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A-Section

# Asymptotic Deficiency of the Estimator of a Parameter of an Autoregressive Process with the Missing Observation

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**Abstract**

Let $\{X_t\}$ be defined by $X_t = \theta X_{t-1} + U_t$ $(t = 1, 2, \ldots)$, where $\{U_t\}$ is a sequence of independently, identically and normally distributed random variables with mean 0 and variance 1 and $X_0$ is a normal random variable with mean 0 and variance $1/(1-\theta^2)$ and for each $t$ $X_0$ is independent of $U_t$. We assume that $|\theta| < 1$ and consider the maximum likelihood estimator (MLE) of $\theta$ based on the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ in which $X_T$ is missing. It is shown that the bias-adjusted MLE is second order asymptotically efficient. When in the above autoregressive process we assume that $X_0 = 0$, the asymptotic deficiency of the MLE is given.

## 1. Introduction.

In the first order autoregressive (AR) processes the first order and the second order asymptotic efficiency of the MLE was discussed by Akahira [1], [2], [3], [4]. The first order asymptotic efficiency was extended by Kabaila [9] to an autoregressive moving average (ARMA) process when the innovations are not necessarily Gaussian and the second order asymptotic efficiency was done by Taniguchi [11] to a Gaussian ARMA process.

In this paper we consider an AR process $\{X_t\}$ which is defined by $X_t = \theta X_{t-1} + U_t (t = 1, 2, \ldots)$, where $\{U_t\}$ is a sequence of independently, identically any normally distributed random variables with mean 0 and variance 1 and $X_0$ is a normal random variable with mean 0 and variance $1/(1-\theta^2)$ and for each $t$ $X_0$ is independent of $U_t$. We assume $|\theta| < 1$. We consider the MLE based on the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ in which $X_T$ is missing. We shall show that the bias-adjusted MLE is second order asymptotically efficient. Further we assume

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that $X_0 = 0$ in the AR process. We shall obtain the asymptotic deficiency of the MLE $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to the MLEs $\hat{\theta}^{T-1}_{ML}$, $\hat{\theta}^{T}_{ML}$ and $\hat{\theta}^{T+1}_{ML}$ based on the samples $(X_1, \dots, X_{T-1})$,$(X_1, \dots, X_T)$ and $(X_1, \dots, X_{T+1})$, respectively.

## 2. Definitions.

Let $(\mathfrak{X}, \mathfrak{B})$ be a sample space and $\Theta$ be a parameter space, which is assumed to be an open set in a Euclidean 1-space $R^1$. We shall denote by $(\mathfrak{X}^{(T)}, \mathfrak{B}^{(T)})$ the $T$-fold direct products of $(\mathfrak{X}, \mathfrak{B})$. For each $T = 1, 2, \dots$, the points of $\mathfrak{X}^{(T)}$ will be denoted by $\tilde{x}_T = (x_1, \dots, x_T)$. We consider a sequence of classes of probability measures $\{P_{T,\theta}: \theta \in \Theta\}$ $(T = 1, 2, \dots)$ each defined on $(\mathfrak{X}^{(T)}, \mathfrak{B}^{(T)})$ such that for each $T = 1, 2, \dots$ and each $\theta \in \Theta$ the following holds:

$$P_{T,\theta}(\mathrm{B}^{(T)}) = P_{T+1,\theta}(\mathrm{B}^{(T)} \times \chi)$$

for all $\mathrm{B}^{(T)} \in \mathfrak{B}^{(T)}$.

An estimator of $\theta$ is defined to be a sequence $\{\hat{\theta}_T\}$ of $\mathfrak{B}^{(T)}$-measurable functions $\hat{\theta}_T$. For simplicity we may denote an estimator $\hat{\theta}$ instead of $\{\hat{\theta}_T\}$. For an increasing sequence of positive numbers $\{c_T\}$ ($c_T$ tending to infinity) an estimator $\hat{\theta}$ is called consistent with order $\{c_T\}$ (or $\{c_T\}$-consistent for short) if for every $\varepsilon > 0$ and every $\vartheta \in \Theta$ there exist a sufficiently small positive number $\delta$ and sufficiently large positive number $L$ satisfying the following:

$$\lim_{T\to\infty} \sup_{\theta: |\theta - \vartheta| < \delta} P_{T,\theta}\{c_T |\hat{\theta}_T - \theta| \geq L\} < \varepsilon.$$

A $\{c_T\}$-consistent estimator $\hat{\theta}$ is $k$-th order asymptotically median unbiased (or $k$-th order *AMU*) if for any $\vartheta \in \Theta$ there exists a positive number $\delta$ such that

$$\lim_{T\to\infty} \sup_{\theta: |\theta-\vartheta| < \delta} c_T^{k-1} \left| P_{T,\theta}\{\hat{\theta} \leq \theta\} - \frac{1}{2} \right| = 0 ;$$

$$\lim_{T\to\infty} \sup_{\theta: |\theta-\vartheta| < \delta} c_T^{k-1} \left| P_{T,\theta}\{\hat{\theta} \geq \theta\} - \frac{1}{2} \right| = 0.$$

For each $k = 1, 2, \dots$ we denote by $A_k$ the class of the all $k$-th order *AMU* estimators.

We have defined a first (second) order *AMU* estimator $\hat{\theta}^*$ to be first (second) order asymptotically efficient in the class $A_1$ ($A_2$) if for any first (second) order *AMU* estimator $\hat{\theta}$ and any $u > 0$

$$\varliminf_{T\to\infty} [P_{T,\theta}\{c_T |\hat{\theta}^* - \theta| < u\} - P_{T,\theta}\{c_T |\hat{\theta} - \theta| < u\}] \geq 0.$$

$$\left(\varliminf_{T\to\infty} c_T [P_{T,\theta}\{c_T |\hat{\theta}^* - \theta| < u\} - P_{T,\theta}\{c_T |\hat{\theta} - \theta| < u\}] \geq 0\right)$$

(e.g. see Akahira and Takeuchi [7]).

Let $D$ be the class of estimators whose element $\hat{\theta}$ is best asymptotically normal and third order *AMU* and may be asymptotically expanded as

$$c_T(\hat{\theta} - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{1}{\sqrt{T}} Q(\theta) + o_p(\frac{1}{\sqrt{T}}),$$

where $I(\theta)$ is the Fisher information, $Z_1(\theta) = O_p(1)$, $Q(\theta) = O_p(1)$ and $E_\theta[Z_1(\theta)Q(\theta)^2] = o(1)$ with the notation $E_\theta[.]$ of the asymptotic expectation, and the distribution of $c_T(\hat{\theta} - \theta)$ admits an Edgeworth expansion. We have defined an estimator $\hat{\theta}^*$ in $D$ to be third order asymptotically efficient in the class $D$ if for any estimator $\hat{\theta}$ in $D$ and any $u > 0$

$$\varliminf_{T\to\infty} c_T{}^2 [P_{T,\theta}\{c_T \mid \hat{\theta}^* - \theta \mid < u\} - P_{T,\theta}\{c_T \mid \hat{\theta} - \theta \mid < u\}] \geq 0.$$

In the subsequent discussion we shall deal with only the case when $c_T = \sqrt{T}$.

Let $k_T (T = 1, 2, ...)$ be positive numbers such that $d = \lim_{T\to\infty}(k_T - T)$ exists, and the estimators $\hat{\theta}_T$ and $\hat{\theta}^*_{k_T}$ in the class $D$ based on the sample sizes $T$ and $k_T$, respectively, are asymptotically equivalent in the sense that asymptotic distributions of $\sqrt{T}(\hat{\theta}_T - \theta)$ and $\sqrt{T}(\hat{\theta}^*_{k_T} - \theta)$ are equal up to the order $T^{-1}$. Then $d$ is called the asymptotic deficiency of $\hat{\theta}^*_{k_T}$ with respect to $\hat{\theta}_T$ (See Hodges and Lehmann [8]). If we denote by $Q$ and $Q^*$ the terms of the order $T^{-1/2}$ in the stochastic expansions of $\sqrt{T}(\hat{\theta}_T - \theta)$ and $\sqrt{T}(\hat{\theta}^*_{k_T} - \theta)$, respectively, we see that the asymptotic deficiency $d$ of $\hat{\theta}^*_{k_T}$ w.r.t. $\hat{\theta}_T$ is given by $I\{V_\theta(Q^*) - V_\theta(Q)\}$, where $I$ is the Fisher information and $V_\theta$ designates the asymptotic variance (See Akahira [5], [6]).

## 3. Second order asymptotic efficiency.

Let $X_t$ $(t = 1, 2, ...)$ be defined recursively by

(3.1) $X_t = \theta X_{t-1} + U_t$ $(t = 1, 2, ...)$,

where $\{U_t\}$ is a sequence of independently, identically and normally distributed random variables with mean 0 and variance 1 and $X_0$ is a normal random variable with mean 0 and variance $1/(1-\theta^2)$ and for each $t$ $X_0$ is independent of $U_t$. We assume that $\Theta = (-1, 1)$, i.e., $|\theta| < 1$. Then it is easily seen that the process (3.1) is stationary.

Let $\hat{\theta}_{ML}$ be the *MLE* based on the sample $(X_0, X_1, \ldots, X_T)$. Then it is known in Akahira [3] that the stochastic expansion of the *MLE* $\hat{\theta}_{ML}$ is given by

$$(3.2)\quad \sqrt{T}(\hat{\theta}_{ML}-\theta) = -\frac{2\theta}{\sqrt{T}} + \frac{2\theta(1-\theta^2)}{\sqrt{T}} X_0{}^2 + \frac{1-\theta^2}{\sqrt{T}} \sum_{t=1}^{T} U_t X_{t-1}$$

$$+\frac{(1-\theta^2)^2}{T\sqrt{T}}\{-\sum_{t=1}^{T}(X_{t-1}^2 - \frac{1}{1-\theta^2})\}\sum_{t=1}^{T} U_t X_{t-1}$$

$$+\frac{(1-\theta^2)^3\ 3J(\theta)+K(\theta)\}}{2T\sqrt{T}}(\sum_{t=1}^{T} U_t X_{t-1})^2 + o_p(\frac{1}{\sqrt{T}}),$$

where

$$J(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial^2}{\partial\theta^2}\log L(\theta)\}\{\frac{\partial}{\partial\theta}\log L(\theta)\}]\ ;$$

$$K(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial}{\partial\theta}\log L(\theta)\}^3]$$

with the likelihood function $L(\theta)$ of $(X_0, X_1, \dots, X_T)$. It is also shown that the bias-adjusted *MLE* and the bias-adjusted least squares estimator are second order asymptotically efficient.

We consider the sample $(X_0, X_1, \dots, X_{T-1}, X_{T+1})$ in which $X_T$ is missing. The joint density of $(X_0, X_1, \dots, X_{T-1}, X_{T+1})$ is given by

$$L(\theta) = L(\theta : x_0, x_1, \dots, x_{T-1}, x_{T+1})$$

$$= \frac{1}{(2\pi)^{(T+1)/2}} \cdot \sqrt{\frac{1-\theta^2}{1+\theta^2}} \exp[-\frac{1}{2}\{(1-\theta^2)x_0{}^2 + \sum_{t=1}^{T-1}(x_t - \theta x_{t-1})^2$$

$$+\frac{1}{1+\theta^2}(x_{T+1} - \theta^2 x_{T-1})^2\}].$$

We put

$$Z_1(\theta) = \frac{1}{\sqrt{T}}\frac{\partial}{\partial\theta}\log L(\theta)\ ;$$

$$Z_2(\theta) = \frac{1}{\sqrt{T}}\{\frac{\partial^2}{\partial\theta^2}\log L(\theta) - E_\theta[\frac{\partial^2}{\partial\theta^2}\log L(\theta)]\};$$

$$I(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial}{\partial\theta}\log L(\theta)\}^2];$$

$$J(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial^2}{\partial\theta^2}\log L(\theta)\}\{\frac{\partial}{\partial\theta}\log L(\theta)\}]\ ;$$

$$K(\theta) = \frac{1}{T} E_\theta [\{\frac{\partial}{\partial\theta} \log L(\theta)\}^3].$$

Then it is known that the stochastic expansion of the *MLE* $\hat{\theta}_{ML}$ based on the sample $(X_0, X_1, \dots, X_{T-1}, X_{T+1})$ is given by

$$(3.3)\quad \sqrt{T}(\hat{\theta}_{ML} - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{Z_1(\theta)Z_2(\theta)}{I(\theta)^2\sqrt{T}} - \frac{3J(\theta) + K(\theta)}{2I(\theta)^3\sqrt{T}} Z_1(\theta)^2 + o_p(\frac{1}{\sqrt{T}})$$

(e.g. see Akahira [3] and Akahira and Takeuchi [7]).
Since

$$\frac{\partial \log L}{\partial\theta} = -\frac{\theta}{1-\theta^2} - \frac{\theta}{1+\theta^2} + \theta x_0^2 + \sum_{t=1}^{T-1} x_{t-1}(x_t - \theta x_{t-1})$$

$$+ \frac{\theta}{(1+\theta^2)^2}(x_{T+1}^2 + 2x_{T+1}x_{T-1}) - \frac{\theta^3(2+\theta^2)}{(1+\theta^2)^2} x_{T-1}^2 ;$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{1+\theta^2}{(1-\theta^2)^2} - \frac{1-\theta^2}{(1+\theta^2)^2} - \sum_{t=1}^{T-2} x_t^2 + \frac{1-3\theta^2}{(1+\theta^2)^3}(x_{T+1}^2 + 2x_{T+1}x_{T-1})$$

$$- \frac{\theta^2(\theta^4 + 3\theta^2 + 6)}{(1+\theta^2)^3} x_{T-1}^2 ;$$

$$E_\theta(\frac{\partial^2 \log L}{\partial\theta^2}) = -\frac{T-2}{1-\theta^2} + \frac{1-8\theta^2-\theta^4}{(1+\theta^2)^2(1-\theta^2)} - \frac{1+\theta^2}{(1-\theta^2)^2} - \frac{1-\theta^2}{(1+\theta^2)^2},$$

We have

$$(3.4)\quad Z_1(\theta) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T-1} U_t X_{t-1} + \frac{1}{\sqrt{T}}(R_{T-a});$$

$$(3.5)\quad Z_2(\theta) = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T-2} (X_t^2 - \frac{1}{1-\theta^2}) + o_p(\frac{1}{\sqrt{T}}),$$

where

$$R_T = \theta X_0^2 + \frac{\theta}{(1+\theta^2)^2}(X_{T+1}^2 + 2X_{T+1}X_{T-1}) - \frac{\theta^3(2+\theta^2)}{(1+\theta^2)^2} X_{T-1}^2 ;$$

$$a = \frac{\theta}{1-\theta^2} + \frac{\theta}{1+\theta^2} = \frac{2\theta}{1-\theta^4}.$$

Note that $E_\theta(R_T) = a$.

Since

$$X_{T+1} = \theta^2 X_{T-1} + \theta U_T + U_{T+1},$$

we obtain

(3.6) $$R_T = \theta X_0{}^2 + \frac{\theta}{(1+\theta^2)^2}\{2(1+\theta^2)(\theta U_T + U_{T+1})X_{T-1} + (\theta U_T + U_{T+1})^2\}.$$

By (3.3), (3.4) and (3.5) we have

(3.7) $$\sqrt{IT}(\hat{\theta}_{ML} - \theta) = \frac{1}{\sqrt{IT}}\sum_{t=1}^{T-1} U_t X_{t-1} + \frac{1}{\sqrt{IT}}(R_T - a) + \frac{1}{I^{3/2}\sqrt{T}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1} U_t X_{t-1}\right)$$

$$\cdot\left\{-\frac{1}{\sqrt{T}}\sum_{t=1}^{T-2}\left(X_t^2 - \frac{1}{1-\theta^2}\right)\right\} - \frac{3J+K}{2I^{5/2}\sqrt{T}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1} U_t X_{t-1}\right)^2 + o_p\left(\frac{1}{\sqrt{T}}\right),$$

where $I = I(\theta) = 1/(1-\theta^2) + o(1/\sqrt{T})$ and $J$ and $K$ denote $J(\theta)$ and $K(\theta)$. By (3.6) we obtain

$$E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)(R_T - a)\right]$$

$$= E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)(R_T - \theta X_0^2)\right] + E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)(\theta X_0^2 - a)\right]$$

$$= 0;$$

$$E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)\left\{-\sum_{t=1}^{T-2}\left(X_t^2 - \frac{1}{1-\theta^2}\right)\right\}(R_T - a)\right] = 0:$$

$$E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)^2 (R_T - a)\right] = 0.$$

Hence the stochastic expansion (3.7) of the *MLE* $\hat{\theta}_{ML}$ based on $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ can be essentially reduced to the case when the stochastic expansion of the *MLE* based on $(X_0, X_1, \ldots, X_T)$ is given by (3.2).

In a similar way as in Akahira [3] we have established the following:

**Theorem 3.1.** The bias-adjusted *MLE* $\hat{\theta}^*$ $(\in A_2)$ based on the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ is second order asymptotically efficient.

## 4. Asymptotic deficiency of the estimator.

In this section we deal with the case when $X_0 = 0$ in the $AR$ process given by (3.1). Then we shall obtain the asymptotic deficiency of the $MLE$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to the $MLEs$ based on the samples $(X_1, \dots, X_{T-1})$, $(X_1, \dots, X_T)$ and $(X_1, \dots, X_{T+1})$, respectively. By (3.2) it is shown that the bias-adjusted $MLE$ based on the sample $(X_1, \dots, X_T)$ belongs to the class $D$. In a similar way as the independently and identically distributed sample case discussed in Akahira and Takeuchi [7] it is seen that the bias-adjusted $MLE$ is third order asymptotically efficient in the class $D$.

Next we consider the $MLE$ $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$. By (3.7) we have

$$(4.1)\quad \sqrt{IT}(\hat{\theta}_{ML} - \theta) = \frac{Z_1}{\sqrt{I}} + \frac{1}{\sqrt{IT}}Q + o_p\left(\frac{1}{\sqrt{T}}\right),$$

where $Z_1 = Z_1' + \dfrac{1}{\sqrt{T}}(R_T' - a')$ ;

$$Q = \frac{1}{I}Z_1'Z_2 - \frac{3J+K}{2I^2}Z_1'^2,$$

with $Z_1' = \dfrac{1}{\sqrt{T}}\sum_{t=1}^{T-1} U_t X_{t-1}$; $Z_2 = -\dfrac{1}{\sqrt{T}}\sum_{t=1}^{T-2}\left(X_t^2 - \dfrac{1}{1-\theta^2}\right)$;

$$R_T' = \frac{\theta}{(1+\theta^2)^2}\{2(1+\theta^2)(\theta U_T + U_{T+1})X_{T-1} + (\theta U_T + U_{T+1})^2\};$$

$$a' = \frac{\theta}{1+\theta^2}$$

Note that $E_\theta(R_T') = a'$ .

Since

$$E_\theta(Z_1'^3 Z_2^2) = O\left(\frac{1}{\sqrt{T}}\right);\ E_\theta(Z_1'^4 Z_2) = O\left(\frac{1}{\sqrt{T}}\right);\ E_\theta(Z_1'^5) = O\left(\frac{1}{\sqrt{T}}\right);$$

$$E_\theta[Z_1'^2 Z_2^2(R_T' - a')] = O\left(\frac{1}{\sqrt{T}}\right);\ E_\theta[Z_1'^3 Z_2(R_T' - a)] = O\left(\frac{1}{\sqrt{T}}\right);$$

$$E_\theta[Z_1'^4(R_T' - a')] = O\left(\frac{1}{\sqrt{T}}\right),$$

it follows that

$$E_\theta[Z_1 Q^2] = O(\frac{1}{\sqrt{T}}).$$

Hence the bias-adjusted $MLE$ $\hat{\theta}^*_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ belongs to the class $D$.

We consider the estimator $\hat{\theta}^{T-1}_{ML}$ based on the sample $(X_1, \dots, X_{T-1})$ which has the stochastic expansion

$$(4.2) \quad \sqrt{IT}(\hat{\theta}^{T-1}_{ML} - \theta) = \frac{Z'_1}{\sqrt{T}} + \frac{1}{\sqrt{IT}} Q_0 + o_p(\frac{1}{\sqrt{T}}),$$

where $Q_0 = \dfrac{1}{I} Z_1{}' Z_2 - \dfrac{3J+K}{2I^2} Z_1{}'^2$.

Let $\hat{\theta}^T_{ML}$ be the $MLE$ based on the sample $(X_1, \dots, X_T)$ and $\hat{\theta}^{T*}_{ML}$ the bias-adjusted $MLE$. We put $S_T = \sqrt{IT}(\hat{\theta}^T_{ML} - \theta)$. Then $\hat{\theta}^T_{ML}$ has cumulants of the following form:

$$E_\theta(S_T) = \frac{\mu}{\sqrt{T}} + o(\frac{1}{T});$$

$$V_\theta(S_T) = 1 + \frac{\tau}{T} + o(\frac{1}{T});$$

$$E_\theta[\{S_T - E_\theta(S_T)\}^3] = \frac{\beta_3}{\sqrt{T}} + o(\frac{1}{T});$$

$$E_\theta[\{S_T - E_\theta(S_T)\}^4] - 3\{V_\theta(S_T)\}^2 = \frac{\beta_4}{T} + o(\frac{1}{T}).$$

It is noted by Akahira [5] that only the terms of the order of $T^{-1}$ in the cumulants are essentially different between the estimators $\hat{\theta}^{T*}_{ML}$ and $\hat{\theta}^*_{ML}$ since they belong to the class $D$.

Also the Edgeworth expansion of the distribution of $\hat{\theta}^T_{ML}$ is given by

$$(4.3) \quad P_{T,\theta}\{\sqrt{IT}(\hat{\theta}^T_{ML} - \theta) \le u\} = \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2-1)\phi(u)$$
$$- \frac{\beta_4}{24T}(u^3 - 3u)\phi(u) - \frac{\beta_3^2}{72T}(u^5 - 10u^3 + 15u)\phi(u)$$

$$-\frac{\tau+\mu^2}{2T} u\,\phi(u) - \frac{\beta_3\mu}{6T}(u^3-3u)\,\phi(u) + o(\frac{1}{T}),$$

where $\Phi(u) = \int_{-\infty}^{u} \phi(x)dx$ with $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Since by (4.1)

$$E_\theta[Z_1'(R_T'-a')] = 0 ;$$

$$E_\theta[Q(R_T'-a')] = 0,$$

it follows by (4.1), (4.2), (4.3) and a similar way as in Akahira ([5], page 71) that the Edgeworth expansion of the conditional distribution of $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ given $R_T'$ is obtained by

(4.4) $P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}-\theta) \leq u \mid R_T'\}$

$$= P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}^{T-1}_{ML}-\theta) \leq u - \frac{1}{\sqrt{IT}}(R_T'-a') \mid R_T'\}$$

$$= \Phi(u - \frac{1}{\sqrt{IT}}(R_T'-a')) - \frac{\mu}{\sqrt{T}}\phi(u - \frac{1}{\sqrt{IT}}(R_T'-a'))$$

$$- \frac{\beta_3}{6\sqrt{T}}\{(u - \frac{1}{\sqrt{IT}}(R_T'-a'))^2 - 1\}\,\phi(u - \frac{1}{\sqrt{IT}}(R_T'-a'))$$

$$- \frac{\beta_4}{24T}(u^3-3u)\,\phi(u) - \frac{\beta_3^2}{72T}(u^5-10u^3+15u)\,\phi(u)$$

$$- \frac{\tau+u^2+1}{2T} u\,\phi(u) - \frac{\beta_3 u}{6T}(u^3-3u)\,\phi(u) + o(\frac{1}{T})$$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2-1)\,\phi(u) - \frac{\beta_4}{24T}(u^3-3u)\,\phi(u)$$

$$- \frac{\beta_3^2}{72T}(u^5-10u^3+15u)\,\phi(u) - \frac{\tau+\mu^2+1}{2T} u\,\phi(u) - \frac{\beta_3\mu}{6T}(u^3-3u)\,\phi(u)$$

$$- \frac{1}{\sqrt{IT}}(R_T'-a')\,\phi(u) + \frac{1}{2IT}(R_T'-a')^2 u\,\phi(u) - \frac{\mu}{\sqrt{IT}}(R_T'-a')u\phi(u)$$

$$- \frac{\beta_3}{6\sqrt{I}\,T}(R_T'-a')\,u\,(u^2-1)\,\phi(u) + \frac{\beta_3}{3\sqrt{I}\,T}(R_T'-a')\,u\,\phi(u) + o(\frac{1}{T}).$$

By (4.4) we have

$$(4.5)\quad P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}-\theta)\le u\}$$

$$= E_\theta[P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}-\theta)\le u \mid R_T'\}]$$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2-1)\phi(u) - \frac{\beta_4}{24T}(u^3-3u)\phi(u)$$

$$-\frac{\beta_3{}^2}{72T}(u^5-10u^3+15u)\phi(u) - \frac{\tau+\mu^2+1}{2T}u\phi(u) - \frac{\beta_3\mu}{6T}(u^3-3u)\phi(u)$$

$$+\frac{1}{2IT}\{E_\theta(R_T'-a')^2\}u\phi(u) + o(\frac{1}{T}).$$

Since

$$E_\theta(X_t^2) = E_\theta[(\sum_{i=1}^{t}\theta^{t-i}U_i)^2] = \frac{1-\theta^{2\cdot t}}{1-\theta^2};$$

$$E_\theta[(\theta U_T + U_{T+1})^2] = 1+\theta^2\ ;$$

$$E_\theta[(\theta U_T + U_{T+1})^4] = 3(1+\theta^2)^2,$$

it follows that

$$E_\theta(R_T{}'^2) = E[\{\frac{\theta}{(1+\theta^2)^2}(\theta U_T + U_{T+1})(2(1+\theta^2)X_{T-1} + \theta U_T + U_{T+1})\}^2]$$

$$= \frac{\theta^2(\theta^2+7)}{(1+\theta^2)^2(1-\theta^2)}.$$

Hence the variance of $R_T{}'$ is given by

$$(4.6)\quad V_\theta(R_T') = E_\theta(R_T'-a')^2 = E_\theta(R_T{}'^2) - a'^2$$

$$= \frac{2\theta^2(\theta^2+3)}{(1+\theta^2)^2(1-\theta^2)}\ .$$

In a similar way as in Akahira ([5], page 71) we have by (4.3)

$$(4.7)\quad P_{T-1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}^{T-1}-\theta)\le u\}$$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2 - 1)\phi(u) - \frac{\beta_4}{24T}(u^3 - 3u)\phi(u)$$

$$- \frac{\beta_3^2}{72T}(u^5 - 10u^3 + 15u)\phi(u) - \frac{\tau_{-1} + \mu^2}{2T} u\phi(u)$$

$$- \frac{\beta_3\mu}{6T}(u^3 - 3u)\phi(u) + o\left(\frac{1}{T}\right);$$

(4.8) $P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}^{T+1} - \theta) \leq u\}$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2 - 1)\phi(u) - \frac{\beta_4}{24T}(u^3 - 3u)\phi(u)$$

$$- \frac{\beta_3^2}{72T}(u^5 - 10u^3 + 15u)\phi(u) - \frac{\tau_1 + \mu^2}{2T} u\phi(u)$$

$$- \frac{\beta_3\mu}{6T}(u^3 - 3u)\phi(u) + o\left(\frac{1}{T}\right),$$

where $\tau_{-1} = \tau + 1$ and $\tau_1 = \tau - 1$

with $V_\theta(\sqrt{I(T-1)}(\hat{\theta}_{ML}^{T-1} - \theta)) = 1 + \frac{\tau_{-1}}{T-1} + o\left(\frac{1}{T-1}\right);$

$$V_\theta(\sqrt{I(T+1)}(\hat{\theta}_{ML}^{T+1} - \theta)) = 1 + \frac{\tau_1}{T+1} + o\left(\frac{1}{T+1}\right).$$

Note that the difference in the above (4.7) and (4.8) appears in the sixth terms of their right-hand sides. It is seen by (4.3),(4.7)and(4.8) that the asymptotic deficiencies of $\hat{\theta}_{ML}^{T-1}$ and $\hat{\theta}_{ML}^{T+1}$ with respect to $\hat{\theta}_{ML}^{T}$ are equal to 1 and $-1$, respectively, i.e., $\tau_{-1} - \tau = 1$ and $\tau_1 - \tau = -1$.

It follows by (4.3), (4.5), (4.7), (4.8) and Akahira [5] that the asymptotic deficiencies of $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ w.r.t. $\hat{\theta}_{ML}^{T-1}$, $\hat{\theta}_{ML}^{T}$ and $\hat{\theta}_{ML}^{T+1}$ are given by $-V_\theta(R_T')/I$, $1 - \{V_\theta(R_T')/I\}$ and $2 - \{V_\theta(R_T')/I\}$ with (4.6), respectively. Hence we have established the following.

**Theorem 4.1.** The asymptotic deficiencies of $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to $\hat{\theta}_{ML}^{T-1}$, $\hat{\theta}_{ML}^{T}$ and $\hat{\theta}_{ML}^{T+1}$ are given in the table below.

| Estimator $\hat{\theta}$ | Asymptotic deficiency of $\hat{\theta}_{ML}$ w.r.t. $\hat{\theta}$ |
|---|---|
| $\hat{\theta}_{ML}^{T-1} = \hat{\theta}_{ML}^{T-1}(X_1, \dots, X_{T-1})$ | $-\dfrac{2\theta^2(\theta^2+3)}{(1+\theta^2)^2}$ |
| $\hat{\theta}_{ML}^{T} = \hat{\theta}_{ML}^{T}(X_1, \dots, X_T)$ | $\dfrac{5-(\theta^2+2)^2}{(1+\theta^2)^2}$ |
| $\hat{\theta}_{ML}^{T+1} = \hat{\theta}_{ML}^{T+1}(X_1, \dots, X_{T+1})$ | $\dfrac{2(1-\theta^2)}{(1+\theta^2)^2}$ |

**Remark.** It is seen that the loss of informations on $\theta$ from the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to the samples $(X_1, \dots, X_{T-1})$, $(X_1, \dots, X_T)$ and $(X_1, \dots, X_{T+1})$ through the *MLE* are given by the asymptotic deficiencies depending on $\theta$ in the table in Theorem 4.1, respectively.

It is natural that the asymptotic deficiencies of $\hat{\theta}_{ML}$ w.r.t. $\hat{\theta}_{ML}^{T-1}$ and $\hat{\theta}_{ML}^{T+1}$ are negative and positive, respectively since the based sample of $\hat{\theta}_{ML}$ includes that of $\hat{\theta}_{ML}^{T-1}$ and is done in that of $\hat{\theta}_{ML}^{T+1}$. It is also seen that the asymptotic deficiency of $\hat{\theta}_{ML}$ w.r.t. $\hat{\theta}_{ML}^{T}$ is positive if $|\theta| < \sqrt{\sqrt{5}-2} \fallingdotseq 0.486$ and negative if $\sqrt{\sqrt{5}-2} < |\theta| < 1$. The fact means that for the sample $(X_1, \dots, X_{T-1})$, $X_T$ is more informative than $X_{T+1}$ if $|\theta| < \sqrt{\sqrt{5}-2}$ and $X_T$ is less informative than $X_{T+1}$ if $\sqrt{\sqrt{5}-2} < |\theta| < 1$. It seems reasonable in the process (3.1) since it is better to contract the spacing of the observations if $|\theta|$ is small and expand it if $|\theta|$ is big. Further it may be extended to the problem on the optimum spacing of observations from a process ([10]).

In a similar way as the above discussion it may be possible to obtain the asymptotic deficiency of the *MLE* based on the sample $(X_1, \dots, X_i, X_{i+k}, \dots, X_{T+1})$ in which $X_{i+1}, \dots, X_{i+k-1}$ are missing, where $1 \le i < i+k \le T+1$ and that of the *MLE* based on the sample in which any observations except the extremes are missing.

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