ASYMPTOTIC DEFICIENCY OF ESTIMATORS UNDER MODELS WITH NUISANCE PARAMETERS

BY

KEI TAKEUCHI and MASAFUMI AKAHIRA

TECHNICAL REPORT NO. 179
FEBRUARY 1982

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT
MCS 81-04262

Ingram Olkin, Project Director

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Kei Takeuchi[1] and Masafumi Akahira[2]

Abstract

Let $X_1, \ldots, X_n$ be independently and identically distributed random variables with a parameter θ to be estimated and also "shape" parameters $\xi_1, \ldots, \xi_k$. In a "large" model, i.e., with many shape parameters, the trade-off between "accuracy" and "simplicity" is discussed in terms of the concept of asymptotic deficiency.

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The author is visiting Stanford University from September 1981 to May 1982 under a grant of the Ministry of Education of Japan.

Asymptotic Deficiency of Estimators under Models with Nuisance Parameters

Kei Takeuchi and Masafumi Akahira

1. Introduction

In many cases of statistical inference, there is often raised the problem of "model selection", that is, to specify the appropriate model for the observed data. In typical situations we have observations $X_1, \ldots, X_n$, which are assumed to be independently and identically distributed with a parameter θ to be estimated and also "shape" parameters $\xi_1, \ldots, \xi_k$ ([4]). If we choose a "large" model, that is, with many shape parameters, the model will be more accurate, or it will include a distribution which is close to the "true" distribution. On the other hand, however, the presence of many nuisance parameters would increase the error of estimation of θ due to the errors of those estimated nuisance parameters. This problem can not be approached when we only consider the first order asymptotic efficiency, since the presence of nuisance parameters will not affect the asymptotic variance of the estimator of θ , provided that the parameters are orthogonal. Hence we have to consider the second (or the third) order asymptotic expansion and discuss the problem in terms of "asymptotic deficiency". And in this term we may consider the trade-off between "accuracy" and "simplicity" of the model. This problem is similar in nature to those problems discussed by Akaike in his introduction of the AIC, but here we restrict our attention to the estimation of one parameter θ and the results are completely different.

2. Results

Suppose that it is required to estimate an unknown quantity θ (real valued) based on a sample of size n whose values are denoted by $X_1, X_2, \ldots, X_n$. We assume that X_i's are i.i.d. according to some distribution absolutely continuous with respect to a non-atomic σ-finite measure μ. It is too natural to assume that the density function of X_i depends on θ, but the value of θ alone does not necessarily determine the density function completely. Therefore, we may choose one among several "models" in which the density function is assumed to have the form

$$f_i(x,\theta,\eta_i) \ , \quad i = 1,\ldots,k \ ,$$

where η_i's are "nuisance" parameters in each of the models. What we are supposed to do is to choose one of the k-models defined above, and assuming as if the "model" chosen were "true", to estimate θ. In this paper asymptotic properties of such procedures will be discussed.

First let us consider the case where the models are in "hierachical" order, that is, the nuisance parameter η_i has the structure

$$\eta_i = (\xi_1,\ldots,\xi_i) \ , \quad i = 1,\ldots,k \tag{2.1}$$

and the density function can be expressed as

$$f_i(x,\theta,\eta_i) = f_0(x,\ \theta,\ \xi_1,\ldots,\xi_i,\ 0,\ldots,0) \ ,$$

that is, we denote instead of (2.1) that

$$\eta_i = (\xi_1,\ldots,\xi_i,\ 0,\ldots,0) \ .$$

The "true" density is denoted by $p(x,\theta)$ which may not necessarily be within the model. We define the values of parameters (θ_{0i},η_{0i}) by

$$\int \{\log f_i(x,\theta_{0i},\eta_{0i})\}\ p(x,\theta)d\mu$$

$$= \sup_{\theta',\eta_i} \int \{\log f_i(x,\theta',\eta_i)\}\ p(x,\theta)d\mu\ ,$$

that is, the density $f_i(x,\theta_{0i},\eta_{0i})$ is the one which is closest to the true density within the i-th model.

We assume the following:

(A.2.1) The models are "unbiased" in the sense that $\theta_{0i} = \theta$ for all values of θ.

(A.2.2) $\eta_{0i} = (\xi_1^0,\ldots,\xi_i^0,\ 0,\ldots,0)$, that is, η_{0i} is determined by the first i coordinates of $\eta_{0k} = (\xi_1^0,\ldots,\xi_k^0)$.

In order to simplify the notation we introduce the k-th model for which the density function is denoted by

$$f(x,\ \theta,\ \xi_1,\ldots,\xi_k)$$

with the condition that

$$f(x,\ \theta,\ \xi_1,\ldots,\xi_{k-1},\ 0) = f_0(x,\ \theta,\ \xi_1,\ldots,\xi_{k-1})\ ;$$

$$f(x,\ \theta,\ \xi_1^0,\ldots,\xi_{k-1}^0,\ \xi_k^0) = p(x,\theta)$$

for $k \geq 2$. The "largest" model thus defined includes the "true" distribution and the i-th model assumed corresponds to the hypothesis that $\xi_{i+1} = \ldots = \xi_k = 0$. We assume the "usual" set of regularity conditions:

(A.2.3) $\{x|f(x,\ \theta,\ \xi_1,\ldots,\xi_k) > 0\}$ does not depend on $\theta,\ \xi_1,\ \ldots,\ \xi_k$.

(A.2.4) For almost all $x[\mu]$, $f(x, \theta, \xi_1,\ldots,\xi_k)$ is three times continuously partially differentiable in $\theta, \xi_1, \ldots, \xi_k$.

(A.2.5) For each $\theta \in \Theta$

$$0 < I_{00} = E\left[\left\{\frac{\partial}{\partial\theta}\log f(X_i, \theta_0, \xi_1^0,\ldots,\xi_k^0)\right\}^2\right]$$

$$= -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X_i, \theta_0, \xi_1^0,\ldots,\xi_k^0)\right]$$

$$< \infty ;$$

$$0 < I_{\alpha\alpha} = E\left[\left\{\frac{\partial}{\partial\xi_\alpha}\log f(X_i, \theta_0, \xi_1^0,\ldots,\xi_k^0)\right\}^2\right]$$

$$= -E\left[\frac{\partial^2}{\partial\xi_\alpha^2}\log f(X_i, \theta_0, \xi_1^0,\ldots,\xi_k^0)\right]$$

$$< \infty$$

$(\alpha = 1, \ldots, k)$.

(A.2.6) The parameters are defined to be "orthogonal" in the sense that

$$E\left[\frac{\partial^2}{\partial\theta\partial\xi_\alpha}\log f(X, \theta_0, \xi_1^0,\ldots,\xi_k^0)\right] = 0 \quad (\alpha = 1,\ldots,k) ;$$

$$E\left[\frac{\partial^2}{\partial\xi_\alpha\partial\xi_\beta}\log f(X, \theta_0, \xi_1^0,\ldots,\xi_k^0)\right] = 0 \quad (\alpha, \beta = 1,\ldots,k; \alpha \neq \beta) .$$

This condition is not really restrictive since we may redefine the sequence of parameters to satisfy it.

(A.2.7) There exist

$$J_{000} = E\left[\left\{\frac{\partial^2}{\partial\theta^2} \log f(X, \theta_0, \xi_1^0, \ldots \xi_k^0)\right\}\left\{\frac{\partial}{\partial\theta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\right] ;$$

$$J_{00\alpha} = E\left[\left\{\frac{\partial^2}{\partial\theta^2} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\left\{\frac{\partial}{\partial\xi_\alpha} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\right] ;$$

$$J_{0\alpha 0} = E\left[\left\{\frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\left\{\frac{\partial}{\partial\theta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\right] ;$$

$$J_{0\alpha\beta} = E\left[\left\{\frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\left\{\frac{\partial}{\partial\xi_\beta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\right] ;$$

$$J_{\alpha\alpha 0} = E\left[\left\{\frac{\partial^2}{\partial\xi_\alpha^2} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\left\{\frac{\partial}{\partial\theta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\right] ;$$

$$K_{000} = E\left[\left\{\frac{\partial}{\partial\theta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}^3\right] ;$$

and the following hold:

$$E\left[\frac{\partial^3}{\partial\theta^3} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right] = -3J_{000} - K_{000} ;$$

$$E\left[\frac{\partial^3}{\partial\theta^2\partial\xi_\alpha} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right] = -J_{0\alpha 0} ;$$

$$E\left[\frac{\partial^3}{\partial\theta\partial\xi_\alpha^2} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right] = -J_{0\alpha\alpha} ;$$

$(\alpha = 1, \ldots, k)$.

By (A.2.6) and (A.2.7) we have for $\alpha \neq \beta$

$$E\left[\frac{\partial^3}{\partial\theta\partial\xi_\alpha\partial\xi_\beta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right] + E\left[\left\{\frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\} \cdot \left\{\frac{\partial}{\partial\xi_\beta} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}\right] = 0 ;$$

$$E\left[\frac{\partial^3}{\partial\theta\partial\xi_\alpha\partial\xi_\beta}\log f(X,\ \theta_0,\ \xi_1^0,\dots,\xi_k^0)\right] + E\left[\left\{\frac{\partial^2}{\partial\theta\partial\xi_\beta}\log f(X,\ \theta_0,\ \xi_1^0,\dots,\xi_k^0)\right\}\right.$$

$$\left.\cdot\left\{\frac{\partial}{\partial\xi_\alpha}\log f(X,\ \theta_0,\ \xi_1^0,\dots,\xi_k^0)\right\}\right] = 0 .$$

Hence we obtain

$$J_{0\alpha\beta} = J_{0\beta\alpha} \quad (\alpha,\ \beta = 1,\dots,k) .$$

For each $m = 1, 2, \dots,$ $\hat{\theta}$ is called an m-th order asymptotically median unbiased (or m-th order AMU) estimator if for any $(\theta_0,\ \xi_1^0,\dots,\xi_k^0)$ it holds that

$$\lim_{n\to\infty} n^{(m-1)/2}\left|\Pr\{\hat{\theta} \leqq \theta\} - \frac{1}{2}\right| = 0 ;$$

$$\lim_{n\to\infty} n^{(m-1)/2}\left|\Pr\{\hat{\theta} \geqq \theta\} - \frac{1}{2}\right| = 0$$

for some neighborhood of $(\theta_0,\ \xi_1^0,\dots,\xi_k^0)$.

Here we define classes $\mathbb{C}$ and $\mathbb{D}$ of estimators as follows. We call the class $\mathbb{C}$ the class of estimators $\hat{\theta}$ which are third order AMU and for which the distribution of $\sqrt{n}(\hat{\theta}-\theta)$ admits the Edgeworth expansion up to the order n^{-1} and

$$\sqrt{n}(\hat{\theta}-\theta) = \frac{1}{I\sqrt{n}}\frac{\partial L_n}{\partial\theta} + \frac{1}{\sqrt{n}}Q + o_p\left(\frac{1}{n}\right) ,$$

where I denote the Fisher information and $L_n = \sum_{i=1}^{n}\log f(X_i,\ \theta,\ \xi_1,\dots,\xi_k)$ and Q is a quantity of stochastic order 1. We say the estimator $\hat{\theta}$ belongs to the class $\mathbb{D}$ if in the above we have

$$E\left[\frac{\partial L_n}{\partial \theta} Q^2\right] = 0 \ ,$$

where E stands for the asymptotic mean.

Now we consider the asymptotic case when n tends to be large under the sequence of "contiguous" distributions, that is, the sequence of "true" parameters satisfies the condition that

$$\xi_\alpha = O\left(\frac{1}{\sqrt{n}}\right) \quad (\alpha = 1,\dots,k)$$

and we express it as

$$\xi_\alpha = \frac{t_\alpha}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \quad (\alpha = 1,\dots,k) \ .$$

In reviewing these assumptions one can distinguish for the first $k-1$ parameters $\xi_1, \dots, \xi_{k-1}$ and the last one ξ_k. For the first set, the contiguity assumption is only natural, because otherwise the "smaller" models would be surely rejected by any natural testing procedure. The last assumption implies that the true distribution is close to the model assumed, which is again natural when the sample size is large since otherwise the model would be rejected; although it is difficult to discuss how to construct a consistent test for the hypothesis for the "shape" of the distribution. Here we simply assume it without any further detailed justification.

Let $\hat{\theta}, \hat{\xi}_1, \dots, \hat{\xi}_k$ be MLE's of $\theta_0, \xi_1^0, \dots, \xi_k^0$ under the true model $(\theta_0, \xi_1^0,\dots,\xi_k^0)$. Since

$$\sum_{i=1}^{n} \frac{\partial}{\partial \theta} \log f(X_i, \hat{\theta}, \hat{\xi}_1,\dots,\hat{\xi}_k) = 0 \ ;$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial \xi_\alpha} \log f(X_i, \hat{\theta}, \hat{\xi}_1,\dots,\hat{\xi}_k) = 0 \quad (\alpha = 1,\dots,k) \ ,$$

expanding them in the neighborhood of $(\theta_0, \xi_1^0, \ldots, \xi_k^0)$ we have

(2.2)

$$0 = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i, \hat{\theta}, \hat{\xi}_1, \ldots, \hat{\xi}_k)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\theta^2} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \sqrt{n}(\hat{\theta}-\theta_0)$$

$$+ \frac{1}{n} \sum_{\alpha=1}^{k} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \sqrt{n}(\hat{\xi}_\alpha-\xi_\alpha^0)$$

$$+ \frac{1}{2n\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^3}{\partial\theta^3} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \{\sqrt{n}(\hat{\theta}-\theta_0)\}^2$$

$$+ \frac{1}{2n\sqrt{n}} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \sum_{i=1}^{n} \left\{ \frac{\partial^3}{\partial\theta\partial\xi_\alpha\partial\xi_\beta} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \{n(\hat{\xi}_\alpha-\xi_\alpha^0)(\hat{\xi}_\beta-\xi_\beta^0)\}$$

$$+ \frac{1}{n\sqrt{n}} \sum_{\alpha=1}^{k} \sum_{i=1}^{n} \frac{\partial^3}{\partial\theta^2\partial\xi_\alpha} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \{n(\hat{\theta}-\theta_0)(\hat{\xi}_\alpha-\xi_\alpha^0)\} + o_p\left(\frac{1}{\sqrt{n}}\right) ;$$

(2.3)

$$0 = \sum_{i=1}^{n} \frac{\partial}{\partial\xi_\alpha} \log f(X_i, \hat{\theta}, \hat{\xi}_1, \ldots, \hat{\xi}_k)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial\xi_\alpha} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\xi_\alpha^2} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \sqrt{n}(\hat{\xi}_\alpha-\xi_\alpha^0)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \sqrt{n}(\hat{\theta}-\theta_0)$$

$$+ \frac{1}{2n\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^3}{\partial\xi_\alpha^3} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \{\sqrt{n}(\hat{\xi}_\alpha-\xi_\alpha^0)\}^2$$

$$+ \frac{1}{2n\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^3}{\partial\theta^2\partial\xi_\alpha} \log f(X_i, \theta_0, \xi_1^0, \ldots, \xi_k^0) \right\} \{\sqrt{n}(\hat{\theta}-\theta_0)\}^2$$

((2.3) continued)

$$+ \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^3}{\partial\theta\partial\xi_\alpha^2} \log f(X_i,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0) \right\} \{n(\hat{\theta}-\theta_0)(\hat{\xi}_\alpha-\xi_\alpha^0)\}$$

$$+ o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

$(\alpha = 1,\ldots,k)$.

Putting

$$Z_0 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(X_i,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0) \ ;$$

$$Z_\alpha = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial\xi_\alpha} \log f(X_i,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0) \quad (\alpha = 1,\ldots,k) \ ;$$

$$Z_{00} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\theta^2} \log f(X_i,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0) + I_{00} \right\} ;$$

$$Z_{\alpha\alpha} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\partial^2}{\partial\xi_\alpha^2} \log f(X_i,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0) + I_{\alpha\alpha} \right\} (\alpha = 1,\ldots,k) \ ;$$

$$Z_{0\alpha} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X_i,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0) \quad (\alpha = 1,\ldots,k) \ ,$$

we see that they are asymptotically normal with mean 0. Since from (2.2) and (2.3) we have

$$0 = Z_0 + \frac{1}{\sqrt{n}} (Z_{00} - \sqrt{n}\ I_{00})\ \sqrt{n}(\hat{\theta}-\theta_0) + \frac{1}{\sqrt{n}} \sum_{i=1}^{k} Z_{0\alpha}\ \sqrt{n}(\xi_\alpha-\xi_\alpha^0)$$

$$+ \frac{1}{2\sqrt{n}} (-3J_{000}-K_{000})\{\sqrt{n}(\hat{\theta}-\theta_0)\}^2$$

$$+ \frac{1}{2\sqrt{n}} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} (-J_{0\alpha\beta})\{n(\hat{\xi}_\alpha-\xi_\alpha^0)(\hat{\xi}_\beta-\xi_\beta^0)\}$$

$$+ \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{k} (-J_{0\alpha 0})\{n(\hat{\theta}-\theta_0)(\hat{\xi}_\alpha-\xi_\alpha^0)\} + o_p\left(\frac{1}{\sqrt{n}}\right) ;$$

$$0 = Z_\alpha + \left(\frac{Z_{\alpha\alpha}}{\sqrt{n}} - I_{\alpha\alpha}\right) \sqrt{n}(\xi_\alpha - \xi_\alpha^0) + \frac{Z_{0\alpha}}{\sqrt{n}} \sqrt{n}(\hat{\theta}-\theta_0)$$

$$+ \frac{1}{2\sqrt{n}} (-3J_{\alpha\alpha\alpha} - K_{\alpha\alpha\alpha})\{\sqrt{n}(\hat{\xi}_\alpha - \xi_\alpha^0)\}^2 + \frac{1}{2\sqrt{n}} (-J_{0\alpha 0})\{\sqrt{n}(\hat{\theta}-\theta_0)\}^2$$

$$+ \frac{1}{\sqrt{n}} (-J_{0\alpha\alpha})\{n(\hat{\theta}-\theta_0)(\hat{\xi}_\alpha - \xi_\alpha^0)\} + o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

it follows that

(2.4)

$$\sqrt{n}(\hat{\theta}-\theta_0) = \frac{Z_0}{I_{00}} + \frac{Z_0 Z_{00}}{\sqrt{n}\, I_{00}^2} + \frac{1}{\sqrt{n}\, I_{00}} \sum_{\alpha=1}^{k} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}}$$

$$- \frac{3J_{000} + K_{000}}{2\sqrt{n}\, I_{00}^3} Z_0^2 - \frac{1}{2\sqrt{n}\, I_{00}} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} J_{0\alpha\beta} \frac{Z_\alpha Z_\beta}{I_{\alpha\alpha} I_{\beta\beta}}$$

$$- \frac{Z_0}{\sqrt{n}\, I_{00}^2} \sum_{\alpha=1}^{k} \frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_\alpha + o_p\left(\frac{1}{\sqrt{n}}\right)$$

$$= \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}\, I_{00}^2}\left(Z_0 Z_{00} - \frac{3J_{000} + K_{000}}{2I_{00}} Z_0^2\right)$$

$$+ \frac{1}{\sqrt{n}\, I_{00}}\left(\sum_{\alpha=1}^{k} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}} - \frac{1}{2} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} J_{0\alpha\beta} \frac{Z_\alpha Z_\beta}{I_{\alpha\alpha} I_{\beta\beta}} - \frac{Z_0}{I_{00}} \sum_{\alpha=1}^{k} \frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_\alpha\right)$$

$$+ o_p\left(\frac{1}{\sqrt{n}}\right) .$$

We put

$$Q_0 = \frac{1}{I_{00}^2}\left(Z_0 Z_{00} - \frac{3J_{000} + K_{000}}{2I_{00}} Z_0^2\right) ;$$

(2.5) $$Q_k = \frac{1}{I_{00}}\left(\sum_{\alpha=1}^{k} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}} - \frac{1}{2} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} J_{0\alpha\beta} \frac{Z_\alpha Z_\beta}{I_{\alpha\alpha} I_{\beta\beta}} - \frac{Z_0}{I_{00}} \sum_{\alpha=1}^{k} \frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_\alpha\right) .$$

Then we have

$$E(Q_0 Q_k) = \frac{1}{I_{00}^3}\left(Z_0 Z_{00} - \frac{3J_{000} + K_{000}}{2I_{00}} Z_0^2\right)$$

$$\cdot\left(\sum_{\alpha=1}^{k} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}} - \frac{1}{2}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} J_{0\alpha\beta}\frac{Z_\alpha Z_\beta}{I_{\alpha\alpha} I_{\beta\beta}} - \frac{Z_0}{I_{00}}\sum_{\alpha=1}^{k}\frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_\alpha\right)$$

$$= \sum_{\alpha=1}^{k}\frac{1}{I_{\alpha\alpha}} E(Z_0 Z_\alpha Z_{00} Z_{0\alpha}) - \frac{1}{2}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\frac{J_{0\alpha\beta}}{I_{\alpha\alpha} I_{\beta\beta}} E(Z_0 Z_\alpha Z_\beta Z_{00})$$

$$- \frac{1}{I_{00}}\sum_{\alpha=1}^{k}\frac{J_{0\alpha 0}}{I_{\alpha\alpha}} E(Z_0^2 Z_\alpha Z_{00}) - \frac{3J_{000} + K_{000}}{2I_{00}}\sum_{\alpha=1}^{k}\frac{1}{I_{\alpha\alpha}} E(Z_0^2 Z_\alpha Z_{0\alpha})$$

$$+ \frac{3J_{000} + K_{000}}{4I_{00}}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\frac{J_{0\alpha\beta}}{I_{\alpha\alpha} I_{\beta\beta}} E(Z_0^2 Z_\alpha Z_\beta)$$

$$+ \frac{3J_{000} + K_{000}}{2I_{00}^2}\sum_{\alpha=1}^{k}\frac{J_{0\alpha 0}}{I_{\alpha\alpha}} E(Z_0^3 Z_\alpha) .$$

Since

$$E(Z_0 Z_\alpha Z_{00} Z_{0\alpha}) = J_{000} J_{0\alpha\alpha} + J_{0\alpha 0} J_{00\alpha} ;$$

$$E(Z_0 Z_\alpha Z_\beta Z_{00}) = \begin{cases} 0 & \text{for } \alpha \neq \beta ; \\ I_{\alpha\alpha} J_{000} & \text{for } \alpha = \beta ; \end{cases}$$

$$E(Z_0^2 Z_\alpha Z_{00}) = I_{00} J_{0\alpha\alpha} ;$$

$$E(Z_0^2 Z_\alpha Z_{0\alpha}) = I_{00} J_{00\alpha} ;$$

$$E(Z_0^2 Z_\alpha Z_\beta) = E(Z_0^2) E(Z_\alpha Z_\beta) = \begin{cases} 0 & \alpha \neq \beta ; \\ I_{00} I_{\alpha\alpha} & \text{for } \alpha = \beta ; \end{cases}$$

$$E(Z_0^3 Z_\alpha) = 0 ,$$

we obtain

$$E(Q_0Q_k) = \frac{1}{I_{00}^3}\left\{\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} (J_{000}J_{0\alpha\alpha} + J_{0\alpha 0}J_{00\alpha})\right.$$

$$- \frac{1}{2}\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} J_{000}I_{\alpha\alpha} - \frac{1}{I_{00}}\sum_{\alpha=1}^{k} \frac{J_{0\alpha 0}}{I_{\alpha\alpha}} I_{00}J_{00\alpha}$$

$$- \frac{3J_{000} + K_{000}}{2I_{00}} \sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} I_{00}J_{0\alpha\alpha}$$

$$\left. + \frac{3J_{000} + K_{000}}{4I_{00}} \sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} I_{00}I_{\alpha\alpha}\right\}$$

$$= - \frac{J_{000} + K_{000}}{4I_{00}^3} \sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}} .$$

Since

$$E(Q_0) = - \frac{J_{000} + K_{000}}{2I_{00}^2} ;$$

$$E(Q_k) = \frac{1}{2I_{00}} \sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}} ,$$

it follows that the covariance of Q_0 and Q_k is given by

$$\mathrm{Cov}(Q_0, Q_k) = E(Q_0Q_k) - E(Q_0)E(Q_k)$$

$$= 0 .$$

By a similar discussion to the one parameter case in the previous papers ([3], [6], [7]) we have the following theorems.

Theorem 2.1. Let $\hat{\theta}_0$ be the modified MLE of θ in the class $\mathfrak{C}$ and any other estimator $\hat{\theta}$ of θ in the class $\mathfrak{C}$, under the true model

$(\theta_0, \xi_1^0, \ldots, \xi_k^0)$. If the assumptions (A.2.1) ~ (A.2.7) hold, then the following holds

$$\underline{\lim_{n\to\infty}}\, n[\Pr\{\sqrt{n}|\hat{\theta}_0-\theta_0| < a\} - \Pr\{\sqrt{n}|\hat{\theta}-\theta_0| < a\}] \geqq 0$$

for all $a > 0$.

Theorem 2.2. Let $\hat{\theta}_{00}$ be the modified MLE in the class $\mathbb{D}$ and $\hat{\theta}$ be any other estimator in the class $\mathbb{D}$, under the true model $(\theta_0, \xi_1^0, \ldots, \xi_k^0)$. If the assumptions (A.2.1) ~ (A.2.7) hold, then the following holds

$$\underline{\lim_{n\to\infty}}\, n[\Pr\{-a < \sqrt{n}(\hat{\theta}_{00}-\theta_0) < b\} - \Pr\{-a < \sqrt{n}(\hat{\theta}-\theta_0) < b\}] \geqq 0$$

for all $a > 0$ and all $b > 0$.

Since $J_{0\alpha\beta} = J_{0\beta\alpha}$, it follows that

$$\begin{aligned} Q_k &= \frac{1}{I_{00}}\left(\sum_{\alpha=1}^{k} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}} - \frac{1}{2}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} J_{0\alpha\beta}\frac{Z_\alpha Z_\beta}{I_{\alpha\alpha}I_{\beta\beta}} - \frac{1}{I_{00}}\sum_{\alpha=1}^{k}\frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_0 Z_\alpha\right) \\ &= \frac{1}{I_{00}}\left(\sum_{\alpha=1}^{k}\frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha + \frac{1}{2}\sum_{\alpha=1}^{k}\frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right), \end{aligned}$$

where

$$W_\alpha = Z_{0\alpha} - \sum_{\beta=0}^{\alpha}\frac{J_{0\alpha\beta}}{I_{\beta\beta}} Z_\beta \quad (\alpha = 1,\ldots,k) .$$

Next we shall calculate the value of $E(Q_k^2)$. We have

$$E(Q_k^2) = \frac{1}{I_{00}^2}\left\{E\left(\sum_{\alpha=1}^{k}\frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha\right)^2 + E\left(\sum_{\alpha=1}^{k}\frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha\right)\left(\sum_{\alpha=1}^{k}\frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right) + \frac{1}{4}E\left(\sum_{\alpha=1}^{k}\frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right)^2\right\}. \tag{2.6}$$

Since for $\alpha \leqq \beta$

$$E(Z_\alpha W_\beta) = 0 , \tag{2.7}$$

it follows that

$$E(Z_\alpha^2 W_\alpha^2) = I_{\alpha\alpha} E(W_\alpha^2) ;$$

$$E(Z_\alpha Z_\beta W_\alpha W_\beta) = 0 \quad (\alpha \neq \beta) .$$

Hence we have

$$E\left(\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha\right)^2 = \sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} E(W_\alpha^2) . \tag{2.8}$$

Since

$$E(Z_\alpha W_\alpha Z_\beta^2) = 0 ,$$

it follows that

$$E\left(\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha\right)\left(\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right) = 0 . \tag{2.9}$$

Since

$$E(Z_\alpha^4) = 3(E(Z_\alpha^2))^2 = 3I_{\alpha\alpha}^2 ;$$

$$E(Z_\alpha^2 Z_\beta^2) = I_{\alpha\alpha} I_{\beta\beta} \quad (\alpha \neq \beta) ,$$

it follows that

$$E\left(\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right) = 2 \sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}^2}{I_{\alpha\alpha}^2} + \left(\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}}\right)^2 . \tag{2.10}$$

From (2.6) and (2.8) ~ (2.10) we have

$$(2.11) \qquad E(Q_k^2) = \frac{1}{I_{00}^2}\left\{\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} E(W_\alpha^2) + \frac{1}{2}\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}^2}{I_{\alpha\alpha}^2} + \frac{1}{4}\left(\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}}\right)^2\right\}.$$

Since

$$E(W_\alpha^2) = E\left(Z_{0\alpha} - \sum_{\beta=0}^{\alpha} \frac{J_{0\alpha\beta}}{I_{\beta\beta}} Z_\beta\right)^2$$

$$= M_{0\alpha\cdot 0\alpha} - \sum_{\beta=0}^{\alpha} \frac{J_{0\alpha\beta}^2}{I_{\beta\beta}},$$

where

$$M_{0\alpha\cdot 0\alpha} = E\left[\left\{\frac{\partial^2}{\partial\theta\partial\xi_\alpha} \log f(X, \theta_0, \xi_1^0, \ldots, \xi_k^0)\right\}^2\right]$$

$(\alpha = 1, \ldots, k)$, it follows that

$$(2.12) \qquad E(Q_k^2) = \frac{1}{I_{00}^2}\left\{\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}}\left(M_{0\alpha\cdot 0\alpha} - \sum_{\beta=0}^{\alpha} \frac{J_{0\alpha\beta}^2}{I_{\beta\beta}}\right) + \frac{1}{2}\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}^2}{I_{\alpha\alpha}^2} + \frac{1}{4}\left(\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}}\right)^2\right\}$$

$$= \frac{1}{I_{00}^2}\left\{\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}}\left(M_{0\alpha\cdot 0\alpha} - \frac{J_{0\alpha 0}^2}{I_{00}} - \frac{1}{2}\sum_{\beta=1}^{k} \frac{J_{0\alpha\beta}^2}{I_{\beta\beta}}\right) + \frac{1}{4}\left(\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}}\right)^2\right\}.$$

Since

$$E(Q_k) = \frac{1}{I_{00}} E\left[\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha + \frac{1}{2}\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right]$$

$$= \frac{1}{2I_{00}} \sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}},$$

it follows from (2.11) and (2.12) that the variance of Q_k is given by

$$(2.13) \qquad V(Q_k) = \frac{1}{I_{00}^2}\left\{\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}} E(W_\alpha^2) + \frac{1}{2}\sum_{\alpha=1}^{k} \frac{J_{0\alpha\alpha}^2}{I_{\alpha\alpha}^2}\right\}$$

$$= \frac{1}{I_{00}^2}\sum_{\alpha=1}^{k}\frac{1}{I_{\alpha\alpha}}\left(M_{0\alpha\cdot 0\alpha} - \frac{J_{0\alpha 0}^2}{I_{00}} - \frac{1}{2}\sum_{\beta=1}^{k}\frac{J_{0\alpha\beta}}{I_{\beta\beta}}\right) .$$

Now we assume that we do not know the "true" model, i.e., we assume that $\xi_k^0 \neq 0$. For each $i = 1, \ldots, k$, let $\hat{\theta}^*, \hat{\xi}_1^*, \ldots, \hat{\xi}_i^*$ be the MLEs of $\theta_0, \xi_1^0, \ldots, \xi_i^0$ under the assumed model $(\theta_0, \xi_1^0, \ldots, \xi_i^0, 0, \ldots, 0)$. Since

$$\sum_{j=1}^{n} \frac{\partial}{\partial\theta} \log f(X_j, \hat{\theta}^*, \hat{\xi}_1^*, \ldots, \hat{\xi}_i^*, 0, \ldots, 0) = 0 ;$$

$$\sum_{j=1}^{n} \frac{\partial}{\partial\xi_\alpha} \log f(X_i, \hat{\theta}^*, \hat{\xi}_1^*, \ldots, \hat{\xi}_i^*, 0, \ldots, 0) = 0 \quad (\alpha = 1, \ldots, k) ,$$

expanding them in the neighborhood of $(\theta_0, \xi_1^0, \ldots, \xi_i^0, \xi_{i+1}^0, \ldots, \xi_k^0)$ we have

$$(2.14) \qquad 0 = \sum_{j=1}^{n} \frac{\partial}{\partial\theta} \log f(X_j, \hat{\theta}^*, \hat{\xi}_1^*, \ldots, \hat{\xi}_i^*, 0, \ldots, 0)$$

$$= Z_0 + \frac{1}{\sqrt{n}}(Z_{00} - \sqrt{n}\, I_{00})\sqrt{n}(\hat{\theta}^* - \theta_0) + \frac{1}{\sqrt{n}}\sum_{\alpha=1}^{i} Z_{0\alpha}\sqrt{n}(\hat{\xi}_\alpha^* - \xi_\alpha^0)$$

$$- \frac{1}{\sqrt{n}}\sum_{\alpha=i+1}^{k} Z_{0\alpha}(\sqrt{n}\,\xi_\alpha^0) + \frac{1}{2\sqrt{n}}(-3J_{000} - K_{000})\{\sqrt{n}(\hat{\theta}^* - \theta_0)\}^2$$

$$+ \frac{1}{2\sqrt{n}}\sum_{\alpha=1}^{i}\sum_{\beta=1}^{i}(-J_{0\alpha\beta})\{n(\hat{\xi}_\alpha^* - \xi_\alpha^0)(\hat{\xi}_\beta^* - \xi_\beta^0)\}$$

$$+ \frac{1}{2\sqrt{n}}\sum_{\alpha=i+1}^{k}\sum_{\beta=i+1}^{k}(-J_{0\alpha\beta})(n\,\xi_\alpha^0\,\xi_\beta^0) + \frac{1}{\sqrt{n}}\sum_{\alpha=1}^{i}(-J_{0\alpha 0})\{n(\hat{\theta}^* - \theta_0)(\hat{\xi}_\alpha^* - \xi_\alpha^0)\}$$

$$- \frac{1}{\sqrt{n}}\sum_{\alpha=i+1}^{k}(-J_{0\alpha 0})\{n(\hat{\theta}^* - \theta_0)\xi_\alpha^0\} + o_p\left(\frac{1}{\sqrt{n}}\right) .$$

In a similar way as in case (2.3) we obtain

$$(2.15) \qquad \sqrt{n}(\hat{\xi}^*_\alpha - \xi^0_\alpha) = \frac{Z_\alpha}{I_{\alpha\alpha}} + o_p \frac{1}{\sqrt{n}} \quad (\alpha = 1,\ldots,i) .$$

If $\xi^0_\alpha = t_\alpha/\sqrt{n}$ $(\alpha = i+1,\ldots,k)$, then we have from (2.14) and (2.15)

(2.16)

$$\sqrt{n}(\hat{\theta}^* - \theta_0) = \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}\, I^2_{00}} \left(Z_0 Z_{00} - \frac{3J_{000} + K_{000}}{2\, I_{00}} Z_0^2 \right)$$

$$+ \frac{1}{\sqrt{n}\, I_{00}} \left(\sum_{\alpha=1}^{i} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}} - \frac{1}{2} \sum_{\alpha=1}^{i} \sum_{\beta=1}^{i} J_{0\alpha\beta} \frac{Z_\alpha Z_\beta}{I_{\alpha\alpha} I_{\beta\beta}} - \frac{Z_0}{I_{00}} \sum_{\alpha=1}^{i} \frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_\alpha \right)$$

$$+ \frac{1}{\sqrt{n}\, I_{00}} \sum_{\alpha=i+1}^{k} t_\alpha \left(\frac{J_{0\alpha 0}}{I_{00}} Z_0 - Z_{0\alpha} \right)$$

$$- \frac{1}{2\sqrt{n}\, I_{00}} \sum_{\alpha=i+1}^{k} \sum_{\beta=i+1}^{k} J_{0\alpha\beta}\, t_\alpha t_\beta + o_p\left(\frac{1}{\sqrt{n}}\right)$$

$$= \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}} Q_0 + \frac{1}{\sqrt{n}} Q_i + \frac{1}{\sqrt{n}} L_i - \frac{c}{\sqrt{n}} + o_p\left(\frac{1}{\sqrt{n}}\right)$$

$$= \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}} Q_0 + \frac{1}{\sqrt{n}} (Q_i + L_i - c) + o_p\left(\frac{1}{\sqrt{n}}\right) ,$$

where

$$Q_0 = \frac{1}{I^2_{00}} \left(Z_0 Z_{00} - \frac{3J_{000} + K_{000}}{2I_{00}} Z_0^2 \right) ;$$

$$(2.17) \qquad L_i = \frac{1}{I_{00}} \sum_{\alpha=i+1}^{k} t_\alpha \left(\frac{J_{0\alpha 0}}{I_{00}} Z_0 - Z_{0\alpha} \right) \quad (i = 1,\ldots,k-1), \quad L_k = 0 ;$$

$$c = \frac{1}{2I_{00}} \sum_{\alpha=i+1}^{k} \sum_{\beta=i+1}^{k} J_{0\alpha\beta}\, t_\alpha t_\beta .$$

Since $E(W_\alpha) = 0$ $(\alpha = 1,\ldots,k)$ and

$$(2.18)\qquad Q_i = \frac{1}{I_{00}}\left(\sum_{\alpha=1}^{i} \frac{Z_\alpha Z_{0\alpha}}{I_{\alpha\alpha}} - \frac{1}{2}\sum_{\alpha=1}^{i}\sum_{\beta=1}^{i} J_{0\alpha\beta}\frac{Z_\alpha Z_\beta}{I_{\alpha\alpha}I_{\beta\beta}} - \frac{Z_0}{I_{00}}\sum_{\alpha=1}^{i}\frac{J_{0\alpha 0}}{I_{\alpha\alpha}} Z_\alpha\right)$$

$$= \frac{1}{I_{00}}\left(\sum_{\alpha=1}^{i}\frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha + \frac{1}{2}\sum_{\alpha=1}^{i}\frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right),$$

it follows that

$$(2.19)\qquad E(L_i Q_i) = 0\ ,$$

where $a = \frac{1}{I_{00}}\sum_{\alpha=i+1}^{k} t_\alpha J_{0\alpha 0}$.

Hence it is seen that Q_i and L_i are asymptotically independent. We also have

$$(2.20)\qquad E(L_i^2) = \frac{1}{I_{00}^2} E\left[\left\{\sum_{\alpha=i+1}^{k} t_\alpha\left(\frac{J_{0\alpha 0}}{I_{00}} Z_0 - Z_{0\alpha}\right)\right\}^2\right]$$

$$= \frac{1}{I_{00}^2} E\left[\left(aZ_0 - \sum_{\alpha=i+1}^{k} t_\alpha Z_{0\alpha}\right)^2\right]$$

$$(2.21)\qquad E(L_i Q_i) = -\frac{1}{I_{00}^2}\left\{\frac{1}{I_{00}}\left(\sum_{\alpha=i+1}^{k} t_\alpha J_{0\alpha 0}\right)^2 - \sum_{\alpha=i+1}^{k}\sum_{\beta=i+1}^{k} t_\alpha t_\beta M_{0\alpha\cdot 0\beta}\right\},$$

where

$$M_{0\alpha\cdot 0\beta} = E\left[\left\{\frac{\partial^2}{\partial\theta\partial\xi_\alpha}\log f(X,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0)\right\}\left\{\frac{\partial^2}{\partial\theta\partial\xi_\beta}\log f(X,\ \theta_0,\ \xi_1^0,\ldots,\xi_k^0)\right\}\right]$$

$$(\alpha,\beta = 1,\ldots,k)\ .$$

From (2.13) we obtain

$$(2.22)\qquad V(Q_i) = \frac{1}{I_{00}^2}\sum_{\alpha=1}^{i}\frac{1}{I_{\alpha\alpha}}\left(M_{0\alpha\cdot 0\alpha} - \frac{J_{0\alpha 0}^2}{I_{00}} - \frac{1}{2}\sum_{\beta=1}^{k}\frac{J_{0\alpha\beta}}{I_{\beta\beta}}\right).$$

From (2.20), (2.21) and (2.22) we have

$$(2.23) \qquad V(\theta_i+L_i+c) = V(Q_i) + E(L_i^2) + c^2$$

$$= \frac{1}{I_{00}^2}\left\{\sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}}\left(M_{0\alpha\cdot0\alpha} - \frac{J_{0\alpha0}^2}{I_{00}} - \frac{1}{2}\sum_{\beta=1}^{k} \frac{J_{0\alpha\beta}^2}{I_{\beta\beta}}\right)\right.$$

$$- \frac{1}{I_{00}}\left(\sum_{\alpha=i+1}^{k} t_\alpha J_{0\alpha0}\right)^2 + \sum_{\alpha=i+1}^{k}\sum_{\beta=i+1}^{k} t_\alpha t_\beta M_{0\alpha\cdot0\beta}$$

$$\left. + \frac{1}{4}\left(\sum_{\alpha=i+1}^{k}\sum_{\beta=i+1}^{k} J_{0\alpha\beta}t_\alpha t_\beta\right)^2\right\} .$$

Since

$$E(Z_0LQ_0) = \frac{1}{I_{00}^3} E\left[Z_0\left\{\sum_{\alpha=i+1}^{k} t_\alpha\left(\frac{J_{0\alpha0}}{I_{00}} Z_0 - Z_{0\alpha}\right)\right\}\left(Z_0Z_{00} - \frac{3J_{000} + K_{000}}{2I_{00}} Z_0^2\right)\right]$$

$$= \frac{1}{I_{00}^3}\left\{3J_{000}\sum_{\alpha=1}^{k} t_\alpha J_{0\alpha0} - \sum_{\alpha=i+1}^{k} t_\alpha(I_{00}M_{00\cdot0\alpha} + 2J_{000}J_{0\alpha0})\right\} ,$$

where $a = \sum_{\alpha=i+1}^{k} t_\alpha J_{0\alpha0}/I_{00}$ and $d = (3J_{000}+K_{000})/(2I_{00})$, it follows that $E(Z_0LQ_0)$ is not generally zero. Hence it is seen that the MLEs $\hat{\theta}^*$, $\hat{\xi}_1^*, \ldots, \hat{\xi}_i^*$ belong to the class $\mathbb{C}$ but not the class $\mathbb{D}$.

In (2.17) the term c represents the asymptotic bias due to the "incorrectness" of the assumed model, and since we can not assume that $t_\alpha (\alpha = i+1,\ldots,k)$ are known, there is no way of adjusting the bias. But in many situations we may assume the following:

$$(A.2.8) \quad J_{0\alpha\beta} = 0 \quad (\alpha,\beta = 1,\ldots,k)$$

which simplifies the matter.

The above condition holds true if, for example, θ is the location parameter and the density function is symmetric about the origin while $\xi_1, \ldots, \xi_k$ are all "shape" parameters (including the scale) properly defined.

From (2.4), (2.5), (2.16), and (A.2.8) we have

$$\sqrt{n}(\hat{\theta}-\theta_0) = \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}} Q_0 + \frac{1}{\sqrt{n}} Q_k + o_p\left(\frac{1}{\sqrt{n}}\right) ; \tag{2.24}$$

$$\sqrt{n}(\hat{\theta}^*-\theta_0) = \frac{Z_0}{I_{00}} + \frac{1}{\sqrt{n}} Q_0 + \frac{1}{\sqrt{n}} (Q_i+L_i) + o_p\left(\frac{1}{\sqrt{n}}\right) , \tag{2.25}$$

where

$$Q_i = \frac{1}{I_{00}}\left(\sum_{\alpha=1}^{i} \frac{1}{I_{\alpha\alpha}} Z_\alpha W_\alpha + \frac{1}{2} \sum_{\alpha=1}^{i} \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}^2} Z_\alpha^2\right)$$

with

$$W_\alpha = Z_{0\alpha} - \frac{J_{0\alpha 0}}{I_{00}} Z_0 - \frac{J_{0\alpha\alpha}}{I_{\alpha\alpha}} Z_\alpha \quad (\alpha = 1,\ldots,k)$$

and

$$L_i = \frac{1}{I_{00}} \sum_{\alpha=i+1}^{k} t_\alpha\left(\frac{J_{0\alpha 0}}{I_{00}} Z_0 - Z_{0\alpha}\right) \quad (i = 1,\ldots,k-1), \quad L_k = 0 .$$

Also from (2.13), (2.23), (2.24), and (2.25) we have

$$V(Q_k) = \frac{1}{I_{00}^2} \sum_{\alpha=1}^{k} \frac{1}{I_{\alpha\alpha}}\left(M_{0\alpha\cdot 0\alpha} - \frac{J_{0\alpha 0}^2}{I_{00}} - \frac{J_{0\alpha\alpha}^2}{2I_{\alpha\alpha}}\right) ; \tag{2.26}$$

and

(2.27)

$$
\begin{aligned}
V(Q_i+L_i) &= V(Q_i) + E(L_i^2) \\
&= E\left[\left\{\sum_{\alpha=i+1}^{k} t_\alpha\left(Z_{0\alpha} - \frac{J_{0\alpha 0}}{I_{00}} Z_0\right)\right\}^2\right] \\
&= V(Q_k) + \frac{1}{I_{00}^2} \sum_{\alpha=i+1}^{k} \sum_{\beta=i+1}^{k} t_\alpha t_\beta M_{0\alpha\cdot 0\beta} - \frac{1}{I_{00}^3}\left(\sum_{\alpha=i+1}^{k} t_\alpha J_{0\alpha 0}\right)^2 \\
&= V(Q_k) + \frac{1}{I_{00}} (d_{1i}+d_{2i}) ,
\end{aligned}
$$

where

(2.28) $$d_{1i} = \frac{1}{I_{00}} \sum_{\alpha=i+1}^{k} \sum_{\beta=i+1}^{k} t_\alpha t_\beta M_{0\alpha\cdot 0\beta} \quad (i = 1,\dots,k-1), \quad d_{1k} = 0 ;$$

(2.29) $$d_{2i} = - \frac{1}{I_{00}^2}\left(\sum_{\alpha=i+1}^{k} t_\alpha J_{0\alpha 0}\right)^2 \quad (i = 1,\dots,k-1), \quad d_{2k} = 0 .$$

It is to be remarked that when we consider only the symmetric intervals and calculate the asymptotic value of probability of

$$\Pr\{\sqrt{n}|\hat{\theta}-\theta| < a\}$$

the term in the third cummulant does not affect the value of the probability up to the order n^{-1}.

Hence it follows from (2.26) and (2.27) that in the asymptotic expansion of the probability for symmetric intervals differences are produced only by the term $I_{00}\{V(Q_i+L_i) - V(Q_k)\} = d_{1i}+d_{2i}$ which corresponds to asymptotic deficiency defined by Hodges and Lehamnn [5] (also see [1], [2], and [4]). Further, it is seen from (2.28) and (2.29) that d_{2i} is an increasing function of i and d_{1i} is a decreasing function of i. From the above it is calculated whether it is a plus or a minus to increase the number of parameters in the model. Hence we have the following:

Theorem 2.3. Under the assumptions (A.2.1) ~ (A.2.8), if $\xi_\alpha^0 = t_\alpha/\sqrt{n}$ $(\alpha = i+1,\ldots,k)$, then the asymptotic deficiency of the MLE $\hat{\theta}^*$ under the assumed model $(\theta_0, \xi_1^0,\ldots,\xi_i^0, 0,\ldots,0)$ relative to the MLE $\hat{\theta}$ under the true model $(\theta_0, \xi_1^0,\ldots,\xi_k^0)$ is given by $d_{1i}+d_{2i}$.

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