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Sequential Interval Estimation of a Location Parameter with the Fixed Width in the Non-regular Case

Dedicated to Professor Masafumi Akahira on his 60th birthday

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Abstract: For a location-scale parameter family of distributions with a finite support, a sequential confidence interval with a fixed width is obtained for the location parameter, and its asymptotic consistency and efficiency are shown. Some comparisons with the Chow-Robbins procedure are also done.

Keywords: Coverage probability; Extreme value; Non-regular case; Sequential interval estimation.

Subject Classifications: 62L12; 62F25.

1. INTRODUCTION

Suppose that we are to estimate a location parameter $\theta$ of a sequence of random observations $X_1, X_2, \ldots, X_n, \ldots$ with unknown scale $\xi$. We would like to obtain sequentially a confidence interval of fixed width $2d$ with confidence coefficient $1 - \alpha$. Obviously we can not obtain a fixed sample size procedure if $\xi$ is unknown. There are many works on the fixed-width interval estimation of normal mean (see, e.g. Ghosh et al. (1997)).
Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to the uniform distribution on the interval $(\theta - (\xi/2), \theta + (\xi/2))$, where $\theta(\in \mathbb{R})$ and $\xi$ ($> 0$) are unknown. Let $X_{(1)} := \min_{1 \leq i \leq n} X_i$, $X_{(n)} := \max_{1 \leq i \leq n} X_i$. Then the midrange and the range are $M_n := (X_{(1)} + X_{(n)})/2$, $R_n := X_{(n)} - X_{(1)}$, respectively. Akahira and Koike (2005) considered a stopping rule:

$$
\tau_1 := \inf \left\{ n \geq n_0 \left| \frac{R_n}{n - 1} \leq -\frac{2d}{\log \alpha} \right. \right\},
$$

where $n_0(\geq 2)$ is an initial size of sample. They showed the asymptotic consistency and efficiency of the estimation procedure $(\tau_1, [M_{\tau_1} - d, M_{\tau_1} + d])$.

In this paper, we consider the case of a location-scale parameter family of distributions with a finite support on the interval $(\theta - \xi a, \theta + \xi a)$, where $\theta$ and $\xi$ are unknown, and obtain a sequential confidence interval of $\theta$ with fixed width $2d$ and confidence coefficient $1 - \alpha$, and show its asymptotic consistency and efficiency. Some comparisons with the Chow-Robbins procedure are also done.

2. ASYMPTOTIC DISTRIBUTIONS OF THE EXTREME VALUES

In this section we consider the asymptotic distributions of the extreme values for distributions with a finite support, in a similar way to Akahira (1991) and Akahira and Takeuchi (1995).

Let $Z_1, Z_2, \ldots, Z_{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables according to the density function $f_0(x - \theta)$ ($\theta \in \mathbb{R}$) with respect to the Lebesgue measure. We assume the following conditions:

(A1) $f_0(x)$ has a finite support $(-a, a)$ ($a > 0$), i.e., $f_0(x) > 0$ for $-a < x < a$, and $f_0(x) = 0$ otherwise.

(A2) $f_0(x)$ is continuously differentiable in the open interval $(-a, a)$ and

$$
\lim_{x \to -a+0} f_0(x) = c, \lim_{x \to a-0} f_0(x) = c',
$$

1If the support of $f_0$ is $(-a, b)$ ($a \neq b$), then the normalized midrange does not converge to $\theta$ in probability as $n \to \infty$. 

2
where $c$ and $c'$ are some positive constants.

(A3) $f_0(x)$ satisfies

$$f_0(x) \approx g(x + a)^\gamma \quad (x \to -a + 0),$$

$$f_0(x) \approx g'|x - a|^{\gamma} \quad (x \to a - 0),$$

where $\gamma, g$ and $g'$ are some positive constants.

Putting $Z^{(1)} := \min_{1 \leq i \leq n} Z_i, Z^{(n)} := \max_{1 \leq i \leq n} Z_i, U := n(Z^{(1)} + a - \theta)$ and $V := n(Z^{(n)} - a - \theta)$, we have the following lemma (cf. Akahira (1991), Akahira and Takeuchi (1995)).

**Lemma 1.** Under the conditions (A1) and (A2), the joint (j.) p.d.f. $f_{U,V}^{(n)}(u, v)$ of $(U, V)$ satisfies

$$f_{U,V}^{(n)}(u, v) \to \begin{cases} c d' \exp\{c'v - cu\} & (v < 0 < u), \\ 0 & \text{(otherwise)}. \end{cases}$$

(2.1) as $n \to \infty$.

**Proof.** The j.p.d.f. $f_{U,V}^{(n)}(u, v)$ of $(U, V)$ is

$$f_{U,V}^{(n)}(u, v) = \begin{cases} \frac{n - 1}{n} \left\{ F\left(a + \frac{v}{n}\right) - F\left(-a + \frac{u}{n}\right) \right\}^{n-2} f_0\left(-a + \frac{u}{n}\right) f_0\left(a + \frac{v}{n}\right) & (v < 0 < u), \\ 0 & \text{(otherwise)}, \end{cases}$$

where $F(x) = \int_{-\infty}^x f_0(u)du$. Hence, by its expansion, we have the desired result. \qed

Next, we consider the location-scale parameter family of distributions with a finite support $(\theta - \xi a, \theta + \xi a)$. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with the p.d.f. $(1/\xi) f_0((x - \theta)/\xi)$, where $\theta \in \mathbb{R}$ and $\xi > 0$. Put $Y_i := (X_i - \theta)/\xi$ for each $i = 1, 2, \ldots$, and $Y^{(1)} := \min_{1 \leq i \leq n} Y_i, Y^{(n)} := \max_{1 \leq i \leq n} Y_i$. Letting $S := n(Y^{(1)} + Y^{(n)})/2$ and $T = n(Y^{(1)} - Y^{(n)} + 2a)/2$, we have the asymptotic (as.) j.p.d.f. of $(S, T)$

$$f_{S,T}(s, t) = \begin{cases} 2cc' \exp\{-c'\ln s - (c + c')t\} & (t > |s|), \\ 0 & \text{(otherwise)}. \end{cases}$$

\[\text{If the converging order } \gamma \text{ is different, then the normalized midrange does not converge to } \theta \text{ in probability as } n \to \infty.\]
Then the as. marginal (m.) p.d.f.’s of $S$ and $T$ are given by

$$f_S(s) = \begin{cases} Ke^{-2cs} & (s \geq 0), \\ Ke^{2cs} & (s < 0), \end{cases}$$

**(2.2)**

$$f_T(t) = \begin{cases} \frac{2cc'}{c' - c} \left( e^{-2ct} - e^{-2c't} \right) & (t > 0 \text{ and } c \neq c'), \\ 4c^2te^{-2ct} & (t > 0 \text{ and } c = c'), \\ 0 & (\text{otherwise}), \end{cases}$$

respectively, where $K = 2cc/(c + c')$.

In the case when $\lim_{x \to -a+0} f_0(x) = \lim_{x \to a-0} f_0(x) = 0$, we need another lemma. Putting $U' := n^{1/(\gamma+1)}(Z_{(1)} + a - \theta)$ and $V' := n^{1/(\gamma+1)}(Z_{(n)} - a - \theta)$, we have the following lemma in a similar way to Lemma 1.

**Lemma 2.** Under the conditions (A1) and (A3), the j.p.d.f.

$$f_{U', V'}^{(n)}(u, v)$$

of $(U', V')$ satisfies

$$f_{U', V'}^{(n)}(u, v) \to \begin{cases} gg'(-uv)\gamma \exp\{-\frac{g'}{\gamma+1}(-v)^{\gamma+1} - \frac{g}{\gamma+1}u^{\gamma+1}\} & (v < 0 < u), \\ 0 & (\text{otherwise}). \end{cases}$$

as $n \to \infty$.

The proof is omitted since it is similar to the one of Lemma 1.

From Lemma 2, $U'$ and $(-V')$ are asymptotically, independently distributed according to Weibull distributions.

**3. CONSTRUCTING CONFIDENCE INTERVAL**

In this section we construct a sequential confidence interval for $\theta$. In the first place, we consider the case under the conditions (A1) and (A2). For $0 < \alpha < 1$, let $l_0$ be the solution of $l$ for the equation

$$\frac{c + c'}{ce'}\alpha = \frac{e^{-2cl}}{c} + \frac{e^{-2c'l}}{c'}.$$

\footnote{It can be shown easily that such $l_0$ exists uniquely.}
If \( \xi \) is known, we have from (2.2) that
\[
P\left\{ |M_n - \theta| \leq d \right\} = P\left\{ \frac{n|M_n - \theta|}{\xi} \leq \frac{dn}{\xi} \right\}
\approx \int_{-dn/\xi}^{dn/\xi} f_S(s)ds
\]
\[
= 1 - \frac{cc'}{c + c'} \left( \frac{e^{-2cnd/\xi}}{c} + \frac{e^{-2c'nd/\xi}}{c'} \right),
\]
where \( \approx \) means that the distribution of \( \frac{n|M_n - \theta|}{\xi} \) is approximated by the asymptotic distribution. Letting \( n^* = l_0 \xi/d \), we have for \( n \geq n^* \)
\[
1 - \frac{cc'}{c + c'} \left( \frac{e^{-2cnd/\xi}}{c} + \frac{e^{-2c'nd/\xi}}{c'} \right) \geq 1 - \alpha.
\]
\( n^* \) is referred as the asymptotically optimal size of samples if \( \xi \) is known.

Note that \( n(M_n - \theta)/\xi = S \) and \( R_n/\xi = -(T/n) + 2a \). Now we take as the stopping rule
\[
\tau_2 := \inf \left\{ n \geq n_0 \left| \frac{R_n}{n-1} \leq \frac{2ad}{l_0} \right. \right\}, \tag{3.1}
\]
where \( n_0(\geq 2) \) is an initial size of sample. Then we obtain the asymptotic properties of the estimation procedure \( (\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d]) \) as follows.

**Theorem 1.** For the sequential estimation procedure \( (\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d]) \), the following hold.

(i) \( \lim_{d \to 0^+} P\left\{ |M_{\tau_2} - \theta| \leq d \right\} = 1 - \alpha \) (asymptotic consistency).

(ii) \( \tau_2/n^* \overset{a.s.}{\to} 1 \) (\( d \to 0^+ \)).

(iii) \( E(\tau_2)/n^* \to 1 \) (\( d \to 0^+ \)) (asymptotic efficiency).

**Proof.** (i) From Lemma 1 of Chow and Robbins (1965), the stopping rule \( \tau_2 \) given by (3.1) satisfies
\[
\lim_{d \to 0^+} \frac{d\tau_2}{\xi l_0} = 1 \quad \text{a.s.} \tag{3.2}
\]
Since \( S = n(M_n - \theta)/\xi \) converges in distribution to a distribution with the density given by (2.2) as \( n \to \infty \), it follows from Theorem 1 of Anscombe (1952) that \( \tau_2(M_{\tau_2} - \theta) \) converges in distribution to the same distribution as \( d \to 0^+ \). Hence, since \( d\tau_2/\xi \overset{a.s.}{\to} l_0 \) as \( d \to 0^+ \) from (3.2), it follows that
\[
\lim_{d \to 0^+} P\left\{ |M_{\tau_2} - \theta| \leq d \right\} = \lim_{d \to 0^+} P\left\{ \tau_2|M_{\tau_2} - \theta|/\xi \leq d\tau_2/\xi \right\}
\]
\[
= \int_{-l_0}^{l_0} f_S(s)ds = 1 - \alpha. \tag{3.3}
\]
(ii) From (3.2) and the definition of \( l_0 \), we have \( \tau_2/n^* = \tau_2 d/(l_0 \xi)^{\alpha/2} \) as \( d \to 0+ \).

(iii) From Lemma 2 of Chow and Robbins (1965), we have the desired result. \( \square \)

**Remark.** In particular, if \( c = c' \), then \( l_0 = -\log \alpha/(2c) \) and \( \tau_2 \) given in (3.1) is expressed as 

\[
\tau_2 = \inf \left\{ n \geq n_0 \mid \frac{R_n}{n-1} \leq -\frac{4acd}{\log \alpha} \right\},
\]

which is equal to \( \tau_1 \) when the underlying distribution is uniform distribution on the interval \((\theta - (\xi/2), \theta + (\xi/2))\).

In the second place, we compare this with the Chow-Robbins procedure. Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables with the mean \( \theta \) and the variance \( \sigma^2 \). Let \( \bar{X}_n := \sum_{i=1}^n X_i/n \), \( s_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1) \). Chow and Robbins (1965) considered a stopping rule defined by

\[
\tau_{CR} := \inf \left\{ n \geq n_0 \mid n \geq u_{\alpha/2} \sigma^2/d^2 \right\},
\]

where \( u_{\alpha/2} \) is the upper \( \alpha/2 \) point of \( N(0,1) \) and \( n_0(\geq 2) \) is an initial size of samples. They showed the asymptotic consistency and efficiency of the estimation procedure \((\tau_{CR}, [\bar{X}_{\tau_{CR}} - d, \bar{X}_{\tau_{CR}} + d])\).

Since, from Theorem 2.2 of Akahira and Koike (2005), Theorem 1 and Theorem of Chow and Robbins (1965),

\[
\tau_1 \approx \frac{\log \alpha}{\log (1 - (2d/\xi))}, \quad \tau_2 \approx \frac{-\xi \log \alpha}{2d}, \quad \tau_{CR} \approx \frac{u_{\alpha/2} \sigma^2}{d^2},
\]

as \( d \to 0+ \), we have \( \tau_1/\tau_{CR}, \tau_2/\tau_{CR} \to 0 \) (\( d \to 0+ \)). Therefore \( \tau_1, \tau_2 \) is asymptotically better than \( \tau_{CR} \) in the sense of the average size of sample.

Furthermore, we consider the case under the conditions (A1) and (A3). By putting \( S' := n^{1/(\gamma + 1)}(Y_{(1)} + Y_{(n)})/2 \) and \( T' := n^{1/(\gamma + 1)}(Y_{(1)} - Y_{(n)} + 2a)/2 \), the as.j.p.d.f. of \((S', T')\) and the as.m.p.d.f.’s of \( S' \) and \( T' \) are obtained from Lemma 2. In a similar way to (3.3), we take \( l_0 \) satisfying \( \int_{-l_0}^{l_0} f_{S'}(s) ds = 1 - \alpha \) for the as.m.p.d.f. \( f_{S'}(s) \) of \( S' \).
If \( \xi \) is known, we have
\[
P\{|M_n - \theta| \leq d\} = P\{n^{1/(\gamma+1)}|M_n - \theta|/\xi \leq dn^{1/(\gamma+1)}/\xi\} \\
\approx \int_{-dn^{1/(\gamma+1)}/\xi}^{dn^{1/(\gamma+1)}/\xi} f_S(s)ds,
\]
where \( \approx \) means that the distribution of \( n^{1/(\gamma+1)}|M_n - \theta|/\xi \) is approximated by the asymptotic distribution. The optimal size of sample required for attaining the preassigned coverage probability \( 1 - \alpha \) is the smallest positive integer \( \geq (l_0\xi/d)^{\gamma+1} =: n^{**} \) (say). Define a stopping rule as
\[
\tau_3 := \inf \left\{ n \geq n_0 \mid \frac{R_n}{n^{1/(\gamma+1)}} \leq \frac{2ad}{l_0} \right\},
\]
where \( n_0(\geq 2) \) is an initial size of samples. Then the next theorem follows.

**Theorem 2.** For the sequential estimation procedure \( (\tau_3, [M_{\tau_3} - d, M_{\tau_3} + d]) \), the following hold.

(i) \( \lim_{d \to 0+} P\{|M_{\tau_3} - \theta| \leq d\} = 1 - \alpha \) (asymptotic consistency).

(ii) \( \tau_3/n^{**} \overset{a.s.}{\to} 1 \) \( (d \to 0+) \) (asymptotic efficiency).

Proof. The proof for (i) is similar to the one of Theorem 1 (i). (ii) follows from \( (\tau_3/n^{**})^{1/(\gamma+1)} \overset{a.s.}{\to} 1 \) as \( d \to 0+ \).

(iii) From (ii), by Fatou’s lemma,
\[
\liminf_{d \to 0+} \frac{E(\tau_3)}{n^{**}} \geq E \left( \liminf_{d \to 0+} \frac{\tau_3}{n^{**}} \right) = 1. \tag{3.4}
\]

On the other hand, since \( 0 \leq R_n \leq 2a\xi \) with probability 1 for any \( n \in \mathbb{N} \), we have \( 0 \leq (R_n l_0/(2ad))^{\gamma+1} \leq (2a\xi l_0/(2ad))^{\gamma+1} = (l_0\xi/d)^{\gamma+1} \) with probability 1 for any \( n \in \mathbb{N} \). So, \( 0 \leq (R_n l_0/(2ad))^{\gamma+1} \leq n \) with probability 1 for \( n \) satisfying \( n \geq (l_0\xi/d)^{\gamma+1} + 1 \). Therefore, since \( \tau_3 = \inf \{ n \geq n_0 \mid (R_n l_0/(2ad))^{\gamma+1} \leq n \} \), we have \( \tau_3 \leq \left( \frac{l_0\xi}{d} \right)^{\gamma+1} + 1 \). Then, using the definition of \( n^{**} \), we have
\[
\frac{E(\tau_3)}{n^{**}} \leq \left( \left( \frac{l_0\xi}{d} \right)^{\gamma+1} + 1 \right) \left( \frac{l_0\xi}{d} \right)^{-(\gamma+1)} = 1 + \left( \frac{d}{l_0\xi} \right)^{\gamma+1},
\]
hence
\[
\lim_{d \to 0^+} \sup \frac{E(\tau_3)}{n^{**}} \leq 1. \tag{3.5}
\]
Combining (3.4) and (3.5), we obtain (iii).

From Theorem 2 and Theorem of Chow and Robbins (1965), \(\tau_3 \approx (l_0 \xi / d)^{\gamma+1}\) and \(\tau_{CR} \approx u^2_\alpha/2\sigma^2/d^2\) as \(d \to 0^+\). Therefore,
\[
\frac{\tau_3}{\tau_{CR}} \begin{cases} 
 0(1) & (0 < \gamma < 1), \\
 0(1) & (\gamma = 1), \\
 \to \infty & (\gamma > 1)
\end{cases}
\]
as \(d \to 0^+\). Therefore, \(\tau_3\) is asymptotically better than \(\tau_{CR}\) in the sense of the average size of sample if \(0 < \gamma < 1\).

In this paper, we considered the cases when the values at the endpoints of the support of the p.d.f. are positive simultaneously, or tend to 0 at the same speed. In the meantime, if the either value at the endpoints of the support of the p.d.f. is positive, or tend to 0 at a different speed, then the coefficients of \(n^\gamma(X_{(1)} - a - \theta)\) and \(n^\delta(X_{(n)} - b - \theta)\) converging to nontrivial random variables are different and estimation by using the midrange \(M_n\) is inappropriate.

4. NUMERICAL EXAMPLE

In this section we examine the coverage probability of the procedure \([M_{\tau_2} - d, M_{\tau_2} + d]\) by simulation based on 100000 repetitions. Suppose that \(X_1, X_2, \ldots, X_n, \ldots\) is a sequence of i.i.d. random variables with the p.d.f. \((1/\xi)f_0((x - \theta)/\xi)\), where \(\theta \in \mathbb{R}, \xi > 0\) and \(f_0(\cdot)\) is a trapezoid-shape p.d.f. given by
\[
f_0(x) = \begin{cases} 
 (1/2 - c)x + 1/2 & (x \in (-1, 1)), \\
 0 & (\text{otherwise})
\end{cases}
\]
with \(0 < c < 1\). Note that, \(f_0\) is the p.d.f. of the uniform distribution over \((-1, 1)\) and an asymmetric p.d.f. over \((-1, 1)\) for \(c = 0.5\) and a sufficiently small \(c > 0\), respectively. Since \(M_{\tau_2}\) is location equivariant, we may assume \(\theta = 0\) without loss of generality.

When \(\alpha = 0.10, d = 0.01(0.01)0.05, \xi = 1(1)5\) and \(n_0 = 5\), Tables 1 and 2 show the values of coverage probabilities of the sequential estimation
procedure \((\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])\) for \(c = 0.1\) and \(c = 0.5\), respectively. The result suggests that the estimation procedure is consistent for this case.

**Table 1.** Coverage probabilities of \([M_{\tau_2} - d, M_{\tau_2} + d]\) for \(c = 0.1\)

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<th>(\xi \setminus d)</th>
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<th>0.03</th>
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</tr>
<tr>
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<tr>
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<tr>
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**Table 2.** Coverage probabilities of \([M_{\tau_2} - d, M_{\tau_2} + d]\) for \(c = 0.5\)

<table>
<thead>
<tr>
<th>(\xi \setminus d)</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
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