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Queueing network model for obstetric patient flow in a hospital

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Abstract A queueing network is used to model the flow of patients in a hospital using the observed admission rate of patients and the histogram for the length of stay for patients in each ward. A complete log of orders for every movement of all patients from room to room covering two years was provided to us by the Medical Information Department of the University of Tsukuba Hospital in Japan. We focused on obstetric patients, who are generally hospitalized at random times in a day, and we analyzed the patient flow probabilistically. On admission, each obstetric patient is assigned to a bed in one of the two wards: one for normal delivery and the other for high-risk delivery. Then, the patient may be transferred between the two wards before discharge. We confirm Little's law of queueing theory for the patient flow in each ward. Next, we propose a new network model of M/G/∞ and M/M/m queues to represent the flow of these patients, which is used to predict the probability distribution for the number of patients staying in each ward at the nightly census time. Although our model is a very rough and simplistic approximation of the real patient flow, the predicted probability distribution shows good agreement with the observed data. The proposed method can be used for capacity planning of hospital wards to predict future patient load in each ward.

Keywords Operations research in healthcare · obstetrics · patient flow · Poisson process · Little's law · queueing network · capacity planning

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1 Introduction

The University of Tsukuba Hospital (UTH) is affiliated to the University of Tsukuba, which is located approximately 40 miles northeast of Tokyo, Japan [24]. It is esteemed as a specific functional hospital in the Ibaraki prefecture. Our team, which consisted of hospital staff and OR/MS researchers in the engineering department, conducted a healthcare service innovation project from April 2011 to March 2014. The goal of our project was to develop a web-based software system for controlling admission and bed allocation of all patients using mathematical optimization. In addition, we analyzed the flow of patients from admission to discharge within the hospital using system-scientific techniques [23].

The Medical Information Department of UTH provided us with a complete log of orders for every movement of all patients (with encrypted IDs) over a period of two fiscal years 2010 and 2011. In particular, the logs were collected from the night of April 1 2010 to the night of March 31 2012. Orders for patient movements include admission, discharge, ward transfer, room transfer in the same ward, clinical group transfer, overnight stay outside hospital, and so on. However, they do not include details such as information about the clinical treatment or the condition of the patient. The statistical treatment of these log data with the resulting publication of research findings in our project was approved by the Research Ethics Committee of UTH. In this paper, we focus on the movements of obstetric patients who are generally hospitalized at random times through the year, unlike patients for cardiovascular or brain surgeries, who have a more controlled admission.

Basic performance measures in patient management in a hospital include the *bed utilization* and the mean *length-of-stay* (LoS) per patient. Bed utilization is calculated as the ratio of the mean number of patients staying in each clinical unit or in the whole hospital to the total number of beds in the unit or in the hospital, respectively. The mean LoS is obtained from the hospitalization record of each patient. Alternatively, it can be calculated as the ratio of the mean number of patients present in the facility each day to the patient admission rate obtained using Little's law from the queueing theory of operations research. Taking this a step forward, we propose an approximate queueing network model for patient flow. Using our model, we can predict the probability distribution for the number of patients staying in each ward at the midnight census time using the given number of beds in the ward, the observed rate of patient admissions, and the histogram of LoS for each patient. Our approximation technique is validated against the data observed during the two years. In spite of very rough approximations, we can "explain" the patient flow with acceptable accuracy. Our method can be used for capacity planning of hospital wards to predict future patient load in each ward.

Preliminary reports of the present study are included in the Proceedings of the 2013 Workshop on Visual Analytics in Healthcare [19] and 2014 SRII Global Conference [22]. See also Takagi [21] for the implication of the present study in the framework of *service science*.

Extensive work has been carried out on the application of system-scientific approaches, in particular, queueing theory, to the patient flow in hospitals. For example, surveys [6, 7, 10, 20] and original papers [5, 14–17, 26–28]. In particular, queueing network models have been proposed and studied by Hershey et al. [12, 25]. Among others, the patients in the obstetric and neonatal units are studied in [1, 8]. Other techniques may be discrete-event simulations of semi-Markov processes [4, 13] and agent-based simulations [2]. A book by Hall [11] introduces general queueing-theoretic methods in detail for a hospital, which is a typical service system.

The rest of this paper is organized as follows. In Section 2, we present the formulation and observed data related to the number and LoS of all obstetric patients in the wards of UTH during the two years 2010–2011. These data are analyzed in the framework of Little's law. In Section 3, a new queueing network model for the dominant routes of obstetric patients is proposed. The assumptions of the Poisson arrival process and general or exponential distributions for the LoS used in the model are validated against the observed data using the chi-square test for "goodness-of-fit". In Section 4, we calculate the distributions of the number of patients staying in each ward as well as in the whole hospital per day during the two years 2010–2011 based on the queueing network model. Further, a detailed comparison is drawn between the theoretical results and observed values. We conclude the paper with a brief remark on the success and difficulty in the modeling and a plan for future research in Section 5. The appendix includes the basic formulas for two relevant queueing models $\mathrm{M/G}/\infty$ and $\mathrm{M/M}/m$.

2 Formulation of the obstetric patient flow

The obstetric unit of UTH is called the *Center for Maternal, Fetal, and Neonatal Health* for treatment of normal as well as high-risk childbirth in Tsukuba and southern areas of the Ibaraki prefecture. In this section, we first present the formulation of observed data related to the number and the LoS of all obstetric patients in the wards of UTH during the two years 2010–2011. We then discuss the application of a generic, system-scientific formula called Little's law (in a finite-time domain) to the obstetric patients treated during that period. Little's law is only concerned with the mean values of the statistical quantities. Further, we show the probability distribution for the number of all obstetric patients who stayed in the wards of UTH on each day.

2.1 Wards of obstetric patients

There are two wards, numbered 300 and 30M, primarily used by the obstetric unit in the UTH (The wards in UTH moved to a new site in December 2012. All data in this paper refer to the statistics before this movement.):

- Ward 300 with 26 beds accommodates patients with normal delivery and also serves as a backup place (waiting room) for patients destined to Ward 30M.
- Ward 30M is the maternal and fetal intensive care unit (MFICU), which has six beds, for the treatment of high-risk delivery.

Some other wards are also used by obstetric patients when it is difficult to admit them in Wards 300 and 30M. Similarly, these two wards accommodate patients from other clinical units when their primary wards are overloaded.

From the order log of all patients treated in the period from April 1 2010 to March 31 2012, we extracted the orders for obstetric patients. The resulting set of orders includes those for patients hospitalized before this period and for those who remained in the hospital after this period. We excluded the orders for these patients from our study because we cannot know how many days they stayed in the hospital before or after this period. In other words, we limited ourselves to considering only those obstetric patients who were admitted on and after April 1 2010, and discharged before and on March 31 2012. Thus, we selected the set of orders for $P = 1,956$ patients. Because 2012 was a leap year, there were $T = 731$ days during the period of our study.

2.2 Routes of obstetric patient flow

We denote the sequence of wards in which a patient stays from admission to discharge as the *route* of that patient. We identified a set of $R = 31$ distinct routes for $P = 1,956$ obstetric patients under consideration (Table 1). Let J_r be the number of wards visited by patients of route r. From the data in the order log, we can calculate the following quantities for route r $(= 1,2,\ldots,R)$:

- Number of patients who take route r: $\mathcal{A}_r$
- Set of wards on route r: $W(r)$
- Number of days that patient i of route r spends (LoS) in ward w as the jth ward: $\mathrm{LoS}^w_{(r,j)}(i)$, $i = 1,2,\ldots,P$. Note that $\mathrm{LoS}^w_{(r,j)}(i) = 0$ if $w \notin W(r)$.
- Mean number of days that each patient of route r spends (LoS) in ward w as the jth ward:

$$E[\mathrm{LoS}^w_{(r,j)}] = \frac{1}{\mathcal{A}_r}\sum_{i=1}^{P}\mathrm{LoS}^w_{(r,j)}(i) \qquad (1)$$

- Patient-days of route r in ward w

$$\mathrm{PD}^w_r = \sum_{j=1}^{J_r}\sum_{i=1}^{P}\mathrm{LoS}^w_{(r,j)}(i) = \mathcal{A}_r\sum_{j=1}^{J_r}E[\mathrm{LoS}^w_{(r,j)}] \qquad (2)$$

- Patient-days of route r

$$\mathrm{PD}_r = \sum_{w\in W(r)}\mathrm{PD}^w_r \qquad (3)$$

Summing all routes, we get the total number of patients and the total patient-days as follows:

$$\mathcal{A}^{\mathrm{all}} = \sum_{r=1}^{R}\mathcal{A}_r = P \quad ; \quad \mathrm{PD}^{\mathrm{all}} = \sum_{r=1}^{R}\mathrm{PD}_r \qquad (4)$$

Table 1 shows $\mathcal{A}_r$, $\{w, E[\mathrm{LoS}^w_{(r,j)}]\}$, $j = 1,2,\ldots,J_r$, and PD_r for route r $(= 1,2,\ldots,R)$. Further, it shows that $\mathcal{A}^{\mathrm{all}} = 1,956$ patients and $\mathrm{PD}^{\mathrm{all}} = 15,712$ patient-days for the obstetric patients of all routes. In this table, we identify five dominant routes by an asterisk (*), which are selected as shown in Fig. 2 below.

2.3 Number and LoS of obstetric patients

In Table 2, we show several statistics on the obstetric patients in Ward 300, Ward 30M, and in the whole hospital (all wards) during the two years 2010–2011. In this table, for a patient assigned to Ward 300 and then transferred to Ward 30M, we count the admission to both wards as separate admissions. For the whole hospital, we count only the external arrivals once. Patients admitted and discharged on the same day are counted in the number of admitted patients with LoS of 0 days so that they do not contribute to the count of patient-days.

The numbers in Table 2 are calculated from the order log by using the following formulas for each ward w and for the whole hospital:

(1) Patient-days in ward w
- Sum of the number of days that each patient stays in ward w

$$\mathrm{PD}^w = \sum_{r=1}^{R}\mathrm{PD}^w_r = \sum_{r=1}^{R}\mathcal{A}_r\sum_{j=1}^{J_r}E[\mathrm{LoS}^w_{(r,j)}] \qquad (5)$$

(2) Number of patients staying in ward w
- Number of patients staying in ward w on the tth day: $N^w(t)$, $t = 1,2,\ldots,T$.
- Number of patients staying in the hospital on the tth day: $N^{\mathrm{all}}(t)$, $t = 1,2,\ldots,T$.
- Sum of the number of patients staying in ward w and in the hospital during the two years

$$\mathrm{PD}^w = \sum_{t=1}^{T}N^w(t) \quad ; \quad \mathrm{PD}^{\mathrm{all}} = \sum_{t=1}^{T}N^{\mathrm{all}}(t), \qquad (6)$$

which agrees with PD^w in Eq. (5) and $\mathrm{PD}^{\mathrm{all}}$ in Eq. (4), respectively.
- Mean number of patients staying in ward w and in the hospital per day

$$E[N^w] = \frac{\mathrm{PD}^w}{T} \quad ; \quad E[N^{\mathrm{all}}] = \frac{\mathrm{PD}^{\mathrm{all}}}{T} \qquad (7)$$

- *Bed utilization* of ward w

$$U^w = \frac{E[N^w]}{B^w} = \frac{\mathrm{PD}^w}{B^w T}, \qquad (8)$$

where B^w is the number of beds in ward w.

Table 1 Characteristics of the 31 routes of obstetric patients. Five routes with an asterisk (*) are the dominant routes. The annotation "(4th)", "(5th)", and "(6th)" indicates the 4th, 5th, and 6th ward, respectively, for routes 29, 30, and 31. Wards 300 (mostly for patients of normal delivery) and 30M (for patients of high-risk delivery) are the two wards of primary concern. Other wards such as 130, 401, and 501 belong to other clinical units used temporarily by obstetric patients.

Route	Number of patients	1st ward	Mean LoS	2nd ward	Mean LoS	3rd ward	Mean LoS	Patient-days
1	1	130	1.0000					1
2 *	1,582	300	6.5215					10,317
3	5	401	8.2000					41
4	2	501	16.5000					33
5	1	601	7.0000					7
6	1	830	2.0000					2
7	9	901	1.5556					14
8	7	930	2.7143					19
9 *	81	30M	7.6667					621
10	1	30	1.0000	300	4.0000			5
11	2	130	0.0000	300	5.0000			10
12	1	300	43.0000	30	1.0000			44
13	3	300	4.3333	901	5.0000			28
14	3	300	5.0000	930	4.6667			29
15 *	25	300	2.4000	30M	14.1600			414
16	1	401	1.0000	300	5.0000			6
17	1	430	2.0000	300	3.0000			5
18	2	530	1.0000	300	4.0000			10
19	4	601	0.5000	300	5.5000			24
20	4	901	0.2500	300	5.2500			22
21	3	930	6.6667	300	3.6667			31
22	1	930	1.0000	30M	12.0000			13
23 *	176	30M	10.5909	300	7.1250			3,118
24	3	300	7.0000	400	1.6667	300	3.3333	36
25	1	300	1.0000	401	5.0000	300	2.0000	8
26	1	300	12.0000	930	1.0000	300	2.0000	15
27 *	30	300	3.4000	30M	14.0667	300	6.7000	725
28	1	430	3.0000	400	2.0000	300	6.0000	11
29	2	30M	1.5000	300	12.5000	400	1.5000	
		(4th) 300	7.0000					45
30	1	300	6.0000	30M	18.0000	300	0.0000	
		(4th) 430	2.0000	(5th) 300	6.0000			32
31	1	300	5.0000	30M	4.0000	300	0.0000	
		(4th) 430	7.0000	(5th) 400	5.0000	(6th) 300	5.0000	26
Dominant	1,894							15,195
Total	1,956							15,712

(3) Number of patients arriving in ward w

- Number of patients who arrive in ward w (from outside as well as from other wards in the hospital) on the tth day: $A^w(t)$, $t = 1, 2, \ldots, T$.
- Number of patients who arrive in the hospital from outside on the tth day: $A^{\text{all}}(t)$, where only external arrivals are counted, $t = 1, 2, \ldots, T$.
- Sum of the number of patients arriving in ward w and in the hospital during the two years

$$\mathcal{A}^w = \sum_{t=1}^{T} A^w(t) \quad ; \quad \mathcal{A}^{\text{all}} = \sum_{t=1}^{T} A^{\text{all}}(t), \tag{9}$$

where $\mathcal{A}^{\text{all}}$ agrees with that in Eq. (4).

- Mean number of patients arriving in ward w and in the hospital per day (arrival rate)

$$E[A^w] = \frac{\mathcal{A}^w}{T} \quad ; \quad E[A^{\text{all}}] = \frac{\mathcal{A}^{\text{all}}}{T} \tag{10}$$

(4) Mean LoS

- Mean number of days that a patient stays in ward w and in the hospital

$$E[\text{LoS}^w] = \frac{\text{PD}^w}{\mathcal{A}^w} = \frac{E[N^w]}{E[A^w]},$$
$$E[\text{LoS}^{\text{all}}] = \frac{\text{PD}^{\text{all}}}{\mathcal{A}^{\text{all}}} = \frac{E[N^{\text{all}}]}{E[A^{\text{all}}]} \tag{11}$$

The second equalities in these equations are instances of Little's law shown below.

Table 2 Statistics on the obstetric patients in Ward 300, Ward 30M, and in the whole hospital during two years 2010–2011.

Ward	w	Ward 300	Ward 30M	All
Number of beds	B^w	26	6	
Patient-days	PD^w	12,204	3,298	15,712
Mean number of patients staying each day	$E[N^w]$	16.6949	4.5116	21.4938
Bed utilization	U^w	64.21%	75.19%	
Total number of admitted patients	$\mathcal{A}^w$	1,889	317	1,956
Patients arrival rate per day	$E[A^w]$	2.5841	0.4337	2.6758
Mean LoS in days	$E[\mathrm{LoS}^w]$	6.4606	10.4038	8.0327

2.4 Statistics on the patients in Wards 300 and 30M

Let us calculate the numbers in Table 2. First, we consider Ward 300 (mostly for patients of normal delivery) where there are $B^{300} = 26$ beds. Summing $\mathrm{LoS}^{300}_{(r,j)}(i)$ for all patients using Eq. (5), we get $\mathrm{PD}^{300} = 12,204$. Therefore, the mean number of patients staying in bed on each day is $E[N^{300}] = 12,204/731 = 16.6949$, which leads to bed utilization $U^{300} = 16.695/26 = 0.6421$. On the other hand, since $\mathcal{A}^{300} = 1,889$ patients were admitted during the two years, the arrival rate was $E[A^{300}] = 1,889/731 = 2.5841$ patients/day. Then, the mean LoS for each patient in Ward 300 is computed by Eq. (11) as $E[\mathrm{LoS}^{300}] = \mathrm{PD}^{300}/\mathcal{A}^{300} = 12,204/1,889 = 6.4606$ days. This result is reasonable because, in Japan, a mother with normal delivery usually stays for five days in a hospital after delivery. Therefore, if a baby is born on the day of admission, the mother stays for five nights. However, if a baby is born on the day following the admission, she stays for six days in the hospital. Mothers with abnormal delivery may stay a few more days in the hospital.

Similar calculation for Ward 30M (for patients of high-risk delivery) with $B^{30\mathrm{M}} = 6$ beds yields $\mathrm{PD}^{30\mathrm{M}} = 3,298$. Then, we have $E[N^{30\mathrm{M}}] = 3,298/731 = 4.5116$ and $U^{30\mathrm{M}} = 4.5116/6 = 0.7519$. From $\mathcal{A}^{30\mathrm{M}} = 317$, we get $E[A^{30\mathrm{M}}] = 317/731 = 0.4337$ and $E[\mathrm{LoS}^{30\mathrm{M}}] = 3,298/317 = 10.4038$ days. This is longer than that in Ward 300, because mothers with high-risk delivery are accommodated in Ward 30M.

For all obstetric patients, we have $\mathrm{PD}^{\mathrm{all}} = 15,712$ and $\mathcal{A}^{\mathrm{all}} = 1,956$. Then, we get $E[N^{\mathrm{all}}] = 15,712/731 = 21.4938$ and $E[A^{\mathrm{all}}] = 1,956/731 = 2.6758$, which lead to an overall $E[\mathrm{LoS}^{\mathrm{all}}] = 15,712/1,956 = 8.0327$ days.

2.5 Little's law for obstetric patients

Let us briefly refer to the application of Little's law in queueing theory to the patient flow in a clinical unit of a hospital [10]. If we write

$$L = E[N^w] \quad ; \quad \lambda = E[A^w] \quad ; \quad W = E[\mathrm{LoS}^w]$$

for ward w and

$$L = E[N^{\mathrm{all}}] \quad ; \quad \lambda = E[A^{\mathrm{all}}] \quad ; \quad W = E[\mathrm{LoS}^{\mathrm{all}}]$$

for the whole hospital, both relations in Eq. (11) are examples of Little's law [9, p.10], [11, p.30]:

$$L = \lambda W. \tag{12}$$

The condition for this law to hold for a system in an infinite-time domain is that the system is stable in the sense that the number of customers present in the system does not grow indefinitely. In such a case, Little's law refers to the relationship that holds for the long-run limits of stochastic variables in the equilibrium state. In this paper, however, we confirmed Eq. (12) only for $P = 1,956$ obstetric patients recorded in the order log during the two years 2010–2011. Thus, the confirmation of Little's law verifies the correctness of counting the number of patients each day and the LoS for each patient. Multiplying both sides of Eq. (12) by T, we have the following two expressions for patient-days:

$$\text{Patient-days} = LT = AW, \quad \text{where} \quad A = \lambda T,$$

as illustrated in Eq. (11) for ward w.

2.6 Distribution for the number of patients in wards

Table 2 shows only the mean of the number of patients staying in each ward or in the hospital on a day. However, we can also find the distribution for the number of hospitalized patients. To do so for ward w, we note that the probability distribution for the number of days on which k patients stayed in ward w during the two years 2010–2011 is given by

$$P\{N^w = k\} = \frac{1}{T}\sum_{t=1}^{T} \mathcal{I}\{N^w(t) = k\} \qquad 0 \le k \le B^w, \tag{13}$$

where $N^w(t)$ is defined in Section 2.3, and $\mathcal{I}\{\cdot\}$ is the indicator function. Similarly, we have the probability distribution for the number of days on which k patients stayed in the hospital during the two years as

$$P\{N^{\mathrm{all}} = k\} = \frac{1}{T}\sum_{t=1}^{T} \mathcal{I}\{N^{\mathrm{all}}(t) = k\} \qquad k \geq 0, \tag{14}$$

where $N^{\mathrm{all}}(t)$ is defined in Section 2.3. We note the normalization conditions for the distribution

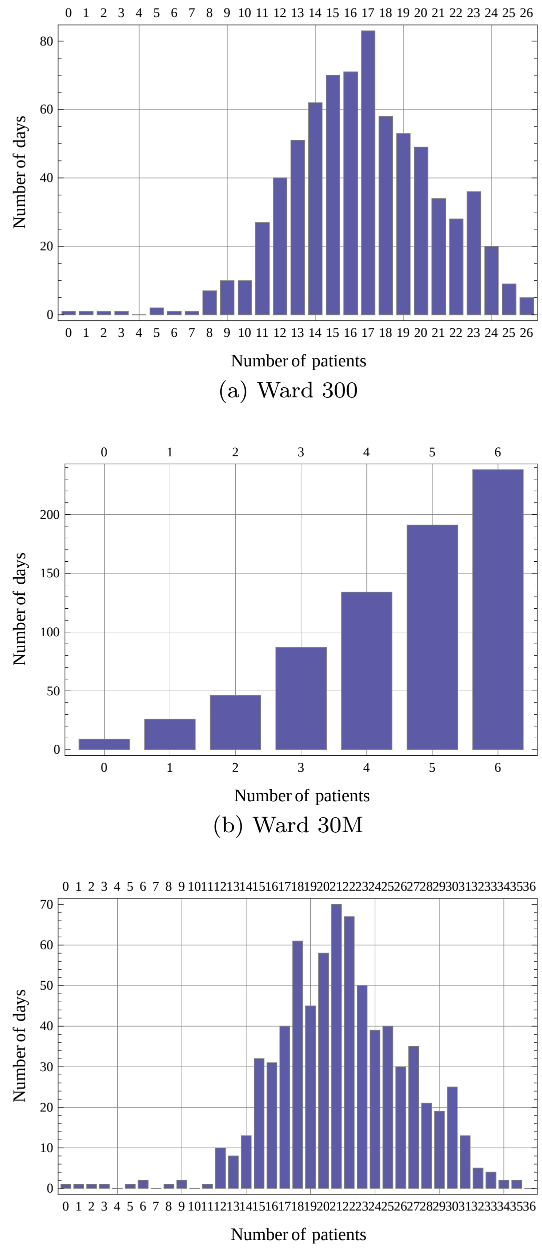
$$\sum_{k=0}^{B^w} P\{N^w = k\} = 1 \quad ; \quad \sum_{k=0}^{\infty} P\{N^{\mathrm{all}} = k\} = 1$$

From Eqs. (13) and (14), we can calculate the mean number of patients in ward w or in the hospital per day as follows:

$$E[N^w] = \sum_{k=1}^{B^w} kP\{N^w = k\};$$
$$E[N^{\mathrm{all}}] = \sum_{k=1}^{\infty} kP\{N^w = k\}, \tag{15}$$

which gives the same numerical values for $E[N^w]$ and $E[N^{\mathrm{all}}]$ as in Eq. (7).

Figure 1 shows $T \cdot P\{N^{300} = k\}$, $T \cdot P\{N^{30\mathrm{M}} = k\}$, and $T \cdot P\{N^{\mathrm{all}} = k\}$, respectively, which are the numbers of days for the number of patients who stayed in Ward 300, Ward 30M, and in the whole hospital (all wards) per day during the two years 2010–2011. These numbers have been obtained from the histogram for the number of days each patient stayed during the two years. It is interesting to observe that the distribution for Ward 300 shown in Fig. 1 (a) is almost zero up to 7, and then, it appears rather symmetric about the mean value 16.695. The likelihood that Ward 300 is fully occupied is very small; the probability that all beds in Ward 300 are occupied on an arbitrary day is $5/731 = 0.00684$, which is less than 1%. However, the distribution for Ward 30M shown in Fig. 1 (b) monotonically increases up to the capacity of six patients. The probability that all beds in Ward 30M are occupied on an arbitrary day is $238/731 = 0.3256$, which is nearly equal to one third. Figure 1 (c) shows the distribution of the number of days during the two years for the number of all obstetric patients staying on each day in the whole hospital. The characteristic of this distribution is similar to that for Ward 300 in Fig. 1 (a), as a large portion of obstetric patients stay in Ward 300.



(a) Ward 300

(b) Ward 30M

(c) All wards

Fig. 1 Distribution of the number of days during the two years 2010–2011 for the number of obstetric patients in Wards 300, 30M, and all wards per day.

3 Queueing network model of the obstetric patient flow

It is very difficult to construct a mathematically precise queueing network model to represent the entire patient flow incorporating all routes given in Table 1. However, in the present case, we can extract five dominant routes in terms of patient-days out of the 31 routes. In addition, a sophisticated model would not be needed for the practical estimation of the number of patients staying in each ward per day.

In this section, we propose the use of two conventional queueing models, M/G/∞ and M/M/m, in our queueing network model for the patient flow of five dominant routes. Naturally, the M/G/∞ queue is used

	Wards	300	30M	Total
Route 2 (1,582 patients)	300	10,317 (84.5%)	0	10,317 (65.7%)
Route 9 (81 patients)	30M	0	621 (18.8%)	621 (4.0%)
Route 15 (25 patients)	300 → 30M	60 (0.5%)	354 (10.7%)	414 (2.6%)
Route 23 (176 patients)	30M → 300	1,254 (10.3%)	1,864 (56.5%)	3,118 (19.8%)
Route 27 (30 patients)	300 → 30M → 300	303 (2.5%)	422 (12.8%)	725 (4.6%)
	5 dominant routes (1,894 patients)	11,934 (97.8%)	3,261 (98.9%)	15,195 (96.7%)
Others (62 patients)	··· 300 ··· 30M ···	270 (2.2%)	37 (1.1%)	517 (3.3%)
	All routes (1,956 patients)	12,204	3,298	15,712

Fig. 2 Dominant routes of obstetric patients in terms of patient-days.

to model Ward 300, which has a sufficient number of beds. (Fig. 1 (a) shows that there is little possibility that all beds are used in Ward 300.) A unique feature of our queueing network model is the capability to use the waiting room of the M/M/m queue to model those set of patients staying in Ward 300 while they are waiting for beds in Ward 30M. Therefore, only the service facility of the M/M/m queue is used to model Ward 30M. We examine the assumption of Poisson processes in the arrival and departure processes from the queues that are to be used to model these wards. Further, we compute the mean value and the distribution for the LoS of patients in the wards on each route. See the appendix for the formulas of queueing theory used in our analysis.

3.1 Selection of dominant routes

Let us start with selecting the dominant routes in the patient flow by means of numerical evaluation. To do so, we calculate the *load* (patient-days) for each route by summing the load of patients over all wards on that route, where the load of patients in a ward is defined as the sum of LoS for all patients who stayed in that ward on that route. The load of each route is shown in Table 1. As a result, Fig. 2 displays the load of obstetric patients on dominant routes 2, 9, 15, 23, and 27 along with the load on all other routes. Since the set of patients on these routes accounts for 96.7% of all loads, we will consider only these patients in our analysis of the obstetric patient flow in the following sections. In terms of the number of patients who take dominant routes, we consider a set of 1,894 patients, which accounts for 96.8% out of a total of 1,956 obstetric patients.

3.2 Network of queues

From the data analysis of the order log provided by the UTH, we found that a dominant portion of the obstetric patient flow consists of five routes 2, 9, 15, 23, and 27 marked by an asterisk (*) in the first column in Table 1. The patients of route 2 are those who have normal delivery. These patients stay only in Ward 300 and simply leave the hospital in approximately six days. The patients of routes 9, 15, 23, and 27 are assumed to be those with high-risk delivery. They are to be treated in Ward 30M. Upon arrival, they are admitted to Ward 30M immediately if beds are available. After giving birth, they either leave the hospital (route 9) or move to Ward 300 possibly for after-birth treatment (route 23). If there are no beds available in Ward 30M when they arrive, they are temporarily accommodated in Ward 300 until beds become available in Ward 30M. Then, they are transferred to Ward 30M. After giving birth there, they either leave the hospital (route 15) or move back to Ward 300 possibly for after-birth treatment (route 27).

Based on the above conjecture, we consider the dominant flow of the obstetric patients shown in Fig. 3, where λ_r denotes the arrival rate of route r patients, and $b^w_{(r,j)}$ denotes the LoS in days that patients of route

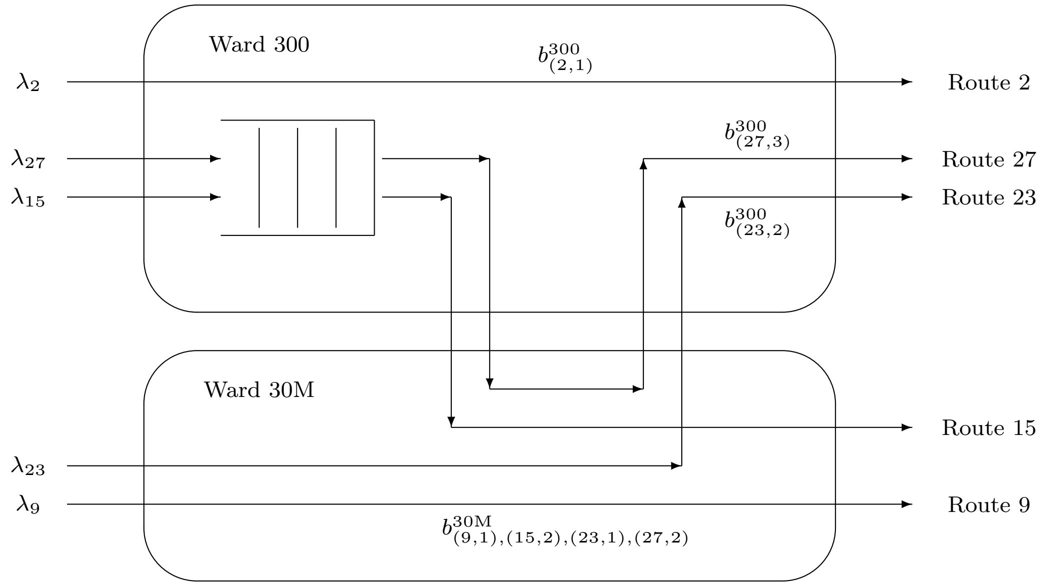


Fig. 3 Dominant flow of the obstetric patients.

r stay in ward w as the jth ward of the route. Numerical values for the parameters of arrival rates and LoS are provided from the order log. This flow is mapped to a queueing network model consisting of an M/G/∞ queue and an M/M/6 queue shown in Fig. 4. Ward 300 is modeled by a combination of the M/G/∞ queue and the waiting room of the M/M/6 queue, while Ward 30M is modeled by the service facility of the M/M/6 queue.

Our model is approximate from a queueing-theoretic viewpoint in the following sense:

- The stochastic process for the patient flow under study is essentially a discrete-time system in which arrivals, departures, and LoS are recorded on a daily basis, while both M/G/∞ and M/M/m queues work in the continuous-time framework.
- The movement of patients of routes 15 and 27 from Ward 300 to Ward 30M, i.e., from the waiting room to the service facility in the same M/M/m queue, is not a Poisson process even in the continuous-time framework.

On the other hand, the residence of patients in each ward is treated as if they were independent while the entrance of patients of route 27 patients twice into Ward 300 clearly violates this assumption. However, such treatment has been proven valid in the continuous-time framework of a Jackson-type queueing network with product-form solution [9, p.210].

Considering these pros and cons, we propose a queueing network model of M/G/∞ and M/M/m queues because of its simplicity in modeling and computation in addition to the belief that mathematical rigor should not be overly expected for practical purposes. The justification of our method is partly provided from the good agreement of the calculated results with the observations as demonstrated below.

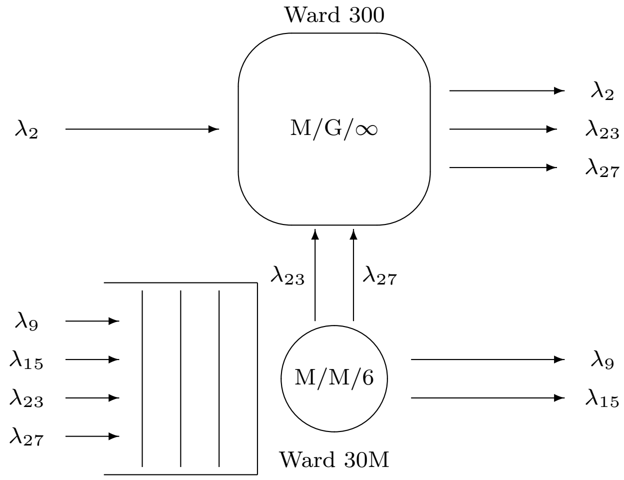


Fig. 4 Queueing network model of the obstetric patient flow.

In building a queueing network model of the obstetric patient flow, we paid special attention to routes 15 and 27 on which 55 patients were first admitted to Ward 300 and then transferred to Ward 30M. We conjecture that Ward 300 is used by patients with high-risk delivery as the "waiting room" for Ward 30M if it is crowded when they arrive. This kind of waiting period is called "administrative days" by Weiss and McClain [26], because the time is spent not for medical reasons but rather for administrative reasons.

Table 3 shows the distribution and the mean for the number of patients staying in Ward 30M exactly before the admission of patients on routes 15 and 27 to

Ward 300, and exactly before the admission of patients on routes 9 and 23 to Ward 30M. Here, the number of patients staying in Ward 30M exactly before the admission of patients consists of the following:

(i) Number of patients in Ward 30M on the preceding day of admission,
(ii) Minus the number of patients who depart from Ward 30M on the day of admission,
(iii) Plus the number of patients who are supposed to enter Ward 30M from other wards on the day of admission.

The time average for the number of patients staying in Ward 30M over the two years is 4.4610 (Section 4.2). In this table, we observe that the bed utilization in Ward 30M is rather higher on those days on which they are admitted to Ward 300 than on average days and on those days on which they are admitted to Ward 30M.

Only this table may not be enough to conclude that all patients of routes 15 and 27 are destined to Ward 30M for high-risk delivery but that they are re-directed to Ward 300 because there is no vacancy in Ward 30M at the arrival time. We are unsure why exactly patients of routes 15 and 27 stay in Ward 300 before entering Ward 30M without clinical information in the given order log. However, our conjecture of using Ward 300 as a waiting room for the intended treatment in Ward 30M was later endorsed to be true by the doctors of the Obstetrics and Gynecology Section of UTH.

3.3 Arrival processes

In order to apply the theoretical queueing network model to real patient flow, it is essential to validate the assumption of the Poisson process for the arrival flow of patients of routes (2,1) to Ward 300; patients of routes (23,2) and (27,3) to Ward 300; patients of routes (9,1), (15,1), (23,1), and (27,1) destined to Ward 30M; and all patients from outside the hospital (see Fig. 4). However, we cannot provide complete validation, because (i) the patients arrive in a discrete-time framework, and (ii) we only have the order log for a finite period of time. Therefore, we content ourselves with examining limited features of the Poisson process in the following.

(1) Poisson distribution for the number of arrivals per day

In a Poisson process, the number of events that occur during a fixed time interval has Poisson distribution. Therefore, we examine the probability distribution for the number of arrivals per day.

From the order log, we can obtain the number of patients of route r who arrive in ward w as the jth ward on the tth day, $A^w_{(r,j)}(t)$, $t = 1, 2, \dots, T$. Then, we get the probability distribution for $A^w_{(r,j)}$, the number of patients of route r who arrive there *per day*:

$$P\{A^w_{(r,j)} = k\} = \frac{1}{T}\sum_{t=1}^{T} \mathcal{I}\{A^w_{(r,j)}(t) = k\}. \tag{16}$$

The mean $E[A^w_{(r,j)}]$ (arrival rate) is given by

$$E[A^w_{(r,j)}] = \frac{1}{T}\sum_{t=1}^{T} A^w_{(r,j)}(t) = \sum_{k=1}^{\infty} kP\{A^w_{(r,j)} = k\}. \tag{17}$$

Figure 5 shows the distribution $T \cdot P\{A^w_{(r,j)} = k\}$, the number of days during the two years 2010–2011 for the number of arrivals per day for the four arrival flows. They are plotted along with the Poisson distributions with the means calculated from the respective observed data. We tested the validity of Poisson distributions against the observed values by the chi-square test using a level of significance $p = 0.05$.

Hereafter, we use the notation $A^{w,w',\dots}_{(r,j),\{(r',j'),(r'',j'')\},\dots}$ to denote the number of patients who arrive at ward w on a single route (r, j), at ward w' on multiple routes $\{(r', j'), (r'', j'')\}$, and so on *per day* such as

$$A^{w,w',\dots}_{(r,j),\{(r',j'),(r'',j'')\},\dots} = A^w_{(r,j)} + A^{w'}_{(r',j'),(r'',j'')} + \cdots .$$

For a single ward in the superscript, the braces for multiple routes in the subscript are omitted.

Figure 5 (a) plots the distribution $T \cdot P\{A^{300}_{(2,1)} = k\}$ for the number of route (2,1) patients who are admitted to Ward 300 per day. The mean of this distribution is calculated as

$$E[A^{300}_{(2,1)}] = 2.1642$$

patients per day. The maximum is seven patients per day. The Poisson distribution with mean 2.1642 is also plotted, which seems to fit with the observed distribution fairly well. The chi-square statistic is 11.6268, which is lower than 14.0671 for the degree-of-freedom 7 at the p-value 0.05. Therefore, we cannot reject the null hypothesis that the set of observed values follows the Poisson distribution at the 0.05 level of significance. In such a case, we would say that the agreement is very good.

Figure 5 (b) plots the distribution $T \cdot P\{A^{300}_{(23,2),(27,3)} = k\}$ for the number of routes (23,2) and (27,3) patients moving from Ward 30M to Ward 300 with the mean

$$E[A^{300}_{(23,2),(27,3)}] = 0.2818$$

Table 3 Distribution and the mean for the number of patients staying in Ward 30M exactly before the admission of patients on routes 15 and 27 to Ward 300 and exactly before the admission of patients on routes 9 and 23 to Ward 30M.

Number of patients in 30M just before admission	0	1	2	3	4	5	6	Total	Mean
Number of patients of route 15 admitted to 300	0	0	1	0	8	6	10	25	4.96
Number of patients of route 27 admitted to 300	0	0	0	4	1	8	17	30	5.27
Number of patients of route 9 admitted to 30M	1	5	11	20	26	18	0	81	3.47
Number of patients of route 23 admitted to 30M	3	11	26	49	53	34	0	176	3.36

patients per day. The Poisson distribution with mean 0.2818 is also plotted, which again seems to fit with the observed distribution very well. The chi-square statistic is 3.6844, which is lower than 7.8147 for the degree-of-freedom 3 at the p-value 0.05.
Figure 5 (c) plots the distribution

$$T \cdot P\{A^{30\text{M}}_{(9,1),(15,1),(23,1),(27,1)} = k\}$$

for the number of routes (9,1), (15,1), (23,1), and (27,1) patients destined to Ward 30M with the mean

$$E[A^{30\text{M}}_{(9,1),(15,1),(23,1),(27,1)}] = 0.4268$$

patients per day. The Poisson distribution with mean 0.4268 is also plotted, which seems to fit with the observed distribution very well. The chi-square statistic is 5.8379, which is lower than 7.8147 for the degree-of-freedom 3 at the p-value 0.05.
Finally, Fig. 5 (d) plots the distribution

$$T \cdot P\{A^{300,30\text{M}}_{(2,1),\{(15,1),(27,1),(9,1),(23,1)\}} = k\}$$

for the number of patients of the five dominant routes admitted to Wards 300 and 30M with the mean

$$E[A^{300,30\text{M}}_{(2,1),\{(15,1),(27,1),(9,1),(23,1)\}}] = 2.5910$$

patients per day. The maximum is eight patients per day. The Poisson distribution with mean 2.5910 is also plotted, which appears to fit with the observed distribution fairly well. However, the chi-square statistic is 30.4890, which is much higher than 15.5073 for the degree-of-freedom 8 at the p-value 0.05. Therefore, we discard the assumption that the observed distribution fits the Poisson distribution. Although this result is somewhat unexpected, the assumption of Poisson distribution for all patients of the dominant routes admitted to Wards 300 and 30M is not used in our queueing network model.

(2) Independence of the numbers of arrivals on two consecutive days
In a Poisson process, the numbers of events that occur during non-overlapping time intervals are independent. Therefore, we examine the auto-correlation for the number of arrivals on two consecutive days. The mean number of patients who arrive in ward w, the jth ward of route r, on each day (arrival rate) during the two years is given in Eq. (17). We can also calculate the variance and covariance as follows:

$$\text{Var}[A^w_{(r,j)}] = \frac{1}{T-1}\sum_{t=1}^{T}\left(A^w_{(r,j)}(t) - E[A^w_{(r,j)}]\right)^2,$$

$$\text{Cov}[A^w_{(r,j)}, A^{w,*}_{(r,j)}] = \frac{1}{T-1}\sum_{t=1}^{T-1}\left(A^w_{(r,j)}(t) - E[A^w_{(r,j)}]\right) \times \left(A^w_{(r,j)}(t+1) - E[A^w_{(r,j)}]\right).$$

In addition, the auto-correlation for the number of patients who arrive in ward w, the jth ward of route r, on each day, $A^w_{(r,j)}$, and that on the next day denoted by $A^{w,*}_{(r,j)}$ is given by

$$\text{Corr}[A^w_{(r,j)}, A^{w,*}_{(r,j)}] = \frac{\text{Cov}[A^w_{(r,j)}, A^{w,*}_{(r,j)}]}{\text{Var}[A^w_{(r,j)}]}.$$

In Table 4, we show the characteristics of the external arrival processes of dominant routes (2,1), (9,1), (15,1), (23,1), and (27,1). First, we note that the mean and the variance of the number of arrivals are nearly equal in all processes. This fact agrees with a property of the Poisson distribution. Second, auto-correlations with a one-day lag in the arrival processes are mostly very small (with an exception of 0.1043 for the route (15,1)). Therefore, we judge that the numbers of arrivals on two consecutive days are almost uncorrelated.

(3) Independence of the bed utilization on a day and the number of arrivals on the next day
We are also interested in the dependence of the total number of patients who stay in a given ward w on each day and the number of patients of route r who arrive in the same ward on the next day. The number of patients staying in ward w on the tth day, $N^w(t)$, is obtained from the order log. Then, we can calculate the mean $E[N^w]$ by Eqs. (7). Similarly, we can also calculate the variance and covariance

$$\text{Var}[N^w] = \frac{1}{T-1}\sum_{t=1}^{T}(N^w(t) - E[N^w])^2,$$

Table 4 Dependency characteristics of the external arrival processes.

Wards Arriving routes	300 (2,1)	30M (9,1)	300 (15,1)	30M (23,1)	300 (27,1)
Total number of admissions	1582	81	25	176	30
Mean of admissions per day	2.1642	0.1108	0.0342	0.2408	0.0410
Variance of admissions per day	2.3730	0.1041	0.0413	0.2598	0.0531
Auto-correlation (1 day lag)	−0.0111	0.0002	0.1043	−0.0181	0.0198
Correlation with patients in 300	−0.0129	−0.0514	0.0333	−0.0638	−0.0169
Correlation with patients in 30M	0.1191	−0.0744	0.0549	−0.1672	0.1022

$$\mathrm{Cov}[N^w, A^{w,*}_{(r,j)}] = \frac{1}{T-1}\sum_{t=1}^{T-1}(N^w(t) - E[N^w]) \times \left(A^w_{(r,j)}(t+1) - E[A^w_{(r,j)}]\right).$$

We also calculate the correlation coefficient between the number of patients who stay in ward w and the number of patients who arrive there as the jth ward of route r on the next day:

$$\mathrm{Corr}[N^w, A^{w,*}_{(r,j)}] = \frac{\mathrm{Cov}[N^w, A^{w,*}_{(r,j)}]}{\sqrt{\mathrm{Var}[N^w]\mathrm{Var}[A^w_{(r,j)}]}}.$$

These correlation coefficients are also shown in Table 4. The correlation between the bed utilization on a day and the number of arrivals on the next day is rather weak in both Wards 300 and 30M. The most negative correlation coefficient (−0.1672) occurs between the number of patients staying in Ward 30M and the number of arrivals of route (23,1) patients there on the next day.

From these results, we may judge that the arrival flows for the queues in the network model can be dealt with as Poisson processes.

3.4 Departure processes

As noted in the appendix, the outputs from M/G/∞ and M/M/m queues are Poisson processes. Therefore, it is noteworthy to check if the departure flows of patients in our network model are Poisson processes.

From the order log, we can obtain the number of patients of route r who leave ward w as the jth ward on the tth day, $D^w_{(r,j)}(t)$, $t = 1, 2, \ldots, T$. Then, we get the probability distribution for $D^w_{(r,j)}$, the number of patients of route r who leave there *per day*:

$$P\{D^w_{(r,j)} = k\} = \frac{1}{T}\sum_{t=1}^{T}\mathcal{I}\{D^w_{(r,j)}(t) = k\} \tag{18}$$

with its mean

$$E[D^w_{(r,j)}] = \frac{1}{T}\sum_{t=1}^{T} D^w_{(r,j)}(t) = \sum_{k=1}^{\infty} kP\{D^w_{(r,j)} = k\}. \tag{19}$$

Again, we use the notation $D^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots}$ to denote the number of patients who depart from ward w on a single route (r,j), from ward w' on multiple routes $\{(r',j'),(r'',j'')\}$, and so on *per day* such as

$$D^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots} = D^w_{(r,j)} + D^{w'}_{(r',j'),(r'',j'')} + \cdots.$$

For a single ward in the superscript, the braces for multiple routes in the subscript are omitted.

Figure 6 shows the distribution $T \cdot P\{D^w_{(r,j)} = k\}$, the number of days during the two years 2010–2011 for the number of patients who depart from Wards 300 and 30M as well as from the hospital per day. Each of them is plotted along with the Poisson distributions with means $E[D^{300}_{(2,1),(23,2),(27,3)}] = 2.4460$, $E[D^{30\mathrm{M}}_{(9,1),(15,2)}] = 0.1450$, and

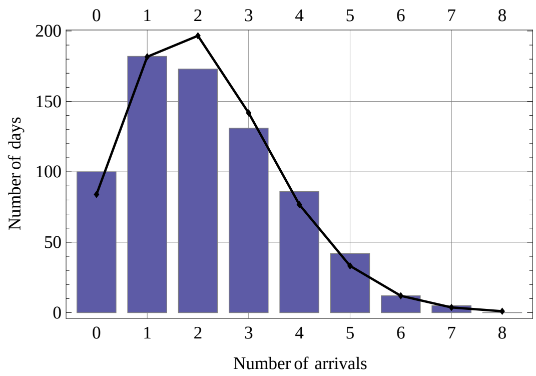
$$E[D^{300,30\mathrm{M}}_{\{(2,1),(15,1),(27,1)\},\{(9,1),(23,1)\}}] = 2.5910,$$

respectively. The agreement between the observed and theoretical values is excellent in all processes. We tested the validity of Poisson distributions against the observed values by the chi-square test at the 0.05 level of significance. The chi-square statistic values are 5.41494, 5.59214, and 8.16896, which are lower than 15.5073, 7.8147, and 15.5073 for the degree-of-freedom 8, 3, and 8 at the p-value 0.05, respectively, in Figs. 6 (a), (b), and (c).

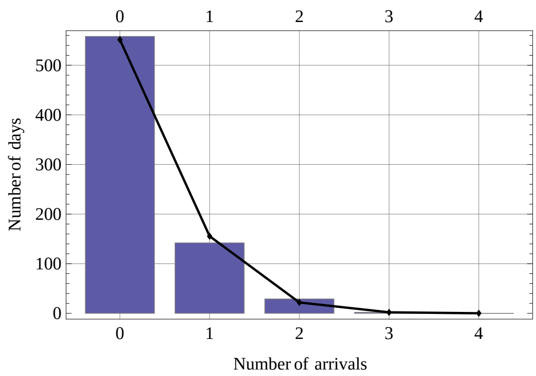
3.5 Length of stay

Finally, we observe the distribution to identify the number of days that a patient stays in each ward as well as in the hospital. From the order log, we can obtain the probability distribution for the number of days (LoS) that a patient of route r spends in ward w as the jth ward

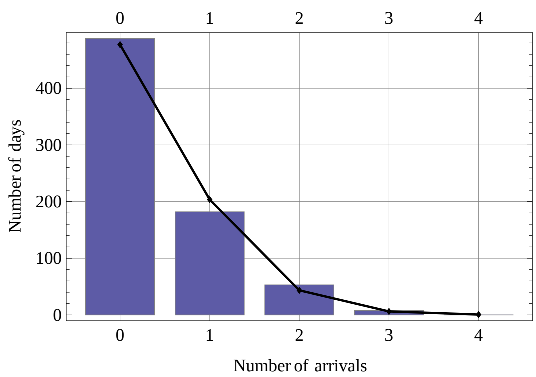
$$P\{\mathrm{LoS}^w_{(r,j)} = k\} = \frac{1}{\mathcal{A}^w_{(r,j)}}\sum_{i=1}^{P}\mathcal{I}\{\mathrm{LoS}^w_{(r,j)}(i) = k\}, \tag{20}$$



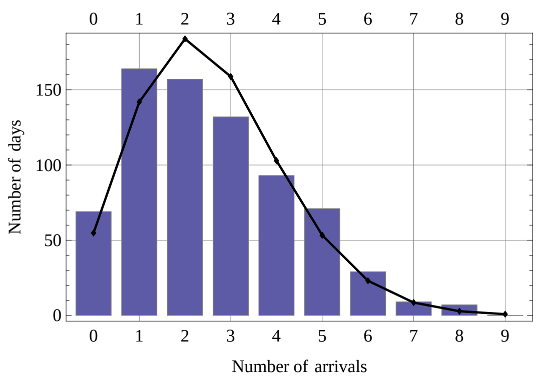
(a) Patients of route (2,1) admitted to Ward 300 and the Poisson distribution with mean 2.1642.



(b) Patients of routes (23,2) and (27,3) moving from Ward 30M to Ward 300 and the Poisson distribution with mean 0.2818.

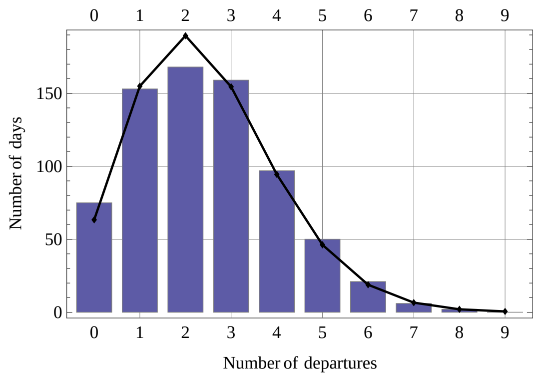


(c) Patients of routes (9,1), (15,1), (23,1), and (27,1) destined to Ward 30M and the Poisson distribution with mean 0.4268.

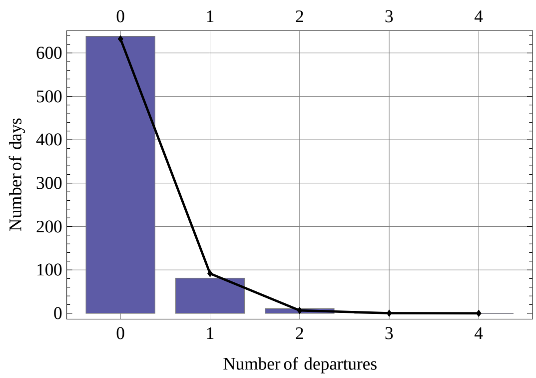


(d) All patients of the five dominant routes admitted to Wards 300 and 30M and the Poisson distribution with mean 2.5910.

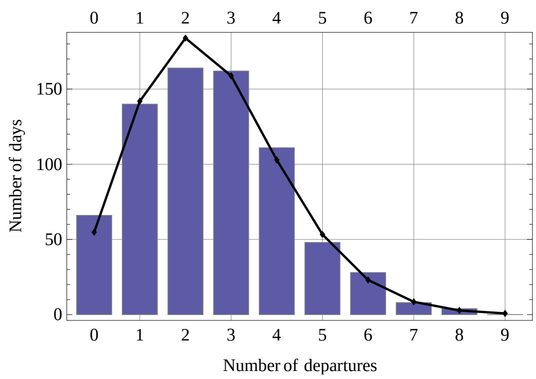
Fig. 5 Distribution of the number of days for the number of arrivals of patients per day during the two years 2010–2011.



(a) Patients of routes (2,1), (23,2), and (27,3) departing from Ward 300 and the Poisson distribution with mean 2.4460.



(b) Patients of routes (9,1) and (15,2) departing from Ward 30M and the Poisson distribution with mean 0.1450.



(c) All patients of the five dominant routes departing from Wards 300 and 30M and the Poisson distribution with mean 2.5910.

Fig. 6 Distribution of the number of days during the two years 2010–2011 for the number of patients departing from Wards 300 and 30M per day.

where

$$\mathcal{A}^w_{(r,j)} = T \cdot E[A^w_{(r,j)}] = \sum_{t=1}^{T} A^w_{(r,j)}(t) \tag{21}$$

is the number of patients of route r who arrive in ward w as the jth ward *during the two years*. The mean LoS of a patient of route r in ward w as the jth ward is given by

$$E[\mathrm{LoS}^w_{(r,j)}] = \sum_{k=1}^{\infty} kP\{\mathrm{LoS}^w_{(r,j)} = k\} = \frac{1}{\mathcal{A}^w_{(r,j)}} \sum_{i=1}^{P} \mathrm{LoS}^w_{(r,j)}(i). \quad (22)$$

Figure 7 shows the distribution for the LoS

$$\mathcal{A}^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots} \cdot P\{\mathrm{LoS}^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots} = k\}$$

of patients in ward w on a single route (r,j), in ward w' on multiple routes $\{(r',j'),(r'',j'')\}$, and so on, where

$$\mathcal{A}^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots} = \mathcal{A}^w_{(r,j)} + \mathcal{A}^{w'}_{(r',j'),(r'',j'')} + \cdots$$

and

$$\mathcal{A}^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots} \cdot E[\mathrm{LoS}^{w,w',\ldots}_{(r,j),\{(r',j'),(r'',j'')\},\ldots}] = \mathcal{A}^w_{(r,j)} \cdot E[\mathrm{LoS}^w_{(r,j)}] + \mathcal{A}^{w'}_{(r',j'),(r'',j'')} \cdot E[\mathrm{LoS}^{w'}_{(r',j'),(r'',j'')}] + \cdots .$$

Figure 7 (a) simply plots the distribution

$$\mathcal{A}^{300}_{(2,1)} \cdot P\{\mathrm{LoS}^{300}_{(2,1)} = k\}$$

for the LoS of route (2,1) patients in Ward 300. There are $\mathcal{A}^{300}_{(2,1)} = 1,582$ patients. The mean is $E[\mathrm{LoS}^{300}_{(2,1)}] = 6.5215$ days. We do not attempt to find any theoretical distribution that may fit with this distribution, because we will not need the distributional form for LoS in our modeling of route (2,1) patients by an M/G/∞ queue.

Figure 7 (b) plots the distribution

$$\mathcal{A}^{300}_{(23,2),(27,3)} \cdot P\{\mathrm{LoS}^{300}_{(23,2),(27,3)} = k\}$$

for the LoS of re-entrant patients on routes (23,2) and (27,3) in Ward 300. There are $\mathcal{A}^{300}_{(23,2),(27,3)} = 206$ patients. The mean is $E[\mathrm{LoS}^{300}_{(23,2),(27,3)}] = 7.0631$ days. Fitting a theoretical distribution is not needed for this distribution either for the same reason as above.

Figure 7 (c) plots the distribution

$$\mathcal{A}^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)} \cdot P\{\mathrm{LoS}^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)} = k\}$$

for the LoS of patients on routes (9,1), (15,2), (23,1), and (27,2) in Ward 30M. There are

$$\mathcal{A}^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)} = 312 \text{ patients.}$$

The mean is $E[\mathrm{LoS}^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}] = 10.4519$ days. The probability density function for the exponential distribution with mean 10.4519 is also plotted in the same coordinates, which appears to agree with the observed distribution very well. We tested the validity of exponential distribution against the observed values by the chi-square test at the 0.05 level of significance. The chi-square statistic is 9.3099, which is lower than 15.5073 for the degree-of-freedom 8 at the p-value 0.05. This agreement is essential to our usage of an M/M/6 queueing model for patients whose high-risk delivery is treated in Ward 30M. (We are not aware of any clinical reason for the exponential distribution of the LoS with high-risk delivery.)

Finally, Fig. 7 (d) shows the distribution

$$\mathcal{A}^{300,30\mathrm{M}}_{\{(2,1),(15,1),(27,1)\},\{(9,1),(23,1)\}} \times P\{\mathrm{LoS}^{300,30\mathrm{M}}_{\{(2,1),(15,1),(27,1)\},\{(9,1),(23,1)\}} = k\}$$

for the LoS of all obstetric patients on the five dominant routes in the whole hospital. There are

$$\mathcal{A}^{300,30\mathrm{M}}_{\{(2,1),(15,1),(27,1)\},\{(9,1),(23,1)\}} = 1,894 \text{ patients.}$$

The distribution concentrates around its mean value

$$E[\mathrm{LoS}^{300,30\mathrm{M}}_{\{(2,1),(15,1),(27,1)\},\{(9,1),(23,1)\}}] = 8.0227 \text{ days.}$$

Fitting a theoretical distribution is not needed for this distribution.

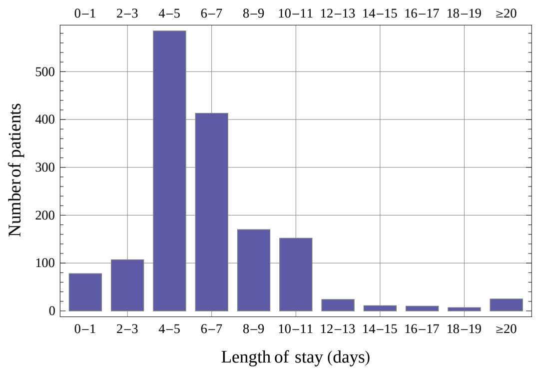
4 Prediction of the number of obstetric patients in Wards 300 and 30M

By now, we have clarified all statistical characteristics for the flow of obstetric patients of the five dominant routes in the framework of a queueing network model. Therefore, we are now in a position to use them in the calculation of the probability distribution for the number of patients staying in Wards 300 and 30M per day. The theoretical results are compared with the observed histogram in Figs. 8 and 9 for Ward 300 and in Fig. 10 for Ward 30M. We tested the fitting of theoretical distributions with the observed values by the chi-square test at the 0.05 level of significance. The agreement looks roughly acceptable in appearance in most cases, which has been confirmed by the chi-square test.

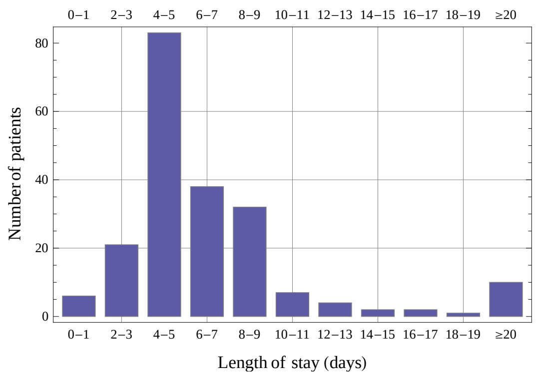
4.1 Number of patients of dominant routes in Ward 300

Let us first consider the number of patients of routes (2,1), (15,1), (23,2), (27,1), and (27,3) who stay in Ward 300 on each day. They are divided into the following three types of patients:

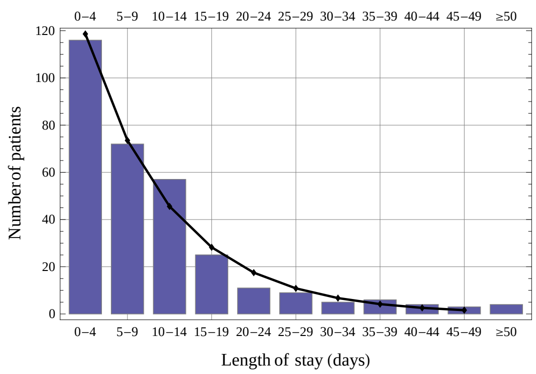
- Patients of route (2,1) going through Ward 300 only,
- Patients of routes (23,2) and (27,3) transferred from Ward 30M,
- Patients of routes (15,1) and (27,1) waiting for beds in Ward 30M.



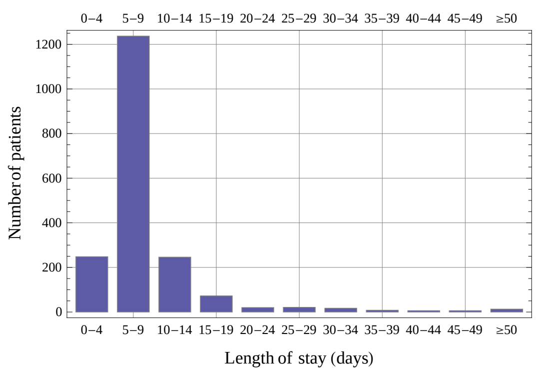
(a) Patients of route (2,1) in Ward 300 with a mean of 6.5215 days.



(b) Patients of routes (23,2) and (27,3) in Ward 300 with a mean of 7.0631 days.



(c) Patients of routes (9,1), (15,2), (23,1), and (27,2) in Ward 30M and the exponential distribution with a mean of 10.4519 days.



(d) All patients of the five dominant routes in Wards 300 and 30M with a mean of 8.0227 days.

Fig. 7 Distribution of the number of patients for the LoS (days) in Wards 300 and 30M.

Therefore, we first find the probability distribution for the number of patients in Ward 300 for each type. We then obtain the probability distribution for the total number of patients in Ward 300 by the convolution of the three distributions.

We know from Fig. 1 (a) that the probability that all beds in Ward 300 are occupied is less than 1%. Therefore, without much error, we can assume that there are a sufficient number of beds in Ward 300, which can accept all patients at any time. Thus, we will use an M/G/∞ queue to model the patient flow of routes (2,1), (23,2), and (27,3) with the observed arrival rates and mean LoS in Ward 300. Then, we apply the convolution technique to the probability distribution for the sum of independent random variables. Note that the numbers of patients of the three types are assumed to be independent of each other in the M/G/∞ model. Furthermore, the LoS of each type of patient does not need to be exponentially distributed in the M/G/∞ model. In addition, patients of routes (15,1) and (27,1) in Ward 300 are modeled by the customers in the waiting room of an M/M/6 queue (a model of Ward 30M) with the Poisson arrival process and exponentially distributed service times of the routes (9,1), (15,2), (23,1), and (27,2) patients.

Now, we use the Poisson distribution in Eq. (27) for the M/G/∞ queue as the probability distribution for the number of route (2,1) patients in Ward 300:

$$P^{300}_{(2,1)}(k) = \frac{[\rho^{300}_{(2,1)}]^k}{k!} e^{-\rho^{300}_{(2,1)}} \qquad k \geq 0, \tag{23}$$

where

$$\begin{aligned}\rho^{300}_{(2,1)} &= E[A^{300}_{(2,1)}] \cdot E[\mathrm{LoS}^{300}_{(2,1)}] = \frac{1,582}{731} \times 6.5215 \\ &= 14.1135.\end{aligned}$$

We also use Eq. (27) as the probability distribution for the number of routes (23,2) and (27,3) patients in Ward 300:

$$P^{300}_{(23,2),(27,3)}(k) = \frac{[\rho^{300}_{(23,2),(27,3)}]^k}{k!} e^{-\rho^{300}_{(23,2),(27,3)}} \quad k \geq 0, \tag{24}$$

where

$$\begin{aligned}\rho^{300}_{(23,2),(27,3)} &= E[A^{300}_{(23,2)}] \cdot E[\mathrm{LoS}^{300}_{(23,2)}] \\ &\quad + E[A^{300}_{(27,3)}] \cdot E[\mathrm{LoS}^{300}_{(27,3)}] \\ &= \frac{176}{731} \times 7.1250 + \frac{30}{731} \times 6.700 = 1.9904.\end{aligned}$$

Theoretical distribution in Eq. (23) and the observed distribution for the number of route (2,1) patients in Ward 300 are plotted in Fig. 8 (a). They seem to agree well in appearance. The chi-square statistic is taken from $k = 3$ to 26 to get a value of 18.6598, which is less than 33.9244 for the degree-of-freedom 22 at the p-value 0.05. Thus, we can say the fitting with the predicted Poisson distribution is well acceptable.

Theoretical distribution in Eq. (24) and the observed distribution for the number of routes (23,2) and (27,3) patients in Ward 300 are plotted in Fig. 8 (b), which also seem to agree well. The chi-square statistic for the range from $k = 0$ to 6 is 10.4275, which is slightly less than 11.0705 for the degree-of-freedom 5 at the p-value 0.05. Therefore, we may barely say that the set of observed values follows the predicted Poisson distribution.

The patients of routes (15,1) and (27,1) in Ward 300 are considered to be staying *in the waiting room* of the M/M/m queue with $m = 6$ that represents Ward 30M. Therefore, the number of these patients has a distribution $P\{L = k\}$ in Eq. (30) in the appendix with

$$\begin{aligned}&\rho^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}\\ &= E[A^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}] \cdot E[\mathrm{LoS}^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}]\\ &= \frac{312}{731} \times 10.4519 = 4.4610\end{aligned}$$

and

$$C(6, \rho^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}) = 0.4099.$$

The theoretical distribution of $T \cdot P\{L = k\}$ and the observed histogram of the number of days during the two years for the number of these patients are plotted in Fig. 8 (c). While we observe that the theoretical distribution captures the basic characteristic of the observed distribution, their mean values are rather different as

$$E[N^{300}_{(15,1),(27,1)}] = \frac{25 \times 2.4 + 30 \times 3.4}{731} = 0.2216,$$

$$E[L] = \frac{4.4610 \times 0.4099}{6 - 4.4610} = 1.1882.$$

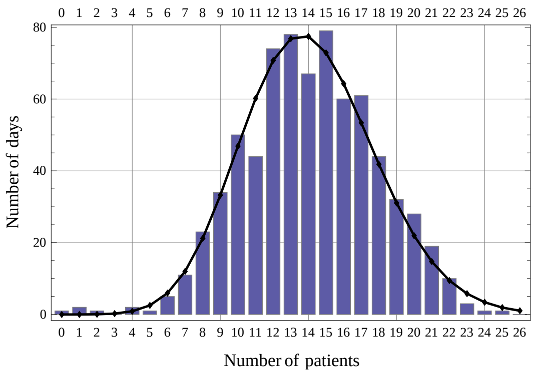
In fact, the chi-square statistic for the range from $k = 0$ to 6 is 103.129, which is much larger than 11.0705 for the degree-of-freedom 5 at the p-value 0.05. This is an instance in our modeling that shows sizable discrepancy between theory and observation.

In addition, the distribution function of the waiting time for the $\mathcal{A}^{300}_{(15,1),(27,1)} = 55$ patients of routes (15,1) and (27,1) who arrive to Ward 300 is given theoretically by $P\{W < t\}$ in Eq. (32) as the waiting time W in the M/M/6 queue with $\rho = 4.4610$ given above. The correspondence between the waiting time W and the LoS is given by

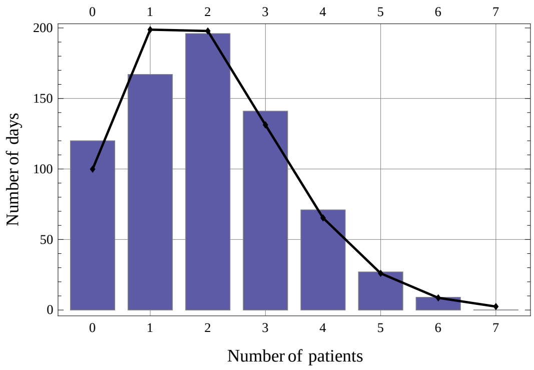
$$P\{\mathrm{LoS}^{300}_{(15,1),(27,1)} = k\} = P\{W < k+1\} - P\{W < k\}\qquad k = 0, 1, 2, \ldots.$$

The theoretical and observed distributions

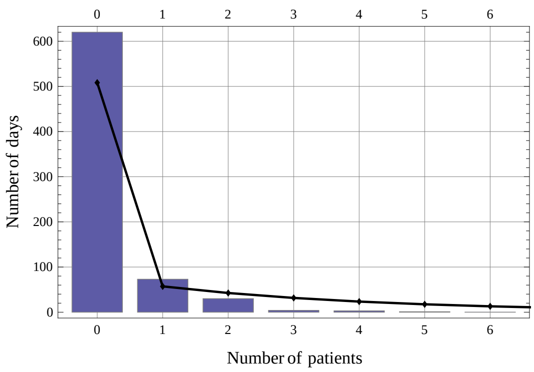
$$\mathcal{A}^{300}_{(15,1),(27,1)} \cdot P\{\mathrm{LoS}^{300}_{(15,1),((27,1)}\}$$



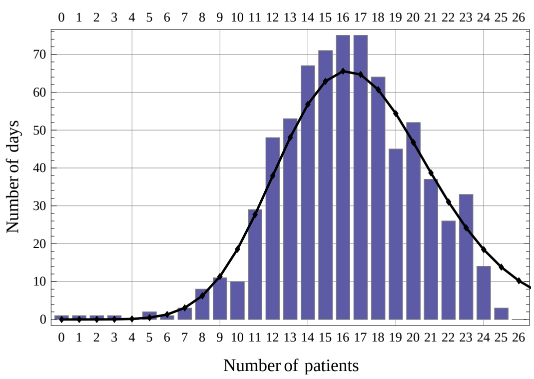
(a) Patients of route (2,1) and the Poisson distribution with mean 14.1135.



(b) Patients of routes (23,2), and (27,3) and the Poisson distribution with mean 1.9904.



(c) Patients of routes (15,1) and (27,1) (observed mean 0.2216) and the distribution in Eq.(30) with $m = 6$ and $\rho = 4.4610$ (theoretical mean 1.1882).



(d) Patients of routes (2,1), (15,1), (23,2), (27,1), and (27,3): observed and theoretical distributions with mean 16.1039.

Fig. 8 Distribution of the number of days during the two years 2010–2011 for the number of patients in Ward 300 with the corresponding theoretical distributions.

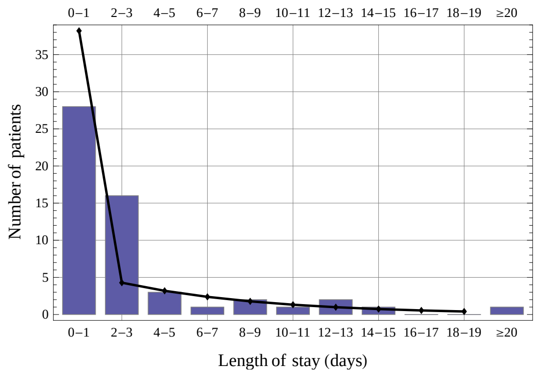


Fig. 9 Distribution of the number of days during the two years 2010–2011 of the number of routes (15,1) and (27,1) patients for the LoS in Ward 300 with a mean of 2.9455 days and the distribution from Eq.(32) with $m = 6$, $\mu = 1/10.4519$, and $\rho = 4.4610$.

are plotted in Fig. 9. We again see that the theoretical distribution barely captures the basic characteristic of the observed distribution. Their mean values also agree fairly well:

$$E[\mathrm{LoS}^{300}_{(15,1),((27,1)}] = 2.9455 \quad ; \quad E[W] = 2.7840.$$

Using $E[A^{300}_{(15,1),(27,1)}] = 0.0752$, we confirm Little's law in the observed data:

$$E[N^{300}_{(15,1),(27,1)}] = E[A^{300}_{(15,1),(27,1)}] \cdot E[\mathrm{LoS}^{300}_{(15,1),((27,1)}].$$

The Little's law $E[L] = \lambda E[W]$ in the theoretical distribution also holds with $\lambda = E[A^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}] = 0.4268$. The sets of patients to which Little's law is applied are different between the observed data and the theoretical model.

Finally, the probability distribution for the total number of patients in Ward 300 is obtained by the convolution of the above-mentioned three distributions. Since the first two are independent Poisson distributions, the convolution of the two generates another Poisson distribution with the mean

$$\rho^{300}_{(2,1)} + \rho^{300}_{(23,2),(27,3)} = 14.1135 + 1.9904 = 16.1039.$$

Therefore, we only have to calculate the convolution of this Poisson distribution with the distribution $P\{L = k\}$ in the form of Eq. (30) with $m = 6$ and $\rho = 4.4610$. Thus, we plot the distribution of the number of days during the two years for the number of routes (2,1), (15,1), (23,2), (27,1), and (27,3) patients in Ward 300 in Fig. 8 (d). The mean number of patients in Ward 300 is given by

$$\begin{aligned} &E[N^{300}_{(2,1),(15,1),(23,2),(27,1),(27,3)}] \\ &= \rho^{300}_{(2,1)} + \rho^{300}_{(23,2),(27,3)} + E[N^{300}_{(15,1),(27,1)}] \\ &= 16.1039 + 0.2216 = 16.3255. \end{aligned}$$

We observe fairly good agreement between the theory and observation except near the capacity 26 of Ward 300. The chi-square statistic for the range from $k = 4$ to 25 is 34.5571, which slightly exceeds 31.4104 for the degree-of-freedom 20 at the p-value 0.05.

4.2 Number of patients of dominant routes in Ward 30M

Next, we consider the number of patients of routes (9,1), (15,2), (23,1), and (27,2) who stay in Ward 30M on each day. In our model, these patients are considered to be *in service* in the M/M/6 queue that models Ward 30M. Therefore, the number of these patients has the distribution $P\{S = k\}$ given in Eq. (33) in the appendix with $m = 6$ and $\rho = 4.4610$, which yields $P_0(6, \rho) = 0.0096$ and $C(6, \rho) = 0.4099$.

The theoretical distribution of $T \cdot P\{S = k\}$ and the observed distribution of the number of days during the two years for the number of these patients are plotted with a solid line graph and a bar chart, respectively, in Fig. 10. The theoretical result still captures major characteristics of the observed values with the same mean values

$$E[N^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}] = E[S] = 4.4610.$$

In this figure, we also plot $T \cdot P\{S^* = k\}$ with a dashed line graph from the theoretical distribution for the number of customers present in the M/M/m/m *Erlang's loss system* [9, p.81]:

$$P\{S^* = k\} = \frac{\rho^k}{k!} \Bigg/ \sum_{j=0}^{m} \frac{\rho^j}{j!} \qquad 0 \le k \le m \tag{25}$$

with $m = 6$ and $\rho = 4.4610$. This model is used by Hershey et al. [12] in their model of a ward with finite capacity. The distributional shape of this model seems to fail to capture the monotonic increase that characterizes the observed data. The mean value predicted by Eq. (25) is $E[S^*] = 3.7863$. The probability that patients are not admitted upon arrival is given as the probability $P\{S^* = m\} = 0.1512$ that all beds are occupied, which is close to the observed ratio

$$\frac{\mathcal{A}^{300}_{(15,1),(27,1)}}{\mathcal{A}^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}} = \frac{55}{312} = 0.1763.$$

4.3 Number of all patients of dominant routes

Finally, we examine the probability distribution for the number of patients of the five dominant routes who

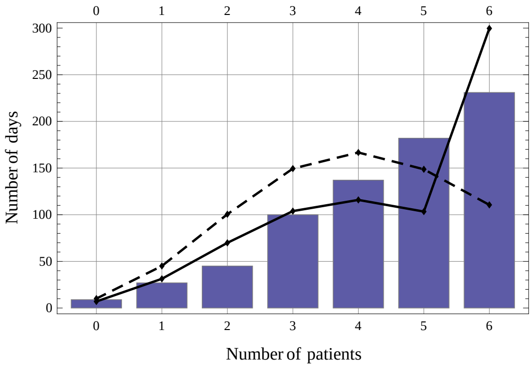


Fig. 10 Distribution of the number of days during the two years 2010–2011 for the number of routes (9,1), (15,2), (23,1), and (27,2) patients in Ward 30M (bar chart), the distribution in Eq. (33) (solid line), and the distribution in Eq. (25) with $m = 6$ and $\rho = 4.4610$ (dashed line).

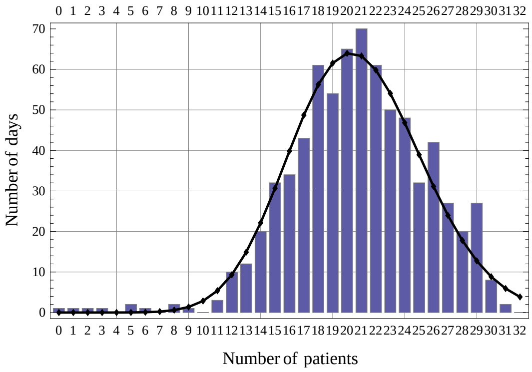


Fig. 11 Distribution of the number of days during the two years 2010–2011 for the number of all patients of dominant routes in Wards 300 and 30M and the Poisson distribution with mean 20.7866.

stayed in Wards 300 and 30M together during the two years. There are $A_{(2,1),(9,1),(15,1),(23,1),(27,1)} = 1,894$ such patients. The distribution for their LoS is shown in Fig. 7 (d), which shows a steep peak at the mean value $E[\mathrm{LoS}_{(2,1),(9,1),(15,1),(23,1),(27,1)}] = 8.0277$ days.

Since all patients are admitted upon arrival, we again apply the Poisson distribution in Eq. (27) for an M/G/∞ queue, and get

$$P_{(2,1),(9,1),(15,1),(23,1),(27,1)}(k) = \frac{\hat{\rho}^k}{k!}e^{-\hat{\rho}} \qquad k \geq 0, \quad (26)$$

where

$$\begin{aligned}\hat{\rho} &= E[A_{(2,1),(9,1),(15,1),(23,1),(27,1)}] \\ &\quad \times E[\mathrm{LoS}_{(2,1),(9,1),(15,1),(23,1),(27,1)}] \\ &= \frac{1,894}{731} \times 8.0227 = 20.7866 \\ &= E[N^{300}_{(2,1),(15,1),(23,2),(27,1),(27,3)}] \\ &\quad + E[N^{30\mathrm{M}}_{(9,1),(15,2),(23,1),(27,2)}].\end{aligned}$$

We note that $T\hat{\rho} = 15,195$ is the patient-days for all patients of the dominant routes during the two years (see Table 1).

Figure 11 plots the distribution of the number of days during the two years for the number of patients of all dominant routes in Wards 300 and 30M and the Poisson distribution with mean 20.7866. We observe excellent agreement between the theory and observation. The chi-square statistic for the range from $k = 11$ to 31 is 30.1521, which almost equals 30.1435 for the degree-of-freedom 19 at the p-value 0.05.

Let us suggest a method to assess the saturation condition using our model. Since the patient flow will be mixed over Wards 300 and 30M when they are crowded, it will be reasonable to consider both wards together with the capacity $m = 26 + 6 = 32$ when we estimate the saturation condition. In order to assess the saturation condition of both wards, we may use Eq. (25), which also holds for the M/G/m/m loss model. With the current load $\hat{\rho} = 20.7866$ and capacity $m = 32$, the blocking probability (the probability that both wards are full) is 0.0053. Therefore, most admission requests are admitted. However, if the admission rate doubles to $\hat{\rho} = 41.5732$, the blocking probability becomes 0.28419, which is probably unacceptable. If $\hat{\rho} = 26.5$, the blocking probability is approximately 5%, which may be barely acceptable. Although it is subjective to determine what is acceptable and unacceptable with respect to the blocking probability, we can say that the current load $\hat{\rho} = 20.7866$ making the blocking probability 0.0053 is fairly well controlled by the standard of UTH.

5 Concluding remarks

In this paper, we applied Little' law of queueing theory and proposed a network model of M/G/∞ and M/M/m queues for the obstetric patient flow in the UTH. Statistical techniques were used to select a set of dominant routes of patient flow. We compared the results of numerical calculation based on our theoretical models with the data extracted from the hospital's order log during the two years, and demonstrated good agreement between them in most cases.

In spite of a very rough approximation, we "explained" the patient flow with acceptable accuracy. The reasons for this success may be as follows:

- The flow of obstetric patients is rather isolated from the flow of patients in other clinical departments as two wards 300 and 30M are almost dedicated to the obstetric unit.

- The process in which obstetric patients arrive and admitted is random and can be modeled as a Poisson process fairly well.
- The probability that all beds are occupied is very small in Ward 300, which makes it possible to use the M/G/∞ queueing model even though the LoS is not exponentially distributed in that ward.
- The LoS in Ward 30M has an exponential distribution, which makes it possible to use the M/M/m queueing model.

On the other hand, the treatment of the patients staying in Ward 300 when waiting for beds in Ward 30M needs further investigation. While Hershey et al. [12] used Erlang's loss system and Weiss and McClain [26] presented a different approach, we have proposed to model the waiting patients as those customers in the waiting room of an M/M/m queue. It is fair to say that none of them have been very successful in predicting the distribution for the number of such patients precisely enough as yet. However, only our model captures the monotonically increasing characteristic in the distribution for the number of patients staying in Ward 30M.

We plan to extend our study of patient flow to those in other clinical units of the UTH. It is also interesting to see if the modeling technique proposed in this paper can be applied to the patient flow in other hospitals.

We showed a method to build a network model of patient flow from the order log usually available in the information department of each hospital. Our model is free of clinical information. It will be useful for capacity planning of hospital wards by determining the optimal allocation of beds among clinical departments in a hospital given the predicted demands of patients in terms of the statistics on the number of arrivals and the LoS for patients of each department.

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A Relevant queueing models

The following queueing models are relatively simple and robust, and the explicit formulas are available for the probability distribution of the number of customers present in the system.

(1) M/G/∞

A model denoted by M/G/∞ in Kendall's notation of queueing theory is simply a system with sufficiently many servers to which customers arrive in a Poisson process and spend a random amount of service time, which is generally distributed probabilistically [9, p.84], [11, p.145]. There is no contention for servers among customers. If λ denotes the arrival rate and b denotes the mean service time, the number N of customers present in the system at an arbitrary time has a Poisson distribution with mean $\rho := \lambda b$:

$$P\{N=k\} = \frac{\rho^k}{k!}e^{-\rho} \qquad k \geq 0. \tag{27}$$

Note that this distribution depends on the service time only through its mean value.

A useful property in modeling is that the output of an M/G/∞ system is a Poisson process [18]. A nice property about the Poisson process is that the superposition of independent Poisson processes forms another Poisson process with added rates. These properties contribute to building a simple and robust model.

(2) M/M/m

A model denoted by M/M/m in Kendall's notation is a queueing system with m servers and a waiting room of infinite capacity to which customers arrive in a Poisson process at rate λ each with the service time exponentially distributed with mean $1/\mu$ [9, p.66], [11, p.142]. Then, the probability distribution for the number N of the customers present in the system at an arbitrary time in the steady state is given by

$$P\{N=k\} = \begin{cases} P_0(m,\rho)\dfrac{\rho^k}{k!} & 0 \leq k \leq m, \\ P_0(m,\rho)\dfrac{m^m}{m!}\left(\dfrac{\rho}{m}\right)^k & k \geq m+1, \end{cases} \tag{28}$$

where $\rho := \lambda/\mu$ and

$$\frac{1}{P_0(m,\rho)} = \sum_{k=0}^{m-1}\frac{\rho^k}{k!} + \frac{\rho^m}{(m-1)!(m-\rho)}. \tag{29}$$

The output of an M/M/m system is also a Poisson process [3].

Each customer in the system is either waiting in the waiting room or being served. The probability distribution and the mean for the number L of customers *in the waiting room* are given by

$$P\{L=k\} = P\{N=m+k\}$$
$$= \begin{cases} 1-\dfrac{\rho}{m}C(m,\rho) & k=0, \\ C(m,\rho)\left(1-\dfrac{\rho}{m}\right)\left(\dfrac{\rho}{m}\right)^k & k \geq 1, \end{cases}$$
$$E[L] = \frac{\rho C(m,\rho)}{m-\rho}, \tag{30}$$

where the *Erlang's C formula* [9, p.70]

$$C(m,\rho) := \frac{\dfrac{\rho^m}{m!}}{\left(1-\dfrac{\rho}{m}\right)\displaystyle\sum_{k=0}^{m-1}\frac{\rho^k}{k!} + \frac{\rho^m}{m!}} \tag{31}$$

gives the probability that an arriving customer waits because all servers are busy.

The probability distribution and the mean for the waiting time (the time that a customer spends in the waiting room) W in the M/M/m queue are given by

$$P\{W=0\} = 1 - C(m,\rho),$$
$$P\{W<t\} = 1 - C(m,\rho)e^{-(m-\rho)\mu t} \qquad t>0,$$
$$E[W] = \frac{C(m,\rho)}{\mu(m-\rho)}. \tag{32}$$

We note that the relation $E[L] = \lambda E[W]$ is an example of Little's law in Eq. (12).

The probability distribution and the mean for the number S of customers *in service* are given by

$$P\{S=k\} = \begin{cases} P\{N=k\} & 0 \le k \le m-1 \\ P\{N \ge m\} & k=m, \end{cases}$$

$$= \begin{cases} P_0(m,\rho)\dfrac{\rho^k}{k!} & 0 \le k \le m-1 \\ C(m,\rho) & k=m, \end{cases}$$

$$E[S] = \rho, \tag{33}$$

where $P_0(m,\rho)$ and $C(m,\rho)$ are given in Eqs. (29) and (31), respectively.