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A Remark on Comparison with a Maximum Likelihood Estimator in Asymptotic Variances in a Non-Regular Case*

Masafumi AKAHIRA**

Abstract

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent and identically distributed random variables with a truncated exponential density. Then it is shown that there exists an estimator whose asymptotic variance is smaller than that of a maximum likelihood estimator.

1. Introduction

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent and identically distributed random variables with a density $f(x-\theta)$ satisfying

$$f(x)=\begin{cases} ce^{-x} & \text{for } 0<x<1\,; \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $c=1/(1-e^{-1})$.

It is easily seen that the maximum likelihood estimator (MLE) $\hat{\theta}_{ML}$ is given by $\hat{\theta}_{ML}=\min_{1\leq i\leq n} X_i$. Then it is shown that there exists an estimator whose asymptotic variance is smaller than that of $\hat{\theta}_{ML}$.

Although the density of only the form (1) is treated in this paper, it is possible to extend it to the density $f(x)$ such that $f(x)$ is continuously differentiable in the interval (α, β),

$$f(x)>0 \quad \text{for} \quad \alpha<x<\beta\,;$$
$$f(x)=0 \quad \text{otherwise,}$$

and $0<\lim_{x\to\alpha+0} f(x)=\lim_{x\to\beta-0} f(x)<\infty$.

2. Results

Let $\mathscr{X}$ be an abstract sample space whose generic point is denoted by x, $\mathscr{B}$ a σ-field of subsets of $\mathscr{X}$ and $\{P_\theta : \theta\in\Theta\}$ a set of probability measures on $\mathscr{B}$, where Θ is called a parameter space. We suppose that $\mathscr{X}=\Theta=R^1$ and $\mathscr{B}$ is a Borel σ-field and for each θ P_θ has the density $f(x-\theta)$ of the form (1). Consider n-fold direct products $(R^n, \mathscr{B}^n)$ of $(R^1, \mathscr{B})$ and the corresponding product measure $P_\theta{}^n$ of P_θ. An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathscr{B}^n$-measurable functions $\hat{\theta}_n$ on R^n into Θ. For simplicity we denote $\{\hat{\theta}_n\}$ by $\hat{\theta}$. A distribution function $F_{\theta,\hat{\theta}}{}^{\mathrm{C}}(\cdot)$ is called to be the asymptotic distribution function of an estimator $\hat{\theta}$ of order $\mathrm{C}=\{c_n\}$ if for each real number y, $F_{\theta,\hat{\theta}}{}^{\mathrm{C}}(y)$ is continuous in θ and for

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** Statistical Laboratory, University of Electro-Communications

any $\vartheta \in \Theta$ there exists a positive number d such that for any continuity point y of $F_{\theta,\hat{\theta}^C}(y)$,

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<d} |P_\theta^n(\{c_n(\hat{\theta}_n-\theta)\leq d\})-F_{\theta,\hat{\theta}^C}(y)|=0$$

(Akahira [1]).

Since

$$\begin{aligned}&\lim_{n\to\infty} P_\theta^n(\{n(\hat{\theta}_{ML}-\theta)\leq y\})\\ &=\lim_{n\to\infty} P_\theta^n(\{\min_{1\leq i\leq n} x_i\leq \theta+yn^{-1}\})\\ &=\begin{cases}1-e^{-cy} & \text{for } y>0\,;\\ 0 & \text{for } y\leq 0,\end{cases}\end{aligned}$$

it follows that the density $f_{\hat{\theta}_{ML}}(y)$ of the asymptotic distribution of $\hat{\theta}_{ML}$ of order $\{n\}$ is given by

$$f_{\hat{\theta}_{ML}}(y)=\begin{cases}ce^{-cy} & \text{for } y>0\,;\\ 0 & \text{for } y\leq 0.\end{cases} \tag{2}$$

Let $\hat{\theta}^*=\max_{1\leq i\leq n} X_i-1$. Then we have

$$\begin{aligned}&\lim_{n\to\infty} P_\theta^n(\{n(\hat{\theta}^*-\theta)\leq y\})\\ &=\lim_{n\to\infty} P_\theta^n(\{\max_{1\leq i\leq n} x_i\leq \theta+1+yn^{-1}\})\\ &=\begin{cases}1 & \text{for } y\geq 0:\\ e^{ce^{-1}y} & \text{for } y<0.\end{cases}\end{aligned}$$

Hence the density $g_{\hat{\theta}^*}(y)$ of the asymptotic distribution of $\hat{\theta}^*$ of order $\{n\}$ is given by

$$g_{\hat{\theta}^*}(y)=\begin{cases}0 & \text{for } y\geq 0\,;\\ ce^{-1}e^{ce^{-1}y} & \text{for } y<0.\end{cases} \tag{3}$$

We define an estimator $\hat{\theta}_a$ by $a\hat{\theta}_{ML}+(1-a)\hat{\theta}^*$, where $4/5\leq a<1$.

We remark that $(\min_{1\leq i\leq n} X_i,\ \max_{1\leq i\leq n} X_i)$ is asymptotically sufficient (Akahira [2]).

Since $\hat{\theta}_{ML}$ and $\hat{\theta}^*$ are asymptotically independent, the density $h_{\hat{\theta}_a}(y)$ of the asymptotic distribution of $\hat{\theta}_a$ of order $\{n\}$ is a convolution of $f_{\hat{\theta}_{ML}}(y)$ and $g_{\hat{\theta}^*}(y)$. It follows from (2) and (3) that

$$h_{\hat{\theta}_a}(y)=\begin{cases}K_a\exp\left\{\dfrac{c}{(1-a)e}y\right\} & \text{for } y\leq 0\,;\\ K_a\exp\left(-\dfrac{c}{a}y\right) & \text{for } y<0,\end{cases}$$

where $K_a=c/\{a+(1-a)e\}$.

Next we shall calculate the asymptotic variances $V_a(Y)$ and $V_{ML}(Y)$ of $\hat{\theta}_a$ and $\hat{\theta}_{ML}$, respectively.

Since

$$E_a(Y)=\frac{K_a\{a^2-e^2(1-a)^2\}}{c^2}\,;$$

$$E_a(Y^2)=\frac{2K_a\{a^3+e^3(1-a)^3\}}{c^3},$$

it follows that

$$V_a(Y)=\frac{a^2+e^2(1-a)^2}{c^2}.$$

Since $e<3$ and $4/5\leq a<1$, we have

$$V_a(Y)<\frac{a^2+9(1-a)^2}{c^2}\leq\frac{1}{c^2}.$$

On the other hand it is easily seen that

$$V_{ML}(Y)=\frac{1}{c^2}.$$

Hence

$$V_a(Y)<V_{ML}(Y).$$

Therefore it is shown that the asymptotic variance of $\hat{\theta}_a$ is smaller than that of $\hat{\theta}_{ML}$.

References

[1] Akahira, M., "Asymptotic theory for estimation of location in non-regular cases, II: Bounds of asymptotic distributions of consistent estimators," Rep. Stat. Appl. Res., JUSE, 22, 99–115 (1975)

[2] Akahira, M., "A remark on asymptotic sufficiency of statistics in non-regular cases," Rep. Univ. Electro-Comm. 27-1, 125–128 (1976)