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**Characterization of anonymous, weakly monotonic and
strategy-proof aggregation functions**

by

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CHARACTERIZATION OF ANONYMOUS, WEAKLY MONOTONIC AND STRATEGY-PROOF AGGREGATION FUNCTIONS

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ABSTRACT. This paper is concerned about the aggregation function which plays a central role in the majority judgement that was recently proposed by Balinski and Laraki as a new voting mechanism. We raise two issues about their aggregation function, named order function, and show that they are resolved by relaxing the strong monotonicity condition imposed on the aggregation function, and that the anonymous, weakly monotonic and strategy-proof aggregation function is completely determined by the set of final grades when the judges split deeply.

1. INTRODUCTION

The problem we consider is how to determine the final grade of an alternative based on the grades reported by a number of judges. Let $N = \{1, 2, \dots, n\}$ denote the set of *judges*, or a *jury*. Each judge reports a grade by choosing one out of m approved words, which is called the *common language* and will be denoted by $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$. We assume that Λ is an ordered set with respect to \preceq such that $\lambda_1 \prec \lambda_2 \prec \dots \prec \lambda_m$. For the sake of simplicity, we will denote the minimum element λ_1 and the maximum element λ_m by α and ω , respectively. This paper is concerned about the aggregation function f defined as follows.

Definition 1.1 (Aggregation function). A function f that provides a grade $f(\mathbf{x}) \in \Lambda$ for the grades $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \Lambda^N$ reported by the jury N , i.e., $f : \Lambda^N \rightarrow \Lambda$, is called an *aggregation function*.

Aggregation function plays a central role in the *majority judgement* that was proposed by Balinski and Laraki as a new voting mechanism in [1, 2]. Let M denote the set of m *alternatives* or *candidates*. Each judge reports a grade to each alternative, which is brought together to an $m \times n$ matrix of grades. This matrix is called a *profile*. The *social grading function* is a function that assigns to each profile an m -dimensional vector consisting of

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grades. If we assume the neutrality and the independence of irrelevant alternatives, we readily see that the grade assigned to alternative i is given by an aggregation function whose input is the i th row of the profile. See Chapter 9 of [2]. Thus the point of discussion moves on to the aggregation function defined in Definition 1.1.

In this paper we first introduce the conditions that Balinski and Laraki imposed on the aggregation function and then raise two issues of their order function in Section 2 and 3. After giving several preliminary lemmas in Section 4, relaxing the conditions, we show the uniqueness of the aggregation function in Section 5. We also discuss the reduction of the parameters of the function and the recovery of the unanimity. Section 6 is devoted to the relationship between the aggregation function and Balinski and Laraki's order function.

2. FUNDAMENTAL CONDITIONS

Among conditions that aggregation functions should satisfy, the following two should be the most fundamental.

Definition 2.1 (Anonymity). The aggregation function f is said to be *anonymous* if

$$f(x_1, x_2, \dots, x_n) = f(x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(n)})$$

for any $(x_1, x_2, \dots, x_n) \in \Lambda^N$ and for any permutation τ on N .

Definition 2.2 (Weak Monotonicity). The aggregation function f is said to be *weakly monotonic* if $x_i \preceq x'_i$ for all $i \in N$ implies $f(x_1, x_2, \dots, x_n) \preceq f(x'_1, x'_2, \dots, x'_n)$.

Anonymity means that the name of judges does not matter, and weak monotonicity means that better grades reported by the jury can yield a better outcome. Throughout this paper we assume that the aggregation function is anonymous and weakly monotonic.

3. CONDITIONS OF BALINSKI AND LARAKI

Balinski and Laraki [2] required the aggregation function to meet three more conditions. Two of them are strong monotonicity and unanimity defined as follows.

Definition 3.1 (Strong Monotonicity). The aggregation function f is said to be *strongly monotonic* if $x_i \prec x'_i$ for all $i \in N$ implies $f(x_1, x_2, \dots, x_n) \prec f(x'_1, x'_2, \dots, x'_n)$.

Definition 3.2 (Unanimity). The aggregation function f is said to be *unanimous* if $f(x, x, \dots, x) = x$ for any $x \in \Lambda$.

To explain the last condition they required, we first define the manipulability of f . Here we write

$$\mathbf{x}/_i x' = (x_1, \dots, x_{i-1}, x', x_{i+1}, \dots, x_n)$$

for $\mathbf{x} = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in \Lambda^N$ and $x' \in \Lambda$.

Definition 3.3 (Manipulability). The aggregation function f is said to be *manipulable* by judge $i \in N$ at $\mathbf{x} \in \Lambda^N$ if there exists a grade $x' \in \Lambda$ such that

$$f(\mathbf{x}) \left\{ \begin{array}{c} \prec \\ \succ \end{array} \right\} f(\mathbf{x}/_i x') \quad \text{when} \quad f(\mathbf{x}) \left\{ \begin{array}{c} \prec \\ \succ \end{array} \right\} x_i. \quad (3.1)$$

The statement (3.1) means that the final grade $f(\mathbf{x})$ is worse (better) than judge i 's grade x_i , which gives him/her an incentive to raise (lower) the final grade in some way, and he/she will gain by misreporting his/her evaluation. If the aggregation function admits such manipulation, judges would be tempted to misreport their evaluations, thereby the credibility of the voting mechanism would be undermined. The last condition that Balinski and Laraki imposed on the aggregation function is the strategy-proofness.

Definition 3.4 (Strategy-proofness). The aggregation function f is said to be *strategy-proof* when it is not manipulable by any judge at any $\mathbf{x} \in \Lambda^N$.

A natural question would be what the aggregation function is like when it satisfies the conditions listed so far. Balinski and Laraki showed that it is the order function.

Definition 3.5 (Order Function). For $k \in \{1, 2, \dots, n\}$, the k th *order function* is the function that gives the k th lowest grade among n grades reported by the jury.

The k th order function permutes the input grades in ascending order as $x_{i_1} \preceq x_{i_2} \preceq \dots \preceq x_{i_n}$ and returns the grade x_{i_k} in the k th position. This definition is different from that in Chapter 10 of Balinski and Laraki [2], where they put the grades in descending order.

Theorem 3.6 (Theorem 10.1 in Balinski and Laraki [2]). *The unique aggregation function satisfying anonymity, weak monotonicity, strong monotonicity, unanimity, and strategy-proofness is the k th order function for some $k \in \{1, 2, \dots, n\}$.*

For the sake of discussion in the succeeding sections, we point out that the unanimity condition is redundant in Balinski and Laraki's framework.

Lemma 3.7. *A strongly monotonic aggregation function is unanimous.*

Proof. By the strong monotonicity we have

$$f(\lambda_1, \lambda_1, \dots, \lambda_1) \prec f(\lambda_2, \lambda_2, \dots, \lambda_2) \prec \dots \prec f(\lambda_m, \lambda_m, \dots, \lambda_m),$$

all of which belong to $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$. Therefore $f(\lambda_k, \lambda_k, \dots, \lambda_k) = \lambda_k$ for $k \in \{1, 2, \dots, m\}$. \square

Although the k th order function is the unique aggregation function that satisfies anonymity, weak monotonicity, strong monotonicity and strategy-proofness, there remain two issues to settle.

The first issue is concerning the number k . As Theorem 3.6 states, the k th order function meets all the desired conditions regardless of the value of k . Some convincing discussion

about which number to select is needed. Balinski and Laraki defined the *middlemost order function* as

$$f(x_1, x_2, \dots, x_n) \begin{cases} = x_{(n+1)/2} & \text{when } n \text{ is odd} \\ \in [x_{n/2}, x_{n/2+1}] & \text{when } n \text{ is even,} \end{cases}$$

where x_i 's are assumed to be put in ascending order. They showed in Chapter 12 of [2] that it minimizes the probability of cheating as well as maximizes the social welfare. When n is even, there still remains the problem of which of $x_{n/2}$ and $x_{n/2+1}$ should be chosen. See also their definition of *majority grade* in the same chapter.

The second issue is that the order function always singles out a grade among those given by the judges, i.e.,

$$f(x_1, x_2, \dots, x_n) \in \{x_1, x_2, \dots, x_n\},$$

no matter how they may split. Even when half of the judges give α and the other half give ω , the order function provides either α or ω , not any grade in between.

Moulin's paper [3] on the facility location problem provides a clue about how to resolve these issues. He assumed that each inhabitant in a one-dimensional city has his/her single-peaked utility function, and considered the location problem of a public institution. He showed that his *generalized majority relation* is the unique function that satisfies some desirable conditions: anonymity, weak monotonicity, Pareto-efficiency and strategy-proofness. The generalized majority relation is a median of the real numbers that inhabitants reported as the peaks of their utility functions as well as several fixed real numbers, which he named *phantom voters*. See also Definition 11.6 and Theorem 11.6 of Moulin [4]. His work is based on the premise that each inhabitant has a single-peaked utility function, however careful reading of his work suggested a possible application of his discussion to Balinski and Laraki's framework with the common language consisting of a finite number of grades. In the following sections we will show that relaxing some of the conditions that Balinski and Laraki required resolves the two issues raised above.

4. PRELIMINARY LEMMAS

The median function defined below plays an essential role in our discussion.

Definition 4.1 (Median). Let $\mathbf{r} = (r_1, r_2, \dots, r_{2k+1})$ be a vector of an odd number of grades of Λ . The $(k+1)$ st lowest grade of $r_1, r_2, \dots, r_{2k+1}$ with respect to the order \preceq defined on Λ is called their *median* and denoted by $\text{med}(r_1, r_2, \dots, r_{2k+1})$ or $\text{med}(\mathbf{r})$.

The following lemma is straightforward from the definition of median.

Lemma 4.2. Let $\mathbf{r} = (r_1, r_2, \dots, r_{2k+1})$ be a vector of an odd number of grades of Λ .

- (1) If $\text{med}(\mathbf{r}) \prec r_i$, then $\text{med}(\mathbf{r}/_i r') = \text{med}(\mathbf{r})$ for all $r' \in \Lambda$ such that $\text{med}(\mathbf{r}) \preceq r'$.
- (2) If $\text{med}(\mathbf{r}) \succ r_i$, then $\text{med}(\mathbf{r}/_i r') = \text{med}(\mathbf{r})$ for all $r' \in \Lambda$ such that $\text{med}(\mathbf{r}) \succeq r'$.
- (3) If $r' \preceq \text{med}(\mathbf{r}) \preceq r''$, then $\text{med}(r', \mathbf{r}, r'') = \text{med}(\mathbf{r})$.

As will be shown in Lemma 6.1, strong monotonicity in Definition 3.1 is the most significant source of the issues. Therefore we assume only anonymity, weak monotonicity and strategy-proofness. The following lemma shows that the median function with several *phantom grades*, which are denoted by γ therein, satisfies all the conditions that we assume.

Lemma 4.3. *Let the aggregation function $f: \Lambda^N \rightarrow \Lambda$ be defined by*

$$f(\mathbf{x}) = \text{med}(\mathbf{x}, \gamma),$$

for arbitrarily fixed, not necessarily distinct, grades $\gamma_0, \gamma_1, \dots, \gamma_k$ of Λ , where $n + k$ is assumed to be an odd number. Then f is anonymous, weakly monotonic and strategy-proof.

Proof. The anonymity and weak monotonicity of f are clear.

To prove the strategy-proofness, suppose that $f(\mathbf{x}) \prec x_i$, i.e., judge i has an incentive to raise the final grade at $\mathbf{x} \in \Lambda^N$. For $x'_i \in \Lambda$ such that $x'_i \preceq x_i$, we readily see that $f(\mathbf{x}/_i x'_i) = \text{med}(\mathbf{x}/_i x'_i, \gamma) \preceq \text{med}(\mathbf{x}, \gamma) = f(\mathbf{x})$ by the weak monotonicity of median. By (1) of Lemma 4.2 we see that replacing x_i by $x'_i \succ x_i$ will not affect the median, i.e., $f(\mathbf{x}/_i x'_i) = \text{med}(\mathbf{x}/_i x'_i, \gamma) = \text{med}(\mathbf{x}, \gamma) = f(\mathbf{x})$. Thus f admits no strategic manipulation. The proof when $f(\mathbf{x}) \succ x_i$ will be done in the same way. \square

Now consider the case of a single judge, i.e., $|N| = 1$. Recall that $\alpha = \lambda_1 = \min_{\preceq} \Lambda$ and $\omega = \lambda_n = \max_{\preceq} \Lambda$.

Lemma 4.4. *Suppose $|N| = 1$ and let $f: \Lambda \rightarrow \Lambda$ be a weakly monotonic aggregation function. Then f is strategy-proof if and only if*

$$f(x) = \text{med}(x, f(\alpha), f(\omega)) \tag{4.1}$$

for all $x \in \Lambda$.

Proof. The “if” part follows Lemma 4.3. To show the “only if” part we will consider the three cases: $f(x) = x$, $f(x) \prec x$ and $f(x) \succ x$.

Firstly, suppose $f(x) = x$. Then by the weak monotonicity we have $f(\alpha) \preceq f(x) = x \preceq f(\omega)$. This means $f(x) = \text{med}(x, f(\alpha), f(\omega))$.

Secondly, suppose $f(x) \prec x$. Since f is strategy-proof, even ω cannot raise the final grade, i.e., $f(\omega) \preceq f(x)$, which together with the weak monotonicity implies $f(\omega) = f(x)$. Then we obtain $f(\alpha) \preceq f(\omega) = f(x) \prec x$. This means $f(x) = \text{med}(x, f(\alpha), f(\omega))$.

When $f(x) \succ x$, we can prove in the same way as above. \square

5. UNIQUENESS OF AGGREGATION FUNCTION

5.1. Main Result. We will prove the main theorem stating that an anonymous, weakly monotonic and strategy-proof aggregation function $f: \Lambda^N \rightarrow \Lambda$ is given as

$$f(x_1, x_2, \dots, x_n) = \text{med}(x_1, x_2, \dots, x_n, \gamma_0, \gamma_1, \dots, \gamma_n)$$

for some $n + 1$ grades $\gamma_0, \gamma_1, \dots, \gamma_n \in \Lambda$. The proof will be based on the induction over the size n of the jury N . Note that we have proved the assertion when $|N| = n = 1$ in

Lemma 4.4 with $\gamma_0 = f(\alpha)$ and $\gamma_1 = f(\omega)$. Now let $\boldsymbol{\lambda}^k$ be a vector consisting of $n - k$ of grade α and k of grade ω , i.e.,

$$\boldsymbol{\lambda}^k = (\underbrace{\alpha, \dots, \alpha}_{n-k}, \underbrace{\omega, \dots, \omega}_k) \in \Lambda^N \quad \text{for } k \in \{0, 1, \dots, n\}. \quad (5.1)$$

Theorem 5.1. *Suppose that the aggregation function $f : \Lambda^N \rightarrow \Lambda$ satisfies anonymity, weak monotonicity and strategy-proofness. Let $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_n)$ be an $(n + 1)$ -vector consisting of*

$$\gamma_k = f(\boldsymbol{\lambda}^k) \quad \text{for } k \in \{0, 1, \dots, n\}. \quad (5.2)$$

Then

$$f(\boldsymbol{x}) = \text{med}(\boldsymbol{x}, \boldsymbol{\gamma}) \quad (5.3)$$

holds for all $\boldsymbol{x} \in \Lambda^N$.

Proof. Since we have seen in Lemma 4.4 that (5.2) and (5.3) hold when $|N| = n = 1$, we assume as the induction hypothesis that an anonymous, weakly monotonic, and strategy-proof aggregation function defined on Λ^N is given by (5.3) together with (5.2). We then add a judge named 0 to the jury N and consider an aggregation function f defined on $\Lambda^{\{0\} \cup N}$.

Firstly, for each $x_0 \in \Lambda$, let $\tau_{x_0} : \Lambda^N \rightarrow \Lambda$ be

$$\tau_{x_0}(\boldsymbol{x}) = f(x_0, \boldsymbol{x}).$$

It is clear that τ_{x_0} is anonymous, weakly monotonic, and strategy-proof as well. According to the induction hypothesis, $\tau_{x_0}(\boldsymbol{x})$ is given by (5.3), i.e.,

$$\tau_{x_0}(\boldsymbol{x}) = \text{med}(\boldsymbol{x}, \gamma_0(x_0), \gamma_1(x_0), \dots, \gamma_n(x_0))$$

for all $\boldsymbol{x} \in \Lambda^N$, where by (5.2)

$$\gamma_k(x_0) = \tau_{x_0}(\boldsymbol{\lambda}^k) = f(x_0, \boldsymbol{\lambda}^k) \quad \text{for } k \in \{0, 1, \dots, n\}.$$

The argument x_0 in parentheses indicates the dependence of γ_k on x_0 . Therefore we see that

$$f(x_0, \boldsymbol{x}) = \text{med}(\boldsymbol{x}, f(x_0, \boldsymbol{\lambda}^0), f(x_0, \boldsymbol{\lambda}^1), \dots, f(x_0, \boldsymbol{\lambda}^n)) \quad (5.4)$$

holds for all $(x_0, \boldsymbol{x}) \in \Lambda^{\{0\} \cup N}$.

Secondly, for each $\boldsymbol{x} \in \Lambda^N$ let $\sigma_{\boldsymbol{x}} : \Lambda \rightarrow \Lambda$ be defined by

$$\sigma_{\boldsymbol{x}}(x_0) = f(x_0, \boldsymbol{x}).$$

It is anonymous, weakly monotonic, and strategy-proof as well. Hence by Lemma 4.4 $\sigma_{\boldsymbol{x}}(x_0)$ is given as

$$\sigma_{\boldsymbol{x}}(x_0) = \text{med}(x_0, \gamma_0(\boldsymbol{x}), \gamma_1(\boldsymbol{x}))$$

for all $x_0 \in \Lambda$, where by (5.2)

$$\gamma_0(\boldsymbol{x}) = \sigma_{\boldsymbol{x}}(\alpha) = f(\alpha, \boldsymbol{x}) \quad \text{and} \quad \gamma_1(\boldsymbol{x}) = \sigma_{\boldsymbol{x}}(\omega) = f(\omega, \boldsymbol{x}).$$

Therefore

$$f(x_0, \mathbf{x}) = \text{med}(x_0, f(\alpha, \mathbf{x}), f(\omega, \mathbf{x}))$$

holds for all $(x_0, \mathbf{x}) \in \Lambda^{\{0\} \cup N}$. Especially we see that

$$f(x_0, \boldsymbol{\lambda}^k) = \text{med}(x_0, f(\alpha, \boldsymbol{\lambda}^k), f(\omega, \boldsymbol{\lambda}^k)) \quad \text{for } k \in \{0, 1, \dots, n\}. \quad (5.5)$$

Substituting (5.5) for $f(x_0, \boldsymbol{\lambda}^k)$ of (5.4) we have

$$f(x_0, \mathbf{x}) = \text{med} \left(\begin{array}{c} \mathbf{x}, \\ \text{med}(x_0, f(\alpha, \boldsymbol{\lambda}^0), f(\omega, \boldsymbol{\lambda}^0)), \\ \text{med}(x_0, f(\alpha, \boldsymbol{\lambda}^1), f(\omega, \boldsymbol{\lambda}^1)), \\ \vdots \\ \text{med}(x_0, f(\alpha, \boldsymbol{\lambda}^n), f(\omega, \boldsymbol{\lambda}^n)) \end{array} \right).$$

For $k \in \{0, 1, \dots, n-1\}$ the vector $(\omega, \boldsymbol{\lambda}^k)$ is a permutation of the vector $(\alpha, \boldsymbol{\lambda}^{k+1})$, hence by the anonymity of f we have

$$f(\omega, \boldsymbol{\lambda}^k) = f(\alpha, \boldsymbol{\lambda}^{k+1}) \quad \text{for } k \in \{0, 1, \dots, n-1\}.$$

Now let

$$\begin{aligned} \boldsymbol{\lambda}^{(k)} &= (\alpha, \boldsymbol{\lambda}^k) = \underbrace{(\alpha, \dots, \alpha)}_{n+1-k}, \underbrace{(\omega, \dots, \omega)}_k && \text{for } k \in \{0, 1, \dots, n\}, \\ \gamma_{(k)} &= f(\boldsymbol{\lambda}^{(k)}) \\ \boldsymbol{\lambda}^{(n+1)} &= (\omega, \boldsymbol{\lambda}^n) = \underbrace{(\omega, \omega, \dots, \omega)}_{n+1} \\ \gamma_{(n+1)} &= f(\boldsymbol{\lambda}^{(n+1)}). \end{aligned}$$

Note that

$$\gamma_{(0)} \preceq \gamma_{(1)} \preceq \dots \preceq \gamma_{(n)} \preceq \gamma_{(n+1)}. \quad (5.6)$$

Then we finally obtain that

$$f(x_0, \mathbf{x}) = \text{med} \left(\begin{array}{c} \mathbf{x}, \\ \text{med}(x_0, \gamma_{(0)}, \gamma_{(1)}), \\ \text{med}(x_0, \gamma_{(1)}, \gamma_{(2)}), \\ \vdots \\ \text{med}(x_0, \gamma_{(n)}, \gamma_{(n+1)}) \end{array} \right). \quad (5.7)$$

To simplify the composite of median functions (5.7), we consider the following three cases concerning the location of x_0 .

Case 1: $x_0 \preceq \gamma_{(0)}$.

We readily see from (5.6)

$$\text{med}(x_0, \gamma_{(k)}, \gamma_{(k+1)}) = \gamma_{(k)} \quad \text{for } k \in \{0, 1, \dots, n\}.$$

Hence by (5.7)

$$f(x_0, \mathbf{x}) = \text{med}(\mathbf{x}, \gamma_{(0)}, \gamma_{(1)}, \dots, \gamma_{(n)}).$$

Since there are $n+1$ of $\gamma_{(k)}$'s, and n of x_k 's, $\gamma_{(0)} \preceq \text{med}(\mathbf{x}, \gamma_{(0)}, \gamma_{(1)}, \dots, \gamma_{(n)}) \preceq \gamma_{(n)}$ holds. Then by (3) of Lemma 4.2 adding x_0 satisfying $x_0 \preceq \gamma_{(0)}$ and $\gamma_{(n+1)}$ satisfying $\gamma_{(n)} \preceq \gamma_{(n+1)}$ does not change the median. Hence we obtain

$$f(x_0, \mathbf{x}) = \text{med}(x_0, \mathbf{x}, \gamma_{(0)}, \gamma_{(1)}, \dots, \gamma_{(n)}, \gamma_{(n+1)}).$$

Case 2: $\gamma_{(n+1)} \preceq x_0$.

In the same way as above we obtain

$$\text{med}(x_0, \gamma_{(k)}, \gamma_{(k+1)}) = \gamma_{(k+1)} \quad \text{for } k \in \{0, 1, \dots, n\},$$

then by (5.7)

$$f(x_0, \mathbf{x}) = \text{med}(\mathbf{x}, \gamma_{(1)}, \gamma_{(2)}, \dots, \gamma_{(n+1)}).$$

We then have $\gamma_{(0)} \preceq \gamma_{(1)} \preceq \text{med}(\mathbf{x}, \gamma_{(1)}, \gamma_{(2)}, \dots, \gamma_{(n+1)}) \preceq \gamma_{(n+1)} \preceq x_0$, hence adding $\gamma_{(0)}$ and x_0 does not change the median, i.e.,

$$f(x_0, \mathbf{x}) = \text{med}(x_0, \mathbf{x}, \gamma_{(0)}, \gamma_{(1)}, \gamma_{(2)}, \dots, \gamma_{(n+1)}).$$

Case 3: $\gamma_{(l)} \prec x_0 \preceq \gamma_{(l+1)}$ for some l such that $0 \leq l \leq n$.

We have

$$\begin{aligned} \text{med}(x_0, \gamma_{(k)}, \gamma_{(k+1)}) &= \gamma_{(k+1)} \quad \text{for } k \in \{0, 1, \dots, l-1\}, \\ \text{med}(x_0, \gamma_{(l)}, \gamma_{(l+1)}) &= x_0, \\ \text{med}(x_0, \gamma_{(k)}, \gamma_{(k+1)}) &= \gamma_{(k)} \quad \text{for } k \in \{l+1, \dots, n\}. \end{aligned}$$

Then by (5.7)

$$f(x_0, \mathbf{x}) = \text{med}(x_0, \mathbf{x}, \gamma_{(1)}, \dots, \gamma_{(l)}, \gamma_{(l+1)}, \dots, \gamma_{(n)}).$$

Note that $\gamma_{(1)}$ is a minimal element and $\gamma_{(n)}$ is a maximal element among $n+1$ elements $\gamma_{(1)}, \dots, \gamma_{(l)}, x_0, \gamma_{(l+1)}, \dots, \gamma_{(n)}$. Therefore $\gamma_{(0)} \preceq \gamma_{(1)} \preceq \text{med}(x_0, \mathbf{x}, \gamma_{(1)}, \dots, \gamma_{(n)}) \preceq \gamma_{(n)} \preceq \gamma_{(n+1)}$, hence adding $\gamma_{(0)}$ and $\gamma_{(n+1)}$ does not change the median.

Thus we have seen that

$$f(x_0, \mathbf{x}) = \text{med}(x_0, \mathbf{x}, \gamma_{(0)}, \gamma_{(1)}, \dots, \gamma_{(n+1)})$$

holds in all cases, and completed the proof. \square

Corollary 5.2. *An aggregation function $f : \Lambda^N \rightarrow \Lambda$ satisfies the anonymity, weak monotonicity and strategy-proofness if and only if*

$$f(\mathbf{x}) = \text{med}(\mathbf{x}, \boldsymbol{\gamma})$$

for some $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_n) \in \Lambda^{n+1}$.

Proof. Straightforward from Lemma 4.3 and Theorem 5.1. \square

5.2. Reduction of Phantom Grades. When the aggregation function has some other properties, we can reduce the number of the phantom grades $\gamma_0, \gamma_1, \dots, \gamma_n$.

Definition 5.3 (Unanimity at Ends). The aggregation function f is said to be *unanimous at ends* when

$$f(\alpha, \alpha, \dots, \alpha) = \alpha \quad \text{and} \quad f(\omega, \omega, \dots, \omega) = \omega$$

hold.

Corollary 5.4. *Let $f : \Lambda^N \rightarrow \Lambda$ be an anonymous, weakly monotonic and strategy-proof aggregation function. Then it is unanimous if and only if it is unanimous at ends.*

Proof. The necessity is trivial. By Theorem 5.1 we have $f(\mathbf{x}) = \text{med}(\mathbf{x}, \boldsymbol{\gamma})$ with $\gamma_k = f(\boldsymbol{\lambda}^k)$ for $k \in \{0, 1, 2, \dots, n\}$. If we assume that f is unanimous at ends, then $\gamma_0 = f(\boldsymbol{\lambda}^0) = f(\alpha, \alpha, \dots, \alpha) = \alpha$ and $\gamma_n = f(\boldsymbol{\lambda}^n) = f(\omega, \omega, \dots, \omega) = \omega$. Therefore $f(\mathbf{x}) = \text{med}(\mathbf{x}, \boldsymbol{\gamma}) = \text{med}(\mathbf{x}, \alpha, \gamma_1, \dots, \gamma_{n-1}, \omega) = \text{med}(\mathbf{x}, \gamma_1, \dots, \gamma_{n-1})$. When $\mathbf{x} = (x, x, \dots, x)$ for some $x \in \Lambda$, we then see that $f(x, x, \dots, x) = \text{med}(\underbrace{x, x, \dots, x}_n, \underbrace{\gamma_1, \dots, \gamma_{n-1}}_{n-1}) = x$, the unanimity of f . \square

Corollary 5.5. *Let $f : \Lambda^N \rightarrow \Lambda$ be an anonymous, weakly monotonic and strategy-proof aggregation function. If f is unanimous at ends or strongly monotonic, there are $n - 1$ phantom grades $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_{n-1}) \in \Lambda^{n-1}$ such that*

$$f(\mathbf{x}) = \text{med}(\mathbf{x}, \boldsymbol{\gamma})$$

holds for all $\mathbf{x} \in \Lambda^N$.

Proof. If f is unanimous at ends or strongly monotonic, we see that $f(\alpha, \alpha, \dots, \alpha) = \alpha$ and $f(\omega, \omega, \dots, \omega) = \omega$. See Lemma 3.7. Then $\gamma_0 = \alpha$ and $\gamma_n = \omega$, hence $\text{med}(\mathbf{x}, \gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) = \text{med}(\mathbf{x}, \alpha, \gamma_1, \dots, \gamma_{n-1}, \omega) = \text{med}(\mathbf{x}, \gamma_1, \dots, \gamma_{n-1})$. This completes the proof. \square

5.3. Resurgence of Unanimity. In this section we discuss how to recover the unanimity when f is not unanimous.

Definition 5.6. The aggregation function is said to be *onto* if $f(\Lambda^N) = \Lambda$.

If f is onto, we readily see that it is unanimous at ends, and then obtain the following corollary from Corollary 5.5.

Corollary 5.7. *Let $f : \Lambda^N \rightarrow \Lambda$ be an anonymous, weakly monotonic and strategy-proof aggregation function. If f is onto, then it is unanimous.*

Definition 5.8. For $f : \Lambda^N \rightarrow \Lambda$ and $\Lambda' = f(\Lambda^N)$, the *restriction* of f to Λ' is the function $f' : (\Lambda')^N \rightarrow \Lambda'$ such that $f'(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in (\Lambda')^N$.

Corollary 5.9. *For an anonymous, weakly monotonic and strategy-proof aggregation function $f : \Lambda^N \rightarrow \Lambda$, let $\Lambda' = f(\Lambda^N)$. Then the restriction of f to Λ' is unanimous.*

Proof. Let α' and ω' be the minimum and maximum elements of Λ' , respectively. Then by the weak monotonicity we have $f(\alpha, \alpha, \dots, \alpha) = \alpha'$ and $f(\omega, \omega, \dots, \omega) = \omega'$. Therefore by Theorem 5.1 we have for $x \in \Lambda'$

$$\begin{aligned} f(x, x, \dots, x) &= \text{med}(x, x, \dots, x, \gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n) \\ &= \text{med}(\alpha', x, x, \dots, x, \gamma_1, \dots, \gamma_{n-1}, \omega') \\ &= \text{med}(\underbrace{x, x, \dots, x}_n, \underbrace{\gamma_1, \dots, \gamma_{n-1}}_{n-1}) = x. \end{aligned}$$

□

Thus by discarding the grades that are not used as a final grade, we can recover the unanimity. We will give another proof of this corollary in Appendix A where the strategy-proofness is used more directly.

6. DERIVATION OF THE ODER FUNCTION

As we have seen, the aggregation function is completely determined by $n + 1$ phantom grades $\gamma_0, \gamma_1, \dots, \gamma_n$, each of which is the final grade when f receives a grade vector consisting of α and ω alone. See (5.1) for the definition of λ^k . Now consider an aggregation function that is unanimous at ends and returns either α or ω when it receives a grade vector λ^k . By the weak monotonicity there is an l such that

$$\gamma_k = \begin{cases} \alpha & \text{for } 0 \leq k \leq l \\ \omega & \text{for } l + 1 \leq k \leq n. \end{cases}$$

Note that $0 \leq l \leq n - 1$ due to the unanimity at ends of f . Then $f(\mathbf{x})$ is given as

$$\begin{aligned} f(\mathbf{x}) &= \text{med}(x, \underbrace{\alpha, \dots, \alpha}_{l+1}, \underbrace{\omega, \dots, \omega}_{n-l}) \\ &= \text{med}(\underbrace{\alpha, \dots, \alpha, x_1, \dots, x_{n-l-1}}_n, x_{n-l}, \underbrace{x_{n-l+1}, \dots, x_n, \omega, \dots, \omega}_n) \\ &= x_{n-l}, \end{aligned}$$

where we assume without loss of generality that $x_1 \preceq x_2 \preceq \dots \preceq x_n$. In other words, f is the $(n - l)$ th order function proposed by Balinski and Laraki.

Corollary 6.1 (Theorem 10.1 in Balinski and Laraki [2]). *Suppose that the aggregation function $f : \Lambda^N \rightarrow \Lambda$ satisfies the strong monotonicity in addition to the anonymity, weak monotonicity and strategy-proofness. Then γ_k is either α or ω for all $k \in \{0, 1, \dots, n\}$, and f is the order function.*

Proof. Note that $f(\mathbf{x}) = \text{med}(\mathbf{x}, \gamma)$ by Theorem 5.1 and that $\gamma_0 = \alpha$ and $\gamma_n = \omega$ by Lemma 3.7 or Corollary 5.5. We will show that $\gamma_k \notin \{\alpha, \omega\}$ for some $k \in \{1, \dots, n-1\}$ leads to a contradiction. Let

$$\mathbf{x} = (\underbrace{\alpha, \dots, \alpha}_{n-k}, \underbrace{\gamma_k, \dots, \gamma_k}_k) \quad \text{and} \quad \mathbf{x}' = (\underbrace{\gamma_k, \dots, \gamma_k}_{n-k}, \underbrace{\omega, \dots, \omega}_k).$$

Then we readily see that $\mathbf{x} \prec \mathbf{x}'$ and both of $\text{med}(\mathbf{x}, \gamma)$ and $\text{med}(\mathbf{x}', \gamma)$ are equal to γ_k . This contradicts the strong monotonicity assumption on f . \square

Thus the strong monotonicity urges f to take either the lowest grade α or the highest grade ω when the judges split deeply, and makes it the order function. Another proof of this corollary will be given in Appendix B where we prove it without using Theorem 5.1.

7. CONCLUSION

In this paper we have proved that the aggregation function is given by the median function with $n+1$ phantom grades when it meets the conditions: anonymity, weak monotonicity and strategy-proofness. We also showed that those phantom grades are the outcomes when judges split deeply. Therefore the jury has only to decide the final grades when the jury's opinion split deeply in order to decide the aggregation function.

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APPENDIX A. PROOF OF COROLLARY 5.9

Now suppose that f is not unanimous, and let Λ' be the image of f , i.e., $\Lambda' = f(\Lambda^N)$. Then by Corollary 5.7 Λ' is a proper subset of Λ . Let α' and ω' be the minimum and maximum elements of Λ' , respectively. Then by the weak monotonicity we have $f(\alpha, \alpha, \dots, \alpha) = \alpha'$ and $f(\omega, \omega, \dots, \omega) = \omega'$. We will show that $f(x, x, \dots, x) \neq x$ for some $x \in \Lambda'$ leads to a contradiction. Now let $s = f(x, x, \dots, x)$ and assume $s \prec x$. The first judge has an incentive to misreport his/her grade as, say ω , to raise the final grade, but due to the strategy-proofness it is not possible. Then $f(\omega, x, \dots, x) = s$. Repeating this argument we obtain $f(\omega, \omega, \dots, \omega) = s$. Since $s \prec x \preceq \omega'$, this contradicts the definition of ω' . When $s \succ x$, we will yield $f(\alpha, \alpha, \dots, \alpha) = s \succ x \succeq \alpha'$, again a contradiction.

APPENDIX B. PROOF OF COROLLARY 6.1

Since we have seen in Lemma 3.7 that $\gamma_0 = \alpha$ and $\gamma_n = \omega$, we will show that $\gamma_k = f(\boldsymbol{\lambda}^k) \in \{\alpha, \omega\}$ for $k \in \{1, \dots, n-1\}$. Let $s = f(\boldsymbol{\lambda}^k)$ and suppose $s \notin \{\alpha, \omega\}$ for some k . By the weak monotonicity we see $f(\boldsymbol{\lambda}^k/1s) \succeq s$. The inequality $f(\boldsymbol{\lambda}^k/1s) \succ s$ would imply that the judge 1 is able to strategically manipulate f at $\boldsymbol{\lambda}^k/1s$ by misreporting his grade as α , hence contradict the strategy-proofness. Then we have $f(\boldsymbol{\lambda}^k/1s) = s$. Repeating this argument $n-k$ times, we obtain

$$f(\underbrace{s, \dots, s}_{n-k}, \underbrace{\omega, \dots, \omega}_k) = s.$$

In the same way we also have

$$f(\underbrace{\alpha, \dots, \alpha}_{n-k}, \underbrace{s, \dots, s}_k) = s.$$

These facts contradict the strong monotonicity of f since $(s, \dots, s, \omega, \dots, \omega) \succ (\alpha, \dots, \alpha, s, \dots, s)$. Thus we obtain $\gamma_k \in \{\alpha, \omega\}$ for all $k \in \{0, 1, \dots, n\}$. The argument before this corollary yields that f is the order function.

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