J. Japan Statist. Soc.
Vol. 19 No. 2 1989 179-196

BEHAVIOUR OF JACKKNIFE ESTIMATORS IN TERMS OF ASYMPTOTIC DEFICIENCY UNDER TRUE AND ASSUMED MODELS

Dedicated to Professor Yukihiro Kodama on his 60th birthday

Masafumi Akahira*

The problem on jackknifing estimators is investigated in the presence of nuisance parameters from the viewpoint of higher order asymptotics. It is shown that the asymptotic deficiency of the jackknife estimator relative to the bias-adjusted maximum likelihood estimator (MLE) is equal to zero under true and assumed models. Moreover, the asymptotic deficiency of the MLE or the jackknife estimator under the assumed model relative to that under the true model is given.

Key words: Jackknife estimator, Maximum likelihood estimator, Asymptotic deficiency, True model, Assumed Model.

1. Introduction

In higher order asymptotics the concept of asymptotic deficiency is very useful for comparing asymptotically efficient estimators (e.g. Akahira, 1986). On the other hand, resampling plans like jackknife and bootstrap have been recently studied by many authors (e.g. Hinkley 1978, Efron 1982). So, it seems to be interesting to investigate the problem on jackknifing estimators in terms of asymptotic deficiency from the viewpoint of higher order asymptotics. In one parameter case, it was shown by Akahira (1983, 1986) that the asymptotic deficiency of the jackknife estimator relative to the bias-adjusted MLE is equal to zero. And also the asymptotic deficiency of the jackknife estimator relative to the estimator in some class was given. Further, the asymptotic deficiency of the bias-adjusted MLE under the assumed model relative to that under the true model was given by Akahira (1986) under the unbiasedness condition.

In this paper, we consider the problem on jacknifing estimators in the presence of the nuisance parameter. It is shown that the jackknife estimator has asymptotic deficiency zero relative to the bias-adjusted MLE under the true and assumed models, which means that the estimators are asymptotically equivalent up to the third order in the sense that their asymptotic distributions are equal up to the order n^{-1} under the models. Further, the asymptotic deficiency of the MLE or the jackknife estimator under the assumed model relative to that under the true model is given.

2. Notations and assumptions

Suppose that $X_1, \cdots, X_n$ are independent and identically distributed (i.i.d.) real random variables with a density function $f(x, \theta, \xi)$ with respect to a σ-finite measure μ,

Received February, 1989. Revised August, 1989. Accepted August, 1989.
* Institute of Mathematics, University of Tsukuba, Ibaraki 305, Japan.

where θ is a real-valued parameter to be estimated and ξ is a real-valued nuisance parameter. We assume the following conditions (A.1) to (A.5).

(A.1) The set $\{x: f(x, \theta, \xi)>0\}$ does not depend on θ and ξ.

(A.2) For almost all $x[\mu]$, $f(x, \theta, \xi)$ is three times continuously differentiable in θ and ξ.

(A.3) For each θ and each ξ

$$0<I_{00}(\theta, \xi)=E[\{l_0(\theta, \xi, X)\}^2]=-E[l_{00}(\theta, \xi, X)]<\infty ,$$
$$0<I_{11}(\theta, \xi)=E[\{l_1(\theta, \xi, X)\}^2]=-E[l_{11}(\theta, \xi, X)]<\infty ,$$

where $l_0(\theta, \xi, x)=(\partial/\partial\theta)l(\theta, \xi, x)$, $l_{00}(\theta, \xi, x)=(\partial^2/\partial\theta^2)l(\theta, \xi, x)$ $l_1(\theta, \xi, x)=(\partial/\partial\xi)l(\theta, \xi, x)$ and $l_{11}(\theta, \xi, x)=(\partial^2/\partial\xi^2)l(\theta, \xi, x)$ with $l(\theta, \xi, x)=\log f(x, \theta, \xi)$.

(A.4) The parameters are defined to be "orthogonal" in the sense that

$$E[l_{01}(\theta, \xi, X)]=0$$

where $l_{01}(\theta, \xi, x)=(\partial^2/\partial\theta\partial\xi)l(\theta, \xi, x)$.

Note that the condition (A.4) is not necessarily restricted, because otherwise we can redefine the parameter $\eta=g(\theta, \xi)$ so that we have the above orthogonality.

(A.5) There exist

$$J_{000}=E[l_{00}(\theta, \xi, X)l_0(\theta, \xi, X)] , \qquad J_{001}=E[l_{00}(\theta, \xi, X)l_1(\theta, \xi, X)] ,$$
$$J_{010}=E[l_{01}(\theta, \xi, X)l_0(\theta, \xi, X)] , \qquad J_{011}=E[l_{01}(\theta, \xi, X)l_1(\theta, \xi, X)] ,$$
$$J_{110}=E[l_{11}(\theta, \xi, X)l_0(\theta, \xi, X)] , \qquad K_{000}=E[\{l_0(\theta, \xi, X)\}^3] ,$$
$$K_{001}=E[\{l_0(\theta, \xi, X)\}^2 l_1(\theta, \xi, X)] ,$$
$$M_{0000}=E[\{l_{00}(\theta, \xi, X)\}^2]-I_{00}^2 ,$$
$$M_{0001}=E[l_{00}(\theta, \xi, X)l_{01}(\theta, \xi, X)] ,$$
$$M_{0101}=E[\{l_{01}(\theta, \xi, X)\}^2] ,$$

and the following holds.

$$E[l_{000}(\theta, \xi, X)]=-3J_{000}-K_{000} , \qquad E[l_{001}(\theta, \xi, X)]=-J_{010} ,$$
$$E[l_{011}(\theta, \xi, X)]=-J_{011} ,$$

where $l_{000}(\theta, \xi, x)=(\partial^3/\partial\theta^3)l(\theta, \xi, x)$, $l_{001}(\theta, \xi, x)=(\partial^3/\partial\theta^2\partial\xi)l(\theta, \xi, x)$ and $l_{011}(\theta, \xi, x)=(\partial^3/\partial\theta\partial\xi^2)l(\theta, \xi, x)$.

From the condition (A.5) it is noted that $K_{001}=J_{010}-J_{001}$. We put

$$Z_0=\frac{1}{\sqrt{n}}\sum_{i=1}^n l_0(\theta, \xi, X_i) , \qquad Z_1=\frac{1}{\sqrt{n}}\sum_{i=1}^n l_1(\theta, \xi, X_i) ,$$

$$Z_{00}=\frac{1}{\sqrt{n}}\sum_{i=1}^n \{l_{00}(\theta, \xi, X_i)+I_{00}\} , \qquad Z_{01}=\frac{1}{\sqrt{n}}\sum_{i=1}^n l_{01}(\theta, \xi, X_i) ,$$

3. Asymptotic deficiency of the jackknife estimator under the true model

In this section we consider the true model where $\theta=\theta_0$ and $\xi=\xi_0$. Henceforth, for simplicity we denote by (θ, ξ) the true model (θ_0, ξ_0) omitting subscript zero.

Let $\hat{\theta}^*$ and $\hat{\xi}^*$ be the maximum likelihood estimators (MLEs) of θ and ξ based on a sample $X_1, \cdots, X_n$ of size n, respectively. Then, we have the following.

THEOREM 3.1. *Assume that the conditions (A.1) to (A.5) hold. Then the MLE $\hat{\theta}^*$ of θ has the following stochastic expansion under the true model (θ, ξ).*

$$\sqrt{n}(\hat{\theta}^*-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}Q_0+\frac{1}{\sqrt{n}}Q_1+o_p\left(\frac{1}{\sqrt{n}}\right), \tag{3.1}$$

where

$$Q_0=\frac{1}{I_{00}^2}\left(Z_{00}Z_0-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^2\right) \tag{3.2}$$

and

$$Q_1=\frac{1}{I_{00}I_{11}}Z_1\left(Z_{01}-\frac{J_{010}}{I_{00}}Z_0-\frac{J_{011}}{I_{11}}Z_1\right)+\frac{J_{011}}{2I_{00}I_{11}^2}Z_1^2\,. \tag{3.3}$$

The proof is given in the paper by Akahira and Takeuchi (1982) and also in section 4.2 of the monograph by Akahira (1986). Let $\hat{\theta}_0^*$ be the MLE bias-adjusted so that

$$E[\sqrt{n}(\hat{\theta}_0^*-\theta)]=o(1/\sqrt{n})$$

under the true model (θ, ξ). Since the asymptotic covariance of Q_0 and Q_1 is equal to $o(1)$, *i.e.*, symbolically Cov $(Q_0, Q_1)=o(1)$, we have the following.

THEOREM 3.2. *Assume that the conditions (A.1) to (A.5) hold. Then the asymptotic deficiency of the bias-adjusted MLE $\hat{\theta}_0^*$ under the true model (θ, ξ) is given by*

$$\begin{aligned} d&=I_{00}\{V(Q_0)+V(Q_1)\}\\ &=\frac{1}{I_{00}^3}(I_{00}M_{0000}-J_{000}^2)+\frac{(J_{000}+K_{000})^2}{2I_{00}^3}+\frac{1}{I_{00}I_{11}}\left(M_{0101}-\frac{J_{010}^2}{I_{00}}-\frac{J_{011}^2}{I_{11}}\right)+\frac{J_{011}^2}{2I_{00}I_{11}^2}+o(1)\,, \end{aligned}$$

where $V(\cdot)$ designates the asymptotic variance.

The proof is given in the paper by Akahira and Takeuchi (1982) and also in section 4.2 of the monograph by Akahira (1986). It is noted that the term

$$M_{0101}-(J_{010}^2/I_{00})-(J_{011}^2/I_{11})$$

in the above asymptotic deficiency, d, is equal to zero if and only if

$$l_{01}(\theta, \xi, x)=a_0(\theta, \xi)+a_1(\theta, \xi)l_0(\theta, \xi, x)+a_2(\theta, \xi)l_1(\theta, \xi, x) \quad a.e.\,[\mu]\,,$$

where $a_i(\theta, \xi)$ $(i=0, 1, 2)$ are certain functions of θ and ξ, which are independent of x.

Next, we shall obtain asymptotic deficiency of the jackknife estimator of θ in the true model (θ, ξ). We partition the sample $X_1, \cdots, X_n$ into g blocks $\mathcal{X}_1, \cdots, \mathcal{X}_g$ of size h each such that $n=gh$, that is,

$$\{X_1, \cdots, X_n\}=\bigcup_{i=1}^{g}\mathcal{X}_i \quad \text{and} \quad \mathcal{X}_i\cap\mathcal{X}_j=\phi \quad \text{for } i\neq j\,.$$

For each $i=1, \ldots, g$, we denote $\{j: X_j\in\mathcal{X}_i\}$ by I_i. Further, for each $i=1, \cdots, g$ let $\hat{\theta}^{(i)}$ be the MLE of θ based on the sample of size $(g-1)h$, where the i-th block, $\mathcal{X}_i$, of size h is deleted. For each i we put $\tilde{\theta}_i=g\hat{\theta}^*-(g-1)\hat{\theta}^{(i)}$. Then we consider the jackknife estimator

$$\tilde{\theta}=\frac{1}{g}\sum_{i=1}^{g}\tilde{\theta}_i\,.$$

From Theorem 3.1 we have for each i

$$\sqrt{n-h}(\hat{\theta}^{(i)}-\theta)=\frac{Z_0^{(i)}}{I_{00}}+\frac{1}{\sqrt{n-h}}Q_0^{(i)}+\frac{1}{\sqrt{n-h}}Q_1^{(i)}+o_p\left(\frac{1}{\sqrt{n}}\right), \tag{3.4}$$

where

$$Z_0^{(i)}=\frac{1}{\sqrt{n-h}}\sum_{k\notin I_i} l_0(\theta, \xi, X_k), \tag{3.5}$$

$$Q_0^{(i)}=\frac{1}{I_{00}^2}\left(Z_{00}^{(i)}Z_0^{(i)}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^{(i)2}\right), \tag{3.6}$$

$$Q_1^{(i)}=\frac{1}{I_{00}I_{11}}Z_1^{(i)}\left(Z_{01}^{(i)}-\frac{J_{010}}{I_{00}}Z_0^{(i)}-\frac{J_{011}}{I_{11}}Z_1^{(i)}\right)+\frac{J_{011}}{2I_{00}I_{11}^2}Z_1^{(i)2} \tag{3.7}$$

with

$$Z_{00}^{(i)}=\frac{1}{\sqrt{n-h}}\sum_{k\notin I_i}\{l_{00}(\theta, \xi, X_k)+I_{00}\}, \qquad Z_1^{(i)}=\frac{1}{\sqrt{n-h}}\sum_{k\notin I_i} l_1(\theta, \xi, X_k)$$

and

$$Z_{01}^{(i)}=\frac{1}{\sqrt{n-h}}\sum_{k\notin I_i} l_{01}(\theta, \xi, X_k).$$

Then we have the following.

THEOREM 3.3. *Assume that the conditions (A.1) to (A.5) hold. The the stochastic expansion of the jackknife estimator $\tilde{\theta}$ of θ under the ture model (θ, ξ) is given by*

$$\sqrt{n}(\tilde{\theta}-\theta)=\frac{Z_0}{I_{00}}+\frac{\sqrt{n}}{n-h}(\tilde{Q}_0+\tilde{Q}_1)+o_p\left(\frac{1}{\sqrt{n}}\right), \tag{3.8}$$

and, moreover, if $h=o(n)$, then

$$\sqrt{n}(\tilde{\theta}-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}(\tilde{Q}_0+\tilde{Q}_1)+o_p\left(\frac{1}{\sqrt{n}}\right)$$

where

$$\tilde{Q}_0=\frac{1}{I_{00}^2}\left(W_{000}-\frac{3J_{000}+K_{000}}{2I_{00}}W_{00}\right), \tag{3.9}$$

$$\tilde{Q}_1=\frac{1}{I_{00}I_{11}}\left(W_{011}-\frac{J_{010}}{I_{00}}W_{01}-\frac{J_{011}}{I_{11}}W_{11}\right)+\frac{J_{011}}{2I_{00}I_{11}^2}W_{11} \tag{3.10}$$

with

$$W_{00}=\frac{1}{n}\sum_{i\neq j}\sum_{k\in I_i} l_0(\theta, \xi, X_k)\sum_{k\in I_j} l_0(\theta, \xi, X_k),$$

$$W_{01}=\frac{1}{n}\sum_{i\neq j}\sum_{k\in I_i} l_0(\theta, \xi, X_k)\sum_{k\in I_j} l_1(\theta, \xi, X_k),$$

$$W_{11}=\frac{1}{n}\sum_{i\neq j}\sum_{k\in I_i} l_1(\theta, \xi, X_k)\sum_{k\in I_j} l_1(\theta, \xi, X_k),$$

$$W_{000}=\frac{1}{n}\sum_{i\neq j}\sum_{k\in I_i} l_0(\theta, \xi, X_k)\sum_{k\in I_j}\{l_{00}(\theta, \xi, X_k)+I_{00}\},$$

$$W_{011}=\frac{1}{n}\sum_{i\neq j}\sum_{k\in I_i} l_1(\theta, \xi, X_k) \sum_{k\in I_j} l_{01}(\theta, \xi, X_k) .$$

The proof is given in section 6. From the above it is seen that

$$E[\sqrt{n}(\bar{\theta}-\theta)]=o(1/\sqrt{n}) ,$$

hence it is not necessary to make a bias-adjustment of $\bar{\theta}$. This is essentially different from that of the MLE $\hat{\theta}^*$. From Theorems 3.2 and 3.3 we have the following.

THEOREM 3.4. *Assume that the conditions (A.1) to (A.5) hold. If $h=o(n)$, then the asymptotic deficiency of the jackknife estimator $\bar{\theta}$ under the ture model (θ, ξ) is given by*

$$\bar{d}=I_{00}\{V(\tilde{Q}_0)+V(\tilde{Q}_1)\}=I_{00}\{V(Q_0)+V(Q_1)\}$$

whose value has been further obtained in Theorem 3.2, and so the asymptotic deficiency of the jackknife estimator $\bar{\theta}$ relative to the bias-adjusted MLE $\hat{\theta}_0^*$ under the ture model is equal to zero.

The proof of this theorem is also given in section 6.

REMARK 3.1. In the stochastic expansion (3.8) of the jackknife estimator $\bar{\theta}$ we consider the case when $h=cn$, with $0<c\leqq 1/2$. Since

$$\sqrt{n}/(n-h)=1/\{(1-c)\sqrt{n}\} ,$$

if follows from (4.2) that, under the true model (θ, ξ),

$$\sqrt{n}(\bar{\theta}-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{(1-c)\sqrt{n}}(\tilde{Q}_0+\tilde{Q}_1)+o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

hence the asymptotic deficiency $\bar{d}_c$ of $\bar{\theta}$ is given by

$$\bar{d}_c=\frac{I_{00}}{(1-c)^2}\{V(\tilde{Q}_0)+V(\tilde{Q}_1)\} ,$$

under the true model.
Then it is seen from (3.11) that, for the asymptotic deficiency $\bar{d}$ of $\bar{\theta}$ in the case $h=o(n)$,

$$\bar{d}\leqq\bar{d}_c \qquad \text{for } 0<c\leqq 1/2 .$$

Hence, if $h=cn$ for $0<c\leqq 1/2$, then it follows from Theorem 3.4 that the asymptotic deficiency of $\bar{\theta}$ relative to the bias-adjusted MLE $\hat{\theta}_0^*$ under the true model (θ, ξ) is obtained by

$$\left\{\frac{1}{(1-c)^2}-1\right\} I_{00}\{V(Q_0)+V(Q_1)\} ,$$

which is nonnegative, where the value of $I_{00}\{V(Q_0)+V(Q_1)\}$ is given in Theorem 3.2. This suggests that if the size, h, of each block of the sample in jackknifing is of the order n, then the jackknife estimator $\bar{\theta}$ is asymptotically worse than the bias-adjusted MLE $\hat{\theta}_0^*$ in the third order, *i.e.*, the order n^{-1}, under the true model.

4. Asymptotic deficiency of the bias-adjusted MLE relative to the jackknife estimator under the assumed model

In this section we consider the assumed model $(\theta_0, 0)$. Henceforth, for simplicity we denote by $(\theta, 0)$ the assumed model $(\theta_0, 0)$ omitting the subscript zero. Let $\hat{\theta}_*$ be the MLE of θ based on a sample $X_1, \cdots, X_n$ of size n under the assumed model. We assume

$$\xi = t/\sqrt{n}$$

under the true (θ, ξ). Then we have the following.

THEOREM 4.1. *Assume that the conditions (A.1) to (A.5) hold. Then under the assumed model $(\theta, 0)$, the MLE $\hat{\theta}_*$ of θ has the following stochastic expansion.*

$$\sqrt{n}(\hat{\theta}_* - \theta) = \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}} Q_0 + \frac{1}{\sqrt{n}}(L - c) + o_p\left(\frac{1}{\sqrt{n}}\right),$$

where Q_0 is given by (3.2),

$$L = \frac{t}{I_{00}}\left(\frac{J_{010}}{I_{00}} Z_0 - Z_{01}\right) \quad \text{and} \quad c = \frac{J_{011}}{2I_{00}} t^2,$$

and $o_p(\cdot)$ is taken under the distribution $P_{\theta,\xi}$ with the density $f(x, \theta, \xi)$.

The proof is given in section 6. It is noted that the linear term, L, of Z_0 and Z_{01} in the order $n^{-1/2}$ is involved in the stochastic expansion of $\hat{\theta}_*$. The following additional assumption is made.

(A.6) $J_{011} = 0$.

The condition (A.6) holds true if, for example, θ is a location parameter, *i.e.* $f(x, \theta, \xi) = f(x - \theta, \xi)$ $a.a. x[\mu]$ and $f(x, \xi)$ has the symmetric property, *i.e.*, $f(x, \xi) = f(-x, \xi)$ $a.a. x[\mu]$.

Let $\hat{\theta}_*^0$ be the MLE bias-adjusted so that

$$E[\sqrt{n}(\hat{\theta}_*^0 - \theta)] = o(1/\sqrt{n})$$

under the assumed model $(\theta, 0)$.

Next we consider the jackknife estimator under the assumed model $(\theta, 0)$. For each $i = 1, \ldots, g$, let $\hat{\theta}_*^{(i)}$ be the MLE of θ based on the sample of size $(g-1)h$, where the i-th block, $\mathcal{X}_i$, of size h is deleted. For each i we put

$$\tilde{\theta}_i^* = g\hat{\theta}_* - (g-1)\hat{\theta}_*^{(i)}.$$

Then the jackknife estimator is defined as

$$\tilde{\theta}_* = \sum_{i=1}^{g} \tilde{\theta}_i^* / g.$$

It is easily seen that

$$E[\sqrt{n}(\tilde{\theta}_* - \theta)] = o(1/\sqrt{n})$$

under the assumed model, hence a bias-adjustment of $\tilde{\theta}_*$ is unnecessary, which is essentially different from that of the MLE $\hat{\theta}_*$. From Theorem 4.1 and (A.6) we have for each i

$$\sqrt{n-h}(\hat{\theta}_*^{(i)} - \theta) = \frac{Z_0^{(i)}}{I_{00}} + \frac{1}{\sqrt{n-h}} Q_0^{(i)} + \frac{1}{\sqrt{n}} L^{(i)} + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{4.1}$$

where $Z_0^{(i)}$ and $Q_i^{(i)}$ are given in (3.5) and (3.6), respectively, and

$$L^{(i)}=\frac{t}{I_{00}}\left(\frac{J_{010}}{I_{00}}Z_0^{(i)}-Z_{01}^{(i)}\right) .$$

Then we have the following.

THEOREM 4.2. *Assume that the conditions (A.1) to (A.6) hold. Then the jackknife estimator, $\bar{\theta}_*$ of θ, has the following stochastic expansion under the assumed model $(\theta, 0)$.*

$$\sqrt{n}(\bar{\theta}_*-\theta)=\frac{Z_0}{I_{00}}+\frac{\sqrt{n}}{n-h}\tilde{Q}_0+\frac{1}{\sqrt{n}}L+o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

where $\tilde{Q}_0$ is given by (3.9), and $o_p(\cdot)$ is taken under the distribution $P_{\theta,\xi}$ with the density $f(x, \theta, \xi)$. Moreover, if $h=o(n)$, then

$$\sqrt{n}(\bar{\theta}_*-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}\tilde{Q}_0+\frac{1}{\sqrt{n}}L+o_p\left(\frac{1}{\sqrt{n}}\right) .$$

The proof of this theorem is also given in section 6. From Theorems 4.1 and 4.2 we have the following.

THEOREM 4.3. *Assume that the conditions (A.1) to (A.6) hold. If $\xi=t/\sqrt{n}$ and $h=o(n)$, then the asymptotic deficiency, $D(a, t)$, of the bias-adjusted MLE, $\hat{\theta}^0_*$, relative to the jackknife estimator $\bar{\theta}_*$, under the assumed model $(\theta, 0)$, is equal to zero so that $\bar{\theta}_*$ has the same asymptotic distribution as $\hat{\theta}^0_*$ at a point a up to the order n^{-1}.*

The proof is given in section 6. From the above we have the following.

THEOREM 4.4. *Assume that the conditions (A.1) to (A.5) hold. Suppose that θ is a location parameter, i.e., $f(x, \theta, \xi)=f(x-\theta, \xi)a.a.x[\mu]$ and $f(x, \xi)$ has the symmetric property, i.e., $f(x, \xi)=f(-x, \xi)a.a.x[\mu]$. If $h=o(n)$, then the asymptotic deficiency of the MLE $\hat{\theta}_*$ (or jackknife estimator $\bar{\theta}_*$) under the assumed model relative to the MLE $\hat{\theta}^*$ (or jackknife estimator $\bar{\theta}$) under the true model is given by*

$$D=\frac{1}{I_{00}^2}\left(t^2-\frac{1}{I_{11}}\right)(I_{00}M_{0101}-J_{010}^2) . \tag{4.2}$$

The proof is given in section 6. It is noted that the symmetric property of f implies unnecessity of bias-corrections of MLEs.

REMARK 4.1. In (4.2), the term t^2 is derived from the asymptotic bias due to the "incorrectness" of the assumed model, and $1/I_{11}$ represents the error due to the presence of unknown nuisance parameter ξ. Since

$$I_{00}M_{0101}-J_{010}^2\geqq 0 ,$$

it follows that the asymptotic deficiency (4.2) can be negative for $t^2<1/I_{11}$.

REMARK 4.2. In a similar manner as the Remark 3.1, it follows from Theorems 4.2, 4.3 and 4.4 that if $h=cn$ for $0<c\leqq 1/2$, then under the same conditions as Theorem 4.4, the asymptotic deficiency of the jackknife estimator $\bar{\theta}_*$ under the assumed model relative to the jackknife estimator $\bar{\theta}$ under the true model is given by

$$\frac{1}{I_{00}^2}\left\{t^2-\frac{1}{(1-c)^2 I_{11}}\right\}(I_{00}M_{0101}-J_{010}^2) .$$

5. Example

Let $f_1(x)$ be a standard normal density, *i.e.*,

$$f_1(x)=(1/\sqrt{2\pi})e^{-x^2/2}$$

for $-\infty<x<\infty$, and $f_0(x)$ be a t-distribution with α-degrees of freedom, i.e., with the density

$$f_0(x)=\frac{1}{\sqrt{\alpha}B\left(\frac{\alpha}{2},\frac{1}{2}\right)\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}$$

for $-\infty<x<\infty$, where $\alpha=1, 2, \cdots$, and $B(a, b)$ denotes the Beta function. Then we consider a mixture $f(x, \xi)$ of densities $f_0(x)$ and $f_1(x)$, defined as

$$f(x, \xi)=C(\xi)\{f_1(x)\}^{\xi}\{f_0(x)\}^{1-\xi} \qquad \text{for } -\infty<x<\infty ,$$

where $0\leqq\xi\leqq1$ and $C(\xi)$ is some constant with $C(0)=C(1)=1$. We also have

$$f(x, \xi)=K(\xi)\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)(\xi-1)/2}e^{-\xi x^2/2} \qquad \text{for } -\infty<x<\infty ,$$

where $0\leqq\xi\leqq1$ and

$$K(\xi)=C(\xi)(1/\sqrt{2\pi})^{\xi}\{\sqrt{\alpha}B(\alpha/2, 1/2)\}^{\xi-1} .$$

It is easily seen that

$$K(0)=\frac{1}{\sqrt{\alpha}B\left(\frac{\alpha}{2},\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\alpha\pi}\Gamma\left(\frac{\alpha}{2}\right)} , \tag{5.1}$$

Suppose that $X_1, \cdots, X_n$ are i.i.d. random variables with the density

$$f(x-\theta, \xi)=K(\xi)\left\{1+\frac{(x-\theta)^2}{\alpha}\right\}^{(\alpha+1)(\xi-1)/2}e^{-\xi(x-\theta)^2/2} \tag{5.2}$$

for $-\infty<x<\infty$.

Then we shall obtain the value of the asymptotic deficiency, D, of the MLE (or jackknife estimator) under the assumed model $(\theta, 0)$ relative to the MLE (or jackknife estimator) under the true model as given by

$$D=\frac{1}{I_{00}^2}\left(t^2-\frac{1}{I_{11}}\right)(I_{00}M_{0101}-J_{010}^2) \tag{5.3}$$

(see Theorem 4.4).

Since

$$\xi=t/\sqrt{n}$$

it is noted that

$$I_{\alpha\alpha}=I_{\alpha\alpha}(\theta, \xi)=I_{\alpha\alpha}(\theta, 0)+o(1) \qquad (\alpha=0, 1) , \tag{5.4}$$

$$M_{0101}=M_{0101}(\theta, \xi)=M_{0101}(\theta, 0)+o(1) , \tag{5.5}$$

(5.6) $$J_{010}=J_{010}(\theta,\xi)=J_{010}(\theta,0)+o(1)\ .$$

In order to obtain the value of deficiency as given by (5.3), it is sufficient to calculate $I_{00}(\theta,0)$, $M_{0101}(\theta,0)$ and $J_{010}(\theta,0)$ instead of I_{00}, M_{0101} and J_{010}, respectively. It should be noted that these variables are independent of θ since θ is the location parameter. Firstly we can write from (5.2)

$$l=l(\theta,\xi,x)=\log f(x-\theta,\xi)$$
$$=\log K(\xi)+\frac{\alpha+1}{2}(\xi-1)\log\left(1+\frac{(x-\theta)^2}{\alpha}\right)-\frac{\xi}{2}(x-\theta)^2\ ,$$

which implies

(5.7) $$l_0(\theta,\xi,x)=\frac{\partial l}{\partial\theta}=-\frac{\alpha+1}{\alpha}(\xi-1)\frac{x-\theta}{1+\frac{(x-\theta)^2}{\alpha}}+\xi(x-\theta)$$

and

(5.8) $$l_{01}(\theta,\xi,x)=\frac{\partial^2 l}{\partial\theta\partial\xi}=-\frac{\alpha+1}{\alpha}\,\frac{x-\theta}{1+\frac{(x-\theta)^2}{\alpha}}+x-\theta\ .$$

Next, we consider the case when $\alpha=3,4,\cdots$.
Since

$$E[l_{01}(\theta,0,X)]=\int_{-\infty}^{\infty}\left(-\frac{\alpha+1}{\alpha}\cdot\frac{x}{1+\frac{x^2}{\alpha}}+x\right)K(0)\frac{1}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx=0\ ,$$

the orthogonality condition (A.4) is satisfied. From (5.1), (5.7) and (5.8) we have

(5.9) $$I_{00}(\theta,0)=E[\{l(\theta,0,X)\}^2]=\int_{-\infty}^{\infty}\left(\frac{\alpha+1}{\alpha}\right)^2\left(\frac{x}{1+\frac{x^2}{\alpha}}\right)^2\frac{K(0)}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx$$
$$=K(0)\frac{(\alpha+1)^2}{\alpha^2}\int_{-\infty}^{\infty}\frac{x^2}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+5)/2}}dx$$
$$=K(0)\frac{(\alpha+1)\sqrt{\pi\alpha}\,\Gamma\left(\frac{\alpha}{2}\right)}{(\alpha+3)\Gamma\left(\frac{\alpha+1}{2}\right)}$$
$$=\frac{\alpha+1}{\alpha+3}$$

(5.10) $$M_{0101}(\theta,0)=E[\{l_{01}(\theta,0,X)\}^2]=\int_{-\infty}^{\infty}\left\{-\frac{\alpha+1}{\alpha}\cdot\frac{x}{1+\frac{x^2}{\alpha}}+x\right\}^2\frac{K(0)}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx$$
$$=K(0)\sqrt{\pi}\left\{\frac{(\alpha+1)\sqrt{\alpha}\,\Gamma\left(\frac{\alpha}{2}\right)}{(\alpha+3)\Gamma\left(\frac{\alpha+1}{2}\right)}-\frac{2\sqrt{\alpha}\,\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}+\frac{\alpha\sqrt{\alpha}\,\Gamma\left(\frac{\alpha-2}{2}\right)}{2\Gamma\left(\frac{\alpha+1}{2}\right)}\right\}$$
$$=\frac{10}{(\alpha-2)(\alpha+3)}\ ,$$

$$(5.11)\quad J_{010}(\theta, 0)=E[l_{01}(\theta, 0, X)l_0(\theta, 0, X)]$$

$$=\int_{-\infty}^{\infty}\left(-\frac{\alpha+1}{\alpha}\cdot\frac{x}{1+\frac{x^2}{\alpha}}+x\right)\frac{\frac{\alpha+1}{\alpha}x}{1+\frac{x^2}{\alpha}}\cdot\frac{K(0)}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx$$

$$=K(0)\left\{-\frac{(\alpha+1)\sqrt{\alpha\pi}\Gamma\left(\frac{\alpha}{2}\right)}{(\alpha+3)\Gamma\left(\frac{\alpha+1}{2}\right)}+\frac{\sqrt{\alpha\pi}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}\right\}$$

$$=\frac{2}{\alpha+3}\,.$$

From (5.3) to (5.6) and (5.9), (5.10), and (5.11) it follows that, in the asymptotic deficiency D,

$$(5.12)\qquad \frac{1}{I_{00}^2}(I_{00}M_{0101}-J_{010}^2)=\frac{6(\alpha+3)}{(\alpha+1)^2(\alpha-2)}=k_\alpha \quad \text{(say)}\,.$$

It is easily seen that $k_\alpha>0$ for $\alpha=3, 4, \cdots$. The numerical calculation for k_α's is given as follows:

Table 1.

α	3	4	5	6	7	8	9	10
k_α	2.251	0.840	0.444	0.276	0.188	0.136	0.103	0.080
α	16	20	25	50	∞			
k_α	0.024	0.017	0.011	0.003	0			

It is noted that, for $\alpha=1, 2$, the value of (5.12) is, formally, infinity.

Next we shall calculate $I_{\alpha\alpha}(\theta, 0)$. We put $k=K(0)$. Since

$$C(\xi)=1\Big/\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\right)^{\xi}k^{1-\xi}e^{-\xi x^2/2}\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)(\xi-1)/2}dx\,,$$

it follows that

$$(5.13)\quad C_\xi(0)=\frac{\partial}{\partial\xi}C(\xi)\Big|_{\xi=0}=\log k\sqrt{2\pi}+\frac{\alpha}{2(\alpha-2)}-\frac{\alpha+1}{2}E_{f_0}\left[\log\left(1+\frac{X^2}{\alpha}\right)\right],$$

$$(5.14)\quad C_{\xi\xi}(0)=\frac{\partial^2}{\partial\xi^2}C(\xi)\Big|_{\xi=0}=(\log k\sqrt{2\pi})^2-\frac{\alpha^2(\alpha+2)}{4(\alpha-2)^2(\alpha-4)}+\frac{\alpha}{\alpha-2}\log k\sqrt{2\pi}$$
$$-\frac{(\alpha+1)^2}{4}E_{f_0}\left[\left\{\log\left(1+\frac{X^2}{\alpha}\right)\right\}^2\right]+\frac{\alpha+1}{2}E_{f_0}\left[X^2\log\left(1+\frac{X^2}{\alpha}\right)\right]$$
$$-(\alpha+1)\left(\log k\sqrt{2\pi}+\frac{\alpha}{\alpha-2}\right)E_{f_0}\left[\log\left(1+\frac{X^2}{\alpha}\right)\right]$$
$$+\frac{(\alpha+1)^2}{2}\left\{E_{f_0}\left[\log\left(1+\frac{X^2}{\alpha}\right)\right]\right\}^2,$$

where $\alpha>4$ and $E_{f_0}[\cdot]$ denotes the expectation under the density $f_0(x)$. From (5.13) and (5.14) we have for $\alpha>4$

$$(5.15)\qquad I_{11}(\theta,0)=C_{\xi}^{2}(0)-C_{\xi\xi}(0)$$
$$=\frac{\alpha^2(\alpha-1)}{2(\alpha-2)^2(\alpha-4)}+\frac{(\alpha+1)^2}{4}V_{f_0}\left(\log\left(1+\frac{X^2}{\alpha}\right)\right)$$
$$+\frac{\alpha(\alpha+1)}{2(\alpha-2)}E_{f_0}\left[\log\left(1+\frac{X^2}{\alpha}\right)\right]-\frac{\alpha+1}{2}E_{f_0}\left[X^2\log\left(1+\frac{X^2}{\alpha}\right)\right],$$

where $V_{f_0}(\cdot)$ denotes the variance under the density $f_0(x)$. In order to obtain the value of (5.15), it is necessary to get

$$(5.16)\qquad E_{f_0}\left[\log\left(1+\frac{X^2}{\alpha}\right)\right]=\int_{-\infty}^{\infty}\frac{k\log\left(1+\frac{x^2}{\alpha}\right)}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx,$$

$$(5.17)\qquad E_{f_0}\left[\left\{\log\left(1+\frac{X^2}{\alpha}\right)\right\}^2\right]=\int_{-\infty}^{\infty}\frac{k\left\{\log\left(1+\frac{x^2}{\alpha}\right)\right\}^2}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx$$

$$(5.18)\qquad E_{f_0}\left[X^2\log\left(1+\frac{X^2}{\alpha}\right)\right]=\int_{-\infty}^{\infty}\frac{kx^2\log\left(1+\frac{x^2}{\alpha}\right)}{\left(1+\frac{x^2}{\alpha}\right)^{(\alpha+1)/2}}dx.$$

Using integration by parts we have for $v=2, 3, \cdots$

$$(5.19)\qquad \int_0^{\infty}y^2(1+y^2)^{-v}\log(1+y^2)dy$$
$$=\frac{1}{2(v-1)}\int_0^{\infty}(1+y^2)^{1-v}\log(1+y^2)dy+\frac{1}{v-1}\int_0^{\infty}y^2(1+y^2)^{-v}dy.$$

Since

$$(5.20)\qquad \int_0^{\infty}(1+y^2)^{1-v}\log(1+y^2)dy$$
$$=\int_0^{\infty}(1+y^2)^{-v}\log(1+y^2)dy+\int_0^{\infty}y^2(1+y^2)^{-v}\log(1+y^2)dy,$$

it follows from (5.19) that

$$(5.21)\qquad \int_0^{\infty}(1+y^2)^{-v}\log(1+y^2)dy$$
$$=\frac{2v-3}{2(v-1)}\int_0^{\infty}(1+y^2)^{1-v}\log(1+y^2)dy-\frac{1}{v-1}\int_0^{\infty}y^2(1+y^2)^{-v}dy$$
$$=\frac{2v-3}{2(v-1)}\int_0^{\infty}(1+y^2)^{1-v}\log(1+y^2)dy-\frac{1}{2(v-1)}B\left(v-\frac{3}{2},\frac{3}{2}\right).$$

Putting

$$I_v=\int_0^{\infty}(1+y^2)^{-v}\log(1+y^2)dy,$$

we obtain from (5.21)

$$I_v=\frac{2v-3}{2(v-1)}I_{v-1}-\frac{1}{2(v-1)}B\left(v-\frac{3}{2},\frac{3}{2}\right), \tag{5.22}$$

and also

$$I_1=\int_0^\infty \frac{\log(1+y^2)}{1+y^2}dy=\pi\log 2. \tag{5.23}$$

From (5.16), (5.22) and (5.23) we can calculate the value of

$$E_{f_0}\left[\log\left(1+\frac{X^2}{\alpha}\right)\right]=2k\sqrt{\alpha}\int_0^\infty \frac{\log(1+y^2)}{(1+y^2)^{(\alpha+1)/2}}dy=2k\sqrt{\alpha}I_{(\alpha+1)/2} \tag{5.24}$$

for $\alpha=2v-1$ $(v=3, 4, 5, \cdots)$. From (5.20) we have

$$\int_0^\infty y^2(1+y^2)^{-v}\log(1+y^2)dy=I_{v-1}-I_v,$$

hence, from (5.18) and (5.22),

$$\begin{aligned}E_{f_0}\left[X^2\log\left(1+\frac{X^2}{\alpha}\right)\right]&=2k\alpha\sqrt{\alpha}\int_0^\infty \frac{y^2\log(1+y^2)}{(1+y^2)^{(\alpha+1)/2}}dy\\&=2k\alpha\sqrt{\alpha}\{I_{(\alpha-1)/2}-I_{(\alpha+1)/2}\}\end{aligned} \tag{5.25}$$

for $\alpha=2v-1$ $(v=3, 4, 5, \cdots)$. Putting

$$J_v=\int_0^\infty (1+y^2)^{-v}\{\log(1+y^2)\}^2dy \quad (v=1, 2, \cdots)$$

we have by a similar procedure as above that

$$J_v=\frac{2v-3}{2(v-1)}J_{v-1}-\frac{1}{v-1}B\left(v-\frac{3}{2},\frac{3}{2}\right) \tag{5.26}$$

for $v=2, 3, \cdots$,
By the transformation $x=1/(1+y^2)$, we obtain

$$J_1=\frac{1}{2}\int_0^1 x^{-1/2}(1-x)^{-1/2}(\log x)^2dx. \tag{5.27}$$

Since

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \qquad \text{for } p,q>0,$$

it follows that

$$\frac{\partial^2}{\partial p^2}B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}(\log x)^2dx, \tag{5.28}$$

and also

$$\begin{aligned}\frac{\partial^2}{\partial p^2}B(p,q)=B(p,q)\Bigg[&\left\{\frac{d^2}{dp^2}\log\Gamma(p)-\frac{d^2}{dp^2}\log\Gamma(p+q)\right\}\\&+\left\{\frac{d}{dp}\log\Gamma(p)-\frac{d}{dp}\log\Gamma(p+q)\right\}^2\Bigg],\end{aligned} \tag{5.29}$$

where

$$\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx\ .$$

Putting

$$\Psi(x)=\frac{d}{dx}\log\Gamma(x)\ ,$$

we have from (5.29) and the numerical table of polygamma functions

$$\frac{\partial^2}{\partial p^2}B(p,q)\Big|_{p=q=1/2}=\pi\left[\left\{\Psi'\left(\frac{1}{2}\right)-\Psi'(1)\right\}+\left\{\Psi\left(\frac{1}{2}\right)-\Psi(1)\right\}^2\right] \tag{5.30}$$
$$\fallingdotseq 5.212\pi\ .$$

From (5.27), (5.28) and (5.30) it follows that

$$J_1=\int_0^\infty\frac{\{\log(1+y^2)\}^2}{1+y^2}dy=\frac{\pi}{2}\left[\Psi'\left(\frac{1}{2}\right)-\Psi'(1)+\left\{\Psi\left(\frac{1}{2}\right)-\Psi(1)\right\}^2\right]$$
$$\fallingdotseq 2.606\pi\ .$$

From (5.17) we obtain

$$E_{f_0}\left[\left\{\log\left(1+\frac{X^2}{\alpha}\right)\right\}^2\right]=2k\sqrt{\alpha}\int_0^\infty\frac{\{\log(1+y^2)\}^2}{(1+y^2)^{(\alpha+1)/2}}dy=2k\sqrt{\alpha}\,J_{(\alpha+1)/2} \tag{5.31}$$

for $\alpha=2v-1$ ($v=3, 4, 5, \cdots$). If $\alpha=5$, then we have from (5.22) to (5.27) and (5.31)

$$E_{f_0}\left[\log\left(1+\frac{X^2}{5}\right)\right]=\frac{16}{3\pi}I_3=\frac{16}{3\pi}\left(\frac{3}{8}I_1-\frac{7}{32}\pi\right)\fallingdotseq 0.220\ ,$$

$$E_{f_0}\left[X^2\log\left(1+\frac{X^2}{5}\right)\right]=\frac{80}{3\pi}(I_2-I_3)=\frac{80}{3\pi}\left(\frac{1}{8}I_1-\frac{\pi}{32}\right)\fallingdotseq 1.477\ ,$$

$$E_{f_0}\left[\left\{\log\left(1+\frac{X^2}{5}\right)\right\}^2\right]=\frac{16}{3\pi}J_3=\frac{16}{3\pi}\left(\frac{3}{8}J_1-\frac{7}{16}\pi\right)\fallingdotseq 2.879$$

hence from (5.15)

$$I_{11}(\theta,0)\fallingdotseq 27.700$$

From (5.3), (5.4), (5.12) and Table 1 it follows that the asymptotic deficiency D is given by

$$D=\frac{1}{I_{00}^2}\left(t^2-\frac{1}{I_{11}}\right)(I_{00}M_{0101}-J_{010}^2)\fallingdotseq 0.444(t^2-0.036)$$

for $\alpha=5$.

6. Proofs

Here, the proofs of theorems in the previous sections, are given.

PROOF OF THEOREM 3.3. First we have

$$\sqrt{n}(\tilde{\theta}-\theta)=g\left\{\sqrt{n}(\hat{\theta}^*-\theta)-\frac{1}{g}\sum_{i=1}^g\sqrt{n}(\hat{\theta}^{(i)}-\theta)\right\}+\frac{1}{g}\sum_{i=1}^g\sqrt{n}(\hat{\theta}^{(i)}-\theta) \tag{6.1}$$
$$=g\sqrt{n}(\hat{\theta}^*-\theta)-\sqrt{\frac{n}{n-h}}(g-1)\frac{1}{g}\sum_{i=1}^g\sqrt{n-h}(\hat{\theta}^{(i)}-\theta)\ .$$

Since

$$\frac{1}{g}\sum_{i=1}^{g} Z_0^{(i)}=\sqrt{\frac{n-h}{n}}Z_0, \qquad \frac{1}{g}\sum_{i=1}^{g} Z_{01}^{(i)}=\sqrt{\frac{n-h}{n}}Z_{01},$$

$$\frac{1}{g}\sum_{i=1}^{g} Z_0^{(i)2}=Z_0^2-\frac{h}{n-h}W_{00}, \qquad \frac{1}{g}\sum_{i=1}^{g} Z_1^{(i)2}=Z_1^2-\frac{h}{n-h}W_{11},$$

$$\frac{1}{g}\sum_{i=1}^{g} Z_0^{(i)}Z_{00}^{(i)}=Z_0Z_{00}-\frac{h}{n-h}W_{000},$$

$$\frac{1}{g}\sum_{i=1}^{g} Z_1^{(i)}Z_{01}^{(i)}=Z_1Z_{01}-\frac{h}{n-h}W_{011},$$

$$\frac{1}{g}\sum_{i=1}^{g} Z_1^{(i)}Z_0^{(i)}=Z_1Z_0-\frac{h}{n-h}W_{01},$$

it follows from (3.4) to (3.7) that

$$\begin{aligned}
(6.2)\quad \frac{1}{g}\sum_{i=1}^{g}\sqrt{n-h}(\hat{\theta}^{(i)}-\theta)&=\frac{1}{g}\sum_{i=1}^{g}\left\{\frac{Z_0^{(i)}}{I_{00}}+\frac{1}{\sqrt{n-h}I_{00}^2}\left(Z_0^{(i)}Z_{00}^{(i)}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^{(i)2}\right)\right.\\
&\quad\left.+\frac{1}{\sqrt{n-h}I_{00}I_{11}}Z_1^{(i)}\left(Z_{01}^{(i)}-\frac{J_{010}}{I_{00}}Z_0^{(i)}-\frac{J_{011}}{2I_{11}}Z_1^{(i)}\right)\right\}+o_p\left(\frac{1}{\sqrt{n}}\right)\\
&=\frac{1}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0+\frac{1}{I_{00}^2\sqrt{n-h}}\left\{Z_0Z_{00}-\frac{h}{n-h}W_{000}-\frac{3J_{000}+K_{000}}{2I_{00}}\left(Z_0^2-\frac{h}{n-h}W_{00}\right)\right\}\\
&\quad+\frac{1}{\sqrt{n-h}I_{00}I_{11}}\left\{Z_1Z_{01}-\frac{h}{n-h}W_{011}-\frac{J_{010}}{I_{00}}\left(Z_1Z_0-\frac{h}{n-h}W_{01}\right)\right.\\
&\quad\left.-\frac{J_{011}}{2I_{11}}\left(Z_1^2-\frac{h}{n-h}W_{11}\right)\right\}+o_p\left(\frac{1}{\sqrt{n}}\right)\\
&=\frac{1}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0+\frac{1}{\sqrt{n-h}}Q_0+\frac{1}{\sqrt{n-h}}Q_1-\frac{h}{(n-h)^{3/2}}\tilde{Q}_0-\frac{h}{(n-h)^{3/2}}\tilde{Q}_1\\
&\quad+o_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

where Q_0, Q_1, $\tilde{Q}_0$, and $\tilde{Q}_1$ are given by (3.2), (3.3), (3.9) and (3.10), respectively. From (6.1) and (6.2) we obtain

$$\begin{aligned}
\sqrt{n}(\bar{\theta}-\theta)&=g\left(\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}Q_0+\frac{1}{\sqrt{n}}Q_1\right)-\sqrt{\frac{n}{n-h}}(g-1)\left\{\frac{1}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0\right.\\
&\quad\left.+\frac{1}{\sqrt{n-h}}(Q_0+Q_1)-\frac{h}{(n-h)^{3/2}}(\tilde{Q}_0+\tilde{Q}_1)\right\}+o_p\left(\frac{1}{\sqrt{n}}\right)\\
&=\frac{Z_0}{I_{00}}+\frac{\sqrt{n}}{n-h}(\tilde{Q}_0+\tilde{Q}_1)+o_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

If $h=o(n)$, then

$$\sqrt{n}(\bar{\theta}-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}(\tilde{Q}_0+\tilde{Q}_1)+o_p\left(\frac{1}{\sqrt{n}}\right).$$

This completes the proof.

PROOF OF THEOREM 3.4. First, it is seen from Akahira and Takeuchi (1982) and Akahira (1986) that the asymptotic deficiency, d, of the jackknife estimator is given by $I_{00}V(\tilde{Q}_0+\tilde{Q}_1)$. Since $E(\tilde{Q}_0)=E(\tilde{Q}_1)=\mathrm{Cov}(\tilde{Q}_0, \tilde{Q}_1)=o(1)$, it follows that

$$\bar{d}=I_{00}\{E(\tilde{Q}_0^2)+E(\tilde{Q}_1^2)\}.$$

Since

$$E(W_{000}^2)=I_{00}M_{0000}+J_{000}^2+o(1)\ ,$$
$$E(W_{000}W_{00})=2I_{00}J_{000}+o(1)\ ,\qquad E(W_{00}^2)=2I_{00}^2+o(1)\ ,$$

it follows that

$$(6.3)\qquad E(\tilde{Q}_0^2)=\frac{1}{I_{00}^4}E\left[\left(W_{000}-\frac{3J_{000}+K_{000}}{2I_{00}}W_{00}\right)^2\right]$$
$$=\frac{1}{I_{00}^4}(I_{00}M_{0000}-J_{000}^2)+\frac{1}{2I_{00}^4}(J_{000}+K_{000})^2+o(1)\ ,$$

Since

$$E(W_{011}^2)=I_{00}M_{0101}+J_{011}^2+o(1)\ ,\qquad E(W_{01}^2)=I_{00}I_{11}+o(1)\ ,$$
$$E(W_{11}^2)=2I_{11}^2+o(1)\ ,\qquad E(W_{011}W_{01})=I_{11}J_{010}+o(1)\ ,\qquad E(W_{01}W_{11})=o(1)$$

and

$$E(W_{011}W_{11})=2I_{11}J_{011}+o(1)\ ,$$

we have

$$(6.4)\qquad E(\tilde{Q}_1^2)=\frac{1}{I_{00}^2I_{11}^2}E\left[\left(W_{011}-\frac{J_{010}}{I_{00}}W_{01}-\frac{J_{011}}{2I_{11}}W_{11}\right)^2\right]$$
$$=\frac{1}{I_{00}^2I_{11}}\left(M_{0101}-\frac{J_{010}^2}{I_{00}}-\frac{J_{011}^2}{2I_{11}}\right)+o(1)$$
$$=\frac{1}{I_{00}^2I_{11}}\left(M_{0101}-\frac{J_{010}^2}{I_{00}}-\frac{J_{011}^2}{I_{11}}\right)+\frac{J_{011}^2}{2I_{00}^2I_{11}^2}+o(1)\ .$$

From (6.3) and (6.4) we obtain

$$\tilde{d}=\frac{1}{I_{00}^3}(I_{00}M_{0000}-J_{000}^2)+\frac{1}{2I_{00}^3}(J_{000}+K_{000})^2+\frac{1}{I_{00}I_{11}}\left(M_{0101}-\frac{J_{010}^2}{I_{00}}-\frac{J_{011}^2}{J_{11}}\right)+\frac{J_{011}^2}{2I_{00}I_{11}^2}+o(1)\ ,$$

which coincides with $I_{00}\{V(Q_0)+V(Q_1)\}$, by Theorem 3.2. This completes the proof.

Proof of Theorem 4.1. Since

$$\sum_{i=1}^{n}l_0(\hat{\theta}_*,0,X_i)=0\ ,$$

we have by the Taylor expansion around (θ,ξ)

$$0=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}l_0(\theta,\xi,X_i)+\frac{1}{n}\sum_{i=1}^{n}l_{00}(\theta,\xi,X_i)\sqrt{n}(\hat{\theta}_*-\theta)$$
$$-\frac{1}{n}\sum_{i=1}^{n}l_{01}(\theta,\xi,X_i)\sqrt{n}\xi+\frac{1}{2n\sqrt{n}}\sum_{i=1}^{n}l_{000}(\theta,\xi,X_i)\{\sqrt{n}(\hat{\theta}_*-\theta)\}^2$$
$$+\frac{1}{2n\sqrt{n}}\sum_{i=1}^{n}l_{011}(\theta,\xi,X_i)n\xi^2-\frac{1}{n\sqrt{n}}\sum_{i=1}^{n}l_{001}(\theta,\xi,X_i)n(\hat{\theta}_*-\theta)\xi+o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$=Z_0+\frac{1}{\sqrt{n}}(Z_{00}-\sqrt{n}I_{00})\sqrt{n}(\hat{\theta}_*-\theta)-\frac{1}{\sqrt{n}}Z_{01}\sqrt{n}\xi$$
$$-\frac{1}{2\sqrt{n}}(3J_{000}+K_{000})\{\sqrt{n}(\hat{\theta}_*-\theta)\}^2-\frac{J_{011}}{2\sqrt{n}}n\xi^2$$

$$+\frac{J_{010}}{\sqrt{n}}n(\hat{\theta}_*-\theta)\xi+o_p\left(\frac{1}{\sqrt{n}}\right),$$

and $o_p(\cdot)$ is taken under the distribution $P_{\theta,\xi}$ with the density $f(x, \theta, \xi)$. Letting

$$\xi=t/\sqrt{n},$$

we obtain

$$0=Z_0+\frac{1}{\sqrt{n}}(Z_{00}-\sqrt{n}I_{00})\sqrt{n}(\hat{\theta}_*-\theta)-\frac{t}{\sqrt{n}}Z_{01}$$
$$-\frac{1}{2\sqrt{n}}(3J_{000}+K_{000})\{\sqrt{n}(\hat{\theta}_*-\theta)\}^2-\frac{J_{011}}{2\sqrt{n}}t^2+\frac{J_{010}}{\sqrt{n}}t\sqrt{n}(\hat{\theta}_*-\theta)+o_p\left(\frac{1}{\sqrt{n}}\right),$$

hence

$$\sqrt{n}(\hat{\theta}_*-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{I_{00}^2\sqrt{n}}Z_0Z_{00}-\frac{3J_{000}+K_{000}}{2I_{00}^3\sqrt{n}}Z_0^2$$
$$+\frac{t}{I_{00}\sqrt{n}}\left(\frac{J_{010}}{I_{00}}Z_0-Z_{01}\right)-\frac{J_{011}}{2I_{00}\sqrt{n}}t^2+o_p\left(\frac{1}{\sqrt{n}}\right).$$

This completes the proof.

PROOF OF THEOREM 4.2. First we have

$$\sqrt{n}(\tilde{\theta}_*-\theta)=g\left\{\sqrt{n}(\hat{\theta}_*-\theta)-\frac{1}{g}\sum_{i=1}^{g}\sqrt{n}(\hat{\theta}_*^{(i)}-\theta)\right\}+\frac{1}{g}\sum_{i=1}^{g}\sqrt{n}(\hat{\theta}_*^{(i)}-\theta) \tag{6.5}$$
$$=g\sqrt{n}(\hat{\theta}_*-\theta)-\sqrt{\frac{n}{n-h}}(g-1)\frac{1}{g}\sum_{i=1}^{g}\sqrt{n-h}(\hat{\theta}_*^{(i)}-\theta).$$

From (4.1) we obtain

$$\frac{1}{g}\sum_{i=1}^{g}\sqrt{n-h}(\hat{\theta}_*^{(i)}-\theta)=\frac{1}{g}\sum_{i=1}^{g}\left\{\frac{Z_0^{(i)}}{I_{00}}+\frac{1}{I_{00}^2\sqrt{n-h}}\left(Z_0^{(i)}Z_{00}^{(i)}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^{(i)2}\right)\right. \tag{6.6}$$
$$\left.+\frac{t}{I_{00}\sqrt{n}}\left(\frac{J_{010}}{I_{00}}Z_0^{(i)}-Z_{01}^{(i)}\right)\right\}+o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$=\frac{1}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0+\frac{1}{I_{00}^2\sqrt{n-h}}\left\{Z_0Z_{00}-\frac{h}{n-h}W_{000}\right.$$
$$\left.-\frac{3J_{000}+K_{000}}{2I_{00}}\left(Z_0^2-\frac{h}{n-h}W_{00}\right)\right\}+\frac{t}{I_{00}\sqrt{n}}\left(\frac{J_{010}}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0\right.$$
$$\left.-\sqrt{\frac{n-h}{n}}Z_{01}\right)+o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$=\frac{1}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0+\frac{1}{\sqrt{n-h}}Q_0-\frac{h}{(n-h)^{3/2}}\tilde{Q}_0+\frac{\sqrt{n-h}}{n}L+o_p\left(\frac{1}{\sqrt{n}}\right),$$

where Q_0 and $\tilde{Q}_0$ are given by (3.2) and (3.9), respectively, and $o_p(\cdot)$ is taken under the distribution $P_{\theta,\xi}$. From (6.5) and (6.6) it follows that

$$\sqrt{n}(\tilde{\theta}_*-\theta)=g\left(\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}Q_0+\frac{1}{\sqrt{n}}L\right)-\sqrt{\frac{n}{n-h}}(g-1)\left\{\frac{1}{I_{00}}\sqrt{\frac{n-h}{n}}Z_0\right.$$
$$\left.+\frac{1}{\sqrt{n-h}}Q_0-\frac{h}{(n-h)^{3/2}}\tilde{Q}_0+\frac{\sqrt{n-h}}{n}L\right\}+o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$=\frac{Z_0}{I_{00}}+\frac{\sqrt{n}}{n-h}\tilde{Q}_0+\frac{1}{\sqrt{n}}L+o_p\left(\frac{1}{\sqrt{n}}\right).$$

If $h=o(n)$, then

$$\sqrt{n}(\tilde{\theta}_*-\theta)=\frac{Z_0}{I_{00}}+\frac{1}{\sqrt{n}}\tilde{Q}_0+\frac{1}{\sqrt{n}}L+o_p\left(\frac{1}{\sqrt{n}}\right) .$$

This completes of the proof.

PROOF OF THEOREM 4.3. First, it follows from Theorem 2.3.1 in Akahira (1986) that for $-\infty<a<\infty$

$$P_{\theta,\xi}\{\sqrt{nI_{00}(\theta,\xi)}(\hat{\theta}^0_*-\theta)\leqq a\}=P_{\theta,\xi}\{\sqrt{nI_{00}(\theta,\xi)}(\tilde{\theta}_*-\theta)\leqq a\}+o(n^{-1})$$

if and only if the asymptotic deficiency $D(a, t)$ of $\hat{\theta}^0_*$ relative to $\tilde{\theta}_*$ is given by

$$\begin{aligned}(6.7)\qquad D(a,t)&=I_{00}[V(Q_0)+V(L)-\{V(\tilde{Q}_0)+V(L)\}]\\&\quad+a\sqrt{I_{00}}\{E(Z_0LQ_0)-E(Z_0L\tilde{Q}_0)\}\\&=I_{00}\{V(Q_0)-V(\tilde{Q}_0)\}+a\sqrt{I_{00}}\{E(Z_0LQ_0)-E(Z_0L\tilde{Q}_0)\} .\end{aligned}$$

Since

$$\begin{aligned}E(Z_0^2Q_0)&=E\left[\frac{1}{I_{00}^2}Z_0^2\left(Z_0Z_{00}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^2\right)\right]\\&=-\frac{3}{2I_{00}}(J_{000}+K_{000})+o(1) ,\end{aligned}$$

$$\begin{aligned}E(Z_0Z_{01}Q_0)&=E\left[\frac{1}{I_{00}^2}Z_0Z_{01}\left(Z_0Z_{00}-\frac{3J_{000}+K_{000}}{2I_{00}}Z_0^2\right)\right]\\&=\frac{1}{I_{00}^2}\left\{I_{00}M_{0001}-\frac{1}{2}J_{010}(5J_{000}+3K_{000})\right\}+o(1) ,\end{aligned}$$

it follows that

$$\begin{aligned}(6.8)\qquad E(Z_0LQ_0)&=\frac{t}{I_{00}}E\left[Z_0\left(\frac{J_{010}}{I_{00}}Z_0-Z_{01}\right)Q_0\right]\\&=\frac{t}{I_{00}^3}(J_{000}J_{010}-I_{00}M_{0001})+o(1) .\end{aligned}$$

Since

$$E(Z_0W_{000})=2I_{00}J_{000}+o(1) ,\qquad E(Z_0^2W_{00})=2I_{00}^2+o(1) ,$$

$$E(Z_0Z_{01}W_{000})=I_{00}M_{0001}+J_{010}J_{000}+o(1) ,\qquad E(Z_0Z_{01}W_{00})=2I_{00}J_{010}+o(1) ,$$

it follows that

$$\begin{aligned}(6.9)\qquad E(Z_0L\tilde{Q}_0)&=\frac{t}{I_{00}^3}E\left[Z_0\left(\frac{J_{010}}{I_{00}}Z_0W_{000}-\frac{J_{010}(3J_{000}+K_{000})}{2I_{00}}Z_0W_{00}\right.\right.\\&\qquad\left.\left.-Z_{01}W_{000}+\frac{3J_{000}+K_{000}}{2I_{00}}Z_{01}W_{00}\right)\right]\\&=\frac{t}{I_{00}^3}(J_{000}J_{010}-I_{00}M_{0001})+o(1) .\end{aligned}$$

Since $V(Q_0)=V(\tilde{Q}_0)+o(1)$, we have from (6.4) to (6.9) $D(a, t)=0$, which completes the proof.

PROOF OF THEOREM 4.4. Since f has the symmetric property, it follows that

$$J_{000}=J_{011}=M_{0001}=0\ , \tag{6.10}$$

hence, by (6.8) and (6.9), $E(Z_0LQ_0)=E(Z_0L\tilde{Q}_0)=o(1)$. Then the asymptotic deficiency $\hat{d}_*(\tilde{d}_*)$ of the MLE $\hat{\theta}_*^0$ (jackknife estimator $\tilde{\theta}_*$) is given by

$$\hat{d}_*=\tilde{d}_*=I_{00}\{V(Q_0)+V(L)\}\ .$$

Since

$$\begin{aligned}V(L)&=E(L^2)+o(1)\\&=E\left[\frac{t^2}{I_{00}^2}\left(\frac{J_{010}}{I_{00}}Z_0-Z_{01}\right)^2\right]+o(1)\\&=\frac{t^2}{I_{00}^3}(I_{00}M_{0101}-J_{010}^2)+o(1)\ ,\end{aligned}$$

it follows from Theorems 3.2, 3.4 and (6.10) that the asymptotic deficiency D of the MLE $\hat{\theta}_*$ (or jackknife estimator $\tilde{\theta}_*$) under the assumed model relative to the MLE $\hat{\theta}^*$ (or jackknife estimator $\tilde{\theta}$) under the true model is given by

$$\begin{aligned}D&=\hat{d}_*-d=\tilde{d}_*-d\\&=I_{00}\{V(Q_0)+V(L)\}-I_{00}\{V(Q_0)+V(Q_1)\}\\&=I_{00}\{V(L)-V(Q_1)\}\\&=\frac{1}{I_{00}^2}\left(t^2-\frac{1}{I_{11}}\right)(I_{00}M_{0101}-J_{010}^2)\ .\end{aligned}$$

This completes the proof.

REFERENCES

Akahira, M. (1983). Asymptotic deficiency of the jackknife estimator. *Austral. J. Statist.* **25** 123–129.

Akahira, M. (1986). *The Structure of Asymptotic Deficiency of Estimators.* Queen's Papers in Pure and Applied Mathematics No. 75, Queen's University Press, Kingston, Ontario, Canada.

Akahira, M. and Takeuchi, K. (1982). On asymptotic deficiency of estimators in pooled samples in the presence of nuisance parameters. *Statistics & Decisions* **1** 17–38.

Efron, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans.* CBMS Regional Conference Series in Applied Mathematics **38**, Society for Industrial and Applied Mathematics, Philadelphia.

Hinkley, D. V. (1977). Improving the jackknife with special reference to correlation estimation. *Biometrika* **65** 13–22.