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A Note on the Second Order Asymptotic Efficiency of Estimators in an Autoregressive Process*

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Abstract

Let $\{X_i\}$ be defined by $X_i = \theta X_{i-1} + U_i$ $(i=1, 2, \cdots)$, where $X_0=0$ and $\{U_i\}$ is a sequence of independent identically distributed real random variables having a density f with mean 0 and variance σ^2. We assume that $|\theta|<1$. The purpose of this paper is to obtain the bound of the second order asymptotic distributions of second order asymptotically median unbiased estimators of θ and the estimator of θ attaining it, that is, the second order asymptotically efficient estimator of θ.

1. Introduction

Let X_i $(i=1, 2, \cdots)$ be defined recursively by

$$X_i = \theta X_{i-1} + U_i \qquad i=1, 2, \cdots,$$

where $X_0=0$ and $\{U_i : i=1, 2, \cdots\}$ is a sequence of independent identically distributed real random variables having a density f with mean 0 and variance σ^2.

We shall define an estimator to be second order asymptotically efficient if the second order asymptotic distribution of it attains the bound of the second order asymptotic distributions of second order asymptotically median unbiased (AMU) estimators of θ. We assume that $|\theta|<1$. The purpose of this paper is to obtain the bound of the second order asymptotic distributions of second order AMU estimators of θ and to show that a modified least squares estimator of θ is second order asymptotically efficient. The approach in this paper is similar to Bahadur [2] dealing with the bound for asymptotic variances.

2. Notations and definitions

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by x, $\mathcal{B}$ a σ-field of subsets of $\mathcal{X}$, and let Θ be a parameter space, which is assumed to be an open set in a Euclidean 1-space R^1. We shall denote by $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ the n-fold direct products of $(\mathcal{X}, \mathcal{B})$. For each $n=1, 2, \cdots$, the point of $\mathcal{X}^{(n)}$ will be denoted by $\tilde{x}_n=(x_1, \cdots, x_n)$. We consider a sequence of classes of probability measures $\{P_{n,\theta} : \theta\in\Theta\}$ $(n=1, 2, \cdots)$ each defined on $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ such that for each $n=1, 2, \cdots$ and each $\theta\in\Theta$ the following holds:

$$P_{n,\theta}(B^{(n)}) = P_{n+1,\theta}(B^{(n)}\times\mathcal{X})$$

for all $B^{(n)}\in\mathcal{B}^{(n)}$.

An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into Θ $(n=1, 2, \cdots)$.

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For an increasing sequence of positive numbers $\{c_n\}$ ($\lim_{n\to\infty} c_n=\infty$) an estimator $\{\hat{\theta}_n\}$ is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and every ϑ of Θ, there exist a sufficiently small positive number δ and a sufficiently large positive number L satisfying the following:

$$\overline{\lim_{n\to\infty}} \sup_{\theta:|\theta-\vartheta|<\delta} P_{n,\theta}(\{c_n|\hat{\theta}_n-\theta|\geq L\})<\varepsilon$$

(Akahira [1]).

In the subsequent discussions we shall deal only with the case when $c_n=\sqrt{n}$. Let $\{\hat{\theta}_n\}$ be a $\{\sqrt{n}\}$-consistent estimator.

Definition 1. *$\{\hat{\theta}_n\}$ is second order asymptotically median unbiased (or second order AMU for short) if for any $\vartheta\in\Theta$ there exists a positive number δ such that*

$$\lim_{n\to\infty} \sup_{\theta\in I_\delta(\vartheta)} \sqrt{n}\left|P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq 0\})-\frac{1}{2}\right|=0$$

where $I_\delta(\vartheta)=(\vartheta-\delta, \vartheta+\delta)$.

Definition 2. *For $\{\hat{\theta}_n\}$ second order asymptotically median unbiased $F_\theta(t)+(1/\sqrt{n})G_\theta(t)$ is called a second order asymptotic distribution of it if*

$$\lim_{n\to\infty} \sqrt{n}\,|P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\})-F_\theta(t)-(1/\sqrt{n})G_\theta(t)|=0.$$

If $\{\hat{\theta}_n\}$ is second order AMU, then $F_{\{\hat{\theta}_n\},\theta}{}^{+}$, $F_{\{\hat{\theta}_n\},\theta}{}^{-}$, $G_{\{\hat{\theta}_n\},\theta}{}^{+}$ and $G_{\{\hat{\theta}_n\},\theta}{}^{-}$ are defined as follows:

$$\lim_{n\to\infty} \sqrt{n}\,|P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\})-F_{\{\hat{\theta}_n\},\theta}{}^{+}(t)-(1/\sqrt{n})G_{\{\hat{\theta}_n\},\theta}{}^{+}(t)|=0 \tag{2.1}$$

for all $t\geq 0$,

$$\lim_{n\to\infty} \sqrt{n}\,|P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\})-F_{\{\hat{\theta}_n\},\theta}{}^{-}(t)-(1/\sqrt{n})G_{\{\hat{\theta}_n\},\theta}{}^{-}(t)|=0 \tag{2.2}$$

for all $t<0$.

Let $\theta_0(\in\Theta)$ be arbitrary but fixed. Consider the problem of testing hypothesis H^+: $\theta=\theta_0+(t/\sqrt{n})$ $(t>0)$ against alternative K: $\theta=\theta_0$. We define $\beta_{\theta_0}{}^{+}(t)$ and $\gamma_{\theta_0}{}^{+}(t)$ as follows:

$$\sup_{\{\phi_n\}\in\Phi_{1/2}} \overline{\lim_{n\to\infty}} \sqrt{n}\,\{E_{n,\theta_0}(\phi_n)-\beta_{\theta_0}{}^{+}(t)-(1/\sqrt{n})\gamma_{\theta_0}{}^{+}(t)\}=0 \tag{2.3}$$

where $\Phi_{1/2}=\{\{\phi_n\}: E_{n,\theta_0+(t/\sqrt{n})}(\phi_n)=1/2+o(1/\sqrt{n}),\ 0\leq\phi_n(\tilde{x}_n)\leq 1$ for all $\tilde{x}_n\in\mathcal{X}^{(n)}$ $(n=1, 2, \cdots)\}$.

Putting $A_{\{\hat{\theta}_n\},\theta}=\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\}$ we have for $t>0$

$$P_{n,\theta_0+(t/\sqrt{n})}(A_{\{\hat{\theta}_n\},\theta_0})=P_{n,\theta_0+(t/\sqrt{n})}(\{\sqrt{n}(\hat{\theta}_n-\theta_0-(t/\sqrt{n}))\leq 0\})=\frac{1}{2}+o\left(\frac{1}{\sqrt{n}}\right).$$

Since a sequence $\{I_{A\{\hat{\theta}_n\},\theta_0}\}$ of the indicators (or characteristic functions) of $A_{\{\hat{\theta}_n\},\theta_0}$ $(n=1, 2, \cdots)$ belongs to $\Phi_{1/2}$, it follows from (2.1) and (2.3) that

$$\overline{\lim_{n\to\infty}} \sqrt{n}\,\{F_{\{\hat{\theta}_n\},\theta_0}{}^{+}(t)+(1/\sqrt{n})G_{\{\hat{\theta}_n\},\theta_0}{}^{+}(t)-\beta_{\theta_0}{}^{+}(t)-(1/\sqrt{n})\gamma_{\theta_0}{}^{+}(t)\}\leq 0 \tag{2.4}$$

for all $t>0$.

Consider next the problem of the testing hypothesis H^-: $\theta=\theta_0+(t/\sqrt{n})$ $(t<0)$ against alternative K: $\theta=\theta_0$. Then we define $\beta_{\theta_0}{}^{-}(t)$ and $\gamma_{\theta_0}{}^{-}(t)$ as follows:

$$\inf_{\{\phi_n\}\in\Phi_{1/2}} \underline{\lim_{n\to\infty}} \sqrt{n}\,\{E_{n,\theta_0}(\phi_n)-\beta_{\theta_0}{}^{-}(t)-(1/\sqrt{n})\gamma_{\theta_0}{}^{-}(t)\}=0. \tag{2.5}$$

In a similar way as the case $t>0$, we have from (2.2) and (2.5)

$$\underline{\lim_{n\to\infty}} \sqrt{n}\,\{F_{\{\hat{\theta}_n\},\theta_0}{}^{-}(t)+(1/\sqrt{n})G_{\{\hat{\theta}_n\},\theta_0}{}^{-}(t)-\beta_{\theta_0}{}^{-}(t)-(1/\sqrt{n})\gamma_{\theta_0}{}^{-}(t)\}\geq 0 \tag{2.6}$$

for all $t<0$.

Since θ_0 is arbitrary, the bound of the second order asymptotic distributions of second order AMU estimators is obtained as follows:

$$\overline{\lim_{n\to\infty}} \sqrt{n}\,\{F_{\{\hat{\theta}_n\},\theta}{}^{+}(t)+(1/\sqrt{n})G_{\{\hat{\theta}_n\},\theta}{}^{+}(t)-\beta_\theta{}^{+}(t)-(1/\sqrt{n})\gamma_\theta{}^{+}(t)\}\leq 0 \text{ for all } t>0;$$

$$\underline{\lim_{n\to\infty}} \sqrt{n}\,\{F_{\{\hat{\theta}_n\},\theta}{}^{-}(t)+(1/\sqrt{n})G_{\{\hat{\theta}_n\},\theta}{}^{-}(t)-\beta_\theta{}^{-}(t)-(1/\sqrt{n})\gamma_\theta{}^{-}(t)\}\geq 0 \text{ for all } t<0.$$

For any $\theta\in\Theta$ letting $\beta_\theta{}^{+}(0)=1/2$ and $\gamma_\theta{}^{+}(0)=0$ we make the following definition.

Definition 3. *For $\{\hat{\theta}_n\}$ second order asymptotically median unbiased it is called second order asymptotically efficient if for each $\theta\in\Theta$*

$$\lim_{n\to\infty} P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\})=\begin{cases}\beta_\theta{}^{+}(t)+\dfrac{1}{\sqrt{n}}\gamma_\theta{}^{+}(t)+o\left(\dfrac{1}{\sqrt{n}}\right) & \textit{for all } t\geq 0,\\[2ex] \beta_\theta{}^{-}(t)+\dfrac{1}{\sqrt{n}}\gamma_\theta{}^{-}(t)+o\left(\dfrac{1}{\sqrt{n}}\right) & \textit{for all } t<0.\end{cases}$$

Throughout the subsequent section we assume that $\mathscr{X}=R^1$ and Θ is an open interval $(-1, 1)$ and consider an autoregressive process $\{X_i\}$ given in Introduction.

3. Second order asymptotic efficiency of estimators

In this section it will be shown that the bound of the second order asymptotic distributions of second order AMU estimators of θ is obtained using the best test statistics and that a modified least squares estimator of θ is second order asymptotically efficient if f is a normal density with mean 0 and variance σ^2. We assume the following:

Assumption (A). *f is continuously differentiable for four times and $f(u)>0$ for all u.*

Let θ_0 be arbitrary but fixed in Θ. Consider the problem of testing hypothesis $H: \theta=\theta_0+(t/\sqrt{n})$ $(t>0)$ against alternative $K: \theta=\theta_0$. Putting $\theta_1=\theta_0+\Delta$ with $\Delta=t/\sqrt{n}$, we define Z_{in} as follows:

$$Z_{in}=\log\frac{f(X_i-\theta_0X_{i-1})}{f(X_i-\theta_1X_{i-1})}.$$

If $\theta=\theta_0$, then we have

$$\begin{aligned}Z_{in}&=\log\frac{f(U_i)}{f(U_i-\Delta X_{i-1})}\\ &=\Delta\psi'(U_i)X_{i-1}-\frac{\Delta^2}{2}\psi''(U_i)X_{i-1}{}^2+\frac{\Delta^3}{6}\psi'''(U_i)X_{i-1}{}^3-\frac{\Delta^4}{24}\psi^{(4)}(U_i{}^*)X_{i-1}{}^4 \qquad (3.1)\end{aligned}$$

where $\psi(u)=\log f(u)$ and for each i, $U_i{}^*$ lies between U_i and $U_i-\Delta X_{i-1}$.

If $\theta=\theta_1$, then we have

$$\begin{aligned}Z_{in}&=\log\frac{f(U_i+\Delta X_{i-1})}{f(U_i)}\\ &=\Delta\psi'(U_i)X_{i-1}+\frac{\Delta^2}{2}\psi''(U_i)X_{i-1}{}^2+\frac{\Delta^3}{6}\psi'''(U_i)X_{i-1}{}^3+\frac{\Delta^4}{24}\psi^{(4)}(U_i{}^{**})X_{i-1}{}^4 \qquad (3.2)\end{aligned}$$

where for each i, $U_i{}^{**}$ lies between U_i and $U_i+\Delta X_{i-1}$.

For the subsequent discussions we assume the following:

Assumption (B). *$d^4\log f(u)/du^4$ $(=\psi^{(4)}(u))$ is a bounded function and $\lim_{u\to\pm\infty} f(u)=\lim_{u\to\pm\infty} f'(u)=\lim_{u\to\pm\infty} f''(u)=0$ and $E[|U_i|^k]<\infty$ $(k=1, 2, 3, 4)$.*

Then we have

$$\int_{-\infty}^{\infty}\psi'''(u)f(u)du=-3\int_{-\infty}^{\infty}\psi''(u)\psi'(u)f(u)du-\int_{-\infty}^{\infty}\{\psi'(u)\}^{3}f(u)du.$$

Put $I=\int_{-\infty}^{\infty}\{\psi'(u)\}^{2}f(u)du$, $J=\int_{-\infty}^{\infty}\psi'(u)\psi''(u)f(u)du$, $K=\int_{-\infty}^{\infty}\{\psi'(u)\}^{3}f(u)du$ and $\mu_3=\int_{-\infty}^{\infty}u^{3}f(u)du.$

Since $E(\psi''(U_i))=-I$, $E_\theta(X_{i-1}{}^2)=\sigma^2\frac{1-\theta^{2(i-1)}}{1-\theta^2}$ and $E_\theta(X_{i-1}{}^3)=\mu_3\frac{1-\theta^{3(i-1)}}{1-\theta^3}$, it follows from (3. 1) that

$$E_{\theta_0}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=\frac{n\varDelta^2\sigma^2I}{2(1-\theta_0{}^2)}-\frac{n\varDelta^3(3J+K)\mu_3}{6(1-\theta_0{}^3)}+o(n\varDelta^3). \tag{3. 3}$$

Similarly we have

$$V_{\theta_0}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=E_{\theta_0}\left[\left\{\sum_{i=1}^{n}(Z_{\text{in}}-E_{\theta_0}(Z_{\text{in}}))\right\}^2\right]$$
$$=\frac{n\varDelta^2\sigma^2I}{1-\theta_0{}^2}-\frac{n\varDelta^3J\mu_3}{1-\theta_0{}^3}+o(n\varDelta^3). \tag{3. 4}$$

Further we have

$$E_{\theta_0}\left[\left\{\sum_{i=1}^{n}Z_{\text{in}}-E_{\theta_0}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)\right\}^3\right]=\frac{\varDelta^3nK\mu_3}{1-\theta_0{}^3}+o(n\varDelta^3) \tag{3. 5}$$

Since $\varDelta=t/\sqrt{n}$, it follows from (3. 3), (3. 4) and (3. 5) that

$$E_{\theta_0}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=\frac{t^2\sigma^2I}{2(1-\theta_0{}^2)}-\frac{t^3\mu_3(3J+K)}{6\sqrt{n}(1-\theta_0{}^3)}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 6}$$

$$V_{\theta_0}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=\frac{t^2\sigma^2I}{1-\theta_0{}^2}-\frac{t^3\mu_3J}{\sqrt{n}(1-\theta_0{}^3)}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 7}$$

$$E_{\theta_0}\left[\left\{\sum_{i=1}^{n}Z_{\text{in}}-E_{\theta_0}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)\right\}^3\right]=\frac{t^3\mu_3K}{\sqrt{n}(1-\theta_0{}^3)}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3. 8}$$

Since $\theta_1=\theta_0+(t/\sqrt{n})$, it follows that for sufficiently large n

$$\frac{1}{1-\theta_1{}^2}=\frac{1}{1-\theta_0{}^2}+\frac{2t\theta_0}{\sqrt{n}(1-\theta_0{}^2)^2}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 9}$$

$$\frac{1}{1-\theta_1{}^3}=\frac{1}{1-\theta_0{}^3}+\frac{3t\theta_0{}^2}{\sqrt{n}(1-\theta_0{}^3)^2}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3. 10}$$

In a similar way as the case $\theta=\theta_0$ we have from (3. 2), (3. 9) and (3. 10)

$$E_{\theta_1}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=-\frac{t^2\sigma^2I}{2(1-\theta_0{}^2)}-\frac{t^3\sigma^2I\theta_0}{\sqrt{n}(1-\theta_0{}^2)^2}-\frac{t^3\mu_3(3J+K)}{6\sqrt{n}(1-\theta_0{}^3)}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 11}$$

$$V_{\theta_1}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=\frac{t^2\sigma^2I}{1-\theta_0{}^2}+\frac{2t^3\sigma^2I\theta_0}{\sqrt{n}(1-\theta_0{}^2)^2}+\frac{\mu_3J}{\sqrt{n}(1-\theta_0{}^3)}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 12}$$

$$E_{\theta_1}\left[\left\{\sum_{i=1}^{n}Z_{\text{in}}-E_{\theta_1}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)\right\}^3\right]=\frac{t^3\mu_3K}{\sqrt{n}(1-\theta_0{}^3)}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3. 13}$$

If

$$E_{\theta}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=\mu+\frac{1}{\sqrt{n}}c_1+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 14}$$

$$V_{\theta}\left(\sum_{i=1}^{n}Z_{\text{in}}\right)=v^2+\frac{1}{\sqrt{n}}c_2+o\left(\frac{1}{\sqrt{n}}\right); \tag{3. 15}$$

$$E_{\theta}\left[\left\{\sum_{i=1}^{n} Z_{in}-E_{\theta}\left(\sum_{i=1}^{n} Z_{in}\right)\right\}^3\right]=\frac{1}{\sqrt{n}}c_3+o\left(\frac{1}{\sqrt{n}}\right), \tag{3.16}$$

then using Gram-Charlier expansion(*) we have

$$P_{n,\theta}\left(\left\{\sum_{i=1}^{n} Z_{in}\leq a\right\}\right)$$
$$=\Phi\left(\frac{a-\mu}{v}\right)-\phi\left(\frac{a-\mu}{v}\right)\left[\frac{c_1}{v\sqrt{n}}+\frac{c_2}{2v^2\sqrt{n}}\left(\frac{a-\mu}{v}\right)+\frac{c_3}{6v^3\sqrt{n}}\left\{\left(\frac{a-\mu}{v}\right)^2-1\right\}\right]+o\left(\frac{1}{\sqrt{n}}\right) \tag{3.17}$$

where $\Phi(x)=\int_{-\infty}^{x}\phi(u)du$ with $\phi(u)=\frac{1}{\sqrt{2\pi}}e^{-(u^2/2)}$.

Next we shall choose a such that

$$P_{n,\theta_1}\left(\left\{\sum_{i=1}^{n} Z_{in}\geq a\right\}\right)=\frac{1}{2}+o\left(\frac{1}{\sqrt{n}}\right).$$

For the purpose putting

$$\Phi\left(\frac{a-\mu}{v}\right)-\phi\left(\frac{a-\mu}{v}\right)\left[\frac{c_1}{v\sqrt{n}}+\frac{c_2}{2v^2\sqrt{n}}\left(\frac{a-\mu}{v}\right)+\frac{c_3}{6v^3\sqrt{n}}\left\{\left(\frac{a-\mu}{v}\right)^2-1\right\}\right]$$
$$=\frac{1}{2}+o\left(\frac{1}{\sqrt{n}}\right)$$
$$=\Phi(0)+o\left(\frac{1}{\sqrt{n}}\right),$$

we have

$$\frac{a-\mu}{v}=O\left(\frac{1}{\sqrt{n}}\right).$$

Since

$$\Phi\left(\frac{a-\mu}{v}\right)-\Phi(0)=\phi(\xi)\frac{a-\mu}{v}=\phi\left(\frac{a-\mu}{v}\right)\left(\frac{c_1}{v\sqrt{n}}-\frac{c_3}{6v^3\sqrt{n}}\right)+o\left(\frac{1}{\sqrt{n}}\right)$$

where ξ lies between 0 and $(a-\mu)/v$, we obtain

$$a=\mu+\frac{c_1}{\sqrt{n}}-\frac{c_3}{6v^2\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.18}$$

From (3. 11)$\sim$(3. 16) and (3. 18) we have

$$a=-\frac{t^2\sigma^2 I}{2(1-\theta_0^2)}-\frac{t^3\mu_3(3J+K)}{6\sqrt{n}(1-\theta_0^3)}-\frac{t^3\sigma^2 I\theta_0}{\sqrt{n}(1-\theta_0^2)^2}-\frac{t^3\mu_3 K(1-\theta_0^2)}{6\sqrt{n}\,t^2\sigma^2 I(1-\theta_0^3)}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.19}$$

On the other hand since

$$P_{n,\theta_0}\left(\left\{\sum_{i=1}^{n} Z_{in}\geq a\right\}\right)=P_{n,\theta_0}\left(\left\{-\left(\sum_{i=1}^{n} Z_{in}-a-\frac{t^2\sigma^2 I}{1-\theta_0^2}\right)\leq\frac{t^2\sigma^2 I}{1-\theta_0^2}\right\}\right),$$

putting

$$W_n=-\left(\sum_{i=1}^{n} Z_{in}-a-\frac{t^2\sigma^2 I}{1-\theta_0^2}\right)$$

we have from (3. 6), (3. 7), (3. 8) and (3. 19)

$$E_{\theta_0}(W_n)=-\frac{t^3\sigma^2 I\theta_0}{\sqrt{n}(1-\theta_0^2)^2}-\frac{t\mu_3 K(1-\theta_0^2)}{6\sqrt{n}\,\sigma^2 I(1-\theta_0^3)}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3.20}$$

* Since Z_{in} are not independently and identically distributed, the usual form of Gram-Charlier expansion may not be applicable but in this particular situation it can be proved that Gram-Charlier expansion holds true.

$$V_{\theta_0}(W_n)=\frac{t^2\sigma^2 I}{1-\theta_0^2}-\frac{t^3\mu_3 J}{\sqrt{n}(1-\theta_0^3)}+o\left(\frac{1}{\sqrt{n}}\right); \tag{3.21}$$

$$E_{\theta_0}[\{W_n-E_{\theta_0}(W_n)\}^3]=-\frac{t^3\mu_3 K}{\sqrt{n}(1-\theta_0^3)}+o\left(\frac{1}{\sqrt{n}}\right). \tag{3.22}$$

From (3.17), (3.20), (3.21) and (3.22) we obtain

$$P_{n,\theta_0}\left(\left\{W_n\leq\frac{t^2\sigma^2 I}{1-\theta_0^2}\right\}\right)$$
$$=\Phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)+\phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\left\{\frac{t^2\sigma\sqrt{I}\,\theta_0}{\sqrt{n}(1-\theta_0^2)^{3/2}}+\frac{t^2\sqrt{1-\theta_0^2}\,\mu_3}{6\sigma\sqrt{I}\sqrt{n}(1-\theta_0^3)}(3J+K)\right\}+o\left(\frac{1}{\sqrt{n}}\right).$$

If $\{\hat{\theta}_n\}$ is second order asymptotically median unbiased, then it follows by (2.1), (2.3), (2.4) and the fundamental lemma of Neyman and Pearson that for all $t>0$

$$\varlimsup_{n\to\infty}\sqrt{n}\left[P_{n,\theta_0}(\{\sqrt{n}(\hat{\theta}_n-\theta_0)\leq t\})-\Phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\right.$$
$$\left.-\phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\left\{\frac{t^2\sigma\sqrt{I}\,\theta_0}{\sqrt{n}(1-\theta_0^2)^{3/2}}+\frac{t^2\sqrt{1-\theta_0^2}\,\mu_3}{6\sigma\sqrt{I}\sqrt{n}(1-\theta_0^3)}(3J+K)\right\}\right]\leq 0.$$

In a similar way as the case $t>0$, we have by (2.2), (2.5), (2.6) and the fundamental lemma of Neyman and Pearson

$$\varliminf_{n\to\infty}\sqrt{n}\left[P_{n,\theta_0}(\{\sqrt{n}(\hat{\theta}_n-\theta_0)\leq t\})-\Phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\right.$$
$$\left.-\phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\left\{\frac{t^2\sigma\sqrt{I}\,\theta_0}{\sqrt{n}(1-\theta_0^2)^{3/2}}+\frac{t^2\sqrt{1-\theta_0^2}\,\mu_3}{6\sigma\sqrt{I}\sqrt{n}(1-\theta_0^3)}(3J+K)\right\}\right]\geq 0.$$

for all $t<0$.

Since θ_0 is arbitrary, we have now established

Theorem 1. *Under Assumptions (A) and (B), the bound of the second order asymptotic distributions of second order AMU estimators $\{\hat{\theta}_n\}$ is given as follows:*

$$\varlimsup_{n\to\infty}\sqrt{n}\left[P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\})-\Phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta^2}}\right)\right.$$
$$\left.-\phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta^2}}\right)\left\{\frac{t^2\sigma\sqrt{I}\,\theta}{\sqrt{n}(1-\theta^2)^{3/2}}+\frac{t^2\sqrt{1-\theta^2}\,\mu_3}{6\sigma\sqrt{I}\sqrt{n}(1-\theta^3)}(3J+K)\right\}\right]\leq 0 \tag{3.23}$$

for all $t\geq 0$;

$$\varliminf_{n\to\infty}\sqrt{n}\left[P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_n-\theta)\leq t\})-\Phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta^2}}\right)\right.$$
$$\left.-\phi\left(\frac{t\sigma\sqrt{I}}{\sqrt{1-\theta^2}}\right)\left\{\frac{t^2\sigma\sqrt{I}\,\theta}{\sqrt{n}(1-\theta^2)^{3/2}}+\frac{t^2\sqrt{1-\theta^2}\,\mu_3}{6\sqrt{I}\sqrt{n}(1-\theta^3)}(3J+K)\right\}\right]\geq 0 \tag{3.24}$$

for all $t<0$.

The least squares estimator $\hat{\theta}_{LS}$ of θ is given by $\left(\sum_{i=2}^{n}X_iX_{i-1}\right)\Big/\sum_{i=2}^{n}X_{i-1}^2$. It is seen that under Assumptions (A) and (B) $\hat{\theta}_{LS}$ is a $\{\sqrt{n}\}$-consistent estimator. We assume that f is a normal density. Then Assumptions (A) and (B) hold.

Since

$$\sqrt{n}(\hat{\theta}_{LS}-\theta)=\frac{(1/\sqrt{n})\sum_{i=2}^{n}U_iX_{i-1}}{(1/n)\sum_{i=2}^{n}X_{i-1}^2}$$

it follows that $\sqrt{n}(\hat{\theta}_{LS}-\theta)\leq t$ if and only if

$$\frac{1}{\sqrt{n}}\sum_{i=2}^{n} U_i X_{i-1} - \frac{t}{n}\sum_{i=2}^{n} X_{i-1}^2 \leq 0.$$

Put

$$Z_n = \frac{1}{\sqrt{n}}\sum_{i=2}^{n} U_i X_{i-1} - \frac{t}{n}\sum_{i=2}^{n} X_{i-1}^2.$$

Since

$$E_\theta(Z_n) = -\frac{t\sigma^2}{1-\theta^2} + o\left(\frac{1}{\sqrt{n}}\right);$$

$$V_\theta(Z_n) = \frac{\sigma^4}{1-\theta^2} - \frac{4\sigma^4\theta t}{\sqrt{n}(1-\theta^2)^2} + o\left(\frac{1}{\sqrt{n}}\right);$$

$$E_\theta[\{Z_n - E_\theta(Z_n)\}^3] = \frac{6\sigma^6\theta}{\sqrt{n}(1-\theta^2)^2} + o\left(\frac{1}{\sqrt{n}}\right),$$

using Gram-Charlier exansion we obtain

$$\begin{aligned} &P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_{LS}-\theta) \leq t\}) \\ &= P_{n,\theta}(\{Z_n \leq 0\}) \\ &= \Phi\left(\frac{t}{\sqrt{1-\theta^2}}\right) + \phi\left(\frac{t}{\sqrt{1-\theta^2}}\right)\left\{\frac{\theta t^2}{\sqrt{n}(1-\theta^2)^{3/2}} + \frac{\theta}{\sqrt{n}\sqrt{1-\theta^2}}\right\} + o\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

As is immediately seen from above, $\hat{\theta}_{LS}$ is not second order AMU. If we define a modified least squares estimator $\hat{\theta}_{LS}^*$ as follows:

$$\hat{\theta}_{LS}^* = \left(1 + \frac{1}{n}\right)\hat{\theta}_{LS},$$

then $\hat{\theta}_{LS}^*$ is second order AMU and

$$P_{n,\theta}(\{\sqrt{n}(\hat{\theta}_{LS}^* - \theta) \leq t\}) = \Phi\left(\frac{t}{\sqrt{1-\theta^2}}\right) + \phi\left(\frac{t}{\sqrt{1-\theta^2}}\right)\frac{\theta t^2}{\sqrt{n}(1-\theta^2)^{3/2}} + o\left(\frac{1}{\sqrt{n}}\right). \quad (3.25)$$

Since $\sigma^2 I = 1$ and $\mu_3 = 0$, it follows from (3.23), (3.24) and (3.25) that the second order asymptotic distribution of $\hat{\theta}_{LS}^*$ attains the bound of the second order asymptotic distributions of second order AMU estimators. Therefore we have now established

Theorem 2. *If f is a normal density with mean 0 and variance σ^2, then $\hat{\theta}_{LS}^*$ is second order asymptotically efficient.*

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