A Mathematical Study of Diffusion

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Abstract

This talk is devoted to the functional analytic approach to the problem of construction of Markov processes in probability theory.

Brief History

Robert Brown

Robert Brown (1773-1858)
Scottish Botanist

Brief History (1)

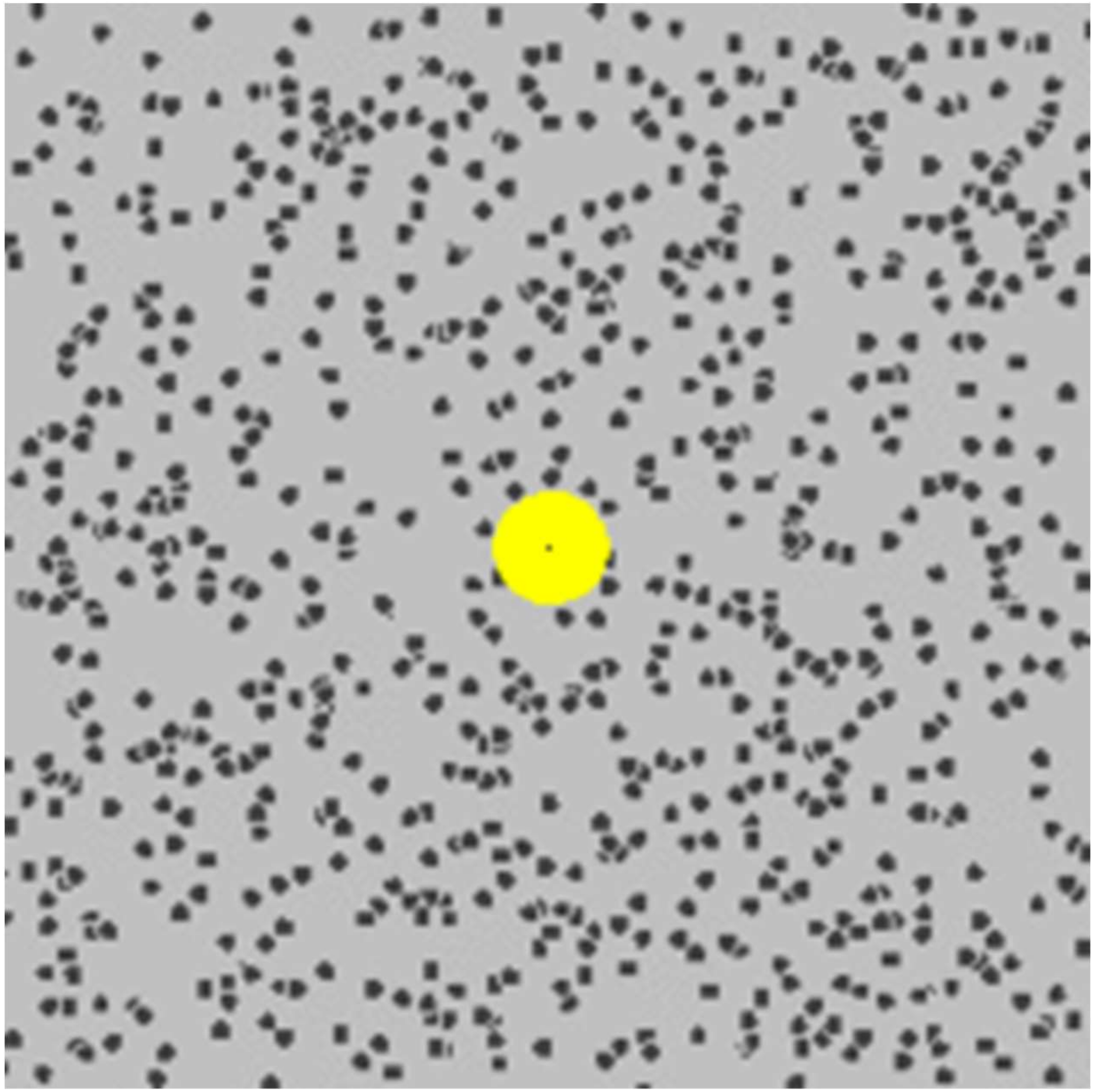
In 1828:

R. Brown observed that pollen grains suspended in water move chaotically, continually changing their direction of motion.

Brief History (2)

The physical explanation of this phenomenon is that a single grain suffers innumerable collisions with the randomly moving molecules of the surrounding water (due to A. Einstein).

A pollen grain suspended in water



Brief History (3)

In 1905:

A mathematical theory for Brownian motion was put forward by A. Einstein.

Albert Einstein

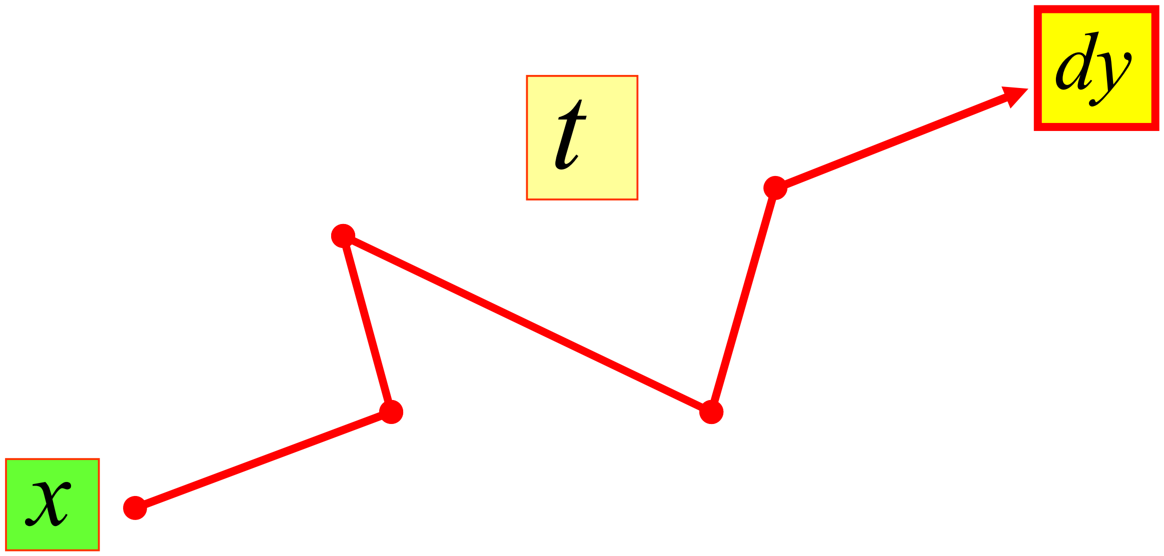
Albert Einstein (1879-1955)
German Physicist
Nobel Laureate in Physics

Einstein's Work (1)

$p(t, x, dy)$ = **the** **probability density function** **that a one-dimensional Brownian particle starting at position** x **will be found at position** y **at time** t.

Transition Density Function

$$\boxed{p(t, x, dy)}$$



Einstein's Work (2)

A. Einstein derived the following formula

from statistical mechanical considerations :

$$\boxed{p(t,x,dy) = \frac{1}{\sqrt{2\pi \mathrm{D} t}} \exp\left[-\frac{(y-x)^2}{2Dt}\right] dy}.$$

D **is a positive constant determined by the radius of the particle, the interaction of the particle with surrounding molecules, temperature and the Boltzmann constant.**

Jean Perrin

Jean Perrin (1870-1942)
French Physicist
Nobel Laureate in Physics

Perrin's Work

Einstein's theory was experimentally tested by J. Perrin between 1906 and 1909.
(Experimental measurement of Avogadro's Number)

Avogadro's Number

$$\boxed{N_A = 6{,}023 \cdot 10^{23}}$$

Brief
Mathematical History

Nobert Wiener

Nobert Wiener (1894-1964)
American Mathematician

Wiener's Work

In 1923:

Brownian motion was put on a firm mathematical foundation for the first time by N. Wiener.

Bird's-Eye View

Mathematical Studies of Brownian Motion

Markov Processes
(Probability)
N. Wiener
E.B. Dynkin
K. Ito

Brownian Motion
(Physics)
A. Einstein
J. Perrin

Diffusion Equations
(P.D.E)
A.N. Kolmogorov

Semigroups
(Functional Analysis)
W. Feller
K. Yosida

Markov Property

The Markov property is that
the prediction of subsequent motion of a particle,
knowing its position at time t, depends
neither on the value of t nor on what has
been observed during the time interval $[0, t]$.
A Markovian particle starts afresh.

One-dimensional case

- **1931: A.N. Kolmogorov (analytic approach)**
- **1952: W. Feller (semigroup approach)**
- **1965: E.B. Dynkin (probabilistic approach)**
- **1965: K. Ito and H.P. McKean, Jr. (probabilistic approach)**

References

- **Kolmogorov**: Math. Ann. 104 (1931), 415-458.
- **Feller**: Ann. Math. 55 (1952), 468-519.
- **Dynkin**: Springer-Verlag, 1965.
- **Ito and McKean, Jr.** : Springer-Verlag, 1965.

Andrey Nikolaevich Kolmogorov

Andrey Nikolaevich Kolmogorov
(1903-1987)
Russian Mathematician

Carl Einar Hille

◆**Carl Einar Hille**

(1894-1980) American Mathematician

Kosaku Yosida

◆**Kosaku Yosida**

(1909-1990) Japanese Mathematician

William Feller

William Feller (1906-1970)
Croatian–American Mathematician

Eugene Borisovich Dynkin

Eugene Borisovich Dynkin
(1924-2014)
Soviet and American Mathematician

Kiyosi Ito

Kiyosi Ito (1915-2008)
Japanese Mathematician
Carl-Friedrich-Gauß-Preis (2006)

References (2)

- **Ikeda and Watanabe**:

Stochastic differential equations and diffusion processes. Second edition, North-Holland Publishing Co., Amsterdam; Kodansha Ltd., Tokyo, 1989.

Bird's-Eye View

Bird's Eye View

Probability Theory (Micro-Scope)	Functional Analysis (Macro-Scope)	Partial Differential Equations (Mezzo-Scope)
Markov Processes	Feller Semigroups	Boundary Value Problems
Markov Property	Semigroup Property	•Waldenfels Operators •Wentzell Conditions

Bird's-Eye View (1)

$$\boxed{p_t(x,dy)} \quad \overset{\textbf{Riesz-Markov-Dynkin}}{\Longleftrightarrow} \quad \boxed{T_t = e^{t\mathfrak{A}}}$$

Laplace $\Updownarrow$ $\Updownarrow$ **Hille - Yosida**

$$\boxed{G_\alpha(x,y)} \quad \underset{\textbf{Riesz-Markov}}{\Longleftrightarrow} \quad \boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}}$$

Bird's-Eye View (2)

$\boxed{T_t = e^{t\mathfrak{A}}}$	Kolmogorov $\Longleftrightarrow$	Parabolic Theory
Hille - Yosida $\Updownarrow$		
$\boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}}$	$\Longleftrightarrow$ Feller	Elliptic Theory

Brownian Motion Case

Bird's-Eye View (1-dimensional case)

$$\boxed{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}} \quad \Leftrightarrow \quad \boxed{e^{t\frac{1}{2}d^2/dx^2}}$$

Laplace $\Updownarrow$ $\qquad$ $\Updownarrow$ **Hille - Yosida**

$$\boxed{\frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}|x-y|}} \quad \Leftrightarrow \quad \boxed{\left(\alpha - \frac{1}{2}\frac{d^2}{dx^2}\right)^{-1}}$$

Bird's-Eye View (2)

$$\boxed{\frac{\partial}{\partial t}-\frac{1}{2}\frac{d^2}{dx^2}} \Leftrightarrow \boxed{\text{Heat Equation}}$$

$$\Updownarrow$$

$$\boxed{\alpha-\frac{1}{2}\frac{d^2}{dx^2}} \Leftrightarrow \boxed{\text{Sturm-Liouville}}$$

Probabilistic Approach

Strategy

(1) Existence theorems for Markov processes (Probability)

(2) Generation theorems for Probabilistic semigroups (Functional Analysis)

(3) Existence and uniqueness theorems for Waldenfels operators with Wentzell boundary conditions (Partial Differential Equations)

From Transition Probabilities to Boundary Value Problem

$$\boxed{\{p_t(x, dy)\}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} (\alpha - \boldsymbol{W})u = f \ \text{ in } I \\ \boldsymbol{L}u = 0 \ \text{ on } \partial I \end{array}}$$

Wiener's Work

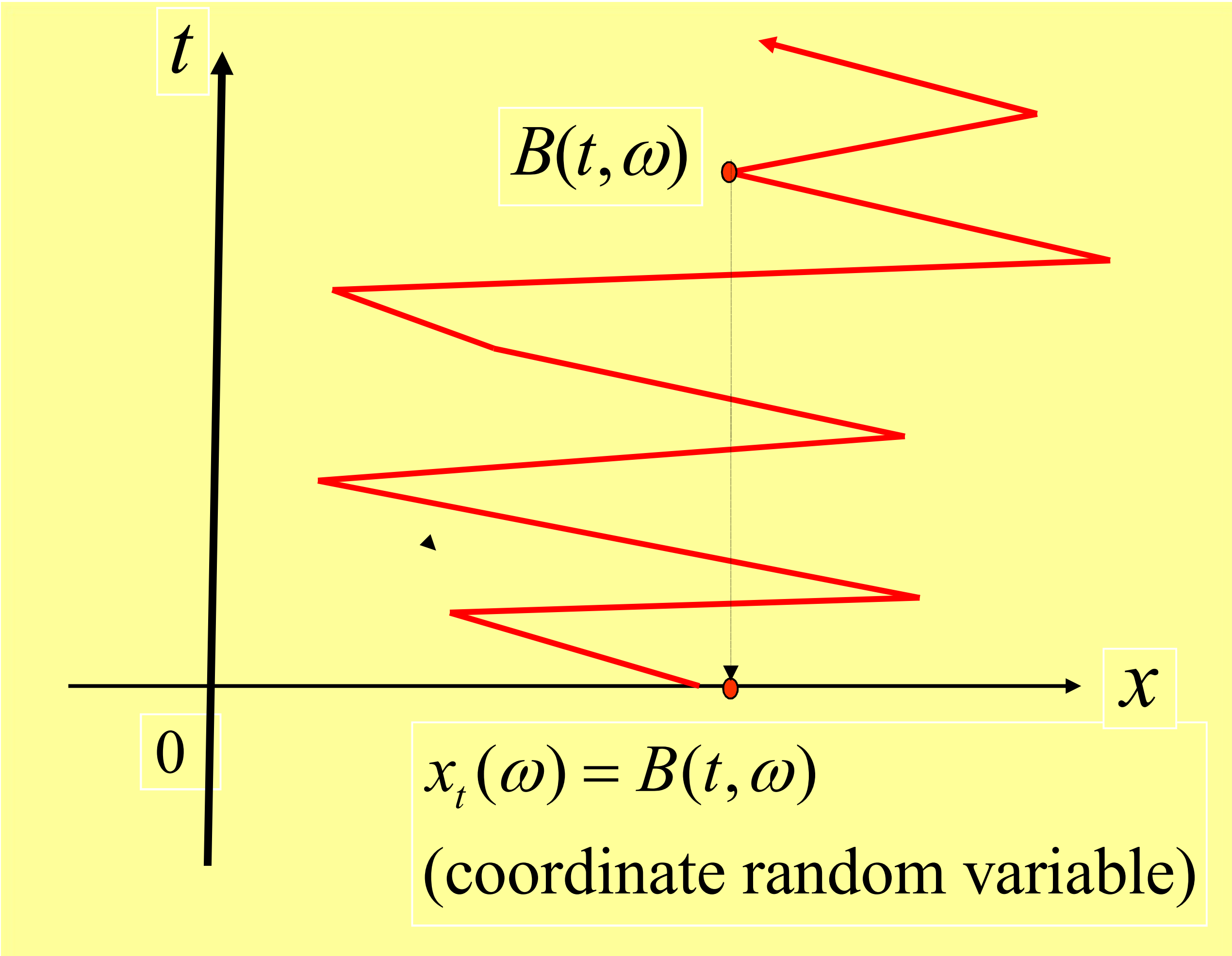
Sample Space

$\Omega = C[0, \infty)$

$=$ **the space of continuous functions of** t

$B(t) =$ **the coordinate random variables in** Ω

Sample Path or Trajectory



Joint Distribution Functions (Micro-Scope)

$$P^x\left(\left\{\omega\in\Omega : a_1 < B(t_1,\omega) < b_1,\, a_1 < B(t_2,\omega) < b_1\right\}\right)$$

$$= \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} p(t_1,x,y_1)\,dy_1\right)p(t_2-t_1,y_1,y_2)\,dy_2$$

$$p(t,x,y)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}$$

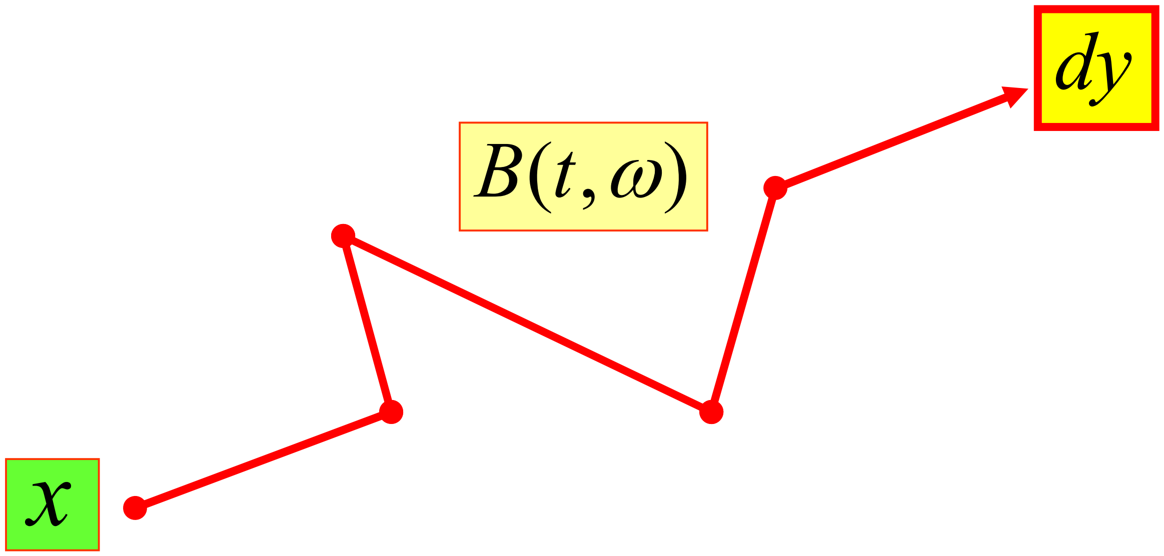
Einstein's Work

$$p(t,x,dy) = \frac{1}{\sqrt{2\pi \mathrm{D} t}} \exp\left[-\frac{(x-y)^2}{2Dt}\right] dy$$

$$\boxed{D=1}$$

Transition Density Function

$$\boxed{p(t,x,y)\,dy = P^x\left(\left\{\omega \in \Omega : B(t,\omega) \in dy\right\}\right)}$$



Chapman-Kolmogorov Equation

(Markov Property)

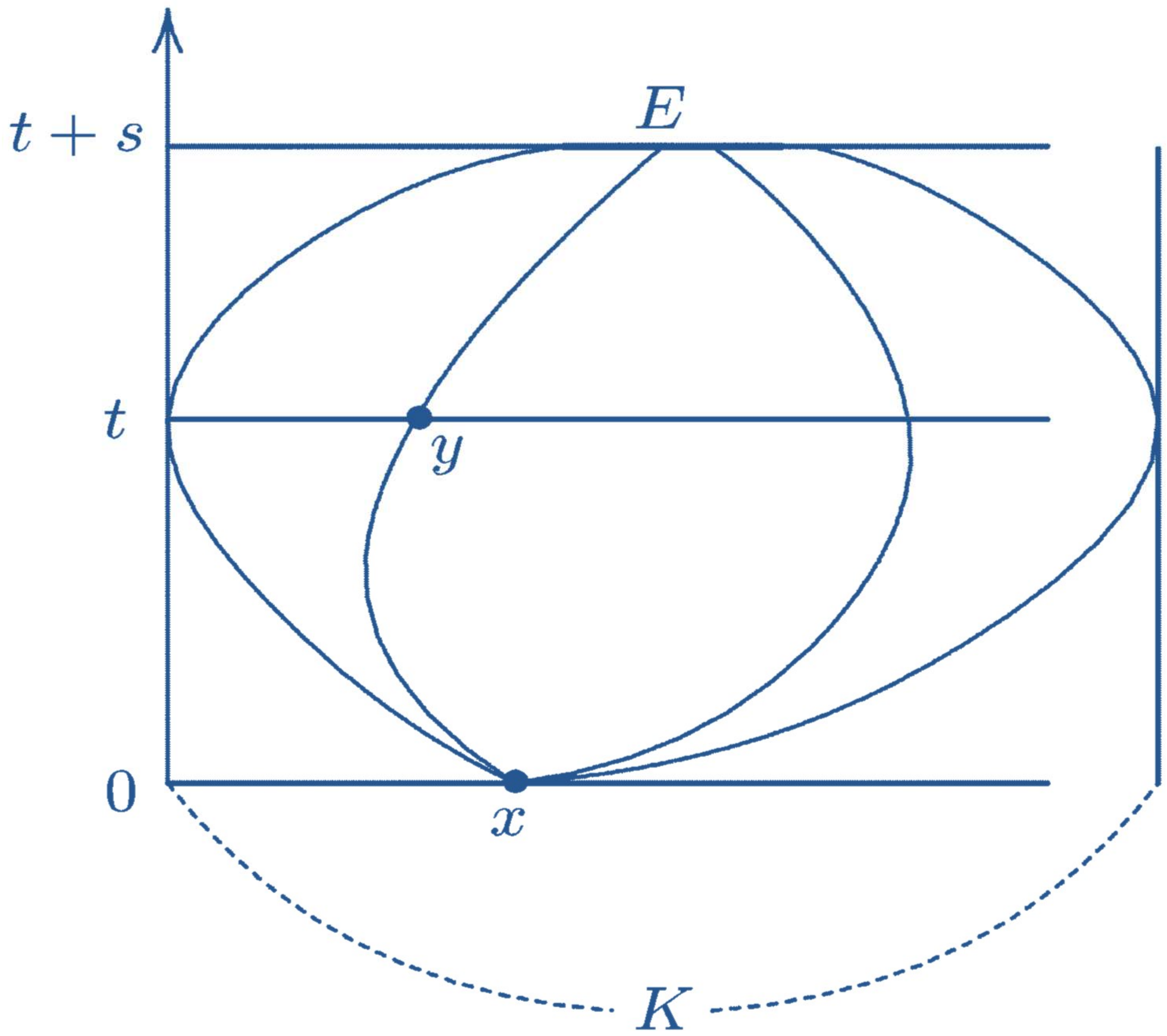
$$\boxed{p_{t+s}(x,E)=\int_K p_s(y,E)\,p_t(x,dy)}$$

$$\boxed{K=\mathbf{R}=(-\infty,\infty)}$$

Probabilistic Meaning of Chapman-Kolmogorov Equation

$$\boxed{p_{t+s}(x,E)=\int_K p_t(x,dy)\,p_s(y,E),\ \ \forall t,s\ge 0}$$

A transition from x to E in time $t+s$ is composed of a transition from x to some y in time t, followed by a transition from y to E in time s.


$t+s$
E
t
y
0
x
K

Probabilistic Transition Semigroup

(Macro-Scope)

$$0 \le p_t(x,\bullet) \le 1, \ \forall t \ge 0, \forall x \in K$$

$$\Rightarrow$$

$$\begin{cases} P_t f(x) = \int_K p_t(x,dy) f(y), \ \forall f \in C(K) \\ P_t : C(K) \to C(K) \end{cases}$$

Probabilistic Transition Semigroup via Expectation

$$
\begin{aligned}
P_t f(x) &= \int_{-\infty}^{\infty} p(t,x,y) f(y)\,dy \\
&= \int_{\Omega} f(B(t,\omega)) P^x(d\omega) \\
&= E^x\left(f(B(t))\right),\ \forall f \in C(K)
\end{aligned}
$$

Semigroup Property
(Markov Property)

$$p_{t+s}(x,E) = \int_K p_t(x,dy)\,p_s(y,E), \quad \forall t,s \geq 0$$

(Chapman - Kolmogorov Equation)

$$\Leftrightarrow$$

$$P_{t+s} = P_t \bullet P_s, \quad \forall t,s \geq 0$$

(Semigroup Property)

Resolvent

$$R_\alpha f(x) = E^x\left(\int_0^\infty e^{-\alpha t} f(B(t))\,dt\right)$$

$$= \int_0^\infty e^{-\alpha t}\left(\int_{-\infty}^\infty f(y)\,P^x\left(\{\omega \in \Omega : B(t,\omega) = y\}\right)dy\right)dt$$

$$= \int_0^\infty e^{-\alpha t}\left(\int_{-\infty}^\infty p(t,x,y)\,f(y)\,dy\right)dt$$

$$= \int_0^\infty e^{-\alpha t} P_t f(x)\,dt$$

Resolvent via Green Kernel

$$
\begin{aligned}
R_\alpha f(x) &= E^x\left(\int_0^\infty e^{-\alpha t} f(B(t))\,dt\right) \\
&= \int_0^\infty e^{-\alpha t}\left(\int_{-\infty}^\infty p(t,x,y) f(y)\,dy\right)dt \\
&= \int_{-\infty}^\infty \left(\int_0^\infty e^{-\alpha t} p(t,x,y)\,dt\right) f(y)\,dy \\
&= \int_{-\infty}^\infty G_\alpha(x,y) f(y)\,dy
\end{aligned}
$$

Abstract Exponential Function

$$P_t = e^{tA}$$

$\exists A$: **infinitesimal generator**

Hille-Yosida Theory

$$D(A) = \left\{ f \in C(K) : \exists \lim_{t \downarrow 0} \frac{P_t f - f}{t} \right\}$$

$$Af = \lim_{t \downarrow 0} \frac{P_t f - f}{t}, \quad \forall f \in D(A)$$

$$\Downarrow$$

$$P_t = e^{tA}$$

Laplace Transform

$$\int_0^\infty e^{-\alpha t} e^{ta}\, dt = \int_0^\infty e^{-(\alpha - a)t}\, dt = \frac{1}{\alpha - a}$$

$$= (\alpha - a)^{-1}$$

Green Operator and Semigroup

$$\boxed{R_\alpha := \int_0^\infty e^{-\alpha t} P_t \, dt = \int_0^\infty e^{-\alpha t} e^{tA} dt = (\alpha I - A)^{-1}}$$

Hille-Yosida Theory

$$\boxed{P_t = e^{tA}}$$

Feller's Work

Characterization of Generator
(Mezzo-Scope)

$$P_t = e^{t\mathfrak{A}}$$

$\mathfrak{A}$: **infinitesimal generator**

$\Downarrow$ **Feller**

$$(1)\ D(\mathfrak{A}) = \left\{ u : \exists \boldsymbol{L} u = 0 \text{ on } \partial I \right\}.$$

$$(2)\ \mathfrak{A} u = \exists \boldsymbol{W} u, \quad \forall u \in D(\mathfrak{A}).$$

Bird's-Eye View (1)

$$\boxed{p_t(x,dy)} \quad \overset{\textbf{Expectation}}{\Longrightarrow} \quad \boxed{P_t = e^{tA}}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Downarrow \textbf{ Laplace}$$

$$\boxed{\text{joint distributions}} \qquad\qquad \boxed{R_\alpha = \left(\alpha I - A\right)^{-1}}$$

Bird's-Eye View (2)

$$\boxed{P_t = e^{tA}} \quad \overset{\textbf{Kolmogorov}}{\Longrightarrow} \quad \boxed{\text{Parabolic Theory}}$$

Hille-Yosida $\Updownarrow$

$$\boxed{(\alpha I - A)^{-1}} \quad \underset{\textbf{Feller}}{\Longrightarrow} \quad \boxed{\text{Elliptic Theory}}$$

From Boundary Value Problem to Transition Probabilities

$$\boxed{\{p_t(x, dy)\}}$$

Feller

$$\boxed{\begin{aligned} &(\alpha - \mathbf{W})u = f \quad \text{in } I \\ &\mathbf{L}u = 0 \quad \text{on } \partial I \end{aligned}}$$

Feller's Analytic Approach

Strategy

(1) Existence and uniqueness theorems for integro-differential operators with general boundary conditions (Ordinary Differential Equations)

(2) Generation theorems for Feller semigroups (Functional Analysis)

(3) Existence theorems for Markov processes (Probability)

From Boundary Value Problem to Transition Probabilities

$$(\alpha - \boldsymbol{W})u = f \quad \text{in } I$$

$$\boldsymbol{L}u = 0 \quad \text{on } \partial I$$

$$\Downarrow$$

$$\{p_t(x, dy)\}$$

First Step

$$(\alpha - \boldsymbol{W})u = f \quad \text{in } I$$

$$\boldsymbol{L}u = 0 \quad \text{on } \partial I$$

Ordinary Differential Equations

$$u = \boldsymbol{G}_\alpha f = (\alpha I - \mathfrak{A})^{-1} f, \forall \alpha > 0$$

Second Step

$$G_\alpha := \int_0^\infty e^{-\alpha t} T_t \, dt = \int_0^\infty e^{-\alpha t} e^{t\mathfrak{A}} dt = \left(\alpha I - \mathfrak{A}\right)^{-1}$$

$$\Downarrow$$

Hille-Yosida Theory

$$\boxed{T_t = e^{t\mathfrak{A}}}$$

Bird's-Eye View (1)

$$\begin{tikzcd}[column sep=huge]
\boxed{P_t = e^{tA}} & \boxed{\text{Parabolic Theory}} \arrow[l, Rightarrow, "\text{Kolmogorov}"'] \\
\boxed{\left(\alpha I - A\right)^{-1}} \arrow[u, Rightarrow] & \boxed{\text{Elliptic Theory}} \arrow[l, Rightarrow, "\text{Feller}"]
\end{tikzcd}$$

Third Step

$$\boxed{T_t = e^{t\mathfrak{A}}}$$

Riesz-Markov-Dynkin

$$\boxed{T_t f(x) = \int_K \exists p_t(x, dy) f(y), \ \forall f \in C(K)}$$

Bird's-Eye View (2)

$$\boxed{p_t(x,dy)} \quad \overset{\textbf{Riesz-Markov-Dynkin}}{\Longleftarrow} \quad \boxed{P_t = e^{tA}}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow \textbf{ Hille - Yosida}$$

$$\boxed{\text{joint distributions}} \qquad\qquad \boxed{\left(\alpha I - A\right)^{-1}}$$

Bird's Eye View

Probability Theory (Micro-Scope)	Functional Analysis (Macro-Scope)	Partial Differential Equations (Mezzo-Scope)
Markov Processes	Feller Semigroups	Boundary Value Problems
Markov Property	Semigroup Property	•Integro-differential Operators •General Boundary Conditions

Theory of Diffusion

Probabilistic Methods

Bird's-Eye View (1)

$$\boxed{p_t(x,dy)} \quad \overset{\textbf{Expectation}}{\Longrightarrow} \quad \boxed{T_t = e^{t\mathfrak{A}}}$$

$\Downarrow$ **Laplace** $\qquad\qquad\qquad\qquad$ $\Downarrow$ **Laplace**

$$\boxed{G_\alpha(x,y)} \quad \Longleftrightarrow \quad \boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}}$$

Bird's-Eye View (2)

$$\boxed{T_t = e^{t\mathfrak{A}}} \quad \overset{\textbf{Kolmogorov}}{\Longrightarrow} \quad \boxed{\text{Parabolic Theory}}$$

Hille - Yosida $\Updownarrow$

$$\boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}} \quad \Longrightarrow \quad \boxed{\text{Elliptic Theory}}$$

Cauchy Process

$$u(x,t) := T_t f(x) = P_t * f(x), \quad x \in \mathbf{R}, t > 0.$$

$$\Rightarrow$$

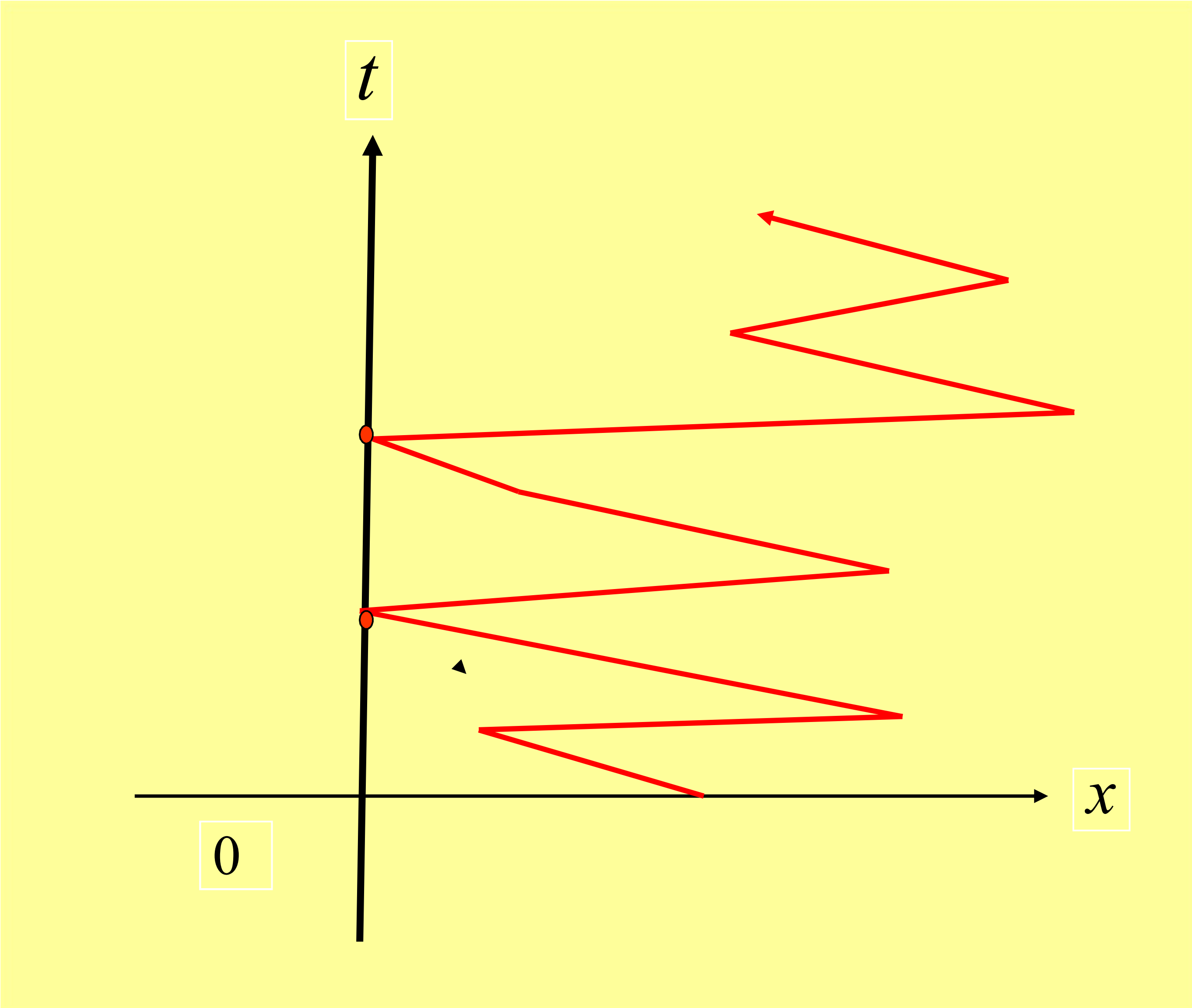
$$\begin{cases} \dfrac{\partial u}{\partial t} + \sqrt{-\dfrac{d^2}{dx^2}}u = 0, \quad \forall x \in \mathbf{R}, \forall t > 0, \\ u\big|_{t=0} = f \end{cases}$$

Augustin Louis Cauchy

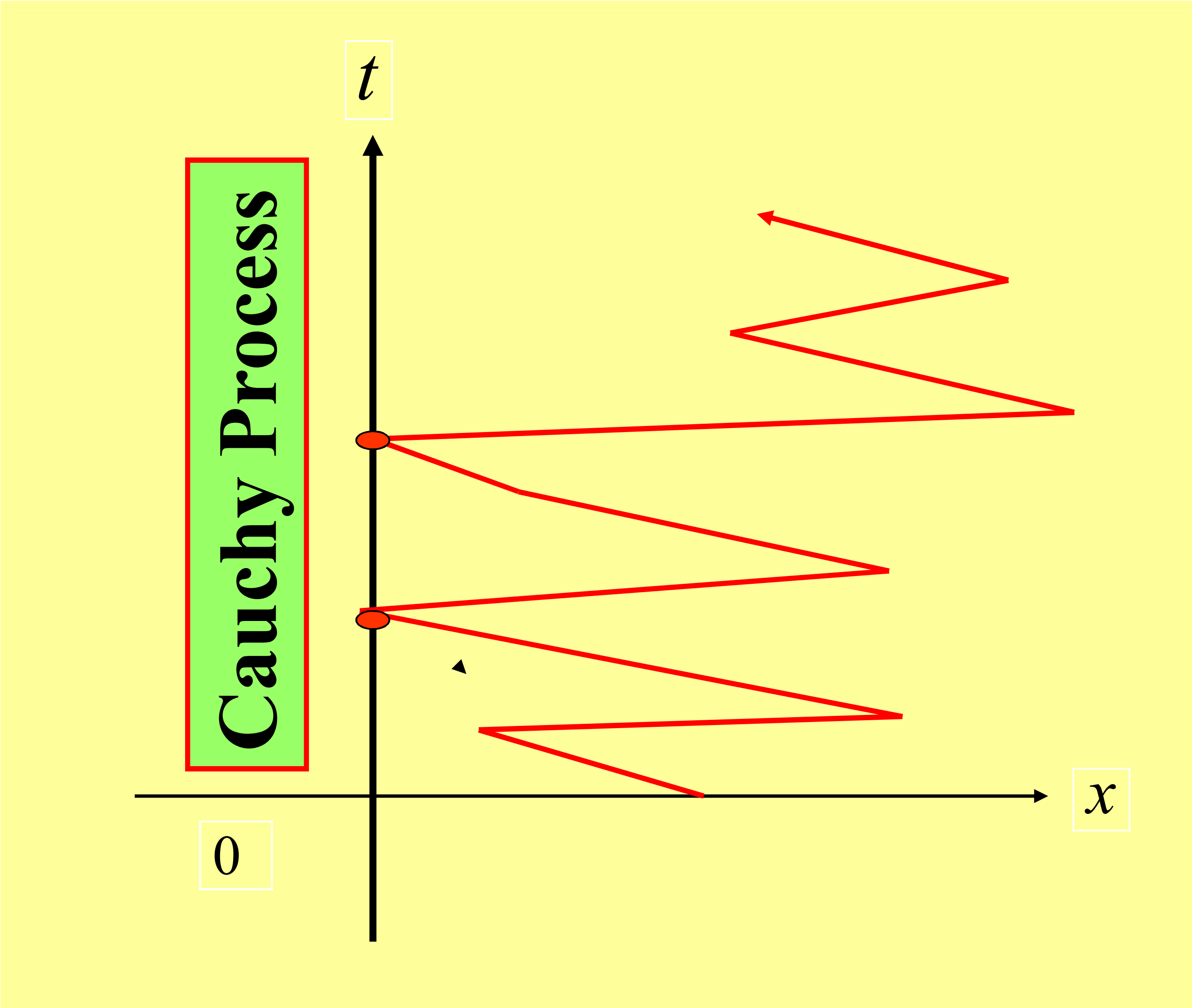
Augustin Louis Cauchy (1789-1857)
French mathematician

Cauchy Process

Cauchy process can be thought as the trace on R of trajectories of two-dimensional reflecting Brownian motion in the half-plane, and it moves by jumps.



t
x
0



Cauchy Process
t
x
0

Bird's-Eye View

$$
\begin{array}{ccc}
\boxed{p_t(x,dy)} & \overset{\textbf{Riesz-Markov-Dynkin}}{\Longrightarrow} & \boxed{T_t = e^{t\mathfrak{A}}} \\
\textbf{Laplace} \Updownarrow & & \Downarrow \textbf{Hille - Yosida} \\
G_\alpha(x,y) & \underset{\textbf{Riesz-Markov}}{\Longleftrightarrow} & \boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}}
\end{array}
$$

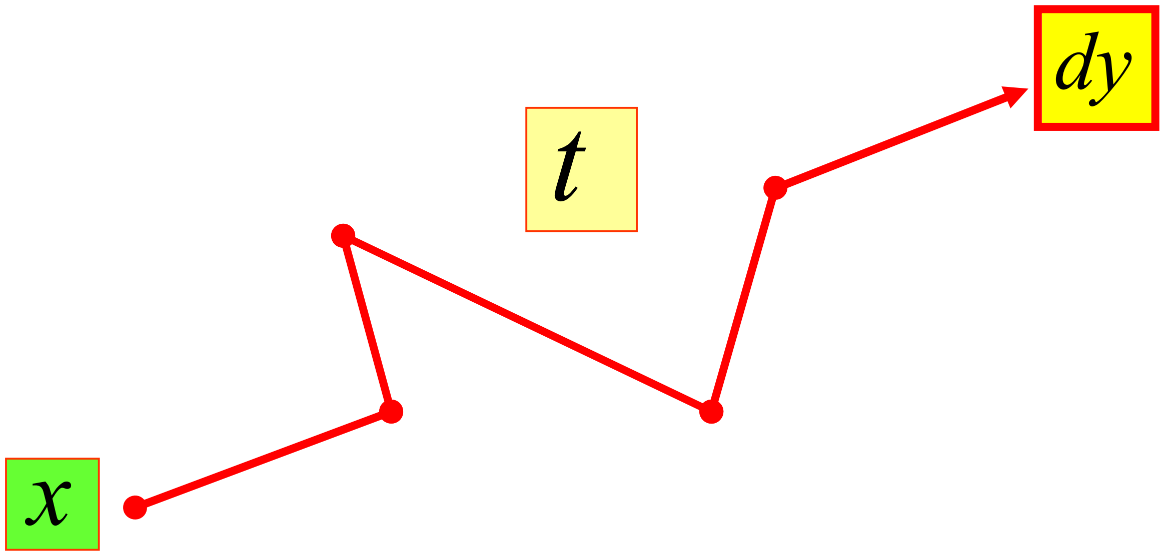
Cauchy Density Function

$$p_t(x, dy) = p(t, x, y)dy :$$

the probability density function that a Markovian particle starting at position x **will be found at position** y **at time** t.

Transition Density Function

$$\boxed{p_t(x, dy) = p(t, x, y)\,dy}$$



Transition probability
(Probability)

$$\boxed{p_t(x, dy) = P_t(x-y)\,dy = p(t, x, y)\,dy}$$

$$\boxed{p(t, x, y) = \frac{1}{\pi}\frac{t}{t^2 + (x-y)^2}}$$

Probabilistic Convolution Semigroup

$$\boxed{\begin{aligned} T_t f(x) &= \int_{-\infty}^{\infty} P_t(x-y) f(y)\, dy, \ \forall f \in BC(\mathbf{R}) \\ &= P_t * f(x) \end{aligned}}$$

$$\boxed{P_t(x-y)\, dy = p_t(x, dy)}$$

Probabilistic Transition Semigroup

$$
\begin{aligned}
T_t f(x) &= \int_{-\infty}^{\infty} p_t(x, dy) f(y) \\
&= \int_{\Omega} f(x_t(\omega)) P^x(d\omega) \\
&= E^x\left(f(x_t)\right), \ \forall f \in BC(\mathbf{R})
\end{aligned}
$$

Fourier transform of Transition Function

$$P_t(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi} e^{-t|\xi|}\, d\xi$$

$$= \frac{1}{\pi}\frac{t}{t^2 + x^2}, \quad t > 0.$$

$$\boxed{P_t(x-y) = p(t,x,y) =, \ \forall t > 0, \forall x, y \in \mathbf{R}.}$$

$$\Downarrow$$

$$\boxed{\widehat{P}_t(\xi) = e^{-t|\xi|}, \quad \forall \xi \in \mathbf{R}, \forall t > 0.}$$

Bird's-Eye View

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{P_t = e^{tA}} \arrow[r, Rightarrow, "\text{Kolmogorov}"] \arrow[d, Leftrightarrow, "\text{Hille-Yosida}"'] & \boxed{\text{Parabolic Theory}} \\
\boxed{\left(\alpha I - A\right)^{-1}} \arrow[r, Rightarrow, "\text{Feller}"'] & \boxed{\text{Elliptic Theory}}
\end{tikzcd}
$$

Heat Equation for the Cauchy Process

$$u(x,t) := T_t f(x) = P_t * f(x), \quad x \in \mathbf{R}, t > 0.$$

$$\Rightarrow$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sqrt{-\dfrac{d^2}{dx^2}}\, u = 0, & \forall x \in \mathbf{R}, \forall t > 0, \\ u\big|_{t=0} = f & \end{cases}$$

Hille-Yosida Theory

$$T_t = e^{t\mathfrak{A}} = e^{-t\sqrt{-d^2/dx^2}}$$

Abstract Exponential Function

Characterization of the Generator

$$\mathfrak{A}f(x) = \frac{1}{\pi}\int_0^\infty \frac{f(x+y)+f(x-y)-2f(x)}{y^2}\,dy,$$

$$\forall f \in C_0^2(\mathbf{R}) \subset D(\mathfrak{A}).$$

$\mathfrak{A}$: **Integral (non - local) Operator**

Probabilistic Meaning of the Generator

$\mathfrak{A}$: Integral (non - local) Operator

$\Updownarrow$

Cauchy process can be thought as the trace on $\mathbf{R}$ of two - dimensional, reflecting Brownian motion in the half - plane, and it moves by jumps.

Fourier Transform Version (1)

$$T_t f(x) = \int_{-\infty}^{\infty} P_t(x-y) f(y)\, dy = P_t * f(x)$$

$$\Leftrightarrow$$

$$\widehat{T_t f}(\xi) = \widehat{P_t * f}(\xi) = e^{-t|\xi|}\, \hat{f}(\xi), \quad \forall t > 0$$

Fourier Transform Version (2)

$$\frac{\widehat{T_t f}(\xi) - \hat{f}(\xi)}{t} = \frac{e^{-t|\xi|} - 1}{t} \hat{f}(\xi)$$

$$\Rightarrow$$

$$\lim_{t\downarrow 0} \frac{\widehat{T_t f}(\xi) - \hat{f}(\xi)}{t} = \left(-|\xi|\right) \hat{f}(\xi)$$

Fourier Transform Version (3)

$$\widehat{\mathfrak{A}f}(\xi) = \lim_{t\downarrow 0} \frac{\widehat{T_t f}(\xi) - \hat{f}(\xi)}{t} = \left(-|\xi|\right)\hat{f}(\xi)$$

$\mathfrak{A}$: **Pseudo - Differential Operator with symbol** $-|\xi|$

Characterization of the Generator (1)

$$T_t = e^{t\mathfrak{A}}$$

$$\mathfrak{A} f(x) = -\sqrt{-\frac{d^2}{dx^2}}\, f(x)$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\, |\xi|\, \widehat{f}(\xi)\, d\xi$$

$\mathfrak{A}$: **Pseudo - Differential Operator with symbol** $-|\xi|$

Characterization of the Generator (2)

$$\mathfrak{A}f(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\,|\xi|\,\widehat{f}(\xi)\,d\xi$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\,|\xi|\left(\int_{-\infty}^{\infty} e^{-iy\xi} f(y)\,dy\right)d\xi$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} e^{i(x-y)\xi}\,|\xi|\,d\xi\right)dy$$

$$= \frac{1}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty} \frac{1}{|x-y|^2} f(y)\,dy$$

Principal Value of the Distribution

Here:

$$\left\langle \text{v.p.}\,\frac{1}{x^2}, \varphi(x) \right\rangle$$

$$= \lim_{\varepsilon \downarrow 0} \int_{|x| \ge \varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2}\, dx$$

$$= \int_0^\infty \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2}\, dx, \quad \forall \varphi \in C_0^\infty(\mathbf{R})$$

(**regularization** of $1/x^2$)

Fourier Transform

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\,|\xi|\,d\xi = -\frac{1}{\pi}\,\mathrm{v.p.}\,\frac{1}{x^2}}$$

$$\Updownarrow$$

$$\boxed{|\xi| = -\frac{1}{\pi}\int_{\mathbf{R}}\left[1-\cos\left(\xi y\right)\right]\frac{1}{y^2}\,dy}$$

Characterization of the Generator (3)

(1) $\mathfrak{A} = -\sqrt{-\dfrac{d^2}{dx^2}} = -(-\Delta)^{1/2}$

(2) **Symbol:** $-|\xi|$

(3) **Distribution kernel:**

$$\frac{1}{\pi}\,\mathrm{v.p.}\,\frac{1}{|x-y|^2}$$

Probabilistic Meaning of Cauchy Process

(1) **Levy measure given by the density function**

$$\nu(y) = \frac{1}{\pi}\frac{1}{|y|^2}$$

(2) $e^{t\mathfrak{A}}$: **probabilistic convolution semigroup**

with Levy measure $\nu(y)dy$

(3) $\mathfrak{A}f = -(-\Delta)^{1/2} f = \nu * f$

Characterization of the Generator (4)

$$
\begin{aligned}
\mathfrak{A} f(x) &= \nu * f(x) \\
&= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{f(x+y) - f(x)}{y^2} dy \\
&= \frac{1}{\pi} \int_0^\infty \frac{f(x+y) + f(x-y) - 2f(x)}{y^2} dy,
\end{aligned}
$$

$\forall f \in C_0^2(\mathbf{R}) \subset D(\mathfrak{A})$.

Continuity of the Generator

(1) $\mathfrak{A} : H^{s,p}_{\mathrm{comp}}(\mathbf{R}) \to H^{s-1,p}_{\mathrm{loc}}(\mathbf{R})$ **continuous**

for $\forall s \geq 1,\ 1 < \forall p < \infty$

(2) $\mathfrak{A} : C^{t}_{\mathrm{comp}}(\mathbf{R}) \to C^{t-1}_{\mathrm{loc}}(\mathbf{R})$ **continuous**

for $\forall t > 1$

Isotropic Stable Processes
and
Partial Differential Equations

Heat Equation for the Stable Process

$$u(x,t) := T_t f(x) = P_t * f(x), \quad x \in \mathbf{R}^n, t > 0.$$

$$\Rightarrow$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{\alpha/2} u = 0, & \forall x \in \mathbf{R}^n, \forall t > 0, \\ u\big|_{t=0} = f \end{cases}$$

Paul Levy

Paul Levy (1886-1971)
French Mathematician

Bird's Eye View

Probability Theory (Micro-Scope)	Functional Analysis (Macro-Scope)	Partial Differential Equations (Mezzo-Scope)
Markov Processes	**Feller Semigroups**	**Boundary Value Problems**
Markov Property	**Semigroup Property**	•**Waldenfels Operators** •**Wentzell Conditions**

Bird's-Eye View (1)

$$\boxed{p_t(x,dy)} \quad \overset{\textbf{Expectation}}{\Longrightarrow} \quad \boxed{T_t = e^{t\mathfrak{A}}}$$

$\Downarrow$ **Laplace** $\qquad\qquad\qquad\qquad$ $\Downarrow$ **Laplace**

$$\boxed{G_\alpha(x,y)} \quad \Longleftrightarrow \quad \boxed{R_\alpha = \left(\alpha I - \mathfrak{A}\right)^{-1}}$$

Bird's-Eye View (2)

$$\begin{array}{ccc} \boxed{T_t = e^{t\mathfrak{A}}} & \overset{\text{Kolmogorov}}{\Longrightarrow} & \boxed{\text{Parabolic Theory}} \\ \text{Hille - Yosida} \Updownarrow & & \\ \boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}} & \underset{\text{Wenzell}}{\Longrightarrow} & \boxed{\text{Elliptic Theory}} \end{array}$$

Stable Process (Transition Probability)

The isotropic α-stable process

$$K = \mathbf{R}^n$$

$$\boxed{0 < \alpha < 2}$$

$$P_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} e^{-t|\xi|^\alpha} d\xi, \quad t > 0.$$

$$p(t,x,y) = P_t(y-x), \quad \forall x, y \in \mathbf{R}^n.$$

$$\boxed{\begin{array}{l} p_t(x,E) = \int_E p(t,x,y)\,dy, \\ \forall t > 0, \forall x \in \mathbf{R}^n, \forall E \in \mathfrak{B}(\mathbf{R}^n). \end{array}}$$

Cauchy Process

(1) **Levy measure** **given by the density function**

$$\nu(y) = \frac{1}{\pi}\frac{1}{|y|^2} \quad (\alpha = 1)$$

(2) $e^{t\mathfrak{A}}$: **probabilistic convolution semigroup**

with Levy measure $\nu(y)dy$

(3) $\mathfrak{A}f = -(-\Delta)^{1/2} f = \nu * f$

Probabilistic Convolution Semigroup

$$T_t f(x) = \int_{\mathbf{R}^n} P_t(x-y) f(y)\, dy, \ \forall f \in C_0(\mathbf{R}^n)$$
$$= P_t * f(x)$$

$$P_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} e^{-t|\xi|^\alpha} d\xi, \ \ \forall t > 0, \forall x \in \mathbf{R}^n.$$

Fourier Transform Version (1)

$$T_t f(x) = \int_{-\infty}^{\infty} P_t(x-y) f(y)\, dy = P_t * f(x)$$

$$\Leftrightarrow$$

$$\widehat{T_t f}(\xi) = \widehat{P_t * f}(\xi) = e^{-t|\xi|^{\alpha}}\ \hat{f}(\xi), \quad \forall t > 0$$

Fourier Transform Version (2)

$$\frac{\widehat{T_t f}(\xi) - \hat{f}(\xi)}{t} = \frac{e^{-t|\xi|^{\alpha}} - 1}{t} \hat{f}(\xi)$$

$$\Rightarrow$$

$$\lim_{t \downarrow 0} \frac{\widehat{T_t f}(\xi) - \hat{f}(\xi)}{t} = \left(-|\xi|^{\alpha}\right) \hat{f}(\xi)$$

Fourier Transform Version (3)

$$\widehat{\mathfrak{A}f}(\xi) = \lim_{t \downarrow 0} \frac{\widehat{T_t f}(\xi) - \hat{f}(\xi)}{t} = \left(-|\xi|^{\alpha}\right)\hat{f}(\xi)$$

$\mathfrak{A}$: **Pseudo - Differential Operator with symbol** $-|\xi|^{\alpha}$

Characterization of the Generator (1)

$$T_t = e^{t\mathfrak{A}}$$

$$\mathfrak{A} f(x) = -\left(-\Delta\right)^{\alpha/2} f(x)$$

$$= -\frac{1}{(2\pi)^n} \int_{R^n} e^{ix\xi} \left|\xi\right|^{\alpha} \widehat{f}(\xi)\, d\xi$$

$$\boxed{\mathfrak{A} = -\left(-\Delta\right)^{\alpha/2}}$$

Characterization of the Generator (2)

$$\mathfrak{A} f(x) = -\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} |\xi|^\alpha \widehat{f}(\xi)\, d\xi$$

$$= -\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} |\xi|^\alpha \left(\int_{\mathbf{R}^n} e^{-iy\xi} f(y)\, dy \right) d\xi$$

$$= -\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} e^{i(x-y)\xi} |\xi|^\alpha\, d\xi \right) f(y)\, dy$$

$$= \frac{2^\alpha}{\pi^{n/2}} \frac{\Gamma((\alpha+n)/2)}{|\Gamma(-\alpha/2)|} \text{ v.p.} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n+\alpha}} f(y)\, dy$$

Characterization of the Generator (3)

(1) $\mathfrak{A} = -(-\Delta)^{\alpha/2}, \quad 0 < \alpha < 2$

(2) **Symbol:** $-|\xi|^{\alpha}$

(3) **Distribution kernel:**

$$\frac{2^{\alpha}}{\pi^{n/2}} \frac{\Gamma((\alpha+n)/2)}{|\Gamma(-\alpha/2)|} \text{ v.p. } \frac{1}{|x-y|^{n+\alpha}}$$

Fourier Transform (1)

$$\boxed{\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix\xi}\,|\xi|^{\alpha}\,d\xi = \frac{2^{\alpha}}{\pi^{n/2}}\frac{\Gamma((\alpha+n)/2)}{\Gamma(-\alpha/2)}\ \text{v.p.}\frac{1}{|x|^{n+\alpha}}}$$

$$\boxed{0 < \alpha < 2}$$

Principal Value of the Distribution

$$\text{v.p.}\,\frac{1}{|x|^{n+\alpha}}, \quad 0<\alpha<2.$$

$$\left\langle \text{v.p.}\,\frac{1}{|x|^{n+\alpha}}, \varphi(x) \right\rangle$$

$$= \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{\varphi(y)-\varphi(0)}{|y|^{n+\alpha}}\, dy, \quad \forall \varphi \in C_0^{\infty}(\mathbf{R}^n)$$

Fourier Transform (2)

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix\xi}\,|\xi|^{\alpha}\,d\xi = \frac{2^{\alpha}}{\pi^{n/2}}\frac{\Gamma((\alpha+n)/2)}{\Gamma(-\alpha/2)}\ \text{v.p.}\,\frac{1}{|x|^{n+\alpha}}$$

$$\Updownarrow$$

$$|\xi|^{\alpha} = \int_{\mathbf{R}^n}\left[1-\cos\left(\xi\cdot y\right)\right]\nu_{\alpha}\left(y\right)dy$$

Probabilistic Meaning of the Pseudo-Differential Operator

(1) **Levy measure given by the density function**

$$\nu_\alpha(y) = \frac{2^\alpha}{\pi^{n/2}} \frac{\Gamma((\alpha+n)/2)}{|\Gamma(-\alpha/2)|} \frac{1}{|y|^{n+\alpha}}, \qquad 0 < \alpha < 2$$

(2) $e^{t\mathfrak{A}}$: **probabilistic convolution semigroup with Levy measure** $\nu_\alpha(y)dy$

(3) $\mathfrak{A}f = -(-\Delta)^{\alpha/2} f = \nu_\alpha * f$

Probabilistic Meaning of the Semigroup

$T_t = e^{t\mathfrak{A}}$: **probabilistic convolution semigroup with Levy measure** $\nu_\alpha(y)dy$

$$
\begin{aligned}
T_t f(x) &= \int_{\mathbf{R}^n} p_t(x, dy) f(y) \\
&= \int_\Omega f(x_t(\omega)) P_x(d\omega) \\
&= E^x\left(f(x_t)\right), \ \forall f \in BC(\mathbf{R}^n)
\end{aligned}
$$

Characterization of the Generator (4)

$$\text{v.p.} \int_{\mathbf{R}^n} \frac{1}{|z|^{n+\alpha}} f(x-z)\,dz$$

$$= \lim_{\varepsilon \downarrow 0} \int_{|z| \geq \varepsilon} \frac{f(x-z) - f(x)}{|z|^{n+\alpha}}\,dz$$

$$= \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{f(x+y) - f(x)}{|y|^{n+\alpha}}\,dy$$

Continuity of the Generator

(1) $\mathfrak{A} = -\left(-\Delta\right)^{\alpha/2} : H^{s,p}_{\text{comp}}(\mathbf{R}^n) \to H^{s-\alpha,p}_{\text{loc}}(\mathbf{R}^n)$

is **continuous** for $\forall s \geq 1,\ 1 < \forall p < \infty.$

(2) $\mathfrak{A} = -\left(-\Delta\right)^{\alpha/2} : C^{t}_{\text{comp}}(\mathbf{R}^n) \to C^{t-\alpha}_{\text{loc}}(\mathbf{R}^n)$

is **continuous** for $\forall t > \alpha.$

Heat Kernel for the Fractional Laplacian

$$u(x,t) := T_t f(x) = P_t * f(x), \quad x \in \mathbf{R}^n, t > 0.$$

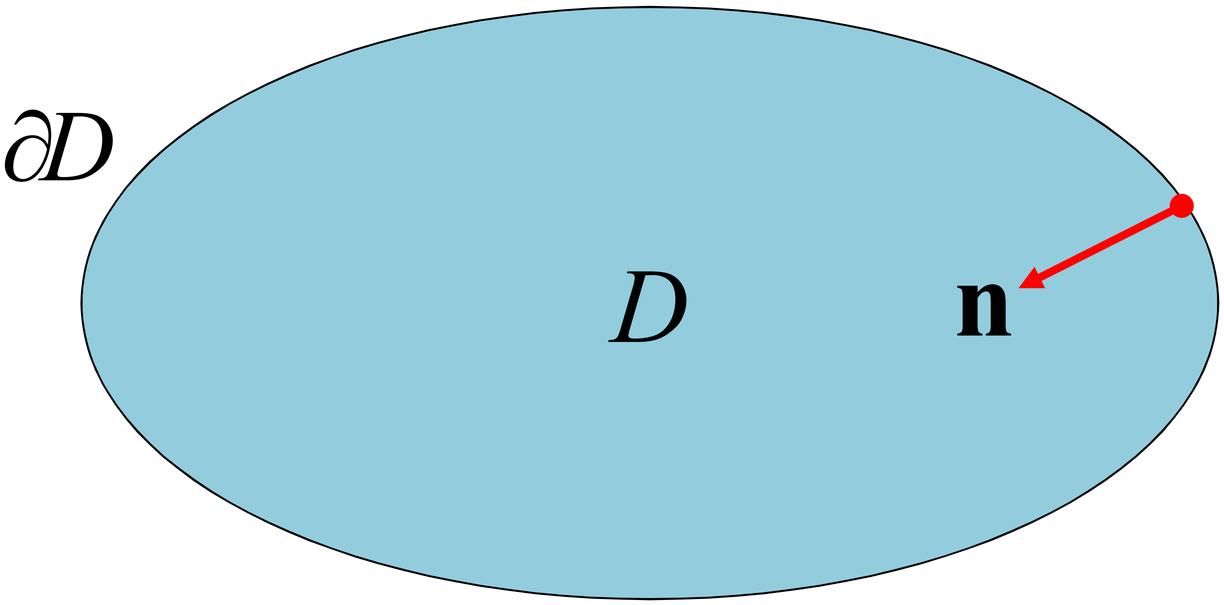
$$\Rightarrow$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{\alpha/2} u = 0, & \forall x \in \mathbf{R}^n, \forall t > 0, \\ u\big|_{t=0} = f \end{cases}$$

Reflecting Diffusion

Bounded Domain with Smooth Boundary

$$\boxed{\mathbf{R}^N, \quad N \geq 2}$$



Function Space

$C(\overline{D}) =$ **the space of real-valued, continuous functions on the closure** $\overline{D} = D \cup \partial D,$

with the maximum norm

$$\|u\| = \max_{x \in \overline{D}} |u(x)|$$

Feller Semigroups

A family of bounded linear operators $\{T_t\}_{t \geq 0}$ is called a Feller semigroup if it satisfies the following three conditions :

(1) $T_{t+s} = T_t \bullet T_s, \ \forall t, s \geq 0.$

(2) $\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\| = 0, \ \forall f \in C(\overline{D}).$

(3) $\forall f \in C(\overline{D}), 0 \leq f \leq 1 \text{ on } \overline{D} \Rightarrow 0 \leq T_t f \leq 1 \text{ on } \overline{D}.$

Main Theorem
(Neumann case)

We define a linear operator

$\mathfrak{A} : C(\overline{D}) \to C(\overline{D})$

as follows :

(a) $D(\mathfrak{A}) = \left\{ u \in C(\overline{D}) : \Delta u \in C(\overline{D}), \dfrac{\partial u}{\partial \mathbf{n}} = 0 \right\}$

(b) $\mathfrak{A} u = \Delta u, \; \forall u \in D(\mathfrak{A})$

Then $\mathfrak{A}$ **generates a Feller semigroup** $e^{t\mathfrak{A}}$.

Neumann Problem
(Mezzo-Scope)

Find a solution u of the Neumann problem

$$\begin{cases} (\alpha - \Delta)u = f & \text{in } D, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases}$$

Here $\alpha > 0$ is a parameter.

Bird's-Eye View (1)

$\boxed{T_t = e^{t\mathfrak{A}}}$	**Initial-value Problem** $\Longleftarrow$	$\boxed{\text{Heat Equation}}$
Hille - Yosida $\Updownarrow$		
$\boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}}$	$\Longleftarrow$ **Potential Theory**	$\boxed{\text{Neumann Problem}}$

Bird's-Eye View (2)

$$
\begin{tikzcd}
\boxed{p_t(x,dy)} \arrow[d, Leftrightarrow, "\text{Laplace}"'] & & \boxed{T_t = e^{t\mathfrak{A}}} \arrow[ll, Rightarrow, "\text{Levy}"'] \arrow[d, Leftrightarrow, "\text{Hille - Yosida}"] \\
G_\alpha(x,y) & & \boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}} \arrow[ll, Rightarrow, "\text{Riesz-Markov}"]
\end{tikzcd}
$$

Riesz-Markov-Dynkin Representation Theorem

$$T_t f(x) = \int_{\overline{D}} \exists!\, p_t(x, dy) f(y), \ \forall f \in C(\overline{D})$$

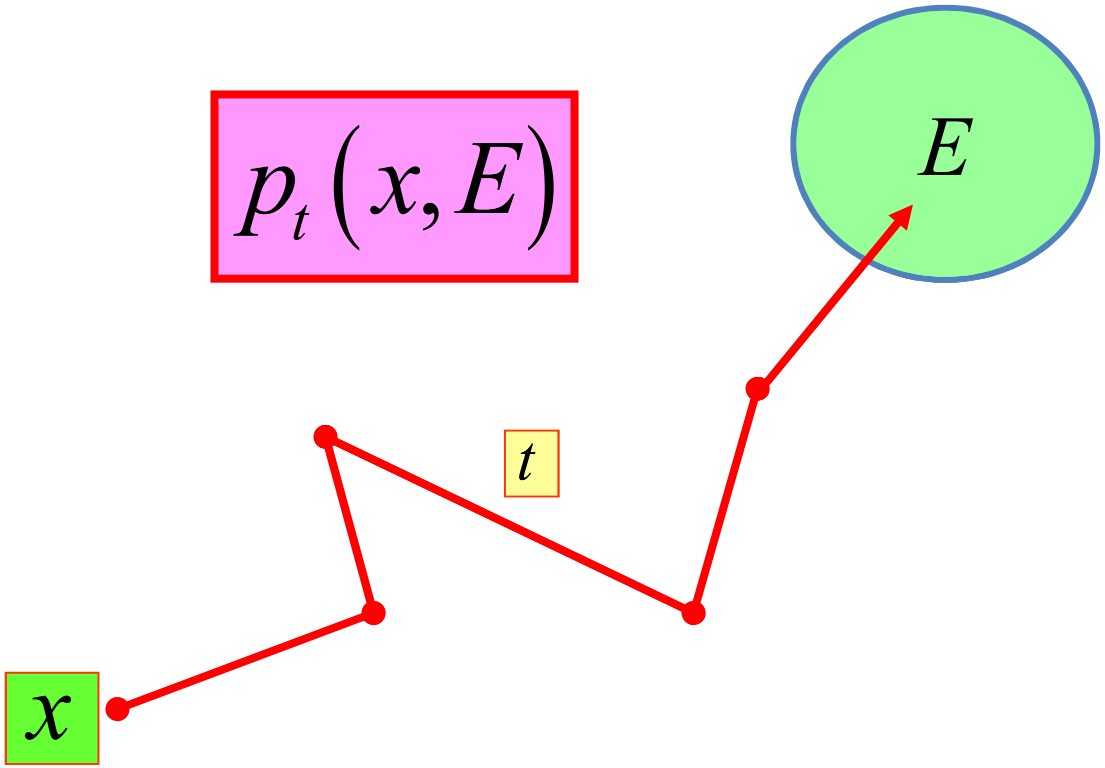
$$\Leftrightarrow$$

$$0 \le p_t(x, \bullet) \le 1, \ \forall t \ge 0, \forall x \in \overline{D}$$

Markov Transition Probability
(Macro-Scope)

$p_t(x, E) =$ **the transition probability that a Markovian particle starting at position x will be found in the set E at time t.**

Transition Probability
(Macro-Scope)



Chapman-Kolmogorov Equation
(Markov Property)

$$\boxed{p_{t+s}(x,E) = \int_{\overline{D}} p_s(y,E)\,p_t(x,dy)}$$

$$\Updownarrow$$

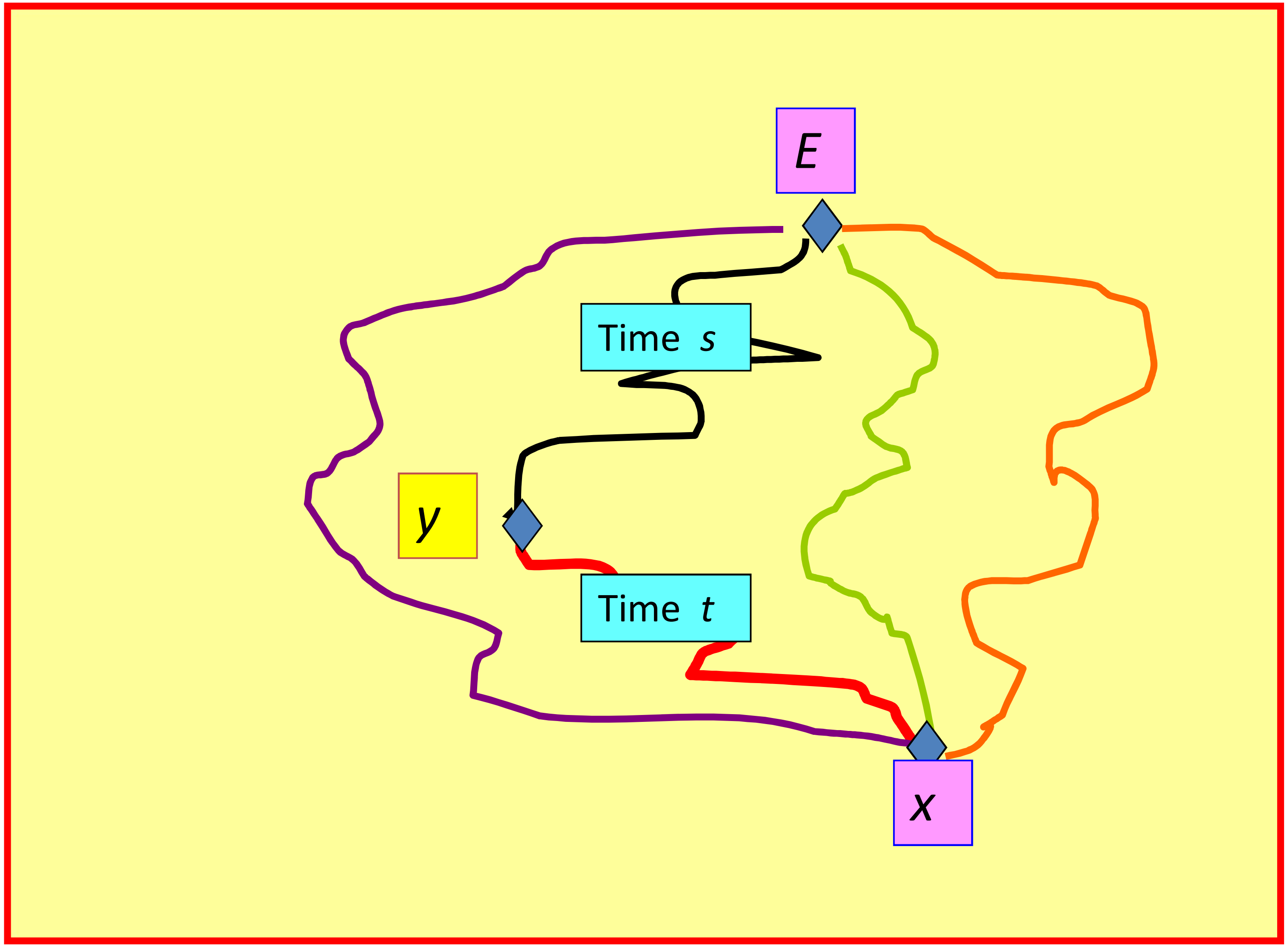
$$\boxed{T_{t+s} = T_t \bullet T_s, \quad \forall t,s \geq 0.}$$

Chapman-Kolmogorov Equation

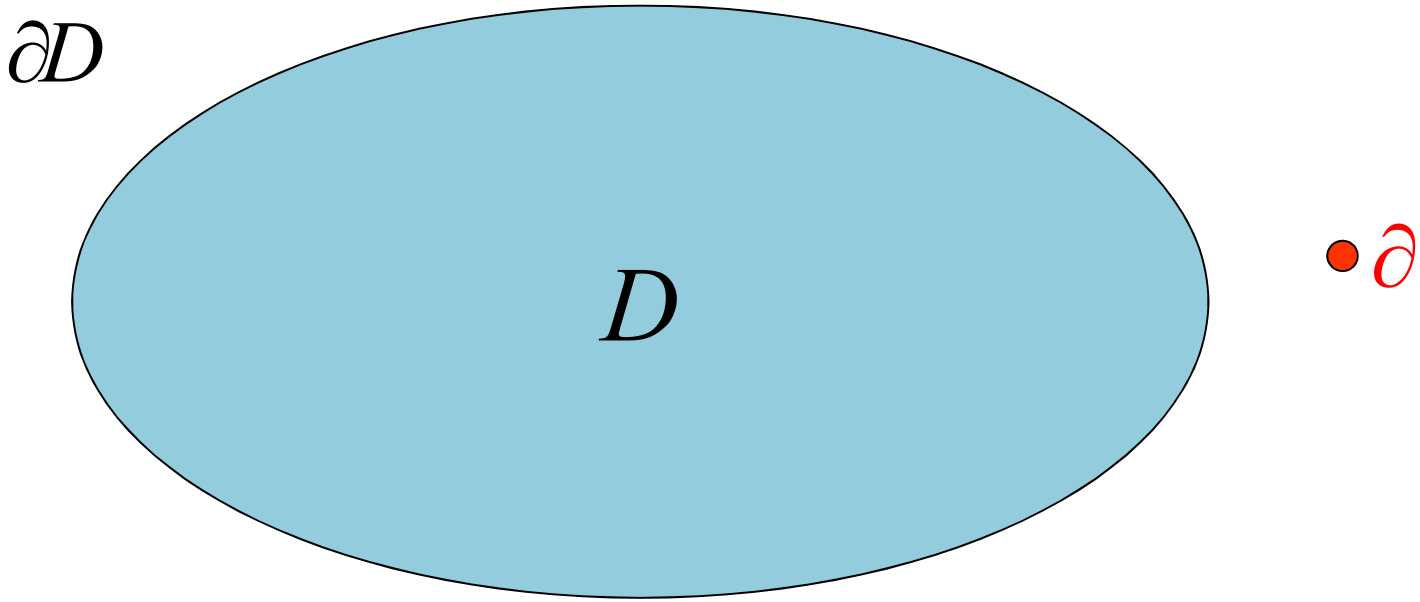
$$\boxed{p_{t+s}(x,E) = \int_{D} p_t(x,dy)\,p_s(y,E), \quad \forall t,s \geq 0}$$

A transition from x **to** E **in time** $t+s$ **is composed of a transition from** x **to some** y **in time** t**, followed by a transition from** y **to** E **in time** s.

Markov Property



Isolated Point (Cemetery)



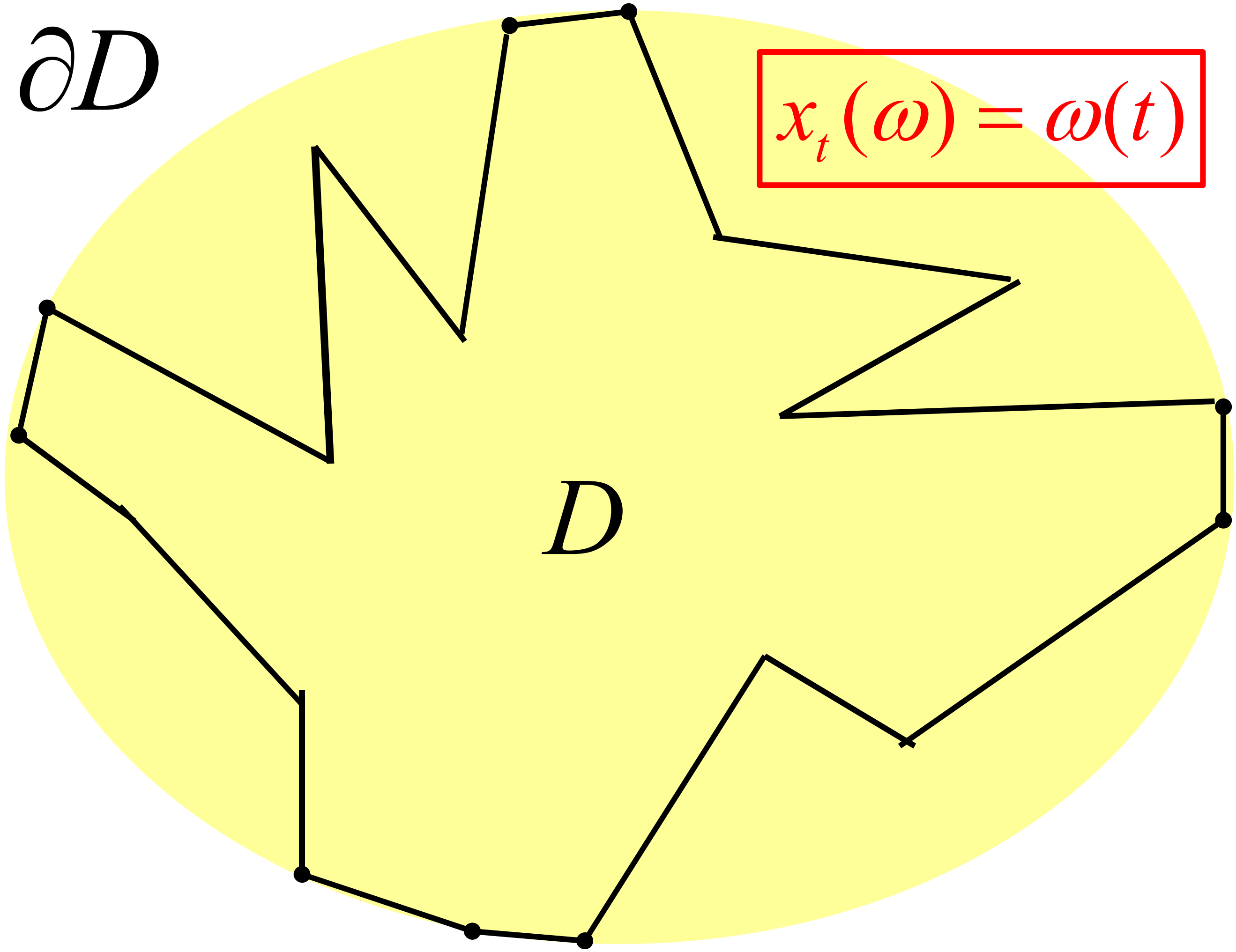
$$\boxed{E := D \cup \partial D \cup \{\partial\}}$$

Reflecting Diffusion Process

$W =$ **the space of right - continuous paths**

$\omega : [0, +\infty] \to \overline{D} \cup \{\partial\}$

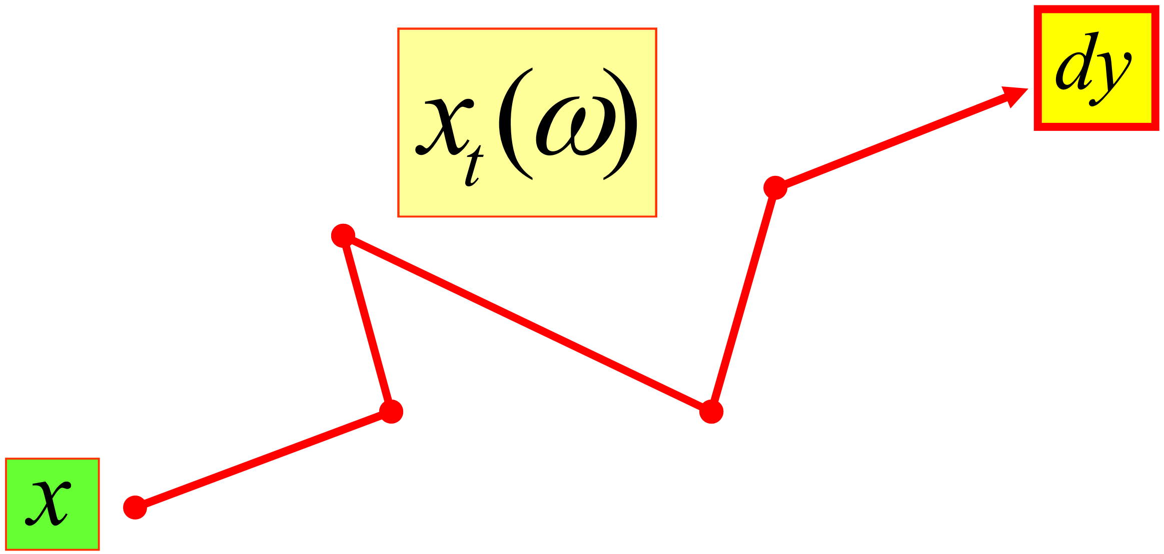
with coordinates $x_t(\omega) = \omega(t)$



∂D
$x_t(\omega) = \omega(t)$
D

Transition Probabilities

$$\boxed{P_x\left(\left\{\omega \in W : x_t(\omega) \in dy\right\}\right) = p_t(x, dy)}$$

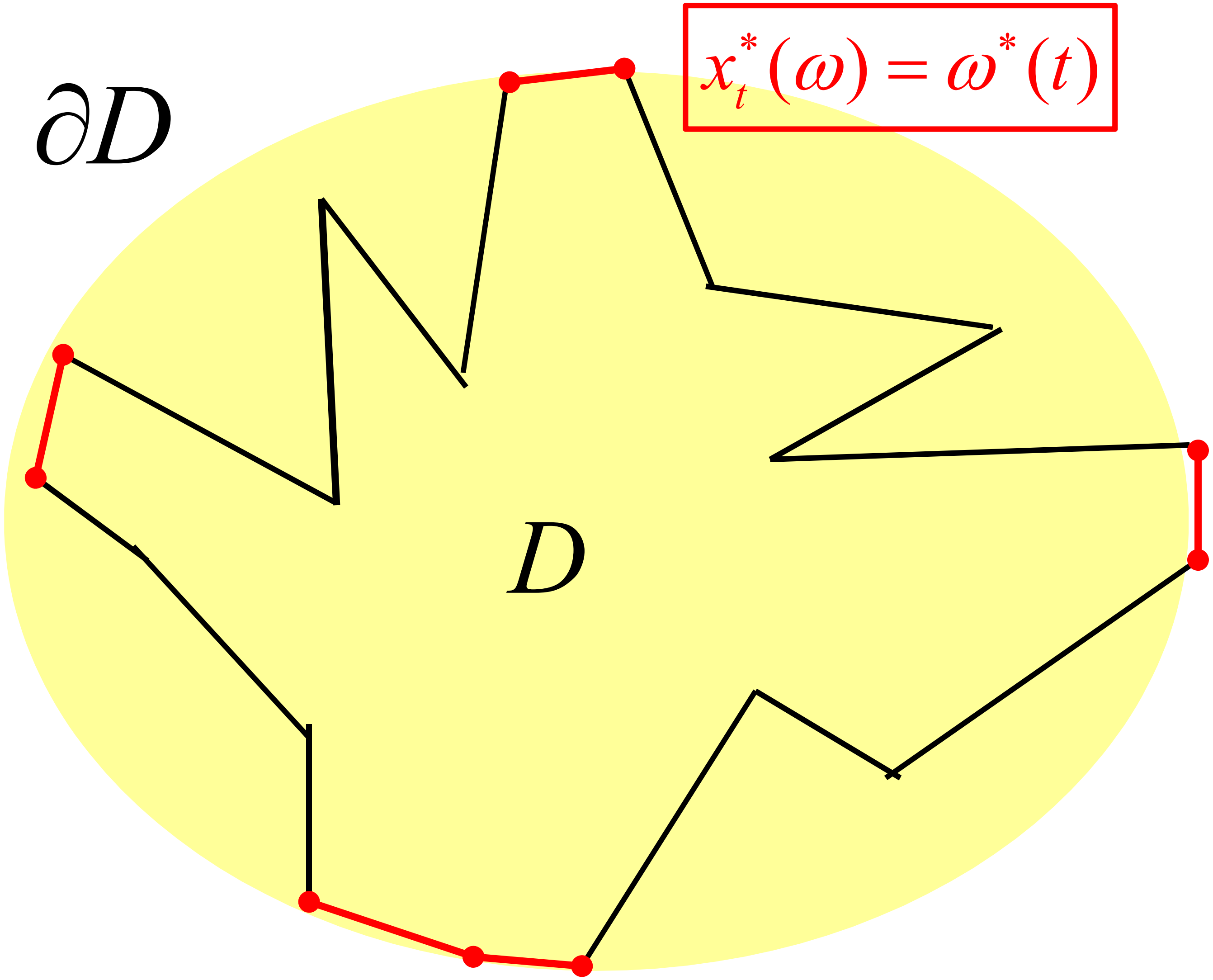


Probabilistic Transition Semigroup

$$
\begin{aligned}
T_t f(x) &= \int_{\overline{D}} p_t(x, dy) f(y) \\
&= \int_W f(x_t(\omega)) P_x(d\omega) \\
&= E_x\big(f(x_t)\big), \ \forall f \in C(\overline{D})
\end{aligned}
$$

Markov Process on the Boundary (1)

A **Markov process on the boundary** ∂D **can be obtained from the trace on** ∂D **of trajectories of the reflecting diffusion process on** $\overline{D} = D \cup \partial D$.


∂D
$x_t^*(\omega) = \omega^*(t)$
D

Markov Process on the Boundary (2)

W^* = **the space of right-continuous paths**

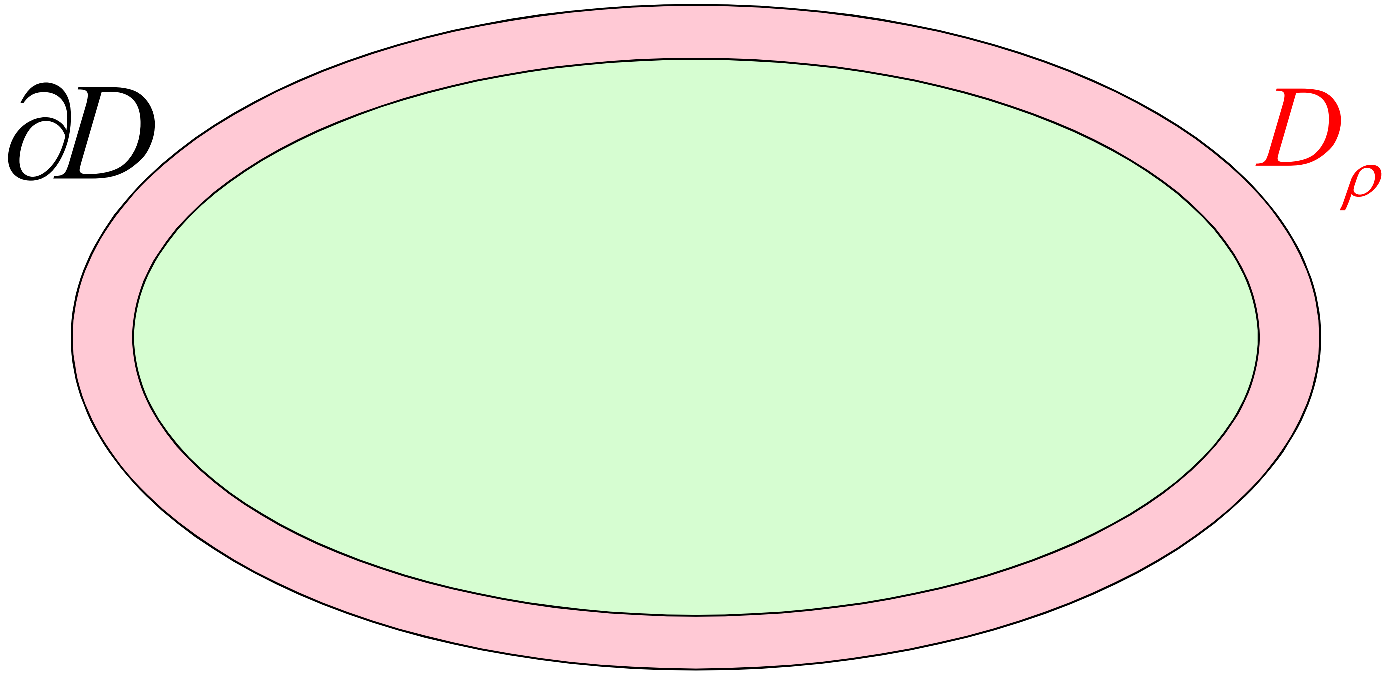
$$\omega^* : [0, +\infty] \to \partial D \cup \{\partial\}$$

with coordinates $x_t^*(\omega) = \omega^*(t)$

World Watch due to Levy

Domain	Trajectories	Watch
Interior D	$x_t(\omega)$	t
Boundary ∂D	$x_t^*(\omega)$	$\tau(t,\omega)$

Local Time on the Boundary (1)



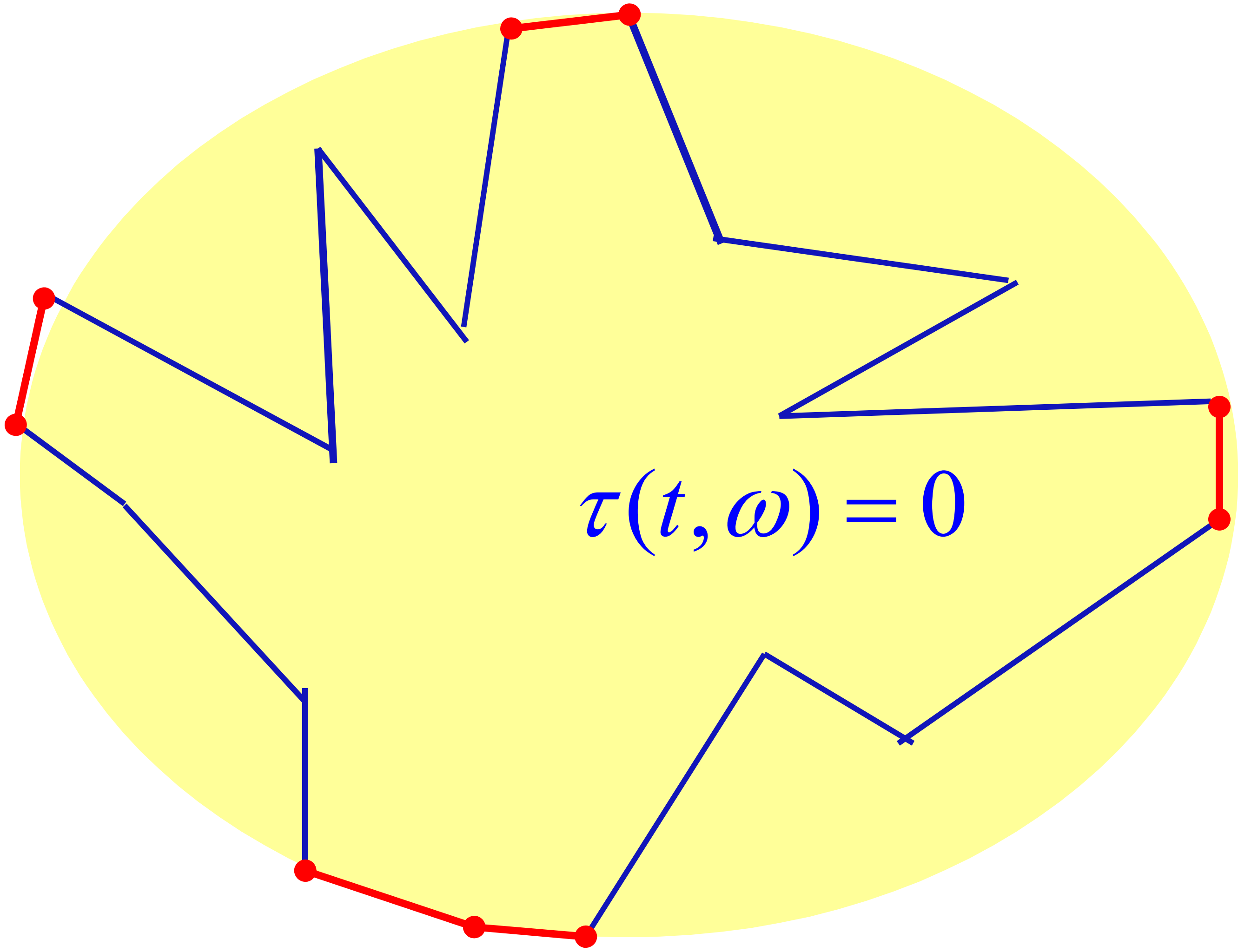
$$\boxed{D_\rho = \{x \in D : \mathrm{dist}(x, \partial D) < \rho\}}$$

Local Time on the Boundary (2)

$$\tau(t,\omega) = \lim_{\rho \downarrow 0} \frac{1}{\rho} \int_0^t \chi_{D_\rho}(x_s(\omega))\, ds, \quad \omega \in W$$

$\tau(t,\omega) =$ **the sojourn time of a path** $x_s(\omega)$ **on** ∂D **up to time** t.

$$\boxed{x^{*}_{\tau(t,\omega)}(\omega^{*}) = x_t(\omega)}$$



Bird's-Eye View

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{p_s^*(x',dy')} \arrow[rr, Rightarrow, "\text{Riesz-Markov-Dynkin}"] \arrow[d, Leftrightarrow, "\text{Laplace}"'] & & \boxed{T_s^* = e^{s\mathfrak{A}^*}} \arrow[d, Rightarrow, "\text{Hille - Yosida}"] \\
G_\beta^*(x',y') \arrow[rr, Leftrightarrow, "\text{Riesz-Markov}"'] & & \boxed{\left(\beta I - \mathfrak{A}^*\right)^{-1}}
\end{tikzcd}
$$

Probabilistic Transition Semigroup

$$
\begin{aligned}
T_s^* \varphi(x') &= \int_{\partial D} p_s^*(x', dy')\varphi(y') \\
&= \int_{W^*} \varphi(x_s^*(\omega^*)) P_{x'}^*\left(d\omega^*\right) \\
&= E_{x'}^*\left(\varphi(x_s^*)\right), \ \forall \varphi \in C(\partial D)
\end{aligned}
$$

Characterization of the Generator (1)

$$T_s^* = e^{s\mathfrak{A}^*}$$

$$\mathfrak{A}^*\varphi(x') = -\left(-\Lambda'\right)^{1/2}\varphi(x')$$

$$= -\frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}} e^{ix'\xi'}|\xi'|\widehat{\varphi}(\xi')\,d\xi'$$

$$\mathfrak{A}^* = -\left(-\Lambda'\right)^{1/2}$$

Characterization of the Generator (2)

$$\mathfrak{A}^*\varphi(x') = -\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}} e^{ix'\xi'}|\xi'|\widehat{\varphi}(\xi')\,d\xi'$$

$$= -\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}} e^{ix\xi}|\xi'|\left(\int_{\mathbf{R}^{n-1}} e^{-iy'\xi'}\varphi(y')\,dy'\right)d\xi'$$

$$= -\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{R}^{n-1}}\left(\int_{\mathbf{R}^{n-1}} e^{i(x'-y')\xi'}|\xi'|\,d\xi'\right)\varphi(y')\,dy'$$

$$= \frac{\Gamma(n/2)}{\pi^{n/2}}\,\text{v.p.}\int_{\mathbf{R}^{n-1}}\frac{1}{|x'-y'|^n}\varphi(y')\,dy'$$

Characterization of the Generator (3)

(1) $\mathfrak{A}^* = -\left(-\Lambda'\right)^{1/2}$

(2) **Principal Symbol:** $-|\xi'|$

(3) **Distribution kernel:**

$$\frac{\Gamma(n/2)}{\pi^{n/2}} \, \text{v.p.} \, \frac{1}{|x'-y'|^n}$$

Remarks

(1) $|\xi'|$: the **length of ξ' with respect to the Riemannian metric of ∂D induced by the natural metric of R^N.**

(2) $|x' - y'|$: **the geodesic distance between x' and y' with respect to the Riemannian metric of ∂D.**

Fourier Transform

$$\frac{1}{(2\pi)^{n-1}} \int_{\mathbf{R}^{n-1}} e^{ix'\xi'} |\xi'| \, d\xi' = \frac{\Gamma(n/2)}{\pi^{n/2}} \, \text{v.p.} \frac{1}{|x'|^n}$$

Principal Value of the Distribution

$$\text{v.p.}\ \frac{1}{|x'|^n}$$

$$\left\langle \text{v.p.}\ \frac{1}{|x'|^n}, \varphi(x') \right\rangle$$

$$= \lim_{\varepsilon \downarrow 0} \int_{|y'| \geq \varepsilon} \frac{\varphi(y') - \varphi(0)}{|y'|^n}\, dy', \quad \forall \varphi \in C_0^\infty(\mathbf{R}^{n-1})$$

Intuitive Meaning of the Generator

$\mathfrak{A}$: **Integral (non - local) Operator**

This Markov process can be thought as the trace on ∂D of the reflecting diffusion, and it moves by jumps.

Multi-Dimensional

General Case

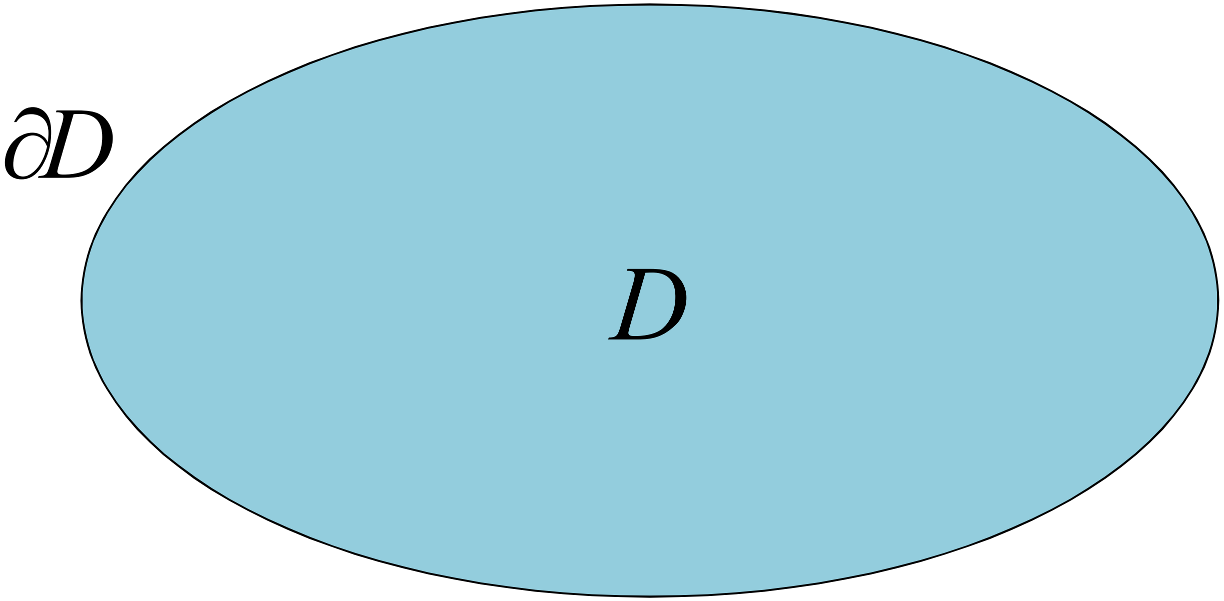
My Work

K. Taira: Semigroups, boundary value problems and Markov processes, 2nd Edition, Springer-Verlag, Springer Monographs in Mathematics, 2014

DOI: 10.1007/978-3-662-43696-7

Bounded Domain with Smooth Boundary

$\boxed{\mathbf{R}^N, \quad N \geq 2}$



Brief History (1)
(multi-dimensional case)

- **1959: A.D. Wentzell (Ventcel')**
- **1964: W.v. Waldenfels**
- **1965: K. Sato and T. Ueno (semigroup approach, abstract setting)**
- **1968: J.M. Bony, P. Courrege and P. Priouret (semigroup approach, non-degenerate case)**

Brief History (2)
(multi-dimensional case)

- **1982: K. Taira (semigroup approach, degenerate case, pseudo-differential operators)**
- **1986: C. Cancelier (semigroup approach, degenerate case, elliptic regularizations)**
- **1988: S. Takanobu and S. Watanabe (stochastic approach, degenerate case)**

References

- **Wentzell:** Theory Prob. and its Appl. 4 (1959), 164-177.
- **Sato and Ueno**: J. Math. Kyoto Univ. 14 (1965), 529-605.
- **Bony, Courrege and Priouret** : Ann. Inst. Fourier 19 (1969), 277-304.
- **Cancelier**: Comm. P. D. E. 11 (1986), 1677-1726.
- **Taira**: Academic Press, 1988.
- **Takanobu and Watanabe**: J. Math. Kyoto Univ. 28 (1988), 71-80.

Analytic Methods

Bird's-Eye View

$\boxed{T_t = e^{t\mathfrak{A}}}$	$\Leftarrow$	Parabolic Theory
$\Uparrow$		
$\boxed{(\alpha I - \mathfrak{A})^{-1}}$	$\Leftarrow$ **Sato-Ueno**	**Elliptic Boundary Value Problems**

Bird's Eye View

Probability Theory (Micro-Scope)	Functional Analysis (Macro-Scope)	Partial Differential Equations (Mezzo-Scope)
Markov Processes	Feller Semigroups	Boundary Value Problems
Markov Property	Semigroup Property	•Waldenfels Operators •Wentzell Conditions

My Work

$$\boxed{\textbf{Feller Semigroup } e^{t\mathfrak{A}}} \quad \Longleftrightarrow \quad \boxed{\text{Parabolic Theory}}$$

$$\Uparrow$$

$$\boxed{\left(\alpha I - \mathfrak{A}\right)^{-1}} \quad \Longleftarrow \quad \boxed{\textbf{Boutet de Monvel Calculus}}$$

My Strategy

(1) Existence and uniqueness theorems for Waldenfels operators with Wentzell boundary conditions (Partial Differential Equations)

(2) Generation theorems for Feller semigroups (Functional Analysis)

(3) Existence theorems for Markov processes (Probability)

Feller Semigroups

A family of bounded linear operators $\{T_t\}_{t\geq 0}$ is called a Feller semigroup if it satisfies the following three conditions:

(1) $T_{t+s} = T_t \bullet T_s, \ \ \forall t, s \geq 0.$

(2) $\lim_{s\downarrow 0} \left\| T_{t+s} f - T_t f \right\| = 0, \ \forall f \in C(\overline{D}).$

(3) $\boxed{\forall f \in C(\overline{D}), 0 \leq f \leq 1 \text{ on } \overline{D} \Rightarrow 0 \leq T_t f \leq 1 \text{ on } \overline{D}}.$

Wentzell's Work (in 1959)

$T_t = e^{t\mathfrak{A}}$: **Feller semigroup**

$\mathfrak{A}$: **infinitesimal generator**

$\Downarrow$

$$(1)\ D(\mathfrak{A}) = \{u : \exists \boldsymbol{L} u = 0 \text{ on } \partial \mathrm{D}\}.$$

$$(2)\ \mathfrak{A} u = \exists \boldsymbol{W} u, \quad u \in D(\mathfrak{A}).$$

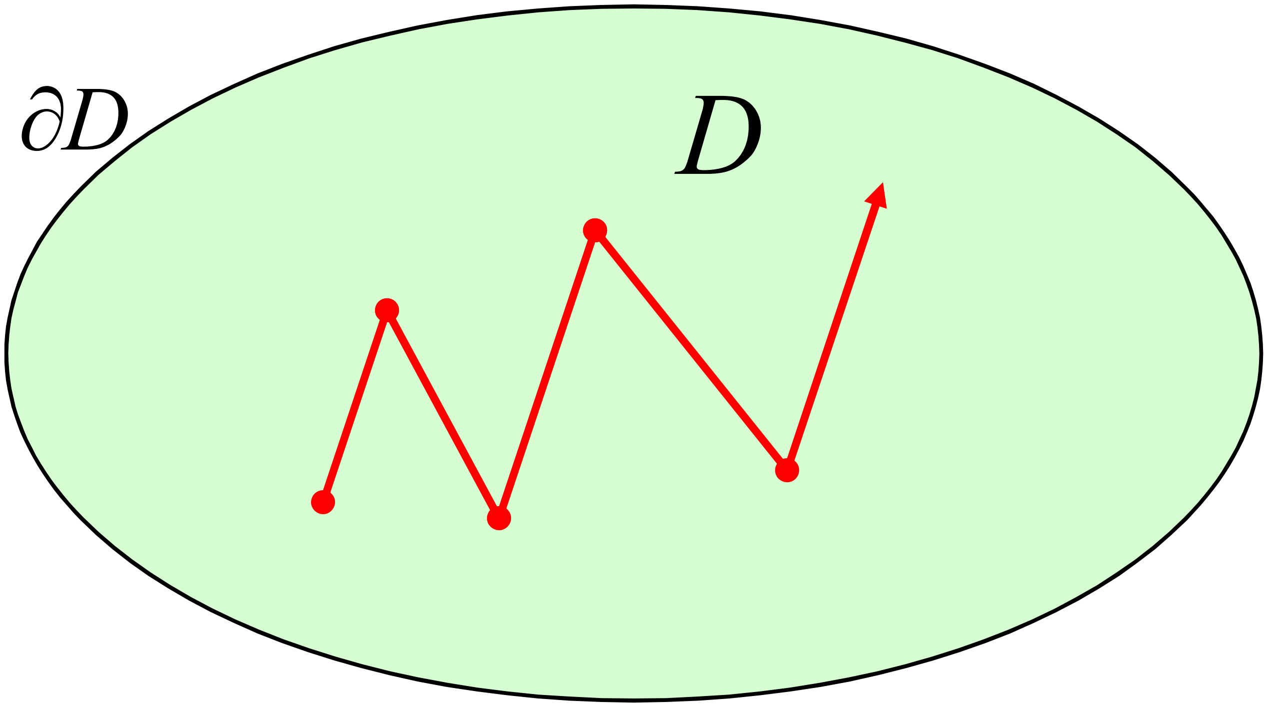
Waldenfels' Work (in 1963)
(integro-differential operator)

$$
\begin{aligned}
Wu &:= Au + Su \\
&= \left(\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \right) \\
&\quad + \int_D s(x, dy) \left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x) \right]
\end{aligned}
$$

Diffusion Operator
(differential operator)

$$\boxed{Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u}$$

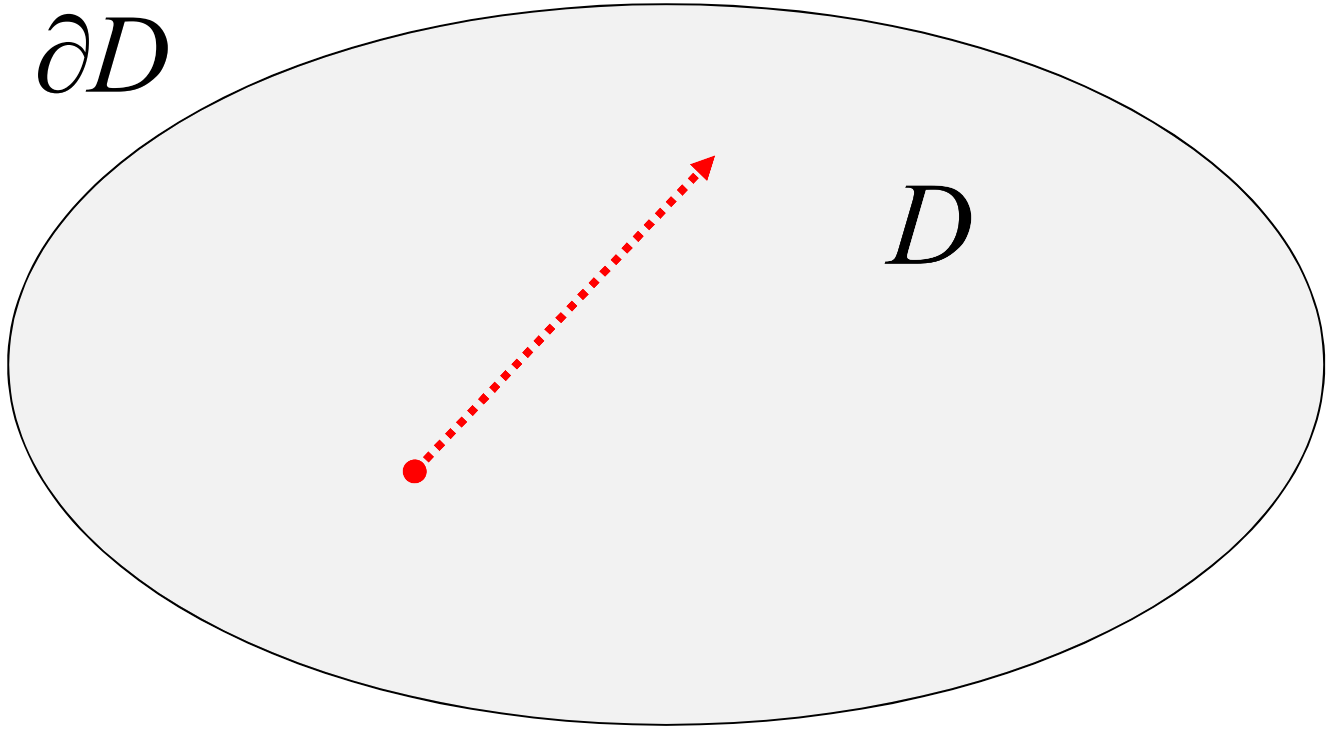
Diffusion Phenomenon
(continuous motion)



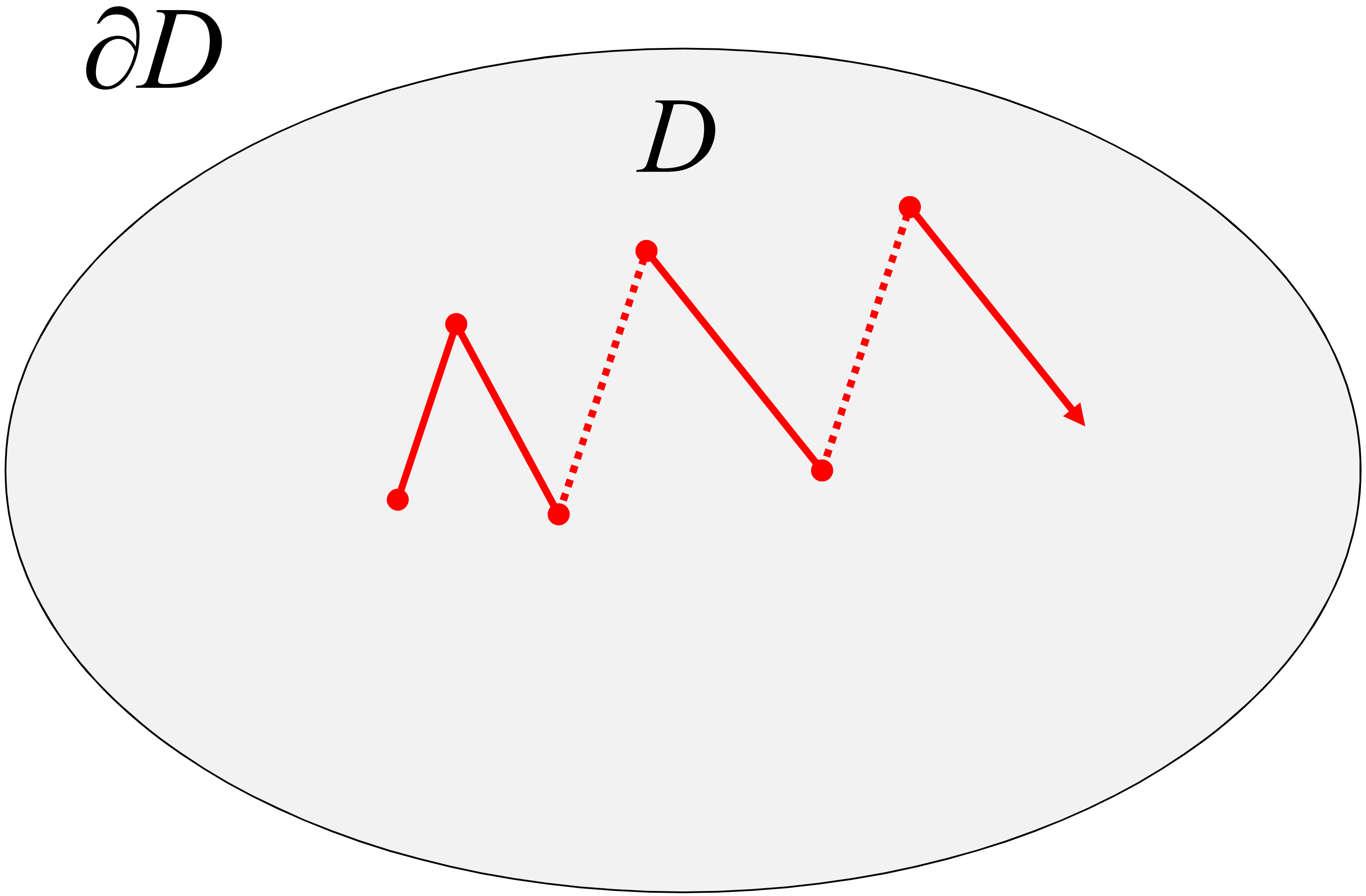
Lévy Operators of First-Order
(Integro-Differential Operator)

$$\boxed{Su = \int_D s(x,dy)\left[u(y) - u(x) - \sum_{j=1}^{N} (y_j - x_j)\frac{\partial u}{\partial x_j}(x) \right]}$$

Jump Phenomenon
(Discontinuous Motion)



General Case



Wentzell's Work (in 1959)
(General Boundary Condition)

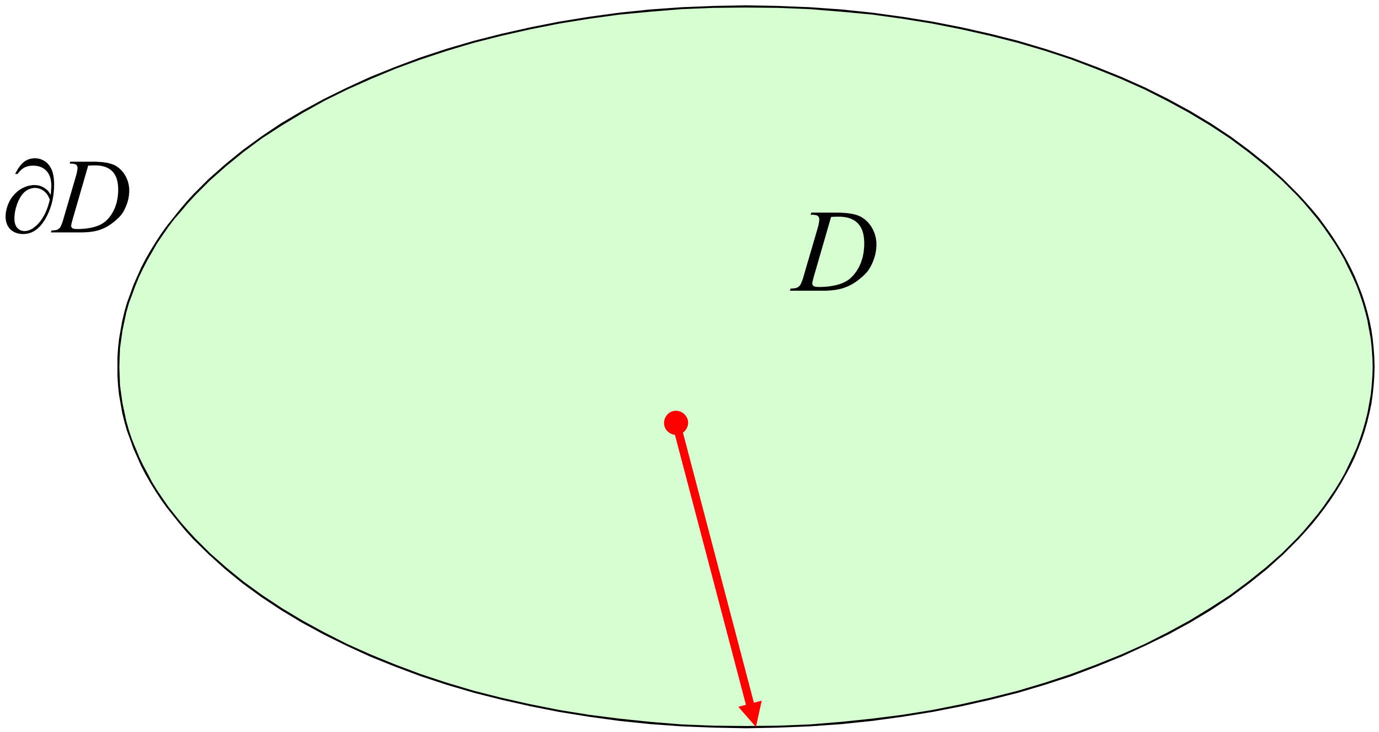
$$\exists \boldsymbol{L}u = \left(\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i} \right)$$

$$\gamma(x')u + \mu(x') \frac{\partial u}{\partial \mathbf{n}} - \delta(x') W u$$

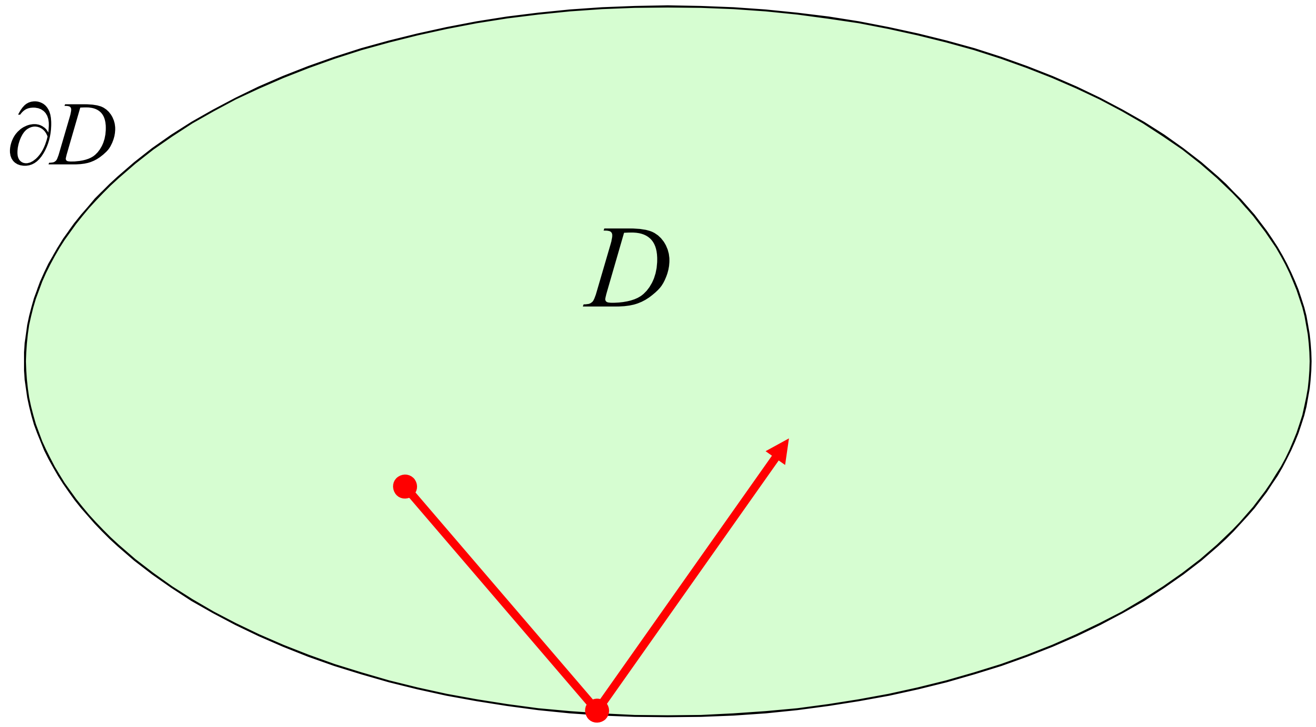
$$+ \int_{\partial D} r(x', dy') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right]$$

$$+ \int_{D} t(x', dy) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right]$$

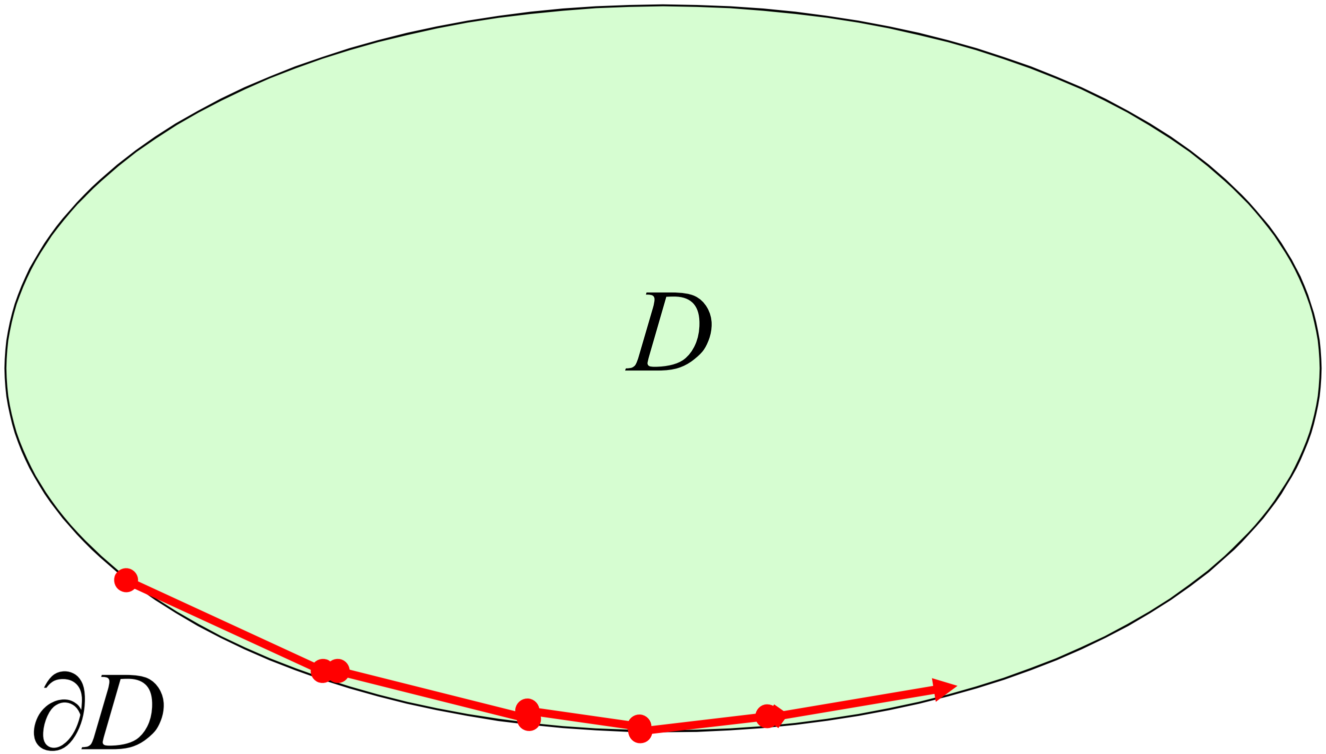
Absorption Phenomenon
(Dirichlet Condition)



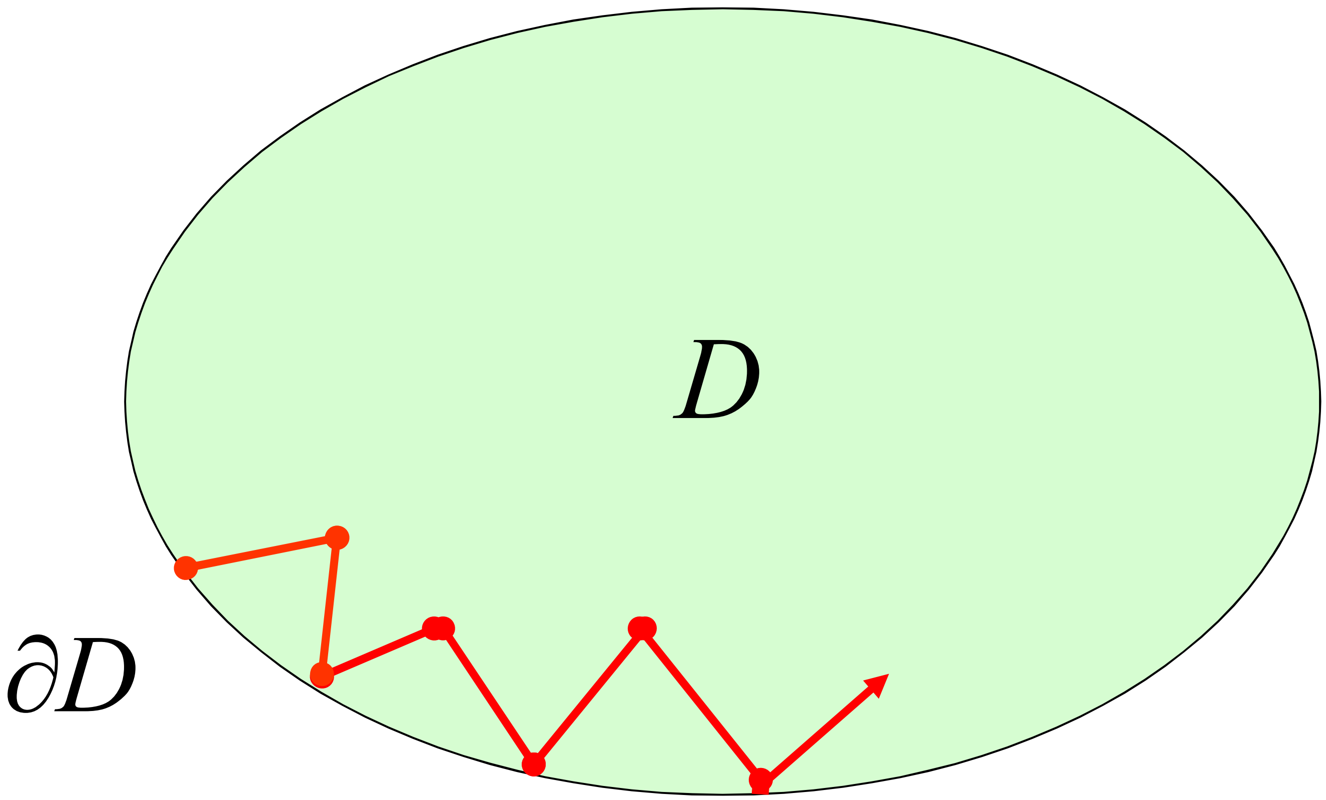
Reflection Phenomenon
(Neumann Condition)



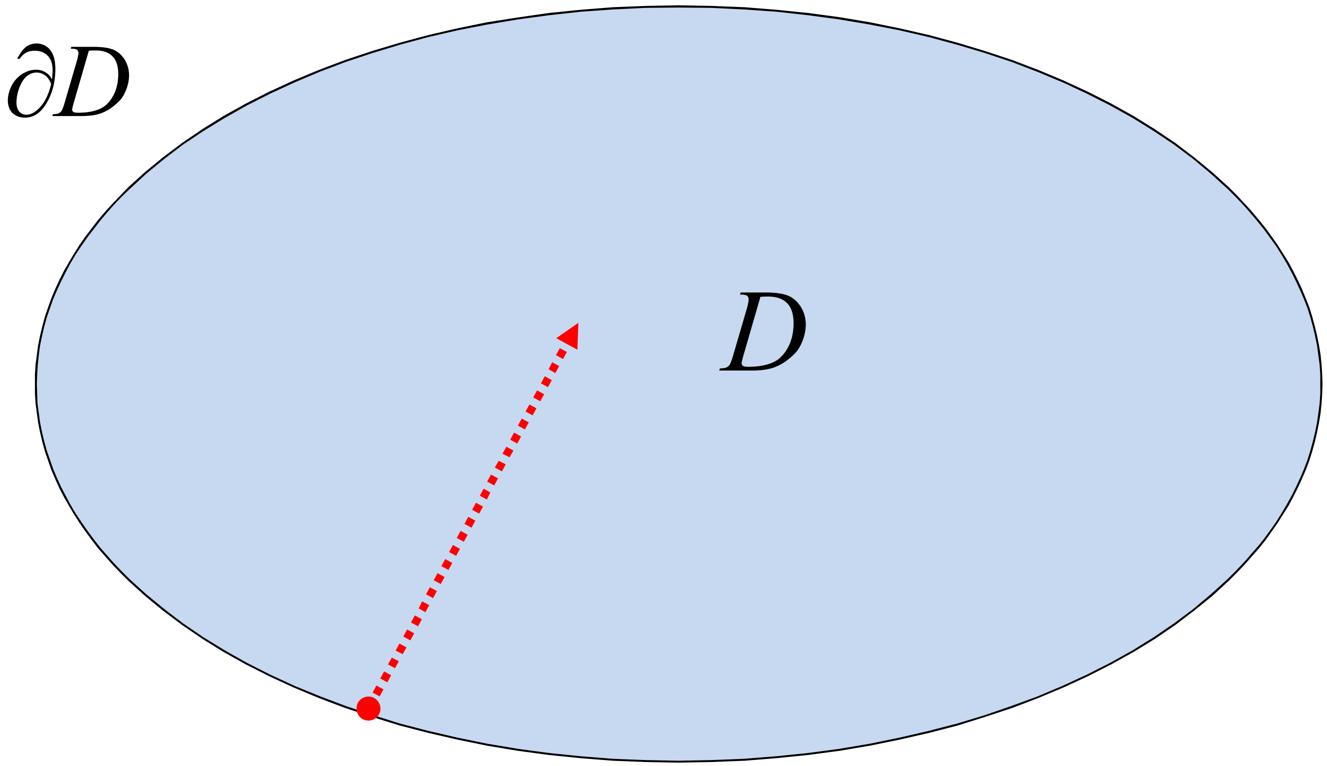
Diffusion on the Boundary



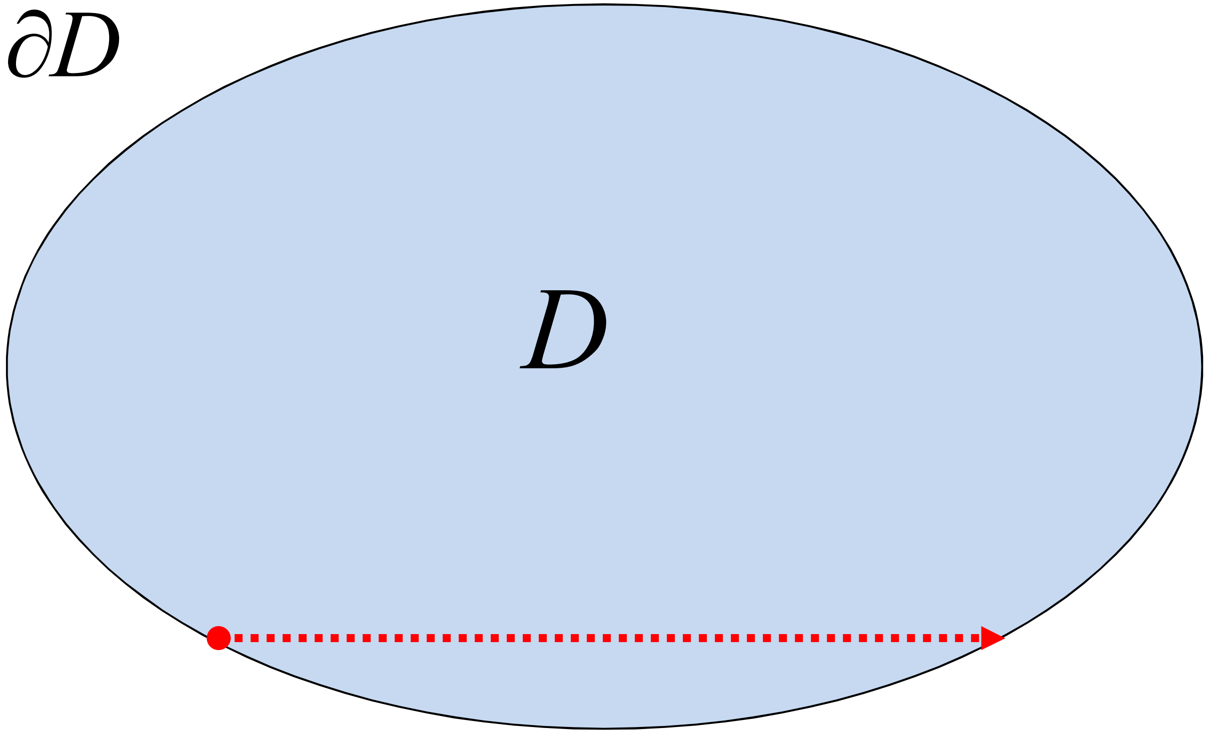
Viscosity Phenomenon



Jump Phenomenon (1)



Jump Phenomenon (2)



Boundary Value Problems

(Mezzo-Scope)

Construction of the Green Operator
(Mezzo-Scope)

$$(\alpha - \boldsymbol{W})u = f \quad \text{in } D$$

$$\boldsymbol{L}u = 0 \quad \text{on } \partial D$$

My Work

$$u = \boldsymbol{G}_\alpha f = \left(\alpha I - \mathfrak{A}\right)^{-1} f$$

Waldenfels Operator

$$
\begin{aligned}
Wu &:= Au + Su \\
&= \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u \\
&\quad + \int_D s(x,y)\left[u(y) - u(x) - \sum_{j=1}^{N}(y_j - x_j)\frac{\partial u}{\partial x_j}(x)\right]dy
\end{aligned}
$$

Diffusion Operator

$$Au = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

Here:

(1) $a^{ij}(x) \in C^\infty(\mathbf{R}^N),\ a^{ij}(x) = a^{ji}(x)$

$$\boxed{\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq \exists\lambda\,|\xi|^2,\ \forall x \in \mathbf{R}^N, \forall \xi \in \mathbf{R}^N}$$

(2) $b^i(x) \in C^\infty(\mathbf{R}^N)$

(3) $c(x) \in C^\infty(\mathbf{R}^N),\ c(x) \leq 0,\ \forall x \in D$

Lévy Operator of First-Order

$$Su = \int_D s(x,y)\left[u(y) - u(x) - \sum_{j=1}^{N}(y_j - x_j)\frac{\partial u}{\partial x_j}(x)\right]dy$$

Here:

(1) $s(x,y)$, **distribution kernel** of

$$S \in L^{2-\kappa}_{cl}\left(\mathbf{R}^N\right),\ \kappa > 0$$

(2) $\boxed{s(x,y) \geq 0,\ \ \forall x \neq y}$

Wentzell Boundary Condition (1)

$$Lu = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i} + \gamma(x')u$$

$$+ \mu(x') \frac{\partial u}{\partial \mathbf{n}} - \delta(x') W u$$

$$+ \int_{\partial D} r(x', y') \left[u(y') - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy'$$

$$+ \int_{D} t(x', y) \left[u(y) - u(x') - \sum_{j=1}^{N-1} (y_j - x_j) \frac{\partial u}{\partial x_j}(x') \right] dy$$

Wentzell Boundary Condition (2)

$$(1)\ \ \alpha^{ij}(x) \in C^{\infty}(\partial D),\ \alpha^{ij}(x') = \alpha^{ji}(x')$$

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\eta_i \eta_j \geq 0,\ \forall x' \in \partial D, \forall \eta' \in T^*_{x'}(\partial D)$$

$$(2)\ \gamma(x') \in C^{\infty}(\partial D),\ \gamma(x') \leq 0,\ \forall x' \in \partial D$$

$$(3)\ \mu(x') \in C^{\infty}(\partial D),\ \mu(x') \geq 0,\ \forall x' \in \partial D$$

$$(4)\ \delta(x') \in C^{\infty}(\partial D),\ \delta(x') \geq 0,\ \forall x' \in \partial D$$

Wentzell Boundary Condition (3)

(1) $r(x', y')$, **distribution kernel of**

$$R \in L_{cl}^{2-\kappa_1}(\partial D), \ \kappa_1 > 0$$

(2) $\boxed{r(x', y') \geq 0, \quad \forall x' \neq y'}$

(3) $t(x, y)$, **distribution kernel of**

$$T \in L_{cl}^{1-\kappa_2}\left(\mathbf{R}^N\right), \ \kappa_2 > 0$$

(4) $\boxed{t(x, y) \geq 0, \quad \forall x \neq y}$

Transversal Condition (1)

$$\int_D t(x', y)\, dy = +\infty \quad \text{if } \mu(x') = \delta(x') = 0$$

$$\frac{1}{\int_D t(x', y)\, dy} = \text{ the sojourn time at } x'$$

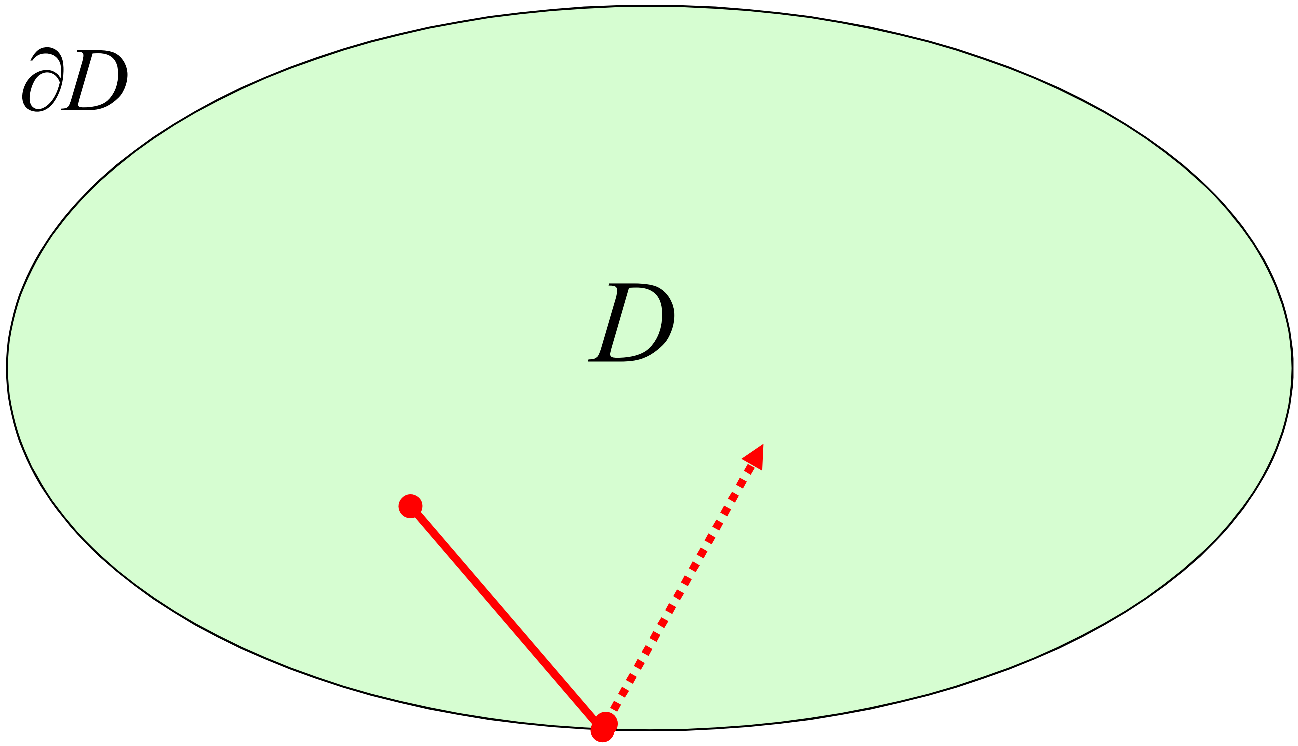
Transversal Condition (2)

Intuitively, the transversal condition implies that a Markovian particle jumps away instantaneously from the points $x' \in \partial D$ where neither reflection nor viscosity phenomenon occurs.

$\Downarrow$

Instantaneous return process

Transversal Condition (3)



Instantaneous return process

Probabilistically, this means that every Markov process on the boundary ∂D is the trace on ∂D of trajectories of some Markov process on the closure $\overline{D} = D \cup \partial D$.

Main Results

Main Theorem (General Case)

We define a linear operator

$$\mathfrak{W} : C(\overline{D}) \to C(\overline{D})$$

as follows:

(a) $D(\mathfrak{W}) = \left\{ u \in C(\overline{D}) : Wu \in C(\overline{D}),\, Lu = 0 \right\}$

(b) $\mathfrak{W} u = Wu = (A + S)u, \ \forall u \in D(\mathfrak{W})$

If L is transversal, then $\mathfrak{W}$ generates a Feller semigroup.

Main Theorem (Dirichlet Case)

We define a linear operator

$\mathfrak{W} : C_0(\overline{D}) \to C_0(\overline{D})$

as follows:

(a) $D(\mathfrak{W}) = \{u \in C_0(\overline{D}) : Wu \in C_0(\overline{D})\}$

(b) $\mathfrak{W}u = Wu = (A + S)u, \ \forall u \in D(\mathfrak{W})$

Then $\mathfrak{W}$ generates a Feller semigroup.

Idea of Proof

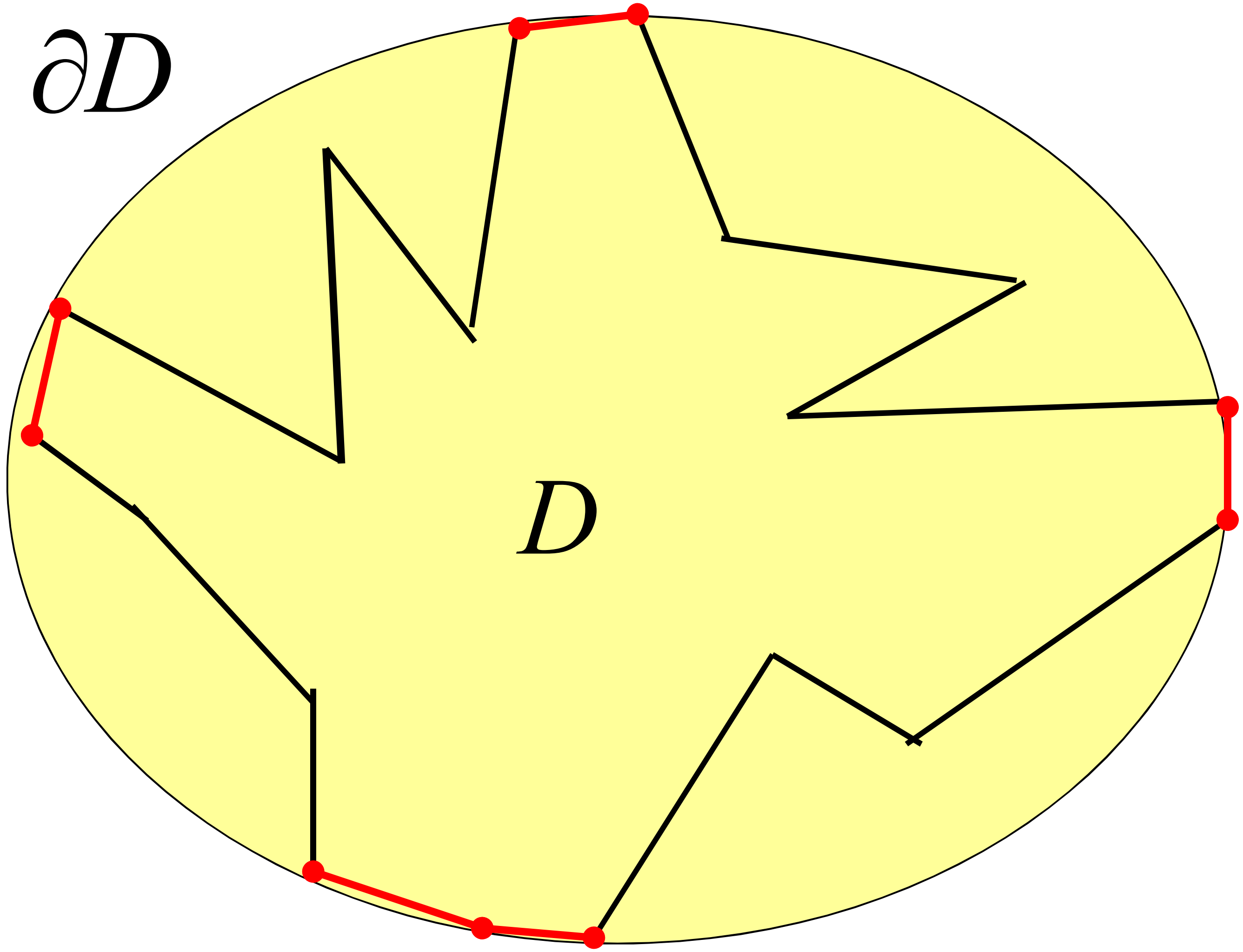
Erik Ivar Fredholm

Erik Ivar Fredholm

Erik Ivar Fredholm (1866-1927)
Swedish Mathematician

Reduction to the Boundary

Probability Theory	Partial Differential Equations
Markov processes on the boundary	Fredholm integral equations on the boundary
Markov processes on the domain	Elliptic Boundary value problems


∂D
D

Markov Process on the Boundary

The **Fredholm operator**

LH_{α}

generates a **Markov process on** ∂D

Green Operator

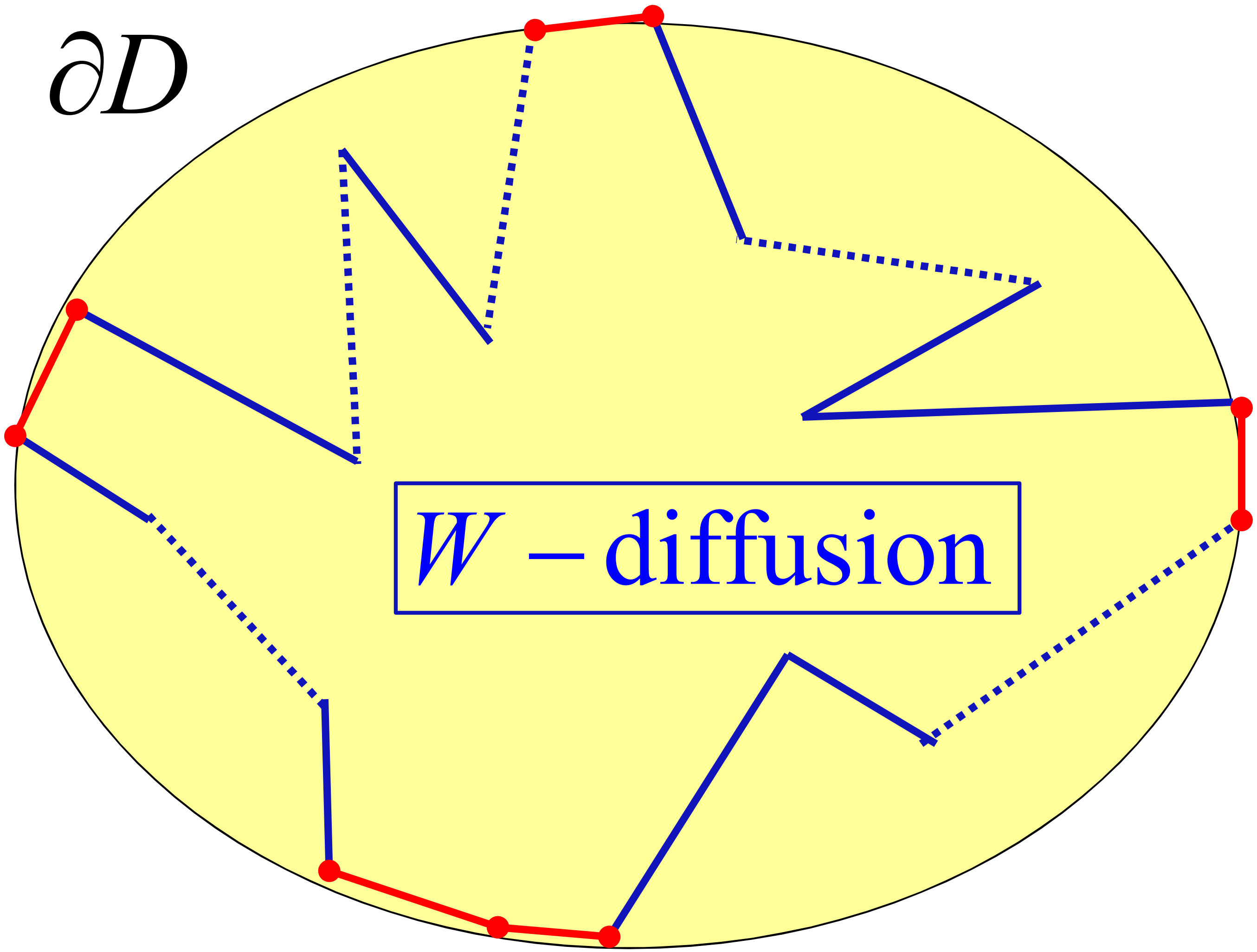
$$u = G_\alpha f$$

$$:= G_\alpha^0 f - H_\alpha \left(\overline{L H_\alpha}^{-1} \left(L G_\alpha^0 f \right) \right)$$

$$G_\alpha f = \left(\alpha I - \mathfrak{W} \right)^{-1} f$$

Probabilistic Meaning of Green Operator

Probabilistically, this formula asserts that if the boundary condition L is transversal on ∂D, then we can piece together a Markov process on ∂D with $W-$diffusion in D to construct a Markov process on the closure $\overline{D} = D \cup \partial D$.



∂D
W – diffusion

Sketch of Proof (1)

The Green operators

$$G_\alpha : C(\overline{D}) \to C(\overline{D}), \quad \forall \alpha > 0$$

are nonnegative.

$$G_\alpha f = (\alpha I - \mathfrak{W})^{-1} f$$

$$\forall f \in C(\overline{D}),\ f \geq 0 \text{ on } \overline{D} \Rightarrow G_\alpha f \geq 0 \text{ on } \overline{D}.$$

Sketch of Proof (2)

The Green operators $G_\alpha : C(\overline{D}) \to C(\overline{D}), \quad \forall \alpha > 0$ **are contractive.**

$$G_\alpha f = (\alpha I - \mathfrak{W})^{-1} f$$

$$\|G_\alpha\| \le \frac{1}{\alpha}, \quad \forall \alpha > 0.$$

Weak Maximum Principle
(Aleksandrov-Bakel'man)

Assume that:

$$u \in C(\bar{D}) \cap W^{2,N}_{\text{loc}}(D),$$

$$(W - \alpha)u(x) \geq 0 \quad \text{for almost all } x \in D.$$

Then:

$$\boxed{\sup_D u \leq \sup_{\partial D} u^+}$$

Strong Maximum Principle

Assume that:

$$u \in C(\overline{D}) \cap W^{2,N}_{\mathrm{loc}}(D),$$

$$(W - \alpha)u(x) \geq 0 \quad \text{for almost all } x \in D,$$

$$m = \sup_D u \geq 0.$$

Then:

$$\exists x_0 \in D \text{ s.t. } u(x_0) = m \Rightarrow u(x) \equiv m, \ \forall x \in D.$$

Hopf Boundary Point Lemma

Assume that:

(1) $u \in C^1(\overline{D}) \cap W^{2,N}_{\text{loc}}(D)$,

$(W - \alpha)u(x) \geq 0$ for almost all $x \in D$.

(2) $\exists x_0' \in \partial D$ such that

$$\begin{cases} u(x_0') = \sup\limits_{D} u = m \geq 0, \\ \quad u(y) < m, \ \forall y \in D. \end{cases}$$

Then:

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0.$$

Sketch of Proof (3)

$$\boxed{\int_D t(x', y)\,dy = +\infty \quad \textbf{if} \quad \mu(x') = \delta(x') = 0}$$

$$\Downarrow$$

$$\boxed{\lim_{\alpha \to +\infty} \left\| \overline{LH}_{\alpha}^{\,-1} \right\| = 0}$$

Sketch of Proof (4)

$$\lim_{\alpha \to +\infty} \left\| \overline{LH_\alpha}^{-1} \right\| = 0$$

$$\Downarrow$$

The domain $D(\mathfrak{W})$ **is dense in** $C(\overline{D})$:

$$\boxed{\lim_{\alpha \to +\infty} \left\| \alpha G_\alpha u - u \right\| = 0, \quad \forall u \in C(\overline{D})}$$

Hille-Yosida Theorem

The operator
$\mathfrak{A} : C(K) \to C(K)$
generates a Feller semigroup
if and only if it satisfies
the following three conditions :

(a) $D(\mathfrak{A})$ **is dense in** $C(K)$.

(b) $\exists ! u \in D(\mathfrak{A})$ s.t. $(\alpha - \mathfrak{A})u = f, \ \forall f \in C(K)$.

(c) $\forall f \in C(K),\ f \geq 0$ in $K \Rightarrow (\alpha - \mathfrak{A})^{-1} f \geq 0$ in K.

(d) $\left\| (\alpha - \mathfrak{A})^{-1} \right\| \leq \dfrac{1}{\alpha}, \ \forall \alpha > 0.$

END