant $a_k > 0$ is the marginal cost of providing an additional unit, and "large" is a sufficiently large number. The remaining constant w_k is the number of all apartment units owned by landlord k. This cost function means that landlord k supplies units up to w_k with the constant marginal cost a_k, while never supplying more than w_k units, since the cost to build a new one is very large relative to market size.

We define a competitive equilibrium in (M, N). Let $p \in \mathbb{R}^T_+$ be the price vector, $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}^m$ be the demand vector and $y \in \mathbb{Z}^T_+$ be the supply vector. A triple (p, x, y) is a *competitive equilibrium* iff

(UM): for all $i \in M$, (i) $I_i - px_i \geq 0$, where $px_i = \sum_{k=1}^T p_k x_{ik}$;
(ii) $u(x_i, I_i - px_i) \geq u(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$ with $I_i - px_i \geq 0$.

[2] Assumption D together with Assumptions A, B and C imply that $u(\mathbf{e}^1, c) > u(\mathbf{e}^2, c) > \cdots > u(\mathbf{e}^T, c)$ for all $c \in \mathbb{R}_+$.

(PM): for all $k \in N$, $p_k y_k - C_k(y_k) \geq p_k y'_k - C_k(y'_k)$ for all $y'_k \in \mathbb{Z}_+$.

(BDS): $\sum_{i \in M} x_i = \sum_{k=1}^{T} y_k \mathbf{e}^k$.

There exists a competitive equilibrium (p, x, y) in (M, N) (Kaneko and Yamamoto, 1986), the maximum and minimum competitive rent vectors (Kaneko et al., 2006; Sai, 2015).[3,4] In our analysis, we focus on the maximum competitive rent vector. This rent vector is calculated by the solution of a certain system of equations called the *rent equation*.[5] The following proposition is necessary to define the rent equation.

Proposition 4.2.1 (Kaneko et al., 2006). *Let (p, x, y) be a competitive equilibrium. Then,*
(1) *If $k < k'$ and $x_i = \mathbf{e}^{k'}$ for some i, then $p_k > p_{k'}$.*
(2) *If $x_i = \mathbf{e}^k$, $x_j = \mathbf{e}^{k'}$ and $I_i > I_j$ for some i, j, then $k \leq k'$.*

This states that in any competitive equilibrium, (1) the price of a better apartment is higher than that of a worse one, and (2) a household with a higher income rents a better apartment. Note that Proposition 4.2.1. (1) does not exclude the case of $y_k = 0$. The following assumption eliminates such a case.

Assumption F. Let (p, x, y) be a competitive equilibrium. There exists some category f such that $y_k > 0$ for $k = 1, ..., f$ and $y_k = 0$ for $k = f + 1, ..., T$.

We call this f the *marginal category.* By Proposition 4.2.1. (1) and Assumption F, we have $p_1 > p_2 > \cdots > p_f$.

Recall that the households $1, ..., m$ are ordered by their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. We define the household with the lowest income in each active category. Let (p, x, y) be a maximum competitive equilibrium. For each category $k \leq f$, we define the household $G(k)$ with the lowest income in the k-th category as

$$G(k) := \sum_{t=1}^{k} y_t.$$

For each k, we call $G(k)$ the *boundary household* of the k-th category.

The rent equation (Kaneko et al., 2006) is defined as the system of equations with un-

[3] A vector $p \in \mathbb{R}^T_+$ is a *competitive price vector* iff (p, x, y) is a competitive equilibrium, and p is the *maximum* (*minimum*) *competitive price vector* iff $p \geq p'$ ($p \leq p'$) for any competitive price vector p'.

[4] Indeed, these existence theorems are guaranteed only under Assumptions A and E.

[5] Instead of the maximum one, we may focus on the minimum competitive rent vector. It follows from Sai (2014) and/or Sai (2015) that the difference between $p^{\max}$ and $p^{\min}$ is rather small when a market is thick with landlords and/or households.

knowns $r_1, ..., r_f$:

$$\left.\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1}) &= u(\mathbf{e}^f, I_{G(f-1)} - r_f), \\ u(\mathbf{e}^{f-2}, I_{G(f-2)} - r_{f-2}) &= u(\mathbf{e}^{f-1}, I_{G(f-2)} - r_{f-1}), \\ &\vdots \\ u(\mathbf{e}^1, I_{G(1)} - r_1) &= u(\mathbf{e}^2, I_{G(1)} - r_2). \end{aligned}\right\} \tag{3.1}$$

Note that the rent equation (3.1) has f unknowns constituted as $f - 1$ equations. Eq. (3.1) states that a household $G(k)$ is indifferent between renting the $k+1$-th apartment at rent r_{k+1} and renting the k-th category at r_k. In Eq. (3.1), if the rent of marginal category r_f is given, the first equation of Eq. (3.1) determines r_{f-1}. In the same manner, the remaining rents $r_{f-2}, ..., r_1$ are recursively determined. We call a solution $(r_1, ..., r_f)$ of Eq. (3.1) a *differential rent vector*. Under our assumptions, if r_f is given with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - r_f)$, then a differential rent vector is uniquely determined and satisfies $r_1 > \cdots > r_{f-1} > r_f$.

We conclude this section by noting the relation between a differential rent vector and a competitive rent vector. Let $p = (p_1, ..., p_T)$ be the maximum competitive rent vector and $(r_1, ..., r_f)$ is a differential rent vector given by $r_f \geq p_f$. Then, it holds that $r_k \geq p_k$ for all $k = 1, ..., f$ (Theorem 3.1 by Sai, 2015). In particular, if $r_f = p_f$ and some condition holds, then $r_k = p_k$ for all $k = 1, ..., f$.[6] Hereafter, we use a differential rent vector for comparative statics.

4.3 The impact of rising income inequalities on competitive rents

4.3.1 Comparative statics

In this section, we study the relation between household income distributions and competitive equilibria. The main purpose is to explain how rising income inequality affects a competitive rent vector. Recall that the set of households $M = \{1, ..., m\}$ is ordered by their income levels as $I_1 \geq \cdots \geq I_m$. Here we consider a new market where only the structure of household incomes change. To be precise, $\{I_1, \cdots, I_m\}$ changes to $\{\widehat{I}_1, \cdots, \widehat{I}_m\}$, but the remaining $M, N,$

[6] These are two conditions by Kaneko et al. (2006), Theorem 2.6: (1) $I_{G(k)} = I_{G(k)+1}$ *for each* $k = 1, ..., f - 1$; (2) $p_k < C_k(y_k + 1) - C_k(y_k)$ *for each* $k = 1, ..., f - 1$.
Even when neither condition holds, a differential rent vector can be an approximation of the maximum competitive rent vector. See Sai (2015), Section 3.1.

$u(\cdot,\cdot)$, $c_k(\cdot)$, f and r_f are unchanged.[7] By assumption E, the supply amount is also unchanged for each category $1, ..., f$, and consequently, the boundary household $G(k) = \sum_{t=1}^{k} w_t$ ($k = 1, ..., f$) remains the same. We consider the following condition on household incomes.

Condition InE (*Increase in Income Inequality*). There exists a household $i^* < m$ such that $I_i < \widehat{I}_i$ for $i \in \{1, \cdots, i^*\}$ and $I_i > \widehat{I}_i$ for $i \in \{i^*+1, \cdots, m\}$, and $\sum_{i \in M}(I_i - \widehat{I}_i) = 0$.

This condition states that in the new market, income increases for the upper households greater than i^*+1 and declines for the lower households less than i^*, preserving the level of gross income.

Let $(r_1, \cdots, r_{f-1}, r_f)$ and $(\widehat{r}_1, \cdots, \widehat{r}_{f-1}, r_f)$ be differential rent vectors in the original and new markets determined by r_f with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, \widehat{I}_{G(f-1)} - r_f)$. In the next theorem, we examine how the new rent vector $(\widehat{r}_1, \cdots, \widehat{r}_{f-1})$ changes under Condition InE (the proof will be given in Section 4.3.2).

Theorem 4.3.1 (*The Possible Cases of Rent Change*). *Under Condition InE, either* (1), (2) *or* (3) *holds*:

(1) $r_k < \widehat{r}_k$ *for* $k = 1, ..., f-1$.

(2) *There exist a category* $k^* (\le f-2)$ *such that* $\begin{cases} r_k < \widehat{r}_k \text{ for } k = 1, ..., k^*-1, \\ r_{k^*} \le \widehat{r}_{k^*}, \\ r_k > \widehat{r}_k \text{ for } k = k^*+1, ..., f-1. \end{cases}$

(3) $r_k > \widehat{r}_k$ *for* $k = 1, ..., f-1$.

This theorem shows three possibilities of rent change when income inequality increases. Theorems 4.3.1. (1) and (3) are straightforward: (1) [(3), respectively] states that rent rises (falls) for every category $1, ..., f-1$ in the new market, and (2) states that rents rise for upper categories $1, ..., k^*$ and fall for lower categories $k^*+1, ..., f-1$. The illustration of (2) is depicted in Fig. 4.1.

One may think Theorems 4.3.1. (1) and (3) are counterintuitive. It is natural that rising income inequality causes a decline in rent for lower categories and a rise in rent for upper categories [case (2)]. Indeed, in the next theorem we show that (1) is an extreme case; on the other hand, we also show that (3) is a common result.

Theorem 4.3.2 (*Location of Household* i^* *and Rent Change*). *Under Condition InE, the following holds*:

[7] We also assume that $\{\widehat{I}_1, \cdots, \widehat{I}_m\}$ satisfies $\widehat{I}_1 \ge \cdots \ge \widehat{I}_m$

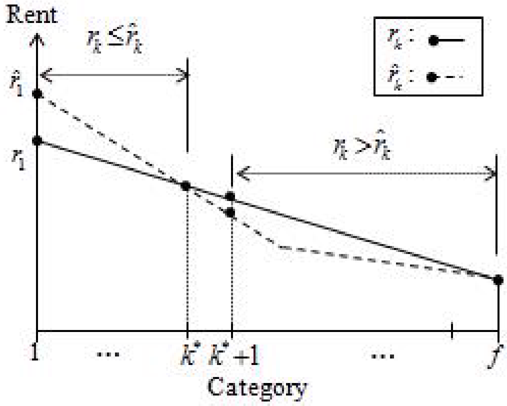

Figure 4.1: An illustration of Theorem 4.3.1. (2).

(1) $G(f-1) \le i^*$ *implies Theorem 4.3.1.(1).*

(2) $G(1) \le i^* < G(f-1)$ *implies Theorem 4.3.1.(2) or (3).*

(3) $i^* < G(1)$ *implies Theorem 4.3.1. (3).*

This theorem characterizes three cases of Theorem 4.3.1 by the location of household i^* of Condition InE. The first inequality $G(f-1) \le i^*$ means that the boundary income of every category rises, i.e., $I_{G(k)} < \widehat{I}_{G(k)}$ for every $k = 1, ..., f-1$. Similarly, the third inequality $G(f-1) \le i^*$ means that $I_{G(k)} > \widehat{I}_{G(k)}$ for every $k = 1, ..., f-1$. Both are extreme cases in that the number of "income declined (increased)" households is extremely small compared with the number of "increased (declined)" households: in the former case, every income declined household is assigned to the marginal category f, and the income declined segment $\{\widehat{I}_{i^*+1}, ..., \widehat{I}_m\}$ is irrelevant to the determination of rents $\widehat{r}_1, ..., \widehat{r}_{f-1}$, and in the latter case, every income increased household is assigned to the 1-st category.

Theorem 4.3.2 implies that when we exclude the two extreme cases above, two possibilities remain, Theorem 4.3.1. (2) or (3), as the result of a rent change (i.e., a counterintuitive result still occurs). Then what factors determine the establishment of either Theorem 4.3.1. (2) or (3)? We discuss this in the next section.

Notice that we assumed that the apartment stock is fixed (Assumption E). In this sense, our study is a short-run equilibrium analysis. One possible example of such a short-term change in household income distribution is that of a government's change in its policy of redistribution. In our result, Theorems 3.1 and 2 imply that strengthening of income redistribution causes either a rise in rent for every category of housing, or a decrease in rent for a few upper categories and a rise in rent for other categories (that is, the supplier share of excess economic rent increases).

We conclude this section by comparing our results with other related studies. Kaneko et al. (2006) studied effects of changes in boundary incomes on a differential rent vector. In particular, they considered the case $\widehat{I}_{G(f-1)} - I_{G(f-1)} \leq \widehat{I}_{G(f-2)} - I_{G(f-2)} \leq \cdots \leq \widehat{I}_{G(1)} - I_{G(1)}$, i.e., the boundary income increment is larger for a better category of apartments.[8,9] Then, according to their Theorem 5.2. (1) and Corollary 6.2. (1) (Kaneko et al., p.160 and p.162), the rent differences form a convex shape.[10]

Määttänen and Terviö (2014) also studied the effect of rising income inequality on housing prices by using the one-sided continuum assignment model. In their model, each agent initially has a house and money, and they exchange them. They assumed that a continuum of agents and housing types, and the homogeneity and normality of utility functions. Their main result (Proposition 4, p.391) is essentially the same as our Theorem 4.3.1 with the exclusion of the case (1).[11] Nevertheless, their analytical method is different from ours because they use calculus for analyses, whereas our model is based on finiteness.

4.3.2 Proofs of Theorem 4.3.1 and Theorem 4.3.2

It suffices to prove Theorem 4.3.2.

Proof of Theorem 4.3.2.(1). Suppose $G(f-1) \leq i^*$, i.e., $I_{G(k)} < \widehat{I}_{G(k)}$ for every $k = 1, ..., f-1$. We prove this by mathematical induction over $f-1, ..., 1$. Let $\delta = \widehat{I}_{G(f-1)} - I_{G(f-1)} > 0$. the rent equation (3.1) and the normality assumption (Assumption B) imply

$$u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1} + \delta) > u(\mathbf{e}^f, I_{G(f-1)} - r_f + \delta),$$

[8] They also considered the opposite case: $\widehat{I}_{G(f-1)} - I_{G(f-1)} \geq \widehat{I}_{G(f-2)} - I_{G(f-2)} \geq \cdots \geq \widehat{I}_{G(1)} - I_{G(1)}$.

[9] Our Condition InE can be applied to their condition as

$$\widehat{I}_{G(f-1)} - I_{G(f-1)} \leq \cdots \leq \widehat{I}_{G(\overline{k})} - I_{G(\overline{k})} < 0 < \widehat{I}_{G(\overline{k}-1)} - I_{G(\overline{k}-1)} \leq \cdots \leq \widehat{I}_{G(1)} - I_{G(1)}$$
$$\text{for some } \overline{k} \in \{2, ..., f-1\}.$$

This is understood as the income inequality *significantly* increases.

[10] To be precise, the rent difference holds

$$0 > \widehat{r}_{f-1} - r_{f-1} > \cdots > \widehat{r}_{k_1} - r_{k_1} = \cdots = \widehat{r}_{k_2} - r_{k_2} < \cdots < \widehat{r}_1 - r_1,$$
$$\text{where } \overline{k} \leq k_2 \leq k_1 \leq f-1,$$

that is, the decrement of $\widehat{r}_k$ to r_k gets larger for category $f-1$, $f-2$, $\cdots$, k_1, becoming maximal for $k_1, \cdots$, k_2 and smaller for $k_2, ..., 1$.

[11] Their condition of an increase in income equality excludes antecedents of Theorem 4.3.2. (1) and (3).

that is,

$$\begin{aligned} u(\mathbf{e}^{f-1}, \widehat{I}_{G(f-1)} - r_{f-1}) &> u(\mathbf{e}^{f}, \widehat{I}_{G(f-1)} - r_{f}) \\ &= u(\mathbf{e}^{f-1}, \widehat{I}_{G(f-1)} - \widehat{r}_{f-1}) \text{ by Eq. (3.1).} \end{aligned}$$

This inequality and the monotonicity (Assumption A) imply $\widehat{I}_{G(f-1)} - r_{f-1} > \widehat{I}_{G(f-1)} - \widehat{r}_{f-1}$, that is, $r_{f-1} < \widehat{r}_{f-1}$.

Suppose $r_k < \widehat{r}_k$ for k with $1 < k \leq f-1$. Then we show this relation also holds for $k-1$. Let $\delta = \widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k)$ and suppose $\delta > 0$. Then, Eq. (3.1) and Assumption B imply

$$u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1} + \delta) > u(\mathbf{e}^{k}, I_{G(k-1)} - r_k + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k) &> u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_{k}) \\ &= u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \text{ by Eq. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $\widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k > \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}$, that is, $\widehat{r}_{k-1} - r_{k-1} > \widehat{r}_k - r_k > 0$.

Suppose the other case $\delta \leq 0$. Then, Assumption A imply $u(\mathbf{e}^k, I_{G(k-1)} - r_k) \geq u(\mathbf{e}^k, \widehat{I}_{G(k-1)} - \widehat{r}_k)$. Since the left hand side equals $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$ and the right hand side equals $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$ by Eq. (3.1), we have $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1}) \geq u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$. Again, by Assumption A, we have $I_{G(k-1)} - r_{k-1} \geq \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}$, that is, $\widehat{r}_{k-1} - r_{k-1} \geq \widehat{I}_{G(k-1)} - I_{G(k-1)} > 0$. Hence we obtain $r_{k-1} < \widehat{r}_{k-1}$. ∎

Proof of (2). Suppose $G(1) \leq i^* < G(f-1)$ and let $k^\circ = \min[k : i^* < G(k)]$. We first prove the inequality $r_k > \widehat{r}_k$ holds for $k = k^\circ, ..., f-1$ by mathematical induction. Let $\delta = I_{G(f-1)} - \widehat{I}_{G(f-1)} > 0$. The rent equation (3.1) and Assumption B imply

$$u(\mathbf{e}^{f-1}, \widehat{I}_{G(f-1)} - \widehat{r}_{f-1} + \delta) > u(\mathbf{e}^{f}, \widehat{I}_{G(f-1)} - r_f + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)} - \widehat{r}_{f-1}) &> u(\mathbf{e}^{f}, I_{G(f-1)} - r_f) \\ &= u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1}) \text{ by Eq. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $I_{G(f-1)} - \widehat{r}_{f-1} > I_{G(f-1)} - r_{f-1}$, that is, $r_{f-1} > \widehat{r}_{f-1}$.

Suppose the inequality $r_k > \widehat{r}_k$ holds for k with $k^\circ < k \le f-1$. We show this also holds for $k-1$. Let $\delta = I_{G(k-1)} - r_k - (\widehat{I}_{G(k-1)} - \widehat{r}_k)$ and suppose $\delta > 0$. Eq. (3.1) and Assumption B imply

$$u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1} + \delta) > u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_{k} + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, I_{G(k-1)} - \widehat{r}_{k-1} - r_k + \widehat{r}_k) &> u(\mathbf{e}^{k}, I_{G(k-1)} - r_k) \\ &= u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $I_{G(k-1)} - \widehat{r}_{k-1} - r_k + \widehat{r}_k > I_{G(k-1)} - r_{k-1}$, that is, $r_{k-1} - \widehat{r}_{k-1} > r_k - \widehat{r}_k > 0$. Hence we obtain $r_{k-1} > \widehat{r}_{k-1}$.

Suppose the other case $\delta \le 0$. Then, Assumption A imply $u(\mathbf{e}^k, \widehat{I}_{G(k-1)} - \widehat{r}_k) \ge u(\mathbf{e}^k, I_{G(k-1)} - r_k)$. Since the left hand side equals $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$ and the right hand side equals $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$ by Eq. (3.1), we have $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \ge u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$. Again, by Assumption A, we have $\widehat{I}_{G(k-1)} - \widehat{r}_{k-1} \ge I_{G(k-1)} - r_{k-1}$, that is, $r_{k-1} - \widehat{r}_{k-1} \ge I_{G(k-1)} - \widehat{I}_{G(k-1)} > 0$. Hence we obtain $r_{k-1} > \widehat{r}_{k-1}$.

From the above discussion, we have $r_k > \widehat{r}_k$ holds for $k = k^\circ, ..., f-1$. We next show either $r_k > \widehat{r}_k$ or $r_k \le \widehat{r}_k$ holds for $k = 1, ..., k^\circ - 1$. Furthermore, we show that once $r_k \le \widehat{r}_k$ appears for some $k^* \le k^\circ - 1$, then it holds that $r_k < \widehat{r}_k$ for $k = 1, ..., k^* - 1$.

Let $\delta = \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ} - (I_{G(k^\circ-1)} - r_{k^\circ})$. By condition InE, we have

$$\widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ} - (I_{G(k^\circ-1)} - r_{k^\circ}) > 0. \tag{3.2}$$

Eq. (3.1) and Assumption B imply

$$u(\mathbf{e}^{k^\circ-1}, I_{G(k^\circ-1)} - r_{k^\circ-1} + \delta) > u(\mathbf{e}^{k^\circ}, I_{G(k^\circ-1)} - r_{k^\circ} + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - r_{k^\circ-1} - \widehat{r}_{k^\circ} + r_{k^\circ}) &> u(\mathbf{e}^{k^\circ}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ}) \\ &= u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1}) \text{ by Eq. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $\widehat{I}_{G(k^\circ-1)} - r_{k^\circ-1} - \widehat{r}_{k^\circ} + r_{k^\circ} > \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1}$, that is,

$$r_{k^\circ} - \widehat{r}_{k^\circ} > r_{k^\circ-1} - \widehat{r}_{k^\circ-1}. \tag{3.3}$$

On the other hand, Eq. (3.2) and Assumption A imply $u(\mathbf{e}^{k^\circ}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ}) > u(\mathbf{e}^{k^\circ}, I_{G(k^\circ-1)} - r_{k^\circ})$. Since the left hand side equals $u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1})$ and the right hand side equals $u(\mathbf{e}^{k^\circ-1}, I_{G(k^\circ-1)} - r_{k^\circ-1})$ by Eq. (3.1), we have $u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1}) > u(\mathbf{e}^{k^\circ-1}, I_{G(k^\circ-1)} - r_{k^\circ-1})$. Again, by assumption A, we have $\widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1} > I_{G(k^\circ-1)} - r_{k^\circ-1}$, that is, $r_{k^\circ-1} - \widehat{r}_{k^\circ-1} > I_{G(k^\circ-1)} - \widehat{I}_{G(k^\circ-1)}$. By this and Eq. (3.3), we have

$$r_{k^\circ} - \widehat{r}_{k^\circ} > r_{k^\circ-1} - \widehat{r}_{k^\circ-1} > I_{G(k^\circ-1)} - \widehat{I}_{G(k^\circ-1)}.$$

Since $r_{k^\circ} > \widehat{r}_{k^\circ}$ and $I_{G(k^\circ-1)} < \widehat{I}_{G(k^\circ-1)}$, there are two cases: $r_{k^\circ-1} > \widehat{r}_{k^\circ-1}$ or $r_{k^\circ-1} \leq \widehat{r}_{k^\circ-1}$. If the latter case, the category k^* of Theorem 4.3.1.(2) is $k^* = k^\circ - 1$.

Let k with $1 < k \leq k^\circ - 1$.

(Case $r_k > \widehat{r}_k$): By Condition InE, $\widehat{I}_{G(k-1)} > I_{G(k-1)}$. Thus, we have $\widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k) > 0$. In the same manner with the above discussion, we have

$$r_k - \widehat{r}_k > r_{k-1} - \widehat{r}_{k-1} > I_{G(k-1)} - \widehat{I}_{G(k-1)},$$

and there may be two cases $r_{k-1} > \widehat{r}_{k-1}$ or $r_{k-1} \leq \widehat{r}_{k-1}$. If the latter case, the category k^* of Theorem 4.3.1.(2) is $k^* = k - 1$.

(Case $r_k \leq \widehat{r}_k$): Suppose that $\delta = \widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k) > 0$. Eq. (3.1) and Assumption B imply

$$u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1} + \delta) > u(\mathbf{e}^k, I_{G(k-1)} - r_k + \delta),$$

that is,

$$
\begin{aligned}
u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k) &> u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_k) \\
&= u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \text{ by Eq. (3.1).}
\end{aligned}
$$

This inequality and Assumption A imply $\widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k > \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}$, that is, $r_{k-1} < \widehat{r}_{k-1}$.

Suppose the other case $\widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k) \leq 0$. This inequality and Assumption A imply $u(\mathbf{e}^k, \widehat{I}_{G(k-1)} - \widehat{r}_k) \leq u(\mathbf{e}^k, I_{G(k-1)} - r_k)$. Since the left hand side equals $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$ and the right hand side equals $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$ we have $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \leq u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$. Again, by Assumption A, $\widehat{I}_{G(k-1)} - \widehat{r}_{k-1} \leq I_{G(k-1)} - r_{k-1}$. Since $\widehat{I}_{G(k-1)} > I_{G(k-1)}$, we obtain $r_{k-1} < \widehat{r}_{k-1}$. ∎

Proof of (3). The proof is the same as the early part of the proof of (2). ∎

4.4 Numerical examples

In the previous section, we gave Theorems 4.3.1 and 4.3.2 for the relation between the distribution of income and housing rents. Theorem 4.3.1 stated that an increase in income inequality causes either (1) a rise in rent for every category, (2) a rise for higher categories and a fall for lower categories, or (3) a fall for every category [also we showed (1) is a special case]. Here, we provide additional observations using two numerical examples.

The first example shows that there is a certain tendency with rent changes depending on the diminishing rate of marginal utility. To be precise, the rent change tends to show case (3) of Theorem 4.3.1 as the *diminishing rate of marginal utility for housing gets larger*; and the rent change tends to show case (2) of Theorem 4.3.1 as the *diminishing rate of marginal utility for composite goods gets larger*. These observations imply that if the *diminishing rate of marginal rate of substitution of housing for composite good* is large (that is, the degree of convexity of the indifference curve is large), then the rent change tends to show case (3) of Theorem 4.3.1.

The second example confirms our Theorem 4.3.2. (2). It shows that an increase in income inequality possibly causes case (2) or (3) of Theorem 4.3.1, under the condition that the other parameters remain the same.

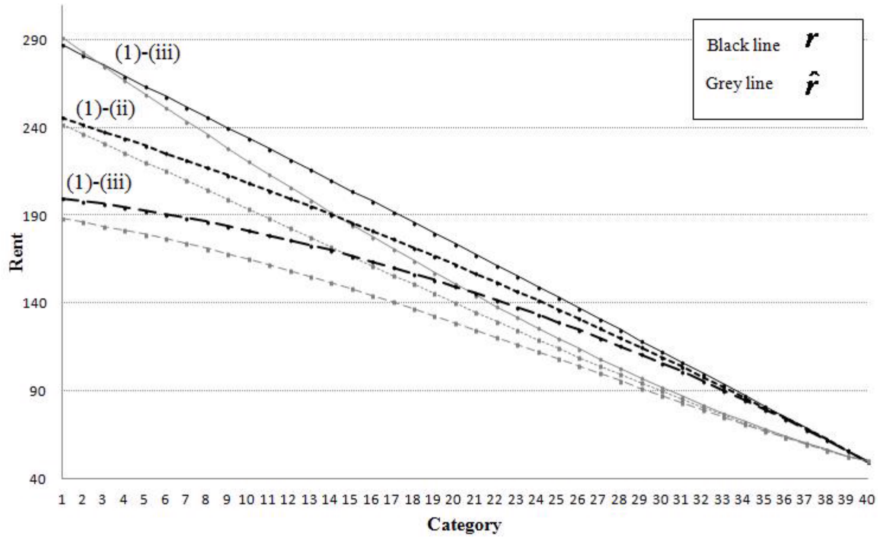

Figure 4.2: Rent distributions when income distribution changes (1).

4.4.1 Diminishing rate of marginal utility and rent change

Suppose that there are $T = 40$ categories of apartments (also suppose $f = T$), an apartment of each category is possibly supplied at most one unit ($w_k = 1$ for all $k = 1, ..., 40$) and the number of households $m = 40$. Then $G(k) = \sum_{t=1}^{k} 1$ for $k = 1, ..., 40$. Each household has the following separable utility function:

$$u(\mathbf{e}^k, c) = h_k + 5c^a \ (k = 0, ..., 40 \text{ and } 0 < a < 1).$$

Let $h_0 = 0$, $h_{40} = 1$ and $\Delta h_k = h_k - h_{k+1}$ ($\Delta h_{40} = h_{40} - h_0$). We consider the following different diminishing rates for Δh_k.

(1) Diminishing rates of marginal utility for h_k:

(i) $\Delta h_{k-1} = \Delta h_k$ (0% diminishing rate);

(ii) $\Delta h_{k-1} = 0.99\Delta h_k$ (1% diminishing rate);

(iii) $\Delta h_{k-1} = 0.97\Delta h_k$ (3% diminishing rate).

A parameter a of $u(\cdot)$ is fixed as $a = 0.5$. We assume that the distribution of household (monthly) income is uniform and its interval changes from $[300, 500]$ to $[100, 700]$. Let the marginal rent $r_{40} = 50$. We then calculate differential rent vectors $(r_1, ..., r_{40})$ and $(\widehat{r}_1, ..., \widehat{r}_{39}, r_{40})$. The rent distributions are shown in Fig. 4.2.

In Fig. 4.2, black lines depict differential rents before the income change, and gray lines depict rents after the income change; solid lines correspond to (1)-(i), fine dotted lines correspond to (1)-(ii), and coarse dotted lines correspond to (1)-(iii). As shown in Fig. 4.2, the decrement of rent when inequality increases gets larger as the diminishing rate for Δh_k increases.

The reason for this tendency could be explained as follows. Let k be a category of apartment. When the marginal utility of dwellings diminishes, for each household the willingness to pay for the net marginal utility received by moving from unit k-th apartment to $k-1$ is smaller than it is in moving from $k+1$ to k. By this reason, in comparison with lower households, an upper household prefers to spend its income on things other than dwellings. As a consequence, if the diminishing marginal rate is large to some extent, the price for a higher category of housing hardly rises enough to supplement the rent decrements in lower categories.

We also consider the following different diminishing rates of marginal utility for composite goods.

(2) Diminishing rates of marginal utility for c^a.

(i) $a = 0.55$;

(ii) $a = 0.5$;

(iii) $a = 0.45$.

A diminishing rate for h_k is fixed as $\Delta h_{k-1} = \Delta h_k$. We assume the same changes for income distribution as (1). The rent distributions with $r_{40} = 50$ are given in Fig. 4.3.

As seen from Fig. 4.3, an increase in the diminishing rate for composite good brings an effect opposite to (1) for higher categories. It is also found that the magnitude of rent difference becomes larger as the diminishing rate becomes smaller.

We can also explain the reason for such a tendency in a manner similar to (1). If the diminishing rate for composite good becomes large, an upper household wants to spend its income on dwellings rather than on composite good. As a consequence, if a diminishing rate is large to some extent, the price for the higher category rises enough to supplement rent decrements in lower categories.

4.4.2 Confirmation of Theorem 4.3.2. (2)

Suppose $T = f = 6$, $w_1 = w_2 = 200$, $w_3 = w_4 = 300$, $w_5 = w_6 = 500$ and $m = \sum_{t=1}^{k} w_t$ (then, $G(k) = \sum_{t=1}^{k} w_t$ for $k = 1, ..., 6$). The utility function is given by $u(\mathbf{e}^k, c) = h_k + \sqrt{c}$

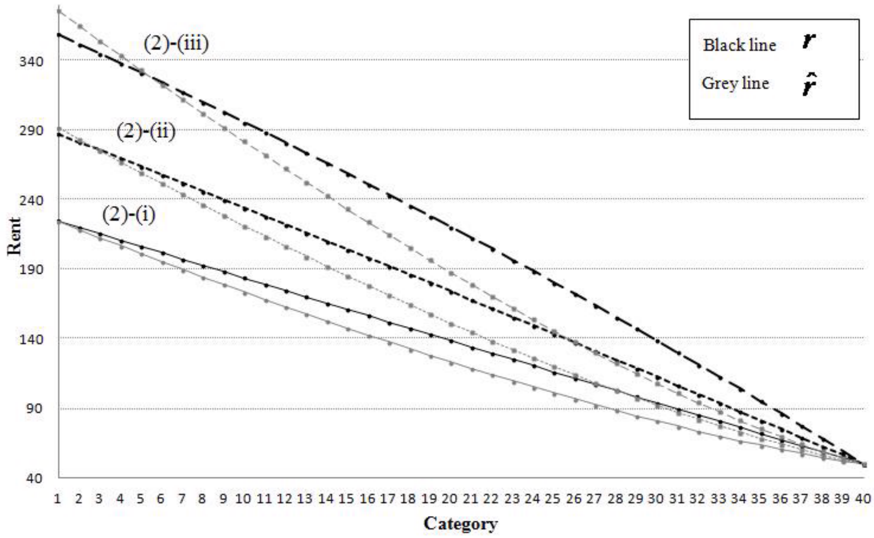

Figure 4.3: Rent distributions when income distribution changes (2).

$(k = 0, ..., 6)$, where $h_1 = 5.1, h_2 = 4.4, h_3 = 3.7, h_4 = 3, h_5 = 2, h_6 = 1$ and $h_0 = 0$. We assume that household income is lognormally distributed.[12] We adopt the following three lognormal distributions: the mean of lognormal distribution is fixed as $E = 330$, and variances are $V_1 = 1000$, $V_2 = 20000$ and $V_3 = 80000$. Fig. 4.4 depicts the probability density distributions for them.

In Fig. 4.4, the highest graph corresponds to (E, V_1) (the initial distribution), the second highest corresponds to (E, V_2) (denoted hats), and the remaining is $(E, V_3) = 80000$ (denoted it by double hats). We generate three sets of 2000 random numbers following each distribution. Table 4.1 gives boundary incomes and Gini coefficients for each income set.

Table 4.1 shows that income inequality increases as the variance increases, and the magnitude of income difference is monotone. The locations of household i^* in Condition InE are $G(3) \leq \widehat{i^*} < G(4)$ and $G(2) \leq \widehat{\widehat{i^*}} < G(3)$ (both hold for (2) of Theorem 4.3.2).

Let $r_f = r_6 = 50$. Calculated differential rent vectors are $(r_1, ..., r_6)$, $(\widehat{r}_1, ..., \widehat{r}_5, r_6)$, $(\widehat{\widehat{r}}_1, ..., \widehat{\widehat{r}}_5, r_6)$ and are illustrated in Table 4.2 and Fig. 4.5.

[12] We say that a (positive) random variable X is lognormally distributed with parameters μ and σ^2 iff $Y = \ln X$ is normally distributed with mean μ and variance σ^2. The lognormal distribution is denoted by $\Lambda(\mu, \sigma^2)$. The probability density function of $X \sim \Lambda(\mu, \sigma^2)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right] \quad (x > 0).$$

The mean E, variance V, median M and mode D of $\Lambda(\mu, \sigma^2)$ are given by $E = \exp(\mu + \frac{1}{2}\sigma^2)$, $V = \exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right]$, $M = \exp(\mu)$ and $D = \exp(\mu - \sigma^2)$. By them, we have $D < M < E$, and thus, $\Lambda(\mu, \sigma^2)$ has a long-tail form. These difinitions and properties are due to Crow and Shimizu (1988). The lognormal distribution is aften used as an approximation of an income distribution.

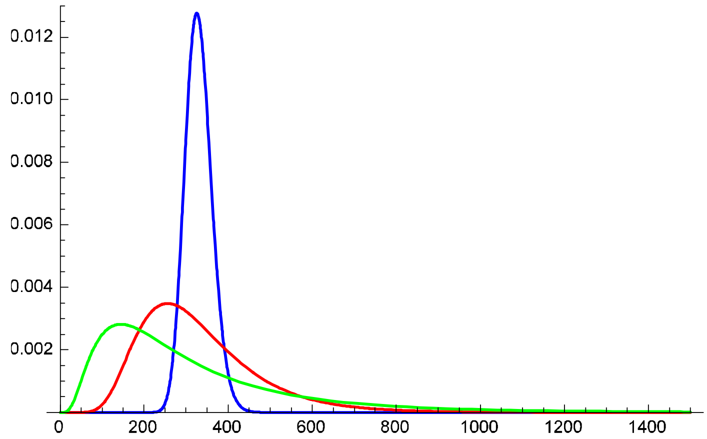

Figure 4.4: Probability density distributions of lognormal distributions.

		Boundary incomes			Differences in $I_{G(k)}$	
k	$G(k)$	$I_{G(k)}$	$\widehat{I}_{G(k)}$	$\widehat{\widehat{I}}_{G(k)}$	$\widehat{I}_{G(k)} - I_{G(k)}$	$\widehat{\widehat{I}}_{G(k)} - \widehat{I}_{G(k)}$
1	200	371.6	514.7	660.5	143.1	145.8
2	400	356.7	433.8	470.1	77.1	36.3
3	700	341.8	356.1	346.0	14.3	−10.1
4	1000	329.3	305.2	264.3	−24.1	−40.9
5	1500	310.1	231.4	167.5	−78.6	−63.9
Gini		0.05	0.22	0.36		

Table 4.1: Changes in boundary incomes

	Changes in r_k			Differences in r_k	
k	r_k	$\widehat{r}_k$	$\widehat{\widehat{r}}_k$	$\widehat{r}_k - r_k$	$\widehat{\widehat{r}}_k - \widehat{r}_k$
1	173.2	177.2	176.1	4.0	−1.0
2	153.0	150.9	114.8	−2.0	−6.1
3	132.5	126.9	119.1	−5.6	−7.8
4	111.8	105.2	97.5	−6.5	−7.7
5	81.3	75.9	70.7	−5.3	−5.3
6	50	50	50	0	0

Table 4.2: Changes in differential rent vectors

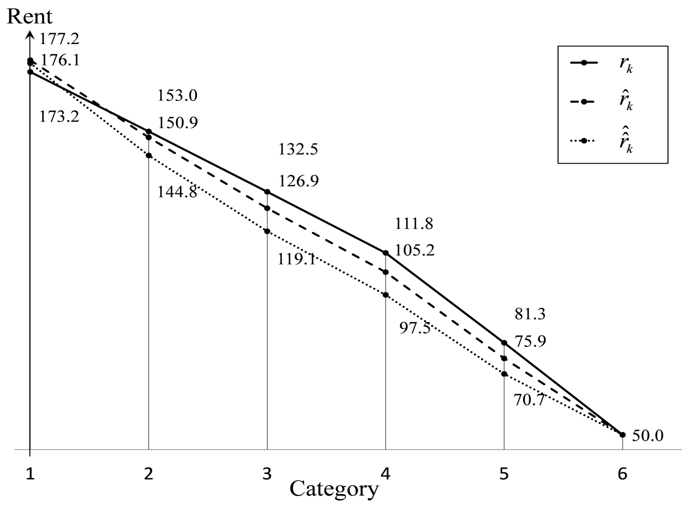

Figure 4.5: Rent distributions when income distribution changes (3).

As seen from Table 4.2 and Fig. 4.5, the first change (from $\{I_i\}_{i\in M}$ to $\{\widehat{I_i}\}_{i\in M}$) causes a rise only for the first category [Theorem 4.3.1. (2) occurs]. On the other hand, the second change causes a decline in rents for every category [Theorem 4.3.1. (3) occurs]. These results are consistent with Theorem 4.3.2. Thus, both Theorem 4.3.1. (2) and (3) may occur when income inequality increases but the other parameters do not change.

4.5 Conclusions

We have studied the comparative statics analysis based on the assignment market model. In particular, we present how rising income inequality affects a competitive rent distribution. The key assumptions of the model are identical utility function and normality for housing quality. A competitive rent vector can be then calculated by a system of equations.

Our main comparative statics result is Theorem 4.3.1, stating that an increase in income inequality affects three cases of the competitive rent vector: (1) rent rises for every category, (2) a rent rises for upper categories and falls for lower categories or (3) rent falls for every category. Further Theorem 4.3.2 implies that (1) is an extreme case, while either (2) or (3) is possible when an inequality increases. We also show the relation between a diminishing rate of marginal utility and a rent change when income inequality increases, using numerical examples.

We conclude this chapter with two remarks about future subjects. In Section 4.4.1, we showed by numerical examples that there is a relation between marginal rate of substitution and effects of increased income inequality for equilibrium rents. One future subject is to show this relation by a proposition. The other subject is about a relaxation of our market model. In Section 4.3.1, we mentioned that since we assumed that the apartment stock is fixed (Assumption E), our study is a short-run equilibrium analysis. On the other hand, a change in income structure is often considered a mid- or long-term structural change. A subject is to relax the fixed apartment stock assumption and compare our result with the result under such a relaxed model.

Appendix A

Additional Results for Chapter 2

A.1 Pareto efficiency of competitive equilibria

In this appendix, we show the Pareto efficiency of competitive equilibria in the GAM model. To show this, we first transform the seller's cost function into the utility function. Let $t = 1, ..., T$ and N_t be the set of all sellers providing indivisible goods t. For each $j \in N_t$, Let $(w_j, I_j) \in \mathbb{Z}_+ \times \mathbb{R}_+$ be the initial endowment of seller j (the value w_j is seller j's the maximum possible supply of indivisible goods t). We transform $c_j : \mathbb{Z}_+ \to \mathbb{R}$ into $u_j : \mathbb{Z}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as

$$u_j(k, d) = \begin{cases} d - c_j(w_j - k) & \text{if } k < w_j, \\ d - c_j(0) & \text{if } k \geq w_j. \end{cases}$$

The profit maximization condition for seller (Definition 2.2.1.(2)) is transformed by the (equivalent) utility maximization condition: for each $t = 1, ..., T$ and $j \in N_t$, $u_j(w_j - y_j, I_j + p_t y_j) \geq u_j(w_j - k, I_j + p_t k)$ for every $k \leq w_j$.

Now, let

$$\mathbb{M} = \{(\delta_{1'}, ..., \delta_{m'}, \delta_1, ..., \delta_n) \in \mathbb{R}^{m'+n} : \textstyle\sum_{i \in M \cup N} \delta_i = \sum_{i \in M \cup N} I_i\},$$

$$\mathbb{I} = \{(x_{1'}, ..., x_{m'}, k_1, ..., k_n) \in \{\mathbf{e}^0, ..., \mathbf{e}^T\}^{m'} \times \textstyle\prod_{j \in N}\{0, ..., w_j\} :$$
$$\textstyle\sum_{i \in M} x_i + \sum_{t=1}^{T} \sum_{j \in N_t} k_j \mathbf{e}^t = \sum_{t=1}^{T} \sum_{j \in N_t} w_j \mathbf{e}^t\}.$$

This $\mathbb{M}$ is the set of all money allocations and $\mathbb{I}$ is the set of all indivisibles allocations in the market.

We then obtain the following assertion.

Theorem A. For any competitive equilibrium $(p, x, y) \in \mathbb{C}$, its allocation $(x_i;\, w_j - y_j;\, I_i - px_i;\, I_j + p_t y_j)_{i\in M, j\in N} \in \mathbb{I} \times \mathbb{M}$ is Pareto efficient.[1]

Proof. Let $(x_i'; k_j'; \delta_i; \delta_j)_{i\in M, j\in N} \in \mathbb{I} \times \mathbb{M}$ be an allocation satisfying

$$u_i(x_i', \delta_i) \geq u_i(x_i, I_i - px_i) \quad \text{for every } i \in M, \tag{A.1}$$

$$u_j(k_j', \delta_j) \geq u_j(w_j - y_j, I_j + p_t y_j) \text{ for every } j \in N_t \text{ and } t = 1, ..., T. \tag{A.2}$$

We show these inequalities hold with equalities.

Let $M_1 = \{i \in M : I_i \geq px_i'\}$ and $M_2 = \{i \in M : I_i < px_i'\}$. For buyer $i \in M_1$, the relation

$$u_i(x_i', \delta_i) \geq u_i(x_i, I_i - px_i) \geq u_i(x_i', I_i - px_i')$$

holds by Eq. (A.1) and utility maximization condition. This and Assumption A1 imply

$$\delta_i \geq I_i - px_i' \text{ for every } i \in M_1, \tag{A.3}$$

hence,

$$\textstyle\sum_{i\in M_1} \delta_i \geq \sum_{i\in M_1} [I_i - px_i']. \tag{A.4}$$

For buyer $i \in M_2$, the relation $\delta_i \geq 0 > I_i - px_i'$ holds by Eq. (A.1) and Assumption A2. Hence,

$$\textstyle\sum_{i\in M_1} \delta_i > \sum_{i\in M_1} [I_i - px_i']. \tag{A.5}$$

For $t = 1, ..., T$ and seller $j \in N_t$, the relation

$$u_j(k_j', \delta_j) \geq u_j(w_j - y_j, I_j + p_t y_j) \geq u_j(k_j', I_j + p_t(w_j - k_j'))$$

holds by Eq. (A.2) and utility maximization condition. This implies

$$\delta_j - c_j(w_j - k_j') \geq I_j + p_t(w_j - k_j') - c_j(w_j - k_j') \text{ for every } j \in N_t \ (t = 1, ..., T),$$

that is,

$$\delta_j \geq I_j + p_t(w_j - k_j') \text{ for every } j \in N_t \ (t = 1, ..., T), \tag{A.6}$$

[1] In the allocation, t satisfies $j \in N_t$.

hence,

$$\sum_{j\in N}\delta_j \geq \sum_{j\in N}[I_j + p_t(w_j - k'_j)]. \tag{A.7}$$

On the other hand, by money balance condition (the definition of $\mathbb{M}$), it holds that

$$\sum_{i\in M\cup N}\delta_i = \sum_{i\in M\cup N} I_i. \tag{A.8}$$

Furthermore, since $\sum_{i\in M} px'_i = \sum_{t=1}^{T}\sum_{j\in N_t} p_t(w_j - k'_j)$, we have

$$\sum_{i\in M\cup N} I_i = \sum_{i\in M_1}(I_i - px'_i) + \sum_{i\in M_2}(I_i - px'_i) + \sum_{t=1}^{T}\sum_{j\in N_t}[I_j + p_t(w_j - k'_j)]. \tag{A.9}$$

Eqs. A.4, 5, 7, 8 and 9 imply $M_2 = \emptyset$. Hence, Eq. A.9 can be rewritten by

$$\sum_{i\in M\cup N}\delta_i = \sum_{i\in M}(I_i - px'_i) + \sum_{t=1}^{T}\sum_{j\in N_t}[I_j + p_t(w_j - k'_j)].$$

This and Eqs. A3, 6 imply

$$\begin{aligned} \delta_i &= I_i - px'_i \text{ for every } i \in M, && \text{(A.10)} \\ \delta_j &= I_j + p_t(w_j - k'_j) \text{ for every } j \in N_t \ (t = 1, ..., T). && \text{(A.11)} \end{aligned}$$

Eqs. A.10 and 11 together with utility maximization condition imply that Eqs. A.1 and 2 hold with equalities. ∎

We can easily show that a maximum (minimum) competitive allocation is seller-optimal (buyer-optimal) Pareto efficient.

Appendix B

Proofs for Chapter 3

B.1 Proof of Theorem 3.2.3

To prove Theorem 3.2.3, we need the following lemma.

Lemma B.1.1. Let (p, x, y) and (p', x', y') be any competitive equilibria and suppose that there is $i \in M$ such that $x_i = \mathbf{e}^k$ and $x'_i = \mathbf{e}^l$, $k \neq l$. Then, $p_k \lesseqqgtr p'_k$ if and only if $p_l \lesseqqgtr p'_l$.

The proof of Lemma B.1.1 needs the following lemma by Sai (2014), p. 45.

Lemma B.1.2 (Sai (2014)). Let (p, x, y) and (p', x', y') be any competitive equilibria and let k be an integer with $1 \leq k \leq T$. Then, $p_k < p'_k$ implies $y_k = y'_k$.

Proof of Lemma B.1.1. (*If part of '<'*) Suppose $p_l < p'_l$. It follows from the supposition and UM of Definition 3.2.1.(1), $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^l, I_i - p_l) > u_i(\mathbf{e}^l, I_i - p'_l) \geq u_i(\mathbf{e}^k, I_i - p'_k)$. Thus we have $u_i(\mathbf{e}^k, I_i - p_k) > u_i(\mathbf{e}^k, I_i - p'_k)$, which implies $p_k < p'_k$. ∎

(*Only if of '<'*) Suppose $p_k < p'_k$. Suppose, on the contrary, $p_l \geq p'_l$. By lemma B.1.2, it holds that $y_k = y'_k$. On the other hand, in equilibrium (p', x', y'), one household i switches his housing choice from k to l. This implies that at least one household $j (\neq i)$ switches his housing choice from $m (\neq k)$ to k. By the supposition and UM, $u_j(\mathbf{e}^m, I_j - p_m) \geq u_j(\mathbf{e}^k, I_j - p_k)$ $> u_j(\mathbf{e}^k, I_j - p'_k) \geq u_j(\mathbf{e}^m, I_j - p'_m)$. This inequality derives $p_m < p'_m$, which implies $m \neq l$. In the same manner with the above discussion, $p_m < p'_m$ implies $y_m = y'_m$, and in equilibrium (p', x', y'), at least one household switches his choice from $n (\neq m)$ to m. This also derives $p_n < p'_n$, $n \neq l$ and $y_n = y'_n$, so this process continues. Since M is finite, this process does not finish even with all the possible household switched. This implies the hypothesis $p_l \geq p'_l$ is false. Thus, we obtain $p_l < p'_l$. ∎

(*If of '='*) Suppose $p_l = p'_l$. By only if part of '$<$', it is enough to show that $p_k \leq p'_k$. It follows from the supposition and UM, $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^l, I_i - p_l) = u_i(\mathbf{e}^l, I_i - p'_l) \geq u_i(\mathbf{e}^k, I_i - p'_k)$. Thus we have $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^k, I_i - p'_k)$, which implies $p_k \leq p'_k$. ∎

(*Only if of '='*): Suppose $p_k = p'_k$. By if part of '$<$', it is enough to show that $p_l \leq p'_l$. Suppose, on the contrary, $p_l > p'_l$. By lemma B.1.2, it holds that $y_l = y'_l$. On the other hand, in equilibrium (p', x', y'), one household i switches his housing choice from k to l. This implies that at least one household $j (\neq i)$ switches his housing choice from l to $m (\neq l)$. By the supposition and UM,$u_j(\mathbf{e}^m, I_j - p'_m) \geq u_j(\mathbf{e}^l, I_j - p'_l) > u_j(\mathbf{e}^l, I_j - p_l) \geq u_j(\mathbf{e}^m, I_j - p_m)$. This inequality derives $p_m > p'_m$, which implies $m \neq k$. In the same manner with the above discussion, $p_m > p'_m$ implies $y_m = y'_m$, and in equilibrium (p', x', y'), at least one household switches his choice from m to $n (\neq m)$. This also derives $p_n > p'_n$, $n \neq k$ and $y_n = y'_n$, so this process continues. Since M is finite, the process does not finish even with all the possible household switched. This implies the hypothesis $p_l > p'_l$ is false Thus, we obtain $p_l \leq p'_l$. ∎

(*If and only if of '>'*): It is immediately derived from "$p_k \leq p'_k$ if and only if $p_l \leq p'_l$. ∎

Proof of Theorem 3.2.3. Let (p, x, y) and (p', x', y') be any competitive equilibria and suppose that $p'_k < p_k$ and $p'_l > p_l$ for some k, l. Then we construct a tuple $(\underline{p}, \underline{x}, \underline{y})$ such that
(m-1): $\underline{p}_k = \min\{p_k, p'_k\}$ for k with $1 \leq k \leq T$;

$$\text{(m-2): for each } i \in M,\ \underline{x}_i = \begin{cases} x_i \text{ if } x_i = \mathbf{e}^k \text{ and } p_k \leq p'_k \text{ for some } k \text{ with } 1 \leq k \leq T, \\ x'_i \text{ if } x'_i = \mathbf{e}^k \text{ and } p_k > p'_k \text{ for some } k \text{ with } 1 \leq k \leq T, \\ \mathbf{0} \qquad \text{otherwise;} \end{cases}$$

(m-3): for $k = 1, \cdots, T$, $\underline{y}_k = y_k$.

Note that the above $\underline{x}$ is well defined: indeed, by Lemma B.1.1, each $i \in M$ chooses at most one category k in $\underline{x}$. In the following, we show that a tuple $(\underline{p}, \underline{x}, \underline{y})$ satisfies competitive equilibrium conditions (UM, PM and BDS).

UM: Let $i \in M$. There are the following three cases.
(Case 1): $\underline{x}_i = x_i = \mathbf{e}^k$. By (m-1), we have the equality $\underline{p}_k = p_k$. It is straightforward that $u_i(\mathbf{e}^k, I_i - \underline{p}_k) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p_m$. Let l be the category which household i chooses in (p', x', y'). By Lemma B.1.1, we have $p_l \leq p'_l$. This inequality together with UM imply $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^l, I_i - p_l) \geq u_i(\mathbf{e}^l, I_i - p'_l) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p'_m$. Thus $(\underline{p}, \underline{x})$ satisfies UM.

(Case 2): $\underline{x}_i = x'_i = \mathbf{e}^k$. By (m-1), we have the equality $\underline{p}_k = p'_k$. It is straightforward that $u_i(\mathbf{e}^k, I_i - \underline{p}_k) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p'_m$. Let l be the category which household i chooses in (p, x, y). By Lemma B.1.1, we have $p_l > p'_l$. This inequality together with UM imply $u_i(\mathbf{e}^k, I_i - p'_k) \geq u_i(\mathbf{e}^l, I_i - p'_l) > u_i(\mathbf{e}^l, I_i - p_l) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p_m$. Thus $(\underline{p}, \underline{x})$ satisfies UM.

(Case 3): $\underline{x}_i = \mathbf{0}$. by (m-2), we have $x_i = x'_i = \mathbf{0}$. Thus $(\underline{p}, \underline{x})$ satisfies UM.

PM and BDS: Let $k \in \mathbb{Z}_+$ with $1 \leq k \leq T$. If $\underline{p}_k = p_k$, landlord k maximizes his profit with production $\underline{y}_k = y_k$. By (m-2), $\underline{x}_i = x_i = \mathbf{e}^k$ for all $i \in M_k$. This implies $\sum_{i \in M_k} \underline{x}_i = \sum_{i \in M_k} x_i = y_k \mathbf{e}^k = \underline{y}_k \mathbf{e}^k$, that is, BDS holds for category k. Otherwise ($\underline{p}_k = p'_k$), the landlord k maximizes his profit with production $\underline{y}_k = y_k = y'_k$ (by Lemma B.1.2). The balance of total demand and supply is inherited from the equilibrium (p', x', y').

The vector $\underline{p}$ satisfies $\underline{p} \leq p$ and $\underline{p} \leq p'$. Since the set of competitive rent vectors is a compact set, there is the minimum competitive rent vector in the market (M, N). In the dual manner, we can also prove the existence of the maximum competitive rent vector. ∎

B.2 Proof of Theorem 3.3.3

Proof of (1). We proof this by mathematical induction over $k = f - 1,\ f - 2\ , ...,\ 1$. Let $k = f - 1$. By utility maximization condition and the upper rent equation (2.1), we have $u(\mathbf{e}^{f-1}, I_{G(f-1)} - p_{f-1}) \geq u(\mathbf{e}^f, I_{G(f-1)} - p_f)$ and $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1}) = u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f)$. Thus, by the condition $\overline{r}_f = p_f$, we have $u(\mathbf{e}^{f-1}, I_{G(f-1)} - p_{f-1}) \geq u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1})$. This imply $I_{G(f-1)} - p_{f-1} \geq I_{G(f-1)} - \overline{r}_{f-1}$, that is, $\overline{r}_{f-1} \geq p_{f-1}$. Suppose that for $k = l$ with $1 < l \leq f - 1$, the inequality $\overline{r}_l \geq p_l$ holds. Let $k = l - 1$. By utility maximization condition and Eq. (2.1), we have $u(\mathbf{e}^{l-1}, I_{G(l-1)} - p_{l-1}) \geq u(\mathbf{e}^l, I_{G(l-1)} - p_l)$ and $u(\mathbf{e}^{l-1}, I_{G(l-1)} - \overline{r}_{l-1}) = u(\mathbf{e}^l, I_{G(l-1)} - \overline{r}_l)$. On the other hand, $\overline{r}_l \geq p_l$ and Assumption A imply $u(\mathbf{e}^l, I_{G(l-1)} - p_l) \geq u(\mathbf{e}^l, I_{G(l-1)} - \overline{r}_l)$. This inequality together with previous inequalities imply $u(\mathbf{e}^{l-1}, I_{G(l-1)} - p_{l-1}) \geq u(\mathbf{e}^{l-1}, I_{G(l-1)} - \overline{r}_{l-1})$. This and Assumption A imply $I_{G(l-1)} - p_{l-1} \geq I_{G(l-1)} - \overline{r}_{l-1}$, that is, $\overline{r}_{l-1} \geq p_{l-1}$. Therefore we have $\overline{r}_k \geq p_k$ for all k with $1 \leq k \leq f - 1$. ∎

Proof of (2). It is proved by the dual manner with (1). Let $k = f - 1$. By utility maximization condition and the lower rent equation (2.2), we have $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - p_{f-1}) \leq u(\mathbf{e}^f, I_{G(f-1)+1} - p_f)$ and $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) = u(\mathbf{e}^f, I_{G(f-1)+1} - \underline{r}_f)$. These together with $\underline{r}_f = p_f$ imply $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - p_{f-1}) \leq u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1})$. Thus, we have $\underline{r}_{f-1} \leq p_{f-1}$. Suppose that for $k = l, 1 < l \leq f-1$, $\underline{r}_l \geq p_l$ and let $k = l-1$. By utility maximization condition and (2.2), we have $u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - p_{l-1}) \leq u(\mathbf{e}^l, I_{G(l-1)+1} - p_l)$ and $u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - \underline{r}_{l-1}) = u(\mathbf{e}^l, I_{G(l-1)+1} - \underline{r}_l)$. On the other hand, $\underline{r}_l \leq p_l$ and Assumption A imply $u(\mathbf{e}^l, I_{G(l-1)+1} - p_l) \leq u(\mathbf{e}^l, I_{G(l-1)+1} - \underline{r}_l)$. This inequality together with previous inequalities imply $u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - p_{l-1}) \leq u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - \underline{r}_{l-1})$. Thus, we have $\underline{r}_{l-1} \leq p_{l-1}$. Therefore we have $\underline{r}_k \leq p_k$ for all k with $1 \leq k \leq f-1$. ∎

B.3 Proof of Theorem 3.3.4.(2)

We first show the following lemma.

Lemma B.3.1 Let (p, x, y) be a maximal competitive equilibrium. For each category $k = 1, ..., f-1$, there exist households $i, j \in M$ such that $x_i = \mathbf{e}^k$, $I_i = I_{G(k)}$ and $x_j = \mathbf{e}^{k+1}$, $I_j = I_{G(k+1)}$.

Proof. This is immediately proved from Lemma 3.3.1.(2). ∎

Proof of Theorem 3.3.4.(2). By Lemma B.3.1, there exist households i, j such that $x_i = \mathbf{e}^k$, $I_i = I_{G(k)}$ and $x_j = \mathbf{e}^{k+1}$, $I_j = I_{G(k+1)}$. utility maximization condition for households $i = G(k)$ and $j = G(k+1)$, it holds that

$$
\begin{aligned}
u(\mathbf{e}^k, I_{G(k)} - p_k) &\geq u(\mathbf{e}^{k+1}, I_{G(k)} - p_{k+1}) \text{ and} \\
u(\mathbf{e}^{k+1}, I_{G(k)+1} - p_{k+1}) &\geq u(\mathbf{e}^k, I_{G(k)+1} - p_k).
\end{aligned}
$$

Suppose that condition (i) of Theorem 3.3.4.(2) holds. Then, by the above inequalities, we have

$$u(\mathbf{e}^k, I_{G(k)+1} - p_k) = u(\mathbf{e}^{k+1}, I_{G(k)+1} - p_{k+1}),$$

that is, the lower rent equation holds.

Suppose that condition (ii) of Theorem 3.3.4.(2) holds. We prove by contradiction. Suppose that there is a category t with $1 \leq t \leq f-1$ such that

$$u(\mathbf{e}^k, I_{G(k)+1} - p_k) = u(\mathbf{e}^{k+1}, I_{G(k)+1} - p_{k+1}) \text{ for } k = 1, ..., t-1;$$

$$u(\mathbf{e}^t, I_{G(t)+1} - p_t) < u(\mathbf{e}^{t+1}, I_{G(t)+1} - p_{t+1}).$$

Then, we can decrease p_t and p_{t-1}, ..., p_1 slightly into p'_t and p'_{t-1}, ..., p'_1 such that

$$\begin{aligned} u(\mathbf{e}^t, I_{G(t)+1} - p'_t) &< u(\mathbf{e}^{t+1}, I_{G(t)+1} - p_{t+1}); \\ p'_t &> C_t(y_t) - C_t(y_t - 1). \end{aligned} \tag{B.1}$$

$$\begin{aligned} &u(\mathbf{e}^k, I_{G(k)+1} - p'_k) = u(\mathbf{e}^{k+1}, I_{G(k)+1} - p'_{k+1}) \text{ and} \\ &p'_k > C_k(y_k) - C_k(y_k - 1) \text{ for } k =, 1, ..., t-1. \end{aligned} \tag{B.2}$$

We now let the new rent vector p^* as

$$p^*_k = \begin{cases} p_k \text{ for } k \geq t+1; \\ p'_k \text{ for } k \leq t. \end{cases}$$

In the following, we show a tuple (p^*, x, y) is also a competitive rent vector: this is a contradictory claim since p is the minimum competitive rent vector. Since (x, y) is a competitive allocation, the balance of total supply and demand condition is satisfied. Furthermore, by the bottoms of Eqs. (B.1) and (B.2), each landlord's profit maximization condition holds with (p^*, y). Utility maximization condition of households is checked by as follows. Let $i \in M$ with $x_i = \mathbf{e}^k$. We easily find $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^{k'}, I_i - p^*_{k'})$ for price unchanged categories $k' = t+1, ..., T$. The remaining part is shown by the following case analysis:

(i) The case of $k \geq t+1$. By the definition of $G(k)$, we have $I_i \leq I_{G(t)+1}$. This together with the top of Eq. (B.1) and Assumption D imply $u(\mathbf{e}^k, I_i - p^*_k) > u(\mathbf{e}^t, I_i - p^*_t)$. Furthermore, this inequality together with the top of Eq. (B.2) and Assumption D imply $u(\mathbf{e}^k, I_i - p^*_k) > u(\mathbf{e}^t, I_i - p^*_t) \geq u(\mathbf{e}^{t-1}, I_i - p^*_{t-1}) \geq \cdots \geq u(\mathbf{e}^1, I_i - p^*_1)$.

(ii) The case of $k < t+1$. Let k' with $k < k' < t+1$. By the definition of $G(k)$, we have $I_i \geq I_{G(k)} \geq I_{G(k)+1}$. This together with Eq. (B.2) and Assumption D imply $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^{k'}, I_i - p^*_{k'})$. Furthermore let k'' with $1 < k'' < k$. By the definition of $G(k)$, we have $I_i \leq I_{G(k-1)+1}$. This together with Eq. (B.2) and Assumption D imply $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^{k''}, I_i - p^*_{k''})$. Combining them, we have $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^l, I_i - p^*_l)$ for all $l = 1, ..., t$. ∎

Appendix C

Additional Results for Chapter 4

C.1 Income inequality and equitability of competitive allocations for households

In this appendix, we briefly mention equitability of competitive allocations in our rental housing market model. As stated in Chapter 1, Svensson (1983), Alkan, Demange and Gale (1991) and Sakai (2007) studied equitability properties in the market with indivisibilities. Their model is different to ours in that they assumed only buyers, the same number of buyers and indivisible units, no initial endowments, and without homogeneous utility function.

We give some notations (definitions are due to Foley, 1967; Varian, 1974). Recall that the consumption set of households are given by $X = \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$. Let an m-tuple $a = (a_1, ..., a_m) \in X^m$ be a consumption allocation. We say that i *envies* j at $a \in X^m$ iff $u(a_j) > u(a_i)$. We say that $a \in X^m$ is the *equitable* (*envy-free*) *allocation* iff $u(a_i) \geq u(a_j)$ for every $i, j \in M$. Note that in our framework, this condition can be translated by $u(a_i) = u(a_j)$ for every $i, j \in M$.

The following proposition holds in our model.

Proposition C.1. *Let (p, x, y) be a competitive equilibrium and let $i, j \in M$. Then, $I_i \geq I_j$ if and only if $u(x_i, I_i - px_i) \geq u(x_j, I_j - px_j)$ (note that $\geq$ is replaced by $\leq, >, <$, or $=$).*

Proof. (*Only If*) By the antecedent $I_i > I_j$ and utility maximization condition, we have $u(x_i, I_i - px_i) \geq u(x_j, I_i - px_j) > u(x_j, I_j - px_j)$. (*If*) Suppose, on the contrary, $I_i \leq I_j$. Then, we obtain the contradictory inequality by utility maximization condition: $u(x_j, I_j - px_j) \geq u(x_i, I_j - px_i) \geq u(x_i, I_i - px_i)$. ∎

This proposition means that if there exist two households having different incomes, then the lower-income household envies the higher-income household in any competitive allocations; conversely, if some household envies the other in a competitive allocation, then the income of the envied household is higher. Furthermore, if incomes of some two households are the same, then their utility levels also the same in any competitive allocations; conversely, if utility levels of some two households are the same in a competitive allocation, then their incomes also the same.

The following corollary follows from the proposition.

Corollary C.2. *Let (p, x, y) be a competitive equilibrium. Then every household has the same income if and only if an m-tuple $((x_1, I_1 - px_1), ..., (x_m, I_m - px_m))$ is an equitable allocation.*

Thus, when the household income distribution has even a little inequality, any competitive allocation does not satisfies equitability (conversely, if a competitive allocation does not satisfies equitability, then the income distribution has an inequality). Theorems 4.3.1,2 and Corollary C.2 imply that rising income inequality tends to cause both dampening the equitability on household allocations and a decline in landlord revenues. Note that since any competitive equilibrium is Pareto efficient, an equitable competitive allocation is a fair allocation.[1] Note also that the only-if part of the corollary holds without identical utility function assumption, whereas the if part does not holds without this assumption. The next example shows a case that income inequality exists but a competitive allocation satisfies equitability.

Example C.3 (*Equitable competitive equilibrium with income inequality exists*). Suppose that there are two households 1 and 2 with incomes $I_1 = 150$ and $I_2 = 100$, two different apartments 1 and 2 (with reservation prices 50 and 36). Suppose that their utility functions are given as

$$u_1(\mathbf{e}^k, c) = \begin{cases} 0 + \sqrt{c} \text{ for } k = 0, \\ 4 + \sqrt{c} \text{ for } k = 1, \\ 1 + \sqrt{c} \text{ for } k = 2, \end{cases} \quad u_2(\mathbf{e}^k, c) = \begin{cases} 0 + \sqrt{c} \text{ for } k = 0, \\ 1 + \sqrt{c} \text{ for } k = 1, \\ 4 + \sqrt{c} \text{ for } k = 2. \end{cases}$$

[1] Svensson (1983) and Sakai (2007) gave a result related to Corollary ??.2. According to them, a consumption allocation $((x_1, c_1), ..., (x_m, c_m)) \in X^m$ is a *Walrasian allocation from equal income* iff there exist $p \in \mathbb{R}^T_+$ and $I \in \mathbb{R}_+$ such that $c_i = I - px_i$ for all $i \in M$ and every household maximizes his utility, where I is the implicit imcome. They showed that the set of equitable allocations coincides with the set of Walrasian allocations from equal income.

This setting explains, for example, the following situation: the apartment 1 is a relatively large one located in a suburban area and the apartment 2 is a small one located in a central city. Household 1 with higher income prefers the apartment 1 to 2, while the household 2 prefers the apartment 2 to 1.

Let $p = (p_1, p_2) = (50, 36)$. Then, $u_1(\mathbf{e}^1, I_1 - p_1) = 14 > u_1(\mathbf{e}^0, I_1) > u_1(\mathbf{e}^2, I_1 - p_2)$ and $u_2(\mathbf{e}^2, I_2 - p_2) = 12 > u_2(\mathbf{e}^0, I_1) > u_2(\mathbf{e}^1, I_1 - p_1)$. Hence, a triple $(p, (\mathbf{e}^1, \mathbf{e}^2), (1, 1))$ is a competitive equilibrium. On the other hand, $u_1(\mathbf{e}^1, I_1 - p_1) = 14 > u_1(\mathbf{e}^2, I_2 - p_2) = 9$ and $u_2(\mathbf{e}^2, I_2 - p_2) = 12 > u_2(\mathbf{e}^1, I_1 - p_1) = 11$. Hence, this equilibrium satisfies equitability but income inequality exists.

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