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On the asymptotic construction of confidence intervals

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Abstract

An asymptotically unbiased confidence interval is constructed from an unbiased test up to the third order, and its application to the location parameter case is described. Further, from the viewpoint of a posterior risk, the upper and lower confidence limits are derived and, in practice, obtained up to the second order in case of the normal, uniform and truncated normal distributions. The relationship between the loss function and a confidence level is also discussed.

1. Introduction

Higher order asymptotics has been extensively investigated by Pfanzagl and Wefelmeyer (1985), Akahira and Takeuchi (1981), Ghosh, Sinha and Wieand (1980), Amari (1985), Akahira (1986) and others (see also Ghosh (1994)). It is known that a bias-adjusted maximum likelihood estimator has the third order asymptotic efficiency, but there is not an uniform result on the third order asymptotic optimality in the case of testing hypothesis.

In this paper, we asymptotically construct an unbiased confidence interval from an unbiased test up to the third order, and apply the result to the location parameter case. A similar discussion is found in Takeuchi (1981). Further, from the viewpoint of a posterior risk, we obtain the upper and lower confidence limits with the minimum posterior risk up to the second order, and construct them in case of the normal, uniform and truncated normal distributions. We also discuss the relationship between the loss function and a confidence level, and, in particular, consider how to determine the level from the loss function.

2. Unbiased confidence intervals

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) real random variables with a density function $f(x,\theta)$ $(\theta \in \Theta)$ with respect to a σ-finite measure μ, where Θ is called a parameter space and assumed to be an open interval of $\mathbf{R}^1$. We assume the following conditions.

(A 1) $\{x|f(x,\theta)>0\}$ does not depend on θ.

(A 2) For almost all $x[\mu]$, $f(x,\theta)$ is four times continuously differentiable in θ.

(A 3) For each θ

$$0 < I(\theta) = E_\theta[\{l^{(1)}(\theta,X)\}^2] = -E_\theta[l^{(2)}(\theta,X)] < \infty,$$

where $l^{(i)}(\theta,x) = (\partial^i/\partial\theta^i)l(\theta,x)$ $(i=1,2,3)$ with $l(\theta,x)=\log f(x,\theta)$, and $I(\theta)$ is three times differentiable in θ.

(A 4) There exist

$J(\theta) = E_\theta[l^{(1)}(\theta,X)\, l^{(2)}(\theta,X)]$, $K(\theta) = E_\theta[\{l^{(1)}(\theta,X)\}^3]$,
$L(\theta) = E_\theta[l^{(1)}(\theta,X)\, l^{(3)}(\theta,X)]$, $M(\theta) = E_\theta[\{l^{(2)}(\theta,X)\}^2] - I^2(\theta)$,
$N(\theta) = E_\theta[\{l^{(1)}(\theta,X)\}^2\, l^{(2)}(\theta,X)] + I^2(\theta)$,
and
$H(\theta) = E_\theta[\{l^{(1)}(\theta,X)\}^4] - 3I^2(\theta)$,
and both of $J(\theta)$ and $K(\theta)$ are differentiable in θ, and
$E_\theta[l^{(3)}(\theta,X)] = -3J(\theta) - K(\theta)$,
$E_\theta[l^{(4)}(\theta,X)] = -H(\theta) - 4L(\theta) - 3M(\theta) - 6N(\theta)$.

In order to obtain unbiased confidence limits we consider an unbiased test φ with level $\alpha + o(1/n)$ of the hypothesis $\theta=\theta_0$ against the alternative hypothesis $\theta\neq\theta_0$, i.e.

$$E_{\theta_0}[\varphi(\mathbf{X})] \le \alpha + o(1/n),$$
$$E_\theta[\varphi(\mathbf{X})] \ge \alpha + o(1/n) \qquad \text{for all } \theta\neq\theta_0,$$

where $\mathbf{X}=(X_1,\ldots,X_n)$. When $(\partial/\partial\theta)E_{\theta_0}(\varphi) = o(1/n)$,

$$E_{\theta_0}\left[\varphi(\mathbf{X})\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i,\theta_0)\right] = o\left(\frac{1}{n}\right) \tag{2.1}$$

provided that the differentiation under the integral sign $E_{\theta_0}(\varphi)$ is allowed. Putting

$$Z(\theta) = \frac{1}{\sqrt{I(\theta)n}}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i,\theta),$$

we have from (2.1)

$$E_{\theta_0}[Z(\theta_0)\varphi(\mathbf{X})] = o(1/n).$$

Now we consider an unbiased test of type

$$\varphi(\mathbf{X}) = \begin{cases} 1 & \text{for } Z(\theta_0) < a_{\theta_0},\ Z(\theta_0) > b_{\theta_0}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where a_{θ_0} and b_{θ_0} are certain constants determined so that $E_{\theta_0}(\varphi) = \alpha + o(1/n)$ and (2.1) hold. Let $g_\theta(z)$ be an asymptotic density function of $Z(\theta_0)$. Since $E_{\theta_0}(\varphi) = \alpha + o(1/n)$, it follows that

$$\int_a^b g_{\theta_0}(z)dz = 1 - \alpha + o(1/n). \tag{2.3}$$

From (2.1) we have

$$\int_a^b z g_{\theta_0}(z)dz = o(1/n). \tag{2.4}$$

First we shall obtain a and b satisfying (2.3) and (2.4). For each $i = 1, \dots, n$, we put

$$Y_i = \frac{1}{\sqrt{I(\theta_0)}} \frac{\partial}{\partial \theta} \log f(X_i, \theta_0).$$

Then, for each $i = 1, \dots, n$, the mean, variance, third and fourth order cumulants of Y_i are given by

$$E_{\theta_0}(Y_i) = 0, \quad V_{\theta_0}(Y_i) = 1, \quad k_{3,\theta_0}(Y_i) = E_{\theta_0}[Y_i^3] = \frac{K(\theta_0)}{I^{3/2}(\theta_0)},$$

and

$$k_{4,\theta_0}(Y_i) = E_{\theta_0}[Y_i^4] - 3\{V_{\theta_0}(Y_i)\}^2 = \frac{H(\theta_0)}{I^2(\theta_0)}.$$

By the Edgeworth expansion of the distribution of $Z(\theta_0)$ we have

$$P_{\theta_0}\{Z(\theta_0) \le z\} = \Phi(z) - \phi(z) \left\{ \frac{K}{6I^{3/2}\sqrt{n}}(z^2 - 1) + \frac{H}{24I^2 n}(z^3 - 3z) + \frac{K^2}{72I^3 n}(z^5 - 10z^3 + 15z) \right\} + o\left(\frac{1}{n}\right),$$

where $\Phi(z) = \int_{-\infty}^z \phi(x)dx$ with $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$, $I = I(\theta_0)$, $K = K(\theta_0)$ and $H = H(\theta_0)$. From the above it follows that the asymptotic density $g_{\theta_0}(z)$ of $Z(\theta_0)$ is given by

$$g_{\theta_0}(z) = \phi(z) + \phi(z) \left\{ \frac{K}{6I^{3/2}\sqrt{n}}(z^3 - 3z) + \frac{H}{24I^2 n}(z^4 - 6z^2 + 3) + \frac{K^2}{72I^3 n}(z^6 - 15z^4 + 45z^2 - 15) \right\} + o\left(\frac{1}{n}\right). \tag{2.5}$$

Then we have the following lemma.

Lemma 2.1. Under the conditions (A 1) to (A 4), the constants a and b satisfying (2.3) and (2.4) are given by

$$a = a(\theta_0) = -u + \frac{K}{6I^{3/2}\sqrt{n}}u^2 + \frac{K^2}{72I^3 n}(4u^3 - 15u) - \frac{H}{24I^2 n}(u^3 - 3u) + o\left(\frac{1}{n}\right),$$
$$b = b(\theta_0) = u + \frac{K}{6I^{3/2}\sqrt{n}}u^2 - \frac{K^2}{72I^3 n}(4u^3 - 15u) + \frac{H}{24I^2 n}(u^3 - 3u) + o\left(\frac{1}{n}\right),$$

where u denotes the upper $100(\alpha/2)\%$ point $u_{\alpha/2}$ of the standard normal distribution.

The proof is given in section 6. In order to obtain the confidence limits for θ we consider the acceptance region of the unbiased test φ with (2.2), i.e.

$$a_{\theta_0} \le Z(\theta_0) \le b_{\theta_0},$$

i.e.

$$a_{\theta_0}\sqrt{I(\theta_0)} \le Z(\theta_0)\sqrt{I(\theta_0)} \le b_{\theta_0}\sqrt{I(\theta_0)}.$$

We put $a_0(\theta) = a_{\theta_0}\sqrt{I(\theta)}$, $b_0(\theta) = b_{\theta_0}\sqrt{I(\theta)}$ and $Z_1(\theta) = Z(\theta)\sqrt{I(\theta)}$. Since, for a neighborhood $U(\theta_0)$ of θ_0

$$\frac{\partial}{\partial\theta}Z_1(\theta) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta^2}\log f(X_i, \theta_0) + o_p(1),$$

it follows that for $\theta \in U(\theta_0)$

$$E_{\theta_0}\left[\frac{\partial}{\partial\theta}Z_1(\theta)\right] = -\sqrt{n}I(\theta_0) + o(1),$$

hence the upper and lower confidence limits $\bar{\theta}$ and $\underline{\theta}$ for θ are obtained from the equations

$$(2.6) \qquad Z_1(\bar{\theta}) = a_0(\bar{\theta}), \qquad Z_1(\underline{\theta}) = b_0(\underline{\theta}).$$

Let $\hat{\theta}_{ML}$ be the maximum likelihood estimator (MLE) of θ. Then we have the following.

Theorem 2.1. Under the conditions (A 1) to (A 4), the upper and lower confidence limits $\bar{\theta}$ and $\underline{\theta}$ for θ are given by

$$\bar{\theta} = \hat{\theta}_{ML} + \Delta(\hat{\theta}_{ML}, u), \qquad \underline{\theta} = \hat{\theta}_{ML} + \Delta(\hat{\theta}_{ML}, -u),$$

where

$$\begin{aligned}\Delta(\theta, u) = {} & \frac{u}{\sqrt{In}} + \frac{Z_2}{I^{3/2}n}u - \frac{3J+K}{6I^2 n}u^2 + \frac{Z_2^2}{I^{5/2}n^{3/2}}u - \frac{15J+4K}{6I^3 n^{3/2}}Z_2 u^2 + \frac{Z_3}{2I^2 n^{3/2}}u^2 \\ & + \left\{\frac{1}{72I^{7/2}n^{3/2}}(36J^2 + 30JK + 5K^2) - \frac{1}{24I^{5/2}n^{3/2}}(H + 4L + 6N)\right\}u^3 \\ & + \left\{\frac{5K^2}{24I^{7/2}n^{3/2}} - \frac{H}{8I^{5/2}n^{3/2}}\right\}u + o(n^{-3/2}),\end{aligned}$$

with $u = u_{\alpha/2}$ and

$$Z_2 = Z_2(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{l^{(2)}(\theta, X_i) + I(\theta)\},$$

$$Z_3 = Z_3(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{l^{(3)}(\theta, X_i) + 3J(\theta) + K(\theta)\},$$

$I = I(\theta)$, $J = J(\theta)$, $K = K(\theta)$, $H = H(\theta)$, $L = L(\theta)$, $M = M(\theta)$, and $N = N(\theta)$.

The proof is given in section 6.

3. Location parameter case

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with a density function $f_0(x-\theta)$ $(\theta \in \mathbf{R}^1)$ with respect to the Lebesgue measure. Then

$$I(\theta) = \int_{-\infty}^{\infty} \{f_0'(x)\}^2 / f_0(x) dx, \tag{3.1}$$

and

$$J(\theta) = (1/2) \int_{-\infty}^{\infty} \{f_0'(x)\}^3 / \{f_0(x)\}^2 dx, \tag{3.2}$$

provided that $\lim_{x \to +\infty} \{f_0'(x)\}^2 / f_0(x) = 0$. Note that

$$J(\theta) = -K(\theta)/2 \tag{3.3}$$

and the above amounts are independent of θ. Since

$$J'(\theta) = L(\theta) + M(\theta) + N(\theta) = 0, \qquad K'(\theta) = H(\theta) + 3N(\theta) = 0,$$

it follows that

$$N(\theta) = -H(\theta)/3, \qquad M(\theta) = -L(\theta) - N(\theta) = -L(\theta) + \{H(\theta)/3\}. \tag{3.4}$$

We also have

$$H(\theta) = \int \frac{\{f_0'(x)\}^4}{\{f_0(x)\}^3} dx - 3I^2,$$

$$L(\theta) = -M(\theta) - N(\theta) = -\int \frac{\{f_0''(x)\}^2}{f_0(x)} dx + 2I^2 + \frac{2H}{3}.$$

We denote $I(\theta)$, $H(\theta)$, $K(\theta)$ and $L(\theta)$ by I, H, K and L, respectively, and using these amounts we have the following.

Corollary 3.1. Under the conditions (A 1) to (A 4), the upper and lower confidence limits $\bar{\theta}$ and $\underline{\theta}$ for θ are given by

$$\bar{\theta} = \hat{\theta}_{ML} + \Delta(\hat{\theta}_{ML}, u), \qquad \underline{\theta} = \hat{\theta}_{ML} + \Delta(\hat{\theta}_{ML}, -u),$$

where

$$\begin{aligned}\Delta(\theta, u) = & \frac{u}{\sqrt{In}} + \frac{Z_2}{I^{3/2}n}u + \frac{K}{12I^2n}u^2 + \frac{Z_2^2}{I^{5/2}n^{3/2}}u + \frac{7KZ_2}{12I^3n^{3/2}}u^2 + \frac{Z_3}{2I^2n^{3/2}}u^2 \\ & - \left(\frac{K^2}{72I^{7/2}n^{3/2}} - \frac{H-4L}{24I^{5/2}n^{3/2}}\right)u^3 + \left(\frac{5K^2}{24I^{7/2}n^{3/2}} - \frac{H}{8I^{5/2}n^{3/2}}\right)u + o\left(\frac{1}{n^{3/2}}\right).\end{aligned}$$

The proof is omitted since it is straightforward from Theorem 2.1. Since $E_\theta(Z_2) = 0$ and $E_\theta(Z_2^2) = M = -L + (H/3)$, it follows from Corollary 3.1 that the expectation of the length of the above confidence interval is given by

(3.5)

$$E_\theta[\bar{\theta} - \underline{\theta}] = \frac{2u}{\sqrt{In}}\left\{1 - \left(\frac{K^2}{72I^3n} - \frac{H-4L}{24I^2n}\right)u^2 + \left(\frac{5K^2}{24I^3n} + \frac{5H-24L}{24I^2n}\right)\right\} + o\left(\frac{1}{n^{3/2}}\right).$$

An estimator $\hat{\theta}_n$ based on a sample $(X_1, \dots, X_n)$ is called to be best asymptotically normal (or BAN for short) if the asymptotic distribution of $\sqrt{nI}(\hat{\theta}_n - \theta)$ is standard normal. Let $\mathbf{D}'$ be the class of the all bias-adjusted BAN estimators $\hat{\theta}_n$ which are third order asymptotically unbiased, i.e. $E_\theta[\sqrt{n}(\hat{\theta}_n - \theta)] = o(1/n)$ and asymptotically expanded as

$$\sqrt{n}(\hat{\theta}_n - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{1}{\sqrt{n}}Q(\theta) + \frac{1}{n}R(\theta) + o_p\left(\frac{1}{n}\right),$$

where $Q(\theta) = O_p(1)$, $R(\theta) = O_p(1)$ and $E_\theta[Z_1(\theta)Q^2(\theta)] = o(1)$, and the distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ admits the Edgeworth expansion up to the order n^{-1}. When the third order asymptotic median unbiasedness is used instead of the third order asymptotic unbiasedness in the class, we denote the class by $\mathbf{D}$ (see Akahira, 1986). For any $\hat{\theta}_n$ in the class $\mathbf{D}'$, the asymptotic distribution of $\sqrt{nI}(\hat{\theta}_n - \theta)$ is, in a similar way to Corollary 2.1.2 in Akahira (1986), given by

(3.6)

$$\begin{aligned}F_{\hat{\theta}_n}(t) = & P_\theta\{\sqrt{nI}(\hat{\theta}_n - \theta) \le t\} \\ = & \Phi(t) - \frac{I\sqrt{I}\beta_3}{6\sqrt{n}}(t^2-1)\phi(t) - \frac{I^2\beta_4}{24n}(t^3 - 3t)\phi(t) - \frac{I^3\beta_3^2}{72n}(t^5 - 10t^3 + 15t)\phi(t) \\ & - \frac{I\tau}{2n}t\phi(t) + o\left(\frac{1}{n}\right),\end{aligned}$$

where

$$\beta_3(\theta) = -\frac{1}{I^3(\theta)}\{3J(\theta) + 2K(\theta)\},$$

$$\beta_4(\theta) = \frac{12}{I^5(\theta)}\{2J(\theta) + K(\theta)\}\{J(\theta) + K(\theta)\} - \frac{1}{I^4(\theta)}\{3H(\theta) + 4L(\theta) + 12N(\theta)\}.$$

Then it follows from (3.6) that the asymptotic density of $\sqrt{In}(\hat{\theta}_n - \theta)$ is given by

$$
\begin{aligned}
(3.7)\quad f_{\hat{\theta}_n}(t) =& \phi(t) + \frac{I\sqrt{I}\beta_3}{6\sqrt{n}}(t^3 - 3t)\phi(t) + \frac{I^2\beta_4}{24n}(t^4 - 6t^2 + 3)\phi(t) \\
&+ \frac{I^3\beta_3^2}{72n}(t^6 - 15t^4 + 45t^2 - 15)\phi(t) + \frac{I\tau}{2n}(t^2 - 1)\phi(t) + o\left(\frac{1}{n}\right). \\
=& \phi(t)\left\{1 + \frac{I\sqrt{I}\beta_3}{6\sqrt{n}}h_3(t) + \frac{I^2\beta_4}{24n}h_4(t) + \frac{I^3\beta_3^2}{72n}h_6(t) + \frac{I\tau}{2n}h_2(t)\right\} + o\left(\frac{1}{n}\right).
\end{aligned}
$$

In order to obtain an asymptotically unbiased confidence limits based on the estimator $\hat{\theta}_n$ in the class $\mathbf{D}'$, it is enough to get the values a and b satisfying the conditions

$$(3.8)\qquad \int_a^b f_{\hat{\theta}_n}(t)dt = 1 - \alpha + o\left(\frac{1}{n}\right),$$

$$(3.9)\qquad f_{\hat{\theta}_n}(a) = f_{\hat{\theta}_n}(b) + + o\left(\frac{1}{n}\right)$$

and put

$$(3.10)\qquad \underline{\theta} = \hat{\theta}_n - b/\sqrt{nI} \qquad \text{and} \qquad \bar{\theta} = \hat{\theta}_n - a/\sqrt{nI}.$$

In a similar way to Section 2, we have the following.

Theorem 3.1. Assume that the conditions (A 1) to (A 4) on the density $f(x, \theta) = f_0(x - \theta)$ hold. Then the upper and lower confidence limits $\bar{\theta}$ and $\underline{\theta}$ are given by (3.10), where

$$(3.11)\qquad a = -u_{\alpha/2} + \frac{b_1}{\sqrt{n}} - \frac{b_2}{n} + o\left(\frac{1}{n}\right), \qquad b = u_{\alpha/2} + \frac{b_1}{\sqrt{n}} + \frac{b_2}{n} + o\left(\frac{1}{n}\right)$$

with

$$b_1 = -\frac{K}{12I\sqrt{I}}(u^2 - 3), \qquad b_2 = \frac{H - 4L}{24I^2}(u^3 - 3u) - \frac{K^2}{144I^3}(2u^3 - 3u) + \frac{I\tau}{2}u,$$

and $\tau = V_\theta(Q(\theta))$.

The proof is given in Section 6. It is shown in Akahira and Takeuchi (1981) that the MLE $\hat{\theta}_{ML}$ of θ has the stochastic expansion

$$
\begin{aligned}
\sqrt{n}(\hat{\theta}_{ML} - \theta) =& \frac{Z_1(\theta)}{I} + \frac{1}{\sqrt{n}}Q_0(\theta) + \frac{1}{I^3 n}\left\{Z_1(\theta)Z_2^2(\theta) + \frac{1}{2}Z_1^2(\theta)Z_3(\theta) - \frac{3(3J+K)}{2I}\right. \\
&\left.\cdot Z_1^2(\theta)Z_2(\theta) - \frac{(3J+K)^2}{2I^2}Z_1^3(\theta) - \frac{1}{6I}(H + 4L + 3M + 6N)Z_1^3(\theta)\right\} + o_p\left(\frac{1}{n}\right),
\end{aligned}
$$

where

$$Q_0(\theta) = \frac{Z_1(\theta)Z_2(\theta)}{I^2} - \frac{3J+K}{2I^3}Z_1^2(\theta).$$

Then a modified MLE

$$\hat{\theta}^*_{ML} = \hat{\theta}_{ML} + \frac{J(\hat{\theta}_{ML}) + K(\hat{\theta}_{ML})}{2nI^2(\hat{\theta}_{ML})}$$

belongs to the class $\mathbf{D}'$. Using the modified MLE $\hat{\theta}^*_{ML}$ instead of $\hat{\theta}_n$ in Theorem 3.1, we have the upper and lower confidence limits (3.10) with (3.11). Comparing the confidence interval $[\underline{\theta},\ \bar{\theta}] = [\hat{\theta}_{ML} + \Delta(\hat{\theta}_{ML}, -u_{\alpha/2}),\ \hat{\theta}_{ML} + \Delta(\hat{\theta}_{ML}, u_{\alpha/2})]$ given in Corollary 3.1 and the confidence interval $[\underline{\theta}^*,\ \bar{\theta}^*] = [\hat{\theta}^*_{ML} - (b/\sqrt{In}),\ \hat{\theta}^*_{ML} - (a/\sqrt{In})]$ with $\hat{\theta}^*_{ML}$ instead of $\hat{\theta}_n$ in Theorem 3.1 with respect to the length of the interval, we have from (3.5) and Theorem 3.1

$$\begin{aligned}
(3.12) \qquad E_\theta[\bar{\theta} - \underline{\theta}] - E_\theta[\bar{\theta}^* - \underline{\theta}^*] &= E_\theta[\bar{\theta} - \underline{\theta}] - \frac{1}{\sqrt{In}}(b - a) \\
&= E_\theta[\bar{\theta} - \underline{\theta}] - \frac{2}{\sqrt{In}}\left(u_{\alpha/2} + \frac{b_2}{n}\right) + o\left(\frac{1}{n\sqrt{n}}\right) \\
&= \frac{2u_{\alpha/2}}{\sqrt{In}}\left(\frac{K^2}{4I^3 n} + \frac{H - 6L}{6I^2 n}\right) + o\left(\frac{1}{n\sqrt{n}}\right) \\
&= \frac{3K^2 + 2H - 12L}{6I^3\sqrt{I}\, n\sqrt{n}} u + o\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}$$

since, by (3.3) and (3.4),

$$\tau = V(Q_0(\theta)) = \frac{1}{I^4}\left\{IM - J^2 + \frac{1}{2}(J + K)^2\right\} = \frac{H - 3L}{3I^3} - \frac{K^2}{8I^4}.$$

Hence

$$E_\theta[\bar{\theta} - \underline{\theta}] \gtreqless E_\theta[\bar{\theta}^* - \underline{\theta}^*] + o(n^{-3/2})$$

if and only if

$$3K^2 + 2H \gtreqless 12L,$$

respectively. Since $H - 3L = M \geq 0$, it follows that $H - 6L \geq 0$ provided that $L \leq 0$. If $L \leq 0$, then, from (3.12), the asymptotic mean of the interval $[\underline{\theta}^*,\ \bar{\theta}^*]$ is asymptotically shorter than that of $[\underline{\theta},\ \bar{\theta}]$. For example, if the density is given by $f_0(x) = 1/\{\pi(1 + x^2)\}$, then $L = -3/4$.

4. Another look of the confidence interval at the minimum posterior risk

In this section, from the viewpoint of the posterior risk we shall construct the confidence intervals. We assume that $\Theta = \mathbf{R}^1$. For any interval $[\underline{\theta},\ \bar{\theta}]$ of Θ, we define a loss function of θ with respect to the interval by

$$L(\theta,\underline{\theta},\bar{\theta}) = \begin{cases} 2(\bar{\theta}-\underline{\theta}) & \text{for } \underline{\theta}\le\theta\le\bar{\theta}, \\ 2(\bar{\theta}-\underline{\theta})+c(\theta-\underline{\theta})^2 & \text{for } \theta<\underline{\theta}, \\ 2(\bar{\theta}-\underline{\theta})+c(\theta-\bar{\theta})^2 & \text{for } \theta>\bar{\theta}, \end{cases} \tag{4.1}$$

where c is some positive constant.

Suppose that $X_1,\dots,X_n$ are i.i.d. random variables with a density function $f(x,\theta)$ ($\theta\in\Theta$) with respect to the Lebesgue measure. Let π be a uniform distribution on the interval $[a,b]$ as a prior distribution, i.e. $\pi(\theta)=(\theta-a)/(b-a)$. Then we shall obtain the confidence limits $\underline{\theta}$ and $\bar{\theta}$ minimizing the posterior risk

$$r(\underline{\theta},\ \bar{\theta}|\ \boldsymbol{x}) = \int_a^b L(\theta,\underline{\theta},\bar{\theta})\frac{\prod_{i=1}^n f(x_i,\theta)}{\int_a^b\prod_{i=1}^n f(x_i,\theta)d\theta}d\theta. \tag{4.2}$$

Theorem 4.1. The confidence limits $\underline{\theta}^*$ and $\bar{\theta}^*$ which minimize the posterior risk (4.2) are given as follows.
(i) If $a<\underline{\theta}<\bar{\theta}<b$, then $\underline{\theta}^*$ and $\bar{\theta}^*$ are given as solutions of $\underline{\theta}$ and $\bar{\theta}$ of the equations

$$\int_a^{\underline{\theta}}(\underline{\theta}-\theta)\prod_{i=1}^n f(x_i,\theta)d\theta = \frac{1}{c}\int_a^b\prod_{i=1}^n f(x_i,\theta)d\theta, \tag{4.3}$$

$$\int_{\bar{\theta}}^b(\theta-\bar{\theta})\prod_{i=1}^n f(x_i,\theta)d\theta = \frac{1}{c}\int_a^b\prod_{i=1}^n f(x_i,\theta)d\theta. \tag{4.4}$$

(ii) If $a<\underline{\theta}<b\le\bar{\theta}$, then $\bar{\theta}^*=b$ and $\underline{\theta}^*$ is given as a solution of $\underline{\theta}$ of the equation (4.3).
(iii) If $\underline{\theta}\le a<\bar{\theta}<b$, then $\underline{\theta}^*=a$ and $\bar{\theta}^*$ is given as a solution of $\bar{\theta}$ of the equation (4.4).
Proof. (i) For $a<\underline{\theta}<\bar{\theta}<b$ we have

$$r(\underline{\theta},\ \bar{\theta}|\ \boldsymbol{x}) = 2(\bar{\theta}-\underline{\theta}) + \frac{c}{\int_a^b\prod_{i=1}^n f(x_i,\theta)d\theta}\left\{\int_a^{\underline{\theta}}(\theta-\underline{\theta})^2\prod_{i=1}^n f(x_i,\theta)d\theta + \int_{\bar{\theta}}^b(\theta-\bar{\theta})^2\prod_{i=1}^n f(x_i,\theta)d\theta\right\}. \tag{4.5}$$

In order to minimize (4.5), it is enough to solve the equations $\partial r(\underline{\theta},\ \bar{\theta}|\ \boldsymbol{x})/\partial\underline{\theta}=0$ and $\partial r(\underline{\theta},\ \bar{\theta}|\ \boldsymbol{x})/\partial\bar{\theta}=0$. It is easily seen that the solutions are given as them of the equations (4.3) and (4.4). The cases (ii) and (iii) can be quite similarly proved.
Corollary 4.1. If the prior distribution is taken as the Lebesgue measure, i.e. $a\to-\infty$ and $b\to\infty$, then the confidence limits $\underline{\theta}^*$ and $\bar{\theta}^*$ minimizing the posterior risk (4.2) are given as solutions of $\underline{\theta}$ and $\bar{\theta}$ of the equations

$$\int_{-\infty}^{\underline{\theta}}(\underline{\theta}-\theta)\prod_{i=1}^n f(x_i,\theta)d\theta\Big/\int_{-\infty}^{\infty}\prod_{i=1}^n f(x_i,\theta)d\theta = 1/c, \tag{4.6}$$

$$\int_{\bar{\theta}}^{\infty}(\theta-\bar{\theta})\prod_{i=1}^n f(x_i,\theta)d\theta\Big/\int_{-\infty}^{\infty}\prod_{i=1}^n f(x_i,\theta)d\theta = 1/c. \tag{4.7}$$

The proof is straightforward from (4.3) and (4.4) in Theorem 4.1. The application of Corollary 4.1 is given in the next section.

5. The application of the confidence limits with the minimum posterior risk

In this section we discuss the normal, uniform and truncated normal cases as the application of Corollary 4.1, where the confidence limits with the minimum posterior risk are given, and also their relationship to the confidence level is considered.

Example 5.1 (Normal case). Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with a normal distribution with mean θ and variance 1. Then it follows from Corollary 4.1 that the confidence limits $\underline{\theta}^*$ and $\bar{\theta}^*$ minimizing the posterior risk (4.2) are given by the solutions of the equations

$$\int_{-\infty}^{\underline{\theta}} (\underline{\theta} - \theta) e^{-(n/2)(\theta - \bar{x})^2} d\theta \Big/ \int_{-\infty}^{\infty} e^{-(n/2)(\theta - \bar{x})^2} d\theta = \frac{1}{c}, \tag{5.1}$$

$$\int_{\bar{\theta}}^{\infty} (\theta - \bar{\theta}) e^{-(n/2)(\theta - \bar{x})^2} d\theta \Big/ \int_{-\infty}^{\infty} e^{-(n/2)(\theta - \bar{x})^2} d\theta = \frac{1}{c}, \tag{5.2}$$

where $\bar{x} = \sum_{i=1}^n x_i/n$. Putting $t = \sqrt{n}(\underline{\theta} - \bar{x})$ and $s = \sqrt{n}(\bar{\theta} - \bar{x})$, we have from (5.1) and (5.2)

$$\begin{aligned} &t\Phi(t) + \phi(t) = \sqrt{n}/c, \\ &-s(1 - \Phi(s)) + \phi(s) = \sqrt{n}/c, \end{aligned} \tag{5.3}$$

hence $t = -s$. Letting $s = d(c)$ be a solution of (5.3), we obtain $\bar{x} \pm (d(c)/\sqrt{n})$ as the confidence limits with minimum posterior risk.

Next we consider the relationship of the above confidence limits to usual confidence level. Since

$$P_\theta \left\{ \bar{X} - \frac{d(c)}{\sqrt{n}} \leq \theta \leq \bar{X} + \frac{d(c)}{\sqrt{n}} \right\} = 2\Phi(d(c)) - 1,$$

we have for $0 < \alpha < 1$, with $\Phi(d(c)) = 1 - (\alpha/2)$, $d(c) = u_{\alpha/2}$, where $\bar{X} = \sum_{i=1}^n X_i/n$ and $u_{\alpha/2}$ denotes the upper $100(\alpha/2)\%$ point of the standard normal distribution. Since from (5.3)

$$-\frac{\alpha}{2} u_{\alpha/2} + \phi(u_{\alpha/2}) = \frac{\sqrt{n}}{c},$$

it is seen that

$$c = \sqrt{n}/\{\phi(u_{\alpha/2}) - (\alpha/2)u_{\alpha/2}\}. \tag{5.4}$$

Conversely, if a value of c is given, then the confidence level $1-\alpha$ is determined from (5.4). Hence we see that a large coefficient of $\sqrt{n}$ in c implies a small α. For example, if $c \fallingdotseq 50\sqrt{n}$ and $c \fallingdotseq 111\sqrt{n}$, then $\alpha \fallingdotseq 0.1$ and 0.05, respectively.

Example 5.2 (Uniform case). Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with a density function

$$f(x,\theta) = \begin{cases} 1 & \text{for } \theta-(1/2) < x < \theta+(1/2), \\ 0 & \text{otherwise.} \end{cases}$$

Putting $U = \max_{1\le i\le n} X_i - (1/2)$ and $V = \min_{1\le i\le n} X_i + (1/2)$, we have

$$\prod_{i=1}^n f(X_i,\theta) = \begin{cases} 1 & \text{for } U < \theta < V, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from Corollary 4.1 that the confidence limits with the minimum posterior risk are given by

$$\underline{\theta}^* = U + \sqrt{\frac{2}{c}(V-U)}, \qquad \bar{\theta}^* = V - \sqrt{\frac{2}{c}(V-U)}.$$

Next, we obtain c such that a conditional level is equal to $1-\alpha$, i.e.

$$P_\theta\left\{U + \sqrt{\frac{2}{c}(V-U)} \le \theta \le V - \sqrt{\frac{2}{c}(V-U)}\,\middle|\, R=r\right\} \ge 1-\alpha,$$

where $R = 1-(V-U)$. Putting $M = (U+V)/2$, we see that the conditional distribution of M given $R=r$ is a uniform one on the interval $\left(\theta - \frac{1-r}{2},\ \theta + \frac{1-r}{2}\right)$. Then

$$P_\theta\left\{U + \sqrt{\frac{2}{c}(V-U)} \le \theta \le V - \sqrt{\frac{2}{c}(V-U)}\,\middle|\, R=r\right\} = 1 - 2\sqrt{\frac{2}{c(1-r)}} \ge 1-\alpha,$$

hence

$$c \ge 8/\{(1-r)\alpha^2\}. \tag{5.5}$$

Conversely, if c is chosen as in (5.5), then the conditional level is equal to $1-\alpha$. For example, if $c = 800/(1-r)$ and $c = 3200/(1-r)$, then the conditional levels are equal to 0.9 and 0.95, respectively.

Further, we obtain c such that an unconditional level is equal to $1-\alpha+O(1/\sqrt{n})$, i.e.

$$P_\theta\left\{U + \sqrt{\frac{2}{c}(V-U)} \le \theta \le V - \sqrt{\frac{2}{c}(V-U)}\right\} \ge 1-\alpha+O\left(\frac{1}{\sqrt{n}}\right).$$

Since the density of R is given by

$$f_R(r) = \begin{cases} n(n-1)r^{n-2}(1-r) & \text{for } 0 < r < 1, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$P_\theta\left\{U+\sqrt{\frac{2}{c}(V-U)}\le\theta\le V-\sqrt{\frac{2}{c}(V-U)}\right\}=1-(n-1)\sqrt{\frac{2\pi}{c}}\frac{\Gamma(n+1)}{\Gamma(n+(3/2))}.$$

Since, by the Stirling formula,

$$\Gamma(n+1)/\Gamma(n+(3/2))=1/\sqrt{n}+O(n^{-3/2}),$$

it follows that

$$\begin{aligned}P_\theta\left\{U+\sqrt{\frac{2}{c}(V-U)}\le\theta\le V-\sqrt{\frac{2}{c}(V-U)}\right\}&=1-\sqrt{\frac{2n}{c}}+O\left(\frac{1}{\sqrt{n}}\right)\\&\ge 1-\alpha+O\left(\frac{1}{\sqrt{n}}\right),\end{aligned}$$

hence

$$c\ge 2n/\alpha^2. \tag{5.6}$$

Conversely, if c is chosen as in (5.6), then the unconditional level is equal to $1-\alpha$. For example, if $c \fallingdotseq 200n$ and $c \fallingdotseq 800n$, then $\alpha \fallingdotseq 0.1$ and $\alpha \fallingdotseq 0.05$, respectively.

Example 5.3 (Truncated normal case). Suppose that $X_1,\dots,X_n$ are i.i.d. random variables with a density function

$$f(x,\theta)=\begin{cases}k\exp\{-(x-\theta)^2/2) & \text{for } -1<x-\theta<1,\\ 0 & \text{otherwise,}\end{cases}$$

where k is some positive constant. Putting $U=\max_{1\le i\le n}X_i-1$ and $V=\min_{1\le i\le n}X_i+1$, we have

$$\prod_{i=1}^n f(X_i,\theta)=\begin{cases}k^n\exp\{-\sum_{i=1}^n(X_i-\theta)^2/2)\} & \text{for } U<\theta<V,\\ 0 & \text{otherwise.}\end{cases}$$

Then it follows from Corollary 4.1 that the confidence limits $\underline{\theta}^*$ and $\bar{\theta}^*$ minimizing the posterior risk (4.2) are given by the solutions of the equations

(5.7)
$$\underline{\theta}\int_u^{\underline{\theta}}\exp\{-n(\theta-\bar{x})^2/2\}d\theta-\int_u^{\underline{\theta}}\theta\exp\{-n(\theta-\bar{x})^2/2\}d\theta=\frac{1}{c}\int_u^v\exp\{-n(\theta-\bar{x})^2/2\}d\theta,$$

(5.8)
$$\int_{\bar{\theta}}^v\theta\exp\{-n(\theta-\bar{x})^2/2\}d\theta-\bar{\theta}\int_{\bar{\theta}}^v\exp\{-n(\theta-\bar{x})^2/2\}d\theta=\frac{1}{c}\int_u^v\exp\{-n(\theta-\bar{x})^2/2\}d\theta.$$

$$\int_u^{\underline{\theta}} \exp\{-n(\theta-\bar{x})^2/2\}d\theta = \sqrt{\frac{2\pi}{n}}\{\Phi(\sqrt{n}(\underline{\theta}-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\},$$

$$\int_u^{\underline{\theta}} \theta\exp\{-n(\theta-\bar{x})^2/2\}d\theta = \sqrt{\frac{2\pi}{n}}\left\{\underline{\theta}\Phi(\sqrt{n}(\underline{\theta}-\bar{x}))-u\Phi(\sqrt{n}(u-\bar{x})) - \frac{1}{\sqrt{n}}\int_{\sqrt{n}(u-\bar{x})}^{\sqrt{n}(\underline{\theta}-\bar{x})}\Phi(t)dt\right\},$$

$$\int_u^{v} \exp\{-n(\theta-\bar{x})^2/2\}d\theta = \sqrt{\frac{2\pi}{n}}\{\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\},$$

it follows that

(5.9)

$$(u-\underline{\theta})\Phi(\sqrt{n}(u-\bar{x}))+\frac{1}{\sqrt{n}}\int_{\sqrt{n}(u-\bar{x})}^{\sqrt{n}(\underline{\theta}-\bar{x})}\Phi(t)dt = \frac{1}{c}\{\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\}.$$

Now we obtain asymptotically the solution of θ of (5.9). Since

$$\int_{\sqrt{n}(u-\bar{x})}^{\sqrt{n}(\underline{\theta}-\bar{x})}\Phi(t)dt = \Phi(\sqrt{n}(u-\bar{x}))\sqrt{n}(\underline{\theta}-u)+\frac{1}{2}\phi(\sqrt{n}(u-\bar{x}))\{\sqrt{n}(\underline{\theta}-u)\}^2+O_p\left(\frac{1}{n}\right),$$

it is seen that

$$\frac{1}{2\sqrt{n}}\phi(\sqrt{n}(u-\bar{x}))\{\sqrt{n}(\underline{\theta}-u)\}^2 = \frac{1}{c}\{\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\}+O_p\left(\frac{1}{n^2}\right),$$

hence

$$\underline{\theta}^* = u+\frac{1}{n}\sqrt{\frac{2n}{c}\frac{\sqrt{n}\{\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\}}{\phi(\sqrt{n}(u-\bar{x}))}}+O_p\left(\frac{1}{n\sqrt{n}}\right). \tag{5.10}$$

In a similar way to the above we have

$$\bar{\theta}^* = v-\frac{1}{n}\sqrt{\frac{2n}{c}\frac{\sqrt{n}\{\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\}}{\phi(\sqrt{n}(v-\bar{x}))}}+O_p\left(\frac{1}{n\sqrt{n}}\right). \tag{5.11}$$

Since

$$\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x})) = \frac{1}{\sqrt{n}}\{n(v-\theta)-n(u-\theta)\}\phi(\sqrt{n}(\bar{x}-\theta))+o_p\left(\frac{1}{\sqrt{n}}\right),$$

$$\phi(\sqrt{n}(u-\bar{x})) = \phi(\sqrt{n}(\bar{x}-\theta))\left\{1+\frac{1}{\sqrt{n}}n(u-\theta)\sqrt{n}(\bar{x}-\theta)+o_p\left(\frac{1}{\sqrt{n}}\right)\right\},$$

it follows that

$$\frac{\sqrt{n}\{\Phi(\sqrt{n}(v-\bar{x}))-\Phi(\sqrt{n}(u-\bar{x}))\}}{\phi(\sqrt{n}(u-\bar{x}))} = n(v-u)\left\{1+O_p\left(\frac{1}{\sqrt{n}}\right)\right\},$$

hence, by (5.10)

$$\bar{\theta}^* = v - \frac{1}{n}\sqrt{\frac{2n}{c}n(v-u)} + O_p\left(\frac{1}{n\sqrt{n}}\right).$$

In a similar way to the above we have from (5.11)

$$\underline{\theta}^* = u + \frac{1}{n}\sqrt{\frac{2n}{c}n(v-u)} + O_p\left(\frac{1}{n\sqrt{n}}\right).$$

Next, we obtain c such that the confidence level is equal to $1-\alpha+O(1/n)$, i.e.

$$(5.12)\qquad P_\theta\left\{U + \frac{1}{n}\sqrt{\frac{2n}{c}n(V-U)} \le \theta \le V - \frac{1}{n}\sqrt{\frac{2n}{c}n(V-U)}\right\} \ge 1-\alpha+O\left(\frac{1}{n}\right)$$

Putting $U_0 = n(U-\theta)$ and $V_0 = n(V-\theta)$, we have

$$\begin{aligned}
(5.13)\qquad & P_\theta\left\{U + \frac{1}{n}\sqrt{\frac{2n}{c}n(V-U)} \le \theta \le V - \frac{1}{n}\sqrt{\frac{2n}{c}n(V-U)}\right\} \\
=& P_0\left\{U_0 \le -\sqrt{\frac{2n}{c}(V_0-U_0)},\quad V_0 \ge \sqrt{\frac{2n}{c}(V_0-U_0)}\right\} \\
=& P_0\left\{U_0^2 \ge \frac{2n}{c}(V_0-U_0),\quad V_0^2 \ge \frac{2n}{c}(V_0-U_0)\right\} \\
=& P_0\left\{U_0^2 \ge \frac{2n}{c}(V_0-U_0),\quad V_0^2 \ge \frac{2n}{c}(V_0-U_0),\quad U_0^2 \ge V_0^2\right\} \\
& + P_0\left\{U_0^2 \ge \frac{2n}{c}(V_0-U_0),\quad V_0^2 \ge \frac{2n}{c}(V_0-U_0),\quad U_0^2 < V_0^2\right\} \\
=& P_0\left\{-\frac{c}{2n}V_0^2 + V_0 \le U_0\right\} + P_0\left\{\frac{c}{2n}U_0^2 + U_0 \ge V_0\right\}.
\end{aligned}$$

Since the asymptotic joint density of (U_0, V_0) is given by

$$g_n(u_0,v_0) = \begin{cases} k^2 e^{-k(v_0-u_0)}\left[1 + \frac{1}{n}\left\{-1 + 2k(v_0-u_0) + \frac{h}{4}((u_0+v_0)^2 + (v_0-u_0)^2)\right.\right. \\ \qquad \left.\left. -\frac{k^2}{2}(v_0-u_0)^2 - \frac{h}{k}(v_0-u_0)\right\}\right] + o\left(\frac{1}{n}\right) \\ \qquad\qquad \text{for } u_0 < \theta < v_0, \\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

where $h = -ke^{-1/2}$ (see Akahira (1991), page 191), it follows that

$$\begin{aligned}
P_0\left\{-\frac{c}{2n}V_0^2 + V_0 \le U\right\} &= \int_0^\infty \int_{-(cv^2/(2n))+v}^0 k^2 e^{-k(v-u)}\,du\,dv + O\left(\frac{1}{n}\right) \\
&= 1 - \frac{1}{2}\sqrt{2k\pi n/c} + O\left(\frac{1}{n}\right), \\
P_0\left\{\frac{c}{2n}U_0^2 + U_0 \ge V_0\right\} &= \int_{-\infty}^0 \int_0^{(cu^2/(2n))+u} k^2 e^{-k(v-u)}\,dv\,du + O\left(\frac{1}{n}\right) \\
&= 1 - \frac{1}{2}\sqrt{2k\pi n/c} + O\left(\frac{1}{n}\right),
\end{aligned}$$

hence, by (5.12) and (5.13)

$$P_\theta\left\{U+\frac{1}{n}\sqrt{\frac{2n}{c}n(V-U)}\le\theta\le V-\frac{1}{n}\sqrt{\frac{2n}{c}n(V-U)}\right\}=2-\sqrt{2k\pi n/c}+O\left(\frac{1}{n}\right)$$
$$\ge 1-\alpha+O\left(\frac{1}{n}\right),$$

which implies

$$c\ge 2k\pi n/(1+\alpha)^2. \tag{5.14}$$

Conversely, if c is chosen as in (5.14), then the confidence level is equal to $1-\alpha+O(1/n)$. For example, if $c \fallingdotseq 3.03n$ and $c \fallingdotseq 3.33$, then $\alpha \fallingdotseq 0.1$ and $\alpha \fallingdotseq 0.05$.

6. Proofs

In this section we give the proofs of Lemma 2.1 and Theorem 2.1.

Proof of Lemma 2.1. From (2.3) and (2.5) we have

$$\begin{aligned}
1-\alpha &= \int_a^b g_{\theta_0}(z)dz \\
&= \int_a^b\left[\phi(z)+\phi(z)\left\{\frac{K}{6I^{3/2}\sqrt{n}}(z^3-3z)+\frac{H}{24I^2n}(z^4-6z^2+3)\right.\right. \\
&\qquad\left.\left.+\frac{K^2}{72I^3n}(z^6-15z^4+45z^2-15)\right\}+o\left(\frac{1}{n}\right)\right]dz \\
&= \Phi(b)-\Phi(a)+\frac{K}{6I^{3/2}\sqrt{n}}\int_a^b h_3(z)\phi(z)dz+\frac{H}{24I^2n}\int_a^b h_4(z)\phi(z)dz \\
&\qquad+\frac{K^2}{72I^3n}\int_a^b h_6(z)\phi(z)dz+o\left(\frac{1}{n}\right),
\end{aligned} \tag{6.1}$$

where

$$h_j(z)=\left(-\frac{d}{dz}\right)^j\phi(z)/\phi(z)\qquad(j=0,1,2,\dots),$$

which are called Hermite polynomials. Since

$$\int_{-\infty}^z h_j(x)\phi(x)dx=-h_{j-1}(z)\phi(z)\qquad(j=1,2,\dots),$$

it follows from (6.1) that

(6.2)

$$\begin{aligned}
1-\alpha=\Phi(b)-\Phi(a)&+\frac{K}{6I^{3/2}\sqrt{n}}\{h_2(a)\phi(a)-h_2(b)\phi(b)\}+\frac{H}{24I^2n}\{h_3(a)\phi(a)-h_3(b)\phi(b)\} \\
&+\frac{K^2}{72I^3n}\{h_5(a)\phi(a)-h_5(b)\phi(b)\}+o\left(\frac{1}{n}\right).
\end{aligned}$$

From (2.4) and (2.5) we also have

$$(6.3)\qquad 0 = \int_a^b z\left[\phi(z) + \phi(z)\left\{\frac{K}{6I^{3/2}\sqrt{n}}(z^3 - 3z) + \frac{H}{24I^2 n}(z^4 - 6z^2 + 3) + \frac{K^2}{72I^3 n}(z^6 - 15z^4 + 45z^2 - 15)\right\} + o\left(\frac{1}{n}\right)\right] dz$$

$$= \int_a^b z\phi(z)dz + \frac{K}{6I^{3/2}\sqrt{n}}\int_a^b zh_3(z)\phi(z)dz + \frac{H}{24I^2 n}\int_a^b zh_4(z)\phi(z)dz + \frac{K^2}{72I^3 n}\int_a^b zh_6(z)\phi(z)dz + o\left(\frac{1}{n}\right).$$

Since

$$\int_{-\infty}^z zh_j(z)\phi(z)dz = -\{h_j(z) + jh_{j-2}(z)\}\phi(z) \qquad (j = 2, 3, \ldots),$$

it follows from (6.3) that

$$(6.4)\qquad 0 = \phi(a) - \phi(b) + \frac{K}{6I^{3/2}\sqrt{n}}\{(h_3(a) + 3h_1(a))\phi(a) - (h_3(b) + 3h_1(b))\phi(b)\} + \frac{H}{24I^2 n}\{(h_4(a) + 4h_2(a))\phi(a) - (h_4(b) + 4h_2(b))\phi(b)\} + \frac{K^2}{72I^3 n}\{(h_6(a) + 6h_4(a))\phi(a) - (h_6(b) + 6h_4(b))\phi(b)\} + o\left(\frac{1}{n}\right).$$

Putting

$$a = -u + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + o\left(\frac{1}{n}\right), \qquad b = u + \frac{b_1}{\sqrt{n}} + \frac{b_2}{n} + o\left(\frac{1}{n}\right),$$

we have

$$\Phi(a) = \Phi(-u) + \frac{a_1}{\sqrt{n}}\phi(u) + \frac{a_2}{n}\phi(u) + \frac{a_1^2}{2n}h_1(u)\phi(u) + o\left(\frac{1}{n}\right),$$

$$\Phi(b) = \Phi(u) + \frac{b_1}{\sqrt{n}}\phi(u) + \frac{b_2}{n}\phi(u) - \frac{b_1^2}{2n}h_1(u)\phi(u) + o\left(\frac{1}{n}\right),$$

$$h_j(a)\phi(a) = h_j(-u)\phi(u) - \frac{a_1}{\sqrt{n}}h_{j+1}(-u)\phi(u) - \frac{a_2}{n}h_{j+1}(-u)\phi(u) + \frac{a_1^2}{2n}h_{j+2}(-u)\phi(u) + o\left(\frac{1}{n}\right) \qquad (j = 2, 3, \ldots),$$

$$h_j(b)\phi(b) = h_j(u)\phi(u) - \frac{b_1}{\sqrt{n}}h_{j+1}(u)\phi(u) - \frac{b_2}{n}h_{j+1}(u)\phi(u) + \frac{b_1^2}{2n}h_{j+2}(u)\phi(u) + o\left(\frac{1}{n}\right) \qquad (j = 2, 3, \ldots),$$

hence, by (6.2), letting $u = u_{\alpha/2}$ we have

$$(6.5)\qquad 0 = \frac{1}{\sqrt{n}}(b_1 - a_1) + \frac{1}{n}(b_2 - a_2) - \frac{1}{2n}(b_1^2 + a_1^2)h_1(u_{\alpha/2}) + \frac{K}{6I^{3/2}n}(a_1 + b_1)h_3(u_{\alpha/2}) - \frac{H}{12I^2 n}h_3(u_{\alpha/2}) - \frac{K^2}{36I^3 n}h_5(u_{\alpha/2}) + o\left(\frac{1}{n}\right).$$

$$\phi(a) = \phi(u) + \frac{a_1}{\sqrt{n}} h_1(u)\phi(u) + \frac{a_2}{n} h_1(u)\phi(u) + \frac{a_1^2}{2n} h_2(u)\phi(u) + o\left(\frac{1}{n}\right),$$
$$\phi(b) = \phi(u) - \frac{b_1}{\sqrt{n}} h_1(u)\phi(u) - \frac{b_2}{n} h_1(u)\phi(u) + \frac{b_1^2}{2n} h_2(u)\phi(u) + o\left(\frac{1}{n}\right),$$

hence, by (6.4)

$$(6.6)\quad 0 = \frac{1}{\sqrt{n}}(a_1 + b_1)h_1(u) + \frac{1}{n}(a_2 + b_2)h_1(u) + \frac{1}{2n}(a_1^2 - b_1^2)h_2(u) - \frac{K}{3I^{3/2}\sqrt{n}} h_3(u)$$
$$- \frac{K}{6I^{3/2}n}(a_1 - b_1)h_4(u) - \frac{K}{I^{3/2}\sqrt{n}} h_1(u) - \frac{K}{2I^{3/2}\sqrt{n}}(a_1 - b_1)h_2(u) + o\left(\frac{1}{n}\right)$$

where $u = u_{\alpha/2}$. From (6.5) we have

$$(6.7)\qquad a_1 = b_1.$$

From (6.7) and the term of the order of $1/\sqrt{n}$ in (6.6) we obtain

$$2a_1 h_1(u) - \frac{K}{3I^{3/2}}\{h_3(u) + 3h_1(u)\} = 0,$$

hence

$$(6.8)\qquad a_1 = \frac{K}{6I^{3/2}} u^2.$$

Since, by (6.6), (6.7) and (6.8)

$$(6.9)\qquad b_2 = -a_2,$$

it follows from (6.5), (6.7) and (6.9) that

$$0 = -2a_2 - a_1^2 h_1(u) + \frac{K}{3I^{3/2}} a_1 h_3(u) - \frac{H}{12I^2} h_3(u) - \frac{K^2}{36I^3} h_5(u),$$

hence

$$(6.10)\qquad a_2 = \frac{K^2}{72I^3}(4u^3 - 15u) - \frac{H}{24I^2}(u^3 - 3u).$$

From (6.7) to (6.10) we have

$$a = -u + \frac{K}{6I^{3/2}\sqrt{n}} u^2 + \frac{K^2}{72I^3 n}(4u^3 - 15u) - \frac{H}{24I^2 n}(u^3 - 3u) + o\left(\frac{1}{n}\right),$$
$$b = u + \frac{K}{6I^{3/2}\sqrt{n}} u^2 - \frac{K^2}{72I^3 n}(4u^3 - 15u) + \frac{H}{24I^2 n}(u^3 - 3u) + o\left(\frac{1}{n}\right).$$

Thus we complete the proof.

Proof of Theorem 2.1. Putting $\hat{\theta} = \hat{\theta}_{ML}$, we see that $Z_1(\hat{\theta}) = 0$. Now we define $Z_2(\theta)$ and $Z_3(\theta)$ as

$$Z_2(\theta) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial \theta^2} \log f(X_i, \theta) + I(\theta) \right\},$$

$$Z_3(\theta) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^3}{\partial \theta^3} \log f(X_i, \theta) + 3J(\theta) + K(\theta) \right\}.$$

Putting $\bar{\theta} := \hat{\theta} + \Delta$, we have by the Taylor expansion

$$\begin{aligned} Z_1(\bar{\theta}) &= Z_1(\hat{\theta}) + Z_1'(\hat{\theta})\Delta + \frac{1}{2} Z_1''(\hat{\theta})\Delta^2 + \frac{1}{6} Z_1'''(\hat{\theta})\Delta^3 + o_p(\sqrt{n}\Delta^3) \\ &= -\sqrt{n} I(\hat{\theta})\Delta + Z_2(\hat{\theta})\Delta - \frac{\sqrt{n}}{2}(3J(\hat{\theta}) + K(\hat{\theta}))\Delta^2 + \frac{1}{2} Z_3(\hat{\theta})\Delta^2 \\ &\quad - \frac{\sqrt{n}}{6}(H(\hat{\theta}) + 4L(\hat{\theta}) + 3M(\hat{\theta}) + 6N(\hat{\theta}))\Delta^3 + o_p(\sqrt{n}\Delta^3), \end{aligned} \tag{6.11}$$

since $(1/n)\sum_{i=1}^{n}(\partial^4/\partial\theta^4) \log f(X_i, \theta)$ converges in probability to $-H(\theta) - 4L(\theta) - 3M(\theta) - 6N(\theta)$. On the other hand we obtain by the Taylor expansion

$$a_0(\bar{\theta}) = a_0(\hat{\theta}) + a_0'(\hat{\theta})\Delta + \frac{1}{2} a_0''(\hat{\theta})\Delta^2 + o_p(\Delta^2). \tag{6.12}$$

Since $I'(\theta) = 2J(\theta) + K(\theta)$, $J'(\theta) = L(\theta) + M(\theta) + N(\theta)$ and $K'(\theta) = H(\theta) + 3N(\theta)$, it follows from Lemma 2.1 that

$$\begin{aligned} a_0'(\theta) &= -u \frac{2J(\theta) + K(\theta)}{2\sqrt{I(\theta)}} + \frac{u^2}{6I^2(\theta)\sqrt{n}} \{(H(\theta) + 3N(\theta))I(\theta) - K(\theta)(2J(\theta) + K(\theta))\} \\ &\qquad + o\left(\frac{1}{n}\right), \\ a_0''(\theta) &= -\frac{u}{2\sqrt{I(\theta)}} \{H(\theta) + 2L(\theta) + 2M(\theta) + 5N(\theta)\} + \frac{u}{4I^{3/2}(\theta)} \{2J(\theta) + K(\theta)\}^2 + o(1). \end{aligned}$$

From (6.12) we have

(6.13)

$$\begin{aligned} a_0(\bar{\theta}) &= -uI^{1/2}(\hat{\theta}) + \frac{K(\hat{\theta})}{6I(\theta)\sqrt{n}} u^2 - \frac{u\Delta\{2J(\hat{\theta}) + K(\hat{\theta})\}}{2I^{1/2}(\hat{\theta})} + \frac{K^2(\hat{\theta})}{72I^{5/2}(\hat{\theta})n}(4u^3 - 5u) \\ &\quad - \frac{H(\hat{\theta})}{24I^{3/2}(\hat{\theta})n}(u^3 - 3u) + \frac{u^2\Delta}{6I^2(\hat{\theta})\sqrt{n}} \{I(\hat{\theta})(H(\hat{\theta}) + 3N(\hat{\theta})) - K(\hat{\theta})(2J(\hat{\theta}) + K(\hat{\theta}))\} \\ &\quad - \frac{u\Delta^2}{4I^{1/2}(\hat{\theta})} \{H(\hat{\theta}) + 2L(\hat{\theta}) + 2M(\hat{\theta}) + 5N(\hat{\theta})\} + \frac{u\Delta^2}{8I^{3/2}(\hat{\theta})} \{2J(\hat{\theta}) + K(\hat{\theta})\}^2 + o\left(\frac{1}{n}\right). \end{aligned}$$

Letting $Z_1(\bar{\theta}) = a_0(\bar{\theta})$, we have from (6.11) and (6.13)

(6.14)

$$\begin{aligned}
\Delta =& \frac{u}{\sqrt{nI(\hat{\theta})}} + \frac{Z_2(\hat{\theta})}{\sqrt{n}I(\hat{\theta})}\Delta - \frac{3J(\hat{\theta})+K(\hat{\theta})}{2I(\hat{\theta})}\Delta^2 + \frac{Z_3(\hat{\theta})}{2\sqrt{n}I(\hat{\theta})}\Delta^2 \\
&- \frac{1}{6I(\hat{\theta})}\{H(\hat{\theta})+4L(\hat{\theta})+3M(\hat{\theta})+6N(\hat{\theta})\}\Delta^3 - \frac{K(\hat{\theta})u^2}{6nI^2(\hat{\theta})} + \frac{2J(\hat{\theta})+K(\hat{\theta})}{2\sqrt{n}I^{3/2}(\hat{\theta})}u\Delta \\
&- \frac{K^2(\hat{\theta})}{72n\sqrt{n}I^{7/2}(\hat{\theta})}(4u^3-15u) + \frac{H(\hat{\theta})}{24n\sqrt{n}I^{5/2}(\hat{\theta})}(u^3-3u) \\
&- \frac{1}{6nI^3(\hat{\theta})}\{I(\hat{\theta})(H(\hat{\theta})+3N(\hat{\theta})) - K(\hat{\theta})(2J(\hat{\theta})+K(\hat{\theta}))\}u^2\Delta \\
&+ \frac{1}{4\sqrt{n}I^{3/2}(\hat{\theta})}\{H(\hat{\theta})+2L(\hat{\theta})+2M(\hat{\theta})+5N(\hat{\theta})\}u\Delta^2 - \frac{(2J(\hat{\theta})+K(\hat{\theta}))^2}{8\sqrt{n}I^{5/2}(\hat{\theta})}u\Delta^2 \\
&+ o_p\left(\frac{\Delta^2}{\sqrt{n}}\right) + o_p(\Delta^3),
\end{aligned}$$

hence, putting $\Delta = u/\sqrt{nI(\hat{\theta})} + \Delta'$ with $\Delta' = O_p(1/n)$, we have

$$\begin{aligned}
\Delta' =& \frac{Z_2(\hat{\theta})}{nI^{3/2}(\hat{\theta})}u - \frac{3J(\hat{\theta})+K(\hat{\theta})}{6nI^2(\hat{\theta})}u^2 + \frac{Z_2^2(\hat{\theta})}{n\sqrt{n}I^{5/2}(\hat{\theta})}u - \frac{3J(\hat{\theta})+K(\hat{\theta})}{6n\sqrt{n}I^3(\hat{\theta})}Z_2(\hat{\theta})u^2 \\
&- \frac{4J(\hat{\theta})+K(\hat{\theta})}{2n\sqrt{n}I^3(\hat{\theta})}Z_2(\hat{\theta})u^2 + \frac{\{4J(\hat{\theta})+K(\hat{\theta})\}\{3J(\hat{\theta})+K(\hat{\theta})\}}{12n\sqrt{n}I^{7/2}(\hat{\theta})}u^3 \\
&+ \frac{Z_3(\hat{\theta})u^2}{2n\sqrt{n}I^2(\hat{\theta})} - \frac{1}{6}\{H(\hat{\theta})+4L(\hat{\theta})+3M(\hat{\theta})+6N(\hat{\theta})\}\frac{u^3}{n\sqrt{n}I^{5/2}(\hat{\theta})} \\
&- \frac{K^2(\hat{\theta})}{72n\sqrt{n}I^{7/2}(\hat{\theta})}(4u^3-15u) + \frac{H(\hat{\theta})}{24n\sqrt{n}I^{5/2}(\hat{\theta})}(u^3-3u) \\
&- \frac{1}{6n\sqrt{n}I^{7/2}(\hat{\theta})}\{I(\hat{\theta})(H(\hat{\theta})+3N(\hat{\theta})) - K(\hat{\theta})(2J(\hat{\theta})+K(\hat{\theta}))\}u^3 \\
&+ \frac{1}{4n\sqrt{n}I^{5/2}(\hat{\theta})}\{H(\hat{\theta})+2L(\hat{\theta})+2M(\hat{\theta})+5N(\hat{\theta})\}u^3 \\
&- \frac{1}{8n\sqrt{n}I^{7/2}(\hat{\theta})}\{2J(\hat{\theta})+K(\hat{\theta})\}^2u^3 + o_p\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned} \tag{6.15}$$

From (6.14) and (6.15) we obtain

$$\begin{aligned}\Delta =& \frac{u}{\sqrt{nI(\hat\theta)}} + \frac{Z_2(\hat\theta)}{nI^{3/2}(\hat\theta)}u - \frac{3J(\hat\theta)+K(\hat\theta)}{6nI^2(\hat\theta)}u^2 + \frac{Z_2^2(\hat\theta)}{n\sqrt{n}I^{5/2}(\hat\theta)}u - \frac{15J(\hat\theta)+4K(\hat\theta)}{6n\sqrt{n}I^3(\hat\theta)}Z_2(\hat\theta)u^2 \\ &+ \frac{Z_3(\hat\theta)}{2n\sqrt{n}I^2(\hat\theta)}u^2 - \left\{\frac{1}{24n\sqrt{n}I^{5/2}(\hat\theta)}(H(\hat\theta)+4L(\hat\theta)+6N(\hat\theta))\right. \\ &\left.-\frac{1}{72n\sqrt{n}I^{7/2}(\hat\theta)}(36J^2(\hat\theta)+30J(\hat\theta)K(\hat\theta)+5K^2(\hat\theta))\right\}u^3 \\ &+\left\{\frac{5K^2(\hat\theta)}{24n\sqrt{n}I^{7/2}(\hat\theta)} - \frac{H(\hat\theta)}{8n\sqrt{n}I^{5/2}(\hat\theta)}\right\}u + o_p\left(\frac{1}{n\sqrt{n}}\right) \\ =&\Delta(\theta,u) \quad \text{(say)}.\end{aligned}$$

Hence we have as the upper confidence limit

$$\bar\theta = \hat\theta + \Delta(\hat\theta, u),$$

where $\hat\theta = \hat\theta_{ML}$ and $u = u_{\alpha/2}$. In a similar way to the above we obtain as the lower confidence limit

$$\underline{\theta} = \hat\theta + \Delta(\hat\theta, -u).$$

Thus we complete the proof.

Proof of Theorem 3.1. From (3.7) and (3.8) we have

(6.16)
$$1-\alpha = \int_a^b \phi(t)\left\{1 + \frac{I\sqrt{I}\beta_3}{6n}h_3(t) + \frac{I^2\beta_4}{24n}h_4(t) + \frac{I^3\beta_3^2}{72n}h_6(t) + \frac{I\tau}{2n}h_2(t) + o\left(\frac{1}{n}\right)\right\}dt.$$

Since

$$\int_{-\infty}^{z} h_j(t)\phi(t)dt = -h_{j-1}(z)\phi(z) \qquad (j=1,2,\dots),$$

it follows from (6.16) that

(6.17)
$$\begin{aligned}1-\alpha =& \Phi(b) - \Phi(a) + \frac{I\sqrt{I}\beta_3}{6\sqrt{n}}\{-h_2(b)\phi(b) + h_2(a)\phi(a)\} + \frac{I^2\beta_4}{24n}\{-h_3(b)\phi(b)+h_3(a)\phi(a)\} \\ &+ \frac{I^3\beta_3^2}{72n}\{-h_5(b)\phi(b)+h_5(a)\phi(a)\} + \frac{I\tau}{2n}\{-h_1(b)\phi(b)+h_1(a)\phi(a)\} + o\left(\frac{1}{n}\right).\end{aligned}$$

From (3.7) and (3.9) we obtain

(6.18)

$$0 = \phi(b) - \phi(a) + \frac{I\sqrt{I}\beta_3}{6\sqrt{n}}\{h_3(b)\phi(b) - h_3(a)\phi(a)\} + \frac{I^2\beta_4}{24n}\{h_4(b)\phi(b) - h_4(a)\phi(a)\}$$
$$+ \frac{I^3\beta_3^2}{72n}\{h_6(b)\phi(b) - h_6(a)\phi(a)\} + \frac{I\tau}{2n}\{h_2(b)\phi(b) - h_2(a)\phi(a)\} + o\left(\frac{1}{n}\right).$$

We put

(6.19) $$a = -u + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + o\left(\frac{1}{n}\right), \qquad b = u + \frac{b_1}{\sqrt{n}} + \frac{b_2}{n} + o\left(\frac{1}{n}\right),$$

where $u = u_{\alpha/2}$. Since

$$\Phi(a) = \Phi(-u) + \frac{a_1}{\sqrt{n}}\phi(u) + \frac{a_2}{n}\phi(u) + \frac{a_1^2}{2n}h_1(u)\phi(u) + o\left(\frac{1}{n}\right),$$
$$\Phi(b) = \Phi(u) + \frac{b_1}{\sqrt{n}}\phi(u) + \frac{b_2}{n}\phi(u) - \frac{b_1^2}{2n}h_1(u)\phi(u) + o\left(\frac{1}{n}\right),$$
$$h_j(a)\phi(a) = h_j(-u)\phi(u) - \frac{a_1}{\sqrt{n}}h_{j+1}(-u)\phi(u) - \frac{a_2}{n}h_{j+1}(-u)\phi(u) + \frac{a_1^2}{2n}h_{j+2}(-u)\phi(u)$$
$$+ o\left(\frac{1}{n}\right) \qquad (j = 2, 3, \dots),$$
$$h_j(b)\phi(b) = h_j(u)\phi(u) - \frac{b_1}{\sqrt{n}}h_{j+1}(u)\phi(u) - \frac{b_2}{n}h_{j+1}(u)\phi(u) + \frac{b_1^2}{2n}h_{j+2}(u)\phi(u)$$
$$+ o\left(\frac{1}{n}\right) \qquad (j = 2, 3, \dots),$$

it follows from (6.17) that

(6.20) $$0 = \frac{1}{\sqrt{n}}(b_1 - a_1) + \frac{1}{n}(b_2 - a_2) - \frac{1}{2n}(b_1^2 + a_1^2)h_1(u) + \frac{I\sqrt{I}\beta_3}{6n}(b_1 + a_1)h_3(u)$$
$$- \frac{I^2\beta_4}{12n}h_3(u) - \frac{I^3\beta_3^2}{36n}h_5(u) - \frac{I\tau}{2n}h_1(u) + o\left(\frac{1}{n}\right).$$

Since

$$\phi(a) = \phi(u) + \frac{a_1}{\sqrt{n}}h_1(u)\phi(u) + \frac{a_2}{n}h_1(u)\phi(u) + \frac{a_1^2}{2n}h_2(u)\phi(u) + o\left(\frac{1}{n}\right),$$
$$\phi(b) = \phi(u) - \frac{b_1}{\sqrt{n}}h_1(u)\phi(u) - \frac{b_2}{n}h_1(u)\phi(u) + \frac{b_1^2}{2n}h_2(u)\phi(u) + o\left(\frac{1}{n}\right),$$

it follows from (6.18) that

(6.21) $$0 = \frac{1}{\sqrt{n}}(a_1 + b_1)h_1(u) + \frac{1}{n}(a_2 + b_2)h_1(u) + \frac{1}{2n}(a_1^2 - b_1^2)h_2(u)$$
$$- \frac{I\sqrt{I}\beta_3}{3\sqrt{n}}h_3(u)\phi(u) - \frac{I\sqrt{I}\beta_3}{6n}(a_1 - b_1)h_4(u)\phi(u) + o\left(\frac{1}{n}\right).$$

From (6.20) we have $a_1 = b_1$. Considering the order of $1/\sqrt{n}$ in (6.21), we obtain

$$a_1 = \frac{I\sqrt{I}\beta_3}{6}(u^2 - 3),$$

hence

$$a_1 = b_1 = \frac{I\sqrt{I}\beta_3}{6}(u^2 - 3). \tag{6.22}$$

Substituting (6.22) in (6.21) we have

$$a_2 = -b_2. \tag{6.23}$$

Using (6.22) and (6.23), we obtain from (6.20)

$$0 = 2b_2 - b_1^2 h_1(u) + \frac{I\sqrt{I}\beta_3}{3} b_1 h_3(u) - \frac{I^2\beta_4}{12} h_3(u) - \frac{I^3\beta_3^2}{36} h_5(u) - \frac{I\tau}{2} h_1(u),$$

hence

$$\begin{aligned} b_2 &= \frac{I^3\beta_3^2}{72}(u^2-3)^2 u - \frac{I^3\beta_3^2}{36}(u^2-3)(u^3-3u) + \frac{I^2\beta_4}{24}(u^3-3u) \\ &\quad + \frac{I^3\beta_3^2}{72n}(u^5 - 10u^3 + 15u) + \frac{I\tau}{2}u \\ &= -\frac{I^3\beta_3^2}{36}(2u^3 - 3u) + \frac{I^2\beta_4}{24}(u^3 - 3u) + \frac{I\tau}{2}u, \end{aligned} \tag{6.24}$$

since $h_1(u) = u$, $h_3(u) = u^3 - 3u$ and $h_5(u) = u^5 - 10u^3 + 15u$. Since, by (3.3) and (3.4)

$$\begin{aligned} \beta_3(\theta) &= -\frac{1}{I^3}(3J + 2K) = -\frac{K}{2I^3}, \\ \beta_4(\theta) &= \frac{12}{I^5}(2J + K)(J + K) - \frac{1}{I^4}(3H + 4L + 12N) = \frac{H - 4L}{I^4}, \end{aligned}$$

it follows from (6.22) and (6.24) that

$$b_1 = -\frac{K}{12I\sqrt{I}}(u^2 - 3), \qquad b_2 = \frac{H - 4L}{24I^2}(u^3 - 3u) - \frac{K^2}{144I^3}(2u^3 - 3u) + \frac{I\tau}{2}u. \tag{6.25}$$

From (6.19), (6.22), (6.23) and (6.25) we get the conclusion.

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