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Sequential Interval Estimation of a Location Parameter with the Fixed Width in the Non-regular Case

Dedicated to Professor Masafumi Akahira on his 60th birthday

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Abstract: For a location-scale parameter family of distributions with a finite support, a sequential confidence interval with a fixed width is obtained for the location parameter, and its asymptotic consistency and efficiency are shown. Some comparisons with the Chow-Robbins procedure are also done.

Keywords: Coverage probability; Extreme value; Non-regular case; Sequential interval estimation.

Subject Classifications: 62L12; 62F25.

1. INTRODUCTION

Suppose that we are to estimate a location parameter \( \theta \) of a sequence of random observations \( X_1, X_2, \ldots, X_n, \ldots \) with unknown scale \( \xi \). We would like to obtain sequentially a confidence interval of fixed width \( 2d \) with confidence coefficient \( 1 - \alpha \). Obviously we can not obtain a fixed sample size procedure if \( \xi \) is unknown. There are many works on the fixed-width interval estimation of normal mean (see, e.g. Ghosh et al. (1997)).
Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to the uniform distribution on the interval $(\theta - (\xi/2), \theta + (\xi/2))$, where $\theta (\in \mathbb{R}^1)$ and $\xi (> 0)$ are unknown. Let $X_{(1)} := \min_{1 \leq i \leq n} X_i$, $X_{(n)} := \max_{1 \leq i \leq n} X_i$. Then the midrange and the range are $M_n := (X_{(1)} + X_{(n)}) / 2$, $R_n := X_{(n)} - X_{(1)}$, respectively. Akahira and Koike (2005) considered a stopping rule:

$$
\tau_1 := \inf \left\{ n \geq n_0 \left| \frac{R_n}{n - 1} \leq -\frac{2d}{\log \alpha} \right. \right\},
$$

where $n_0 (\geq 2)$ is an initial size of sample. They showed the asymptotic consistency and efficiency of the estimation procedure $(\tau_1, [M_{\tau_1} - d, M_{\tau_1} + d])$.

In this paper, we consider the case of a location-scale parameter family of distributions with a finite support on the interval $(\theta - \xi a, \theta + \xi a)$, where $\theta$ and $\xi$ are unknown, and obtain a sequential confidence interval of $\theta$ with fixed width $2d$ and confidence coefficient $1 - \alpha$, and show its asymptotic consistency and efficiency. Some comparisons with the Chow-Robbins procedure are also done.

2. ASYMPTOTIC DISTRIBUTIONS OF THE EXTREME VALUES

In this section we consider the asymptotic distributions of the extreme values for distributions with a finite support, in a similar way to Akahira (1991) and Akahira and Takeuchi (1995).

Let $Z_1, Z_2, \ldots$, be a sequence of independent and identically distributed (i.i.d.) random variables according to the density function $f_0(x - \theta)$ $(\theta \in \mathbb{R}^1)$ with respect to the Lebesgue measure. We assume the following conditions:

(A1) $f_0(x)$ has a finite support $(-a, a)$, i.e., $f_0(x) > 0$ for $-a < x < a$, and $f_0(x) = 0$ otherwise.

(A2) $f_0(x)$ is continuously differentiable in the open interval $(-a, a)$ and

$$
\lim_{x \to -a+0} f_0(x) = c, \quad \lim_{x \to a-0} f_0(x) = c',
$$

1If the support of $f_0$ is $(-a, b)$ $(a \neq b)$, then the normalized midrange does not converge to $\theta$ in probability as $n \to \infty$. 

2
where \( c \) and \( c' \) are some positive constants.

(A3) \( f_0(x) \) satisfies

\[
\begin{align*}
  f_0(x) &\approx g(x + a)^\gamma \quad (x \to -a + 0), \\
  f_0(x) &\approx g'|x - a|^\gamma \quad (x \to a - 0),
\end{align*}
\]

where \( \gamma, g \) and \( g' \) are some positive constants.

Putting \( Z^{(1)} := \min_{1 \leq i \leq n} Z_i, ~ Z^{(n)} := \max_{1 \leq i \leq n} Z_i, ~ U := n(Z^{(1)} + a - \theta) \) and \( V := n(Z^{(n)} - a - \theta) \), we have the following lemma (cf. Akahira (1991), Akahira and Takeuchi (1995)).

**Lemma 1.** Under the conditions (A1) and (A2), the joint (j.) p.d.f. \( f^{(n)}_{U,V}(u,v) \) of \((U, V)\) satisfies

\[
f^{(n)}_{U,V}(u,v) \to \begin{cases} 
  cc' \exp\{c'v - cu\} & (v < 0 < u), \\
  0 & \text{(otherwise).}
\end{cases}
\]

as \( n \to \infty \).

**Proof.** The j.p.d.f. \( f^{(n)}_{U,V}(u,v) \) of \((U, V)\) is

\[
\begin{align*}
  f^{(n)}_{U,V}(u,v) &= \begin{cases} 
    \frac{n-1}{n} \left\{ F\left( a + \frac{u}{n} \right) - F\left( -a + \frac{u}{n} \right) \right\}^{n-2} f_0\left( -a + \frac{u}{n} \right) f_0\left( a + \frac{u}{n} \right) & (v < 0 < u), \\
    0 & \text{(otherwise),}
  \end{cases}
\end{align*}
\]

where \( F(x) = \int_{-\infty}^x f_0(u)du \). Hence, by its expansion, we have the desired result. \( \square \)

Next, we consider the location-scale parameter family of distributions with a finite support \((\theta - \xi a, \theta + \xi a)\). Suppose that \( X_1, X_2, \ldots, X_n, \ldots \) is a sequence of i.i.d. random variables with the p.d.f. \((1/\xi) f_0\left( (x - \theta)/\xi \right)\), where \( \theta \in \mathbb{R} \) and \( \xi > 0 \). Put \( Y_i := (X_i - \theta)/\xi \) for each \( i = 1, 2, \ldots \), and \( Y^{(1)} := \min_{1 \leq i \leq n} Y_i, \ Y^{(n)} := \max_{1 \leq i \leq n} Y_i \). Letting \( S := n(Y^{(1)} + Y^{(n)})/2 \) and \( T = n(Y^{(1)} - Y^{(n)} + 2a)/2 \), we have the asymptotic (as.) j.p.d.f. of \((S, T)\)

\[
\begin{cases} 
  2cc' \exp\{-(c - c')s - (c + c')t\} & (t > |s|), \\
  0 & \text{(otherwise).}
\end{cases}
\]

}\footnote{If the converging order \( \gamma \) is different, then the normalized midrange does not converge to \( \theta \) in probability as \( n \to \infty \).}
Then the as. marginal(m.) p.d.f.’s of \( S \) and \( T \) are given by

\[
\begin{align*}
    f_S(s) &= \begin{cases} 
        Ke^{-2cs} & (s \geq 0), \\
        Ke^{2cs} & (s < 0),
    \end{cases} \\
    f_T(t) &= \begin{cases} 
        \frac{2cc'}{c'-c} \left( e^{-2ct} - e^{-2c't} \right) & (t > 0 \text{ and } c \neq c'), \\
        4c^2te^{-2ct} & (t > 0 \text{ and } c = c'), \\
        0 & (\text{otherwise}),
    \end{cases}
\end{align*}
\]

respectively, where \( K = 2cc'/c + c' \).

In the case when \( \lim_{x \to -a+0} f_0(x) = \lim_{x \to -a-0} f_0(x) = 0 \), we need another lemma. Putting \( U' := n^{1/(\gamma+1)}(Z(1) + a - \theta) \) and \( V' := n^{1/(\gamma+1)}(Z(n) - a - \theta) \), we have the following lemma in a similar way to Lemma 1.

**Lemma 2.** Under the conditions (A1) and (A3), the j.p.d.f. \( f_{U',V'}^{(n)}(u,v) \) of \( U', V' \) satisfies

\[
    f_{U',V'}^{(n)}(u,v) \to \begin{cases} 
        g g'(uv)^\gamma \exp\left\{ -\frac{g'}{\gamma+1}(-v)^{\gamma+1} - \frac{g}{\gamma+1}u^{\gamma+1} \right\} & (v < 0 < u), \\
        0 & (\text{otherwise}),
    \end{cases}
\]

as \( n \to \infty \).

The proof is omitted since it is similar to the one of Lemma 1.

From Lemma 2, \( U' \) and \((-V')\) are asymptotically, independently distributed according to Weibull distributions.

### 3. CONSTRUCTING CONFIDENCE INTERVAL

In this section we construct a sequential confidence interval for \( \theta \). In the first place, we consider the case under the conditions (A1) and (A2). For \( 0 < \alpha < 1 \), let \( l_0 \) be the solution\(^3\) of \( f \) for the equation

\[
    \frac{c + c'}{cc'} \alpha = \frac{e^{-2cl}}{c} + \frac{e^{-2c'l}}{c'}.
\]

\(^3\)It can be shown easily that such \( l_0 \) exists uniquely.
If $\xi$ is known, we have from (2.2) that
\[
P\{ |M_n - \theta| \leq d \} = P\{ n |M_n - \theta|/\xi \leq dn/\xi \}
\approx \int_{-dn/\xi}^{dn/\xi} f_S(s) ds
= 1 - \frac{cc'}{e + c'} \left( e^{-2cn(d/\xi)} + e^{-2c'n(d/\xi)} \right),
\]
where \(\approx\) means that the distribution of \(n |M_n - \theta|/\xi\) is approximated by the asymptotic distribution. Letting \(n^* = l_0 \xi/d\), we have for \(n \geq n^*\)
\[
1 - \frac{cc'}{e + c'} \left( e^{-2cn(d/\xi)} + e^{-2c'n(d/\xi)} \right) \geq 1 - \alpha.
\]
\(n^*\) is referred as the asymptotically optimal size of samples if \(\xi\) is known.

Note that \(n(M_n - \theta)/\xi = S\) and \(R_n/\xi = -(T/n) + 2a\). Now we take as the stopping rule
\[
\tau_2 := \inf \left\{ n \geq n_0 \mid \frac{R_n}{n - 1} \leq \frac{2ad}{l_0} \right\},
\]
where \(n_0 (\geq 2)\) is an initial size of sample. Then we obtain the asymptotic properties of the estimation procedure \((\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])\) as follows.

**Theorem 1.** For the sequential estimation procedure \((\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])\), the following hold.

(i) \(\lim_{d \to 0^+} P\{ |M_{\tau_2} - \theta| \leq d \} = 1 - \alpha\) (asymptotic consistency).
(ii) \(\tau_2/n^* \xrightarrow{a.s.} 1\) \((d \to 0^+)\).
(iii) \(E(\tau_2)/n^* \to 1\) \((d \to 0^+)\) (asymptotic efficiency).

**Proof.** (i) From Lemma 1 of Chow and Robbins (1965), the stopping rule \(\tau_2\) given by (3.1) satisfies
\[
\lim_{d \to 0^+} \frac{d\tau_2}{\xi l_0} = 1 \quad \text{a.s.}
\]
Since \(S = n(M_n - \theta)/\xi\) converges in distribution to a distribution with the density given by (2.2) as \(n \to \infty\), it follows from Theorem 1 of Anscombe (1952) that \(\tau_2(M_{\tau_2} - \theta)\) converges in distribution to the same distribution as \(d \to 0^+\). Hence, since \(d\tau_2/\xi \xrightarrow{a.s.} l_0\) as \(d \to 0^+\) from (3.2), it follows that
\[
\lim_{d \to 0^+} P\{ |M_{\tau_2} - \theta| \leq d \} = \lim_{d \to 0^+} P\{ \tau_2 |M_{\tau_2} - \theta|/\xi \leq d\tau_2/\xi \}
= \int_{-l_0}^{l_0} f_S(s) ds = 1 - \alpha.
\]
(ii) From (3.2) and the definition of \( l_0 \), we have \( \tau_2/n^* = \tau_2d/(l_0\xi)^{\alpha/2}1 \) as \( d \to 0^+ \).

(iii) From Lemma 2 of Chow and Robbins (1965), we have the desired result.

\[\square\]

**Remark.** In particular, if \( c = c' \), then \( l_0 = -\log \alpha/(2c) \) and \( \tau_2 \) given in (3.1) is expressed as

\[
\tau_2 = \inf \left\{ n \geq n_0 \mid \frac{R_n}{n-1} \leq -\frac{4acd}{\log \alpha} \right\},
\]

which is equal to \( \tau_1 \) when the underlying distribution is uniform distribution on the interval \((\theta - (\xi/2), \theta + (\xi/2))\).

In the second place, we compare this with the Chow-Robbins procedure. Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables with the mean \( \theta \) and the variance \( \sigma^2 \). Let \( \bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i \), \( s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \). Chow and Robbins (1965) considered a stopping rule defined by

\[
\tau_{CR} := \inf \left\{ n \geq n_0 \mid n \geq u_{\alpha/2}^2 \sigma^2/d^2 \right\},
\]

where \( u_{\alpha/2} \) is the upper \( \alpha/2 \) point of \( N(0,1) \) and \( n_0(\geq 2) \) is an initial size of samples. They showed the asymptotic consistency and efficiency of the estimation procedure \((\tau_{CR}, [\bar{X}_{\tau_{CR}} - d, \bar{X}_{\tau_{CR}} + d])\).

Since, from Theorem 2.2 of Akahira and Koike (2005), Theorem 1 and Theorem of Chow and Robbins (1965),

\[
\tau_1 \approx \frac{\log \alpha}{\log (1 - (2d/\xi))} \approx \frac{-\xi \log \alpha}{2d}, \quad \tau_2 \approx l_0\xi/d, \quad \tau_{CR} \approx u_{\alpha/2}^2 \sigma^2/d^2,
\]

as \( d \to 0^+ \), we have \( \tau_1/\tau_{CR}, \tau_2/\tau_{CR} \to 0 \) \((d \to 0^+)\). Therefore \( \tau_1, \tau_2 \) is asymptotically better than \( \tau_{CR} \) in the sense of the average size of sample.

Furthermore, we consider the case under the conditions (A1) and (A3). By putting \( S' := n^{1/(\gamma+1)}(Y_{(1)} + Y_{(n)})/2 \) and \( T' := n^{1/(\gamma+1)}(Y_{(1)} - Y_{(n)} + 2a)/2 \), the as.j.p.d.f. of \((S', T')\) and the as.m.p.d.f.’s of \( S' \) and \( T' \) are obtained from Lemma 2. In a similar way to (3.3), we take \( l_0 \) satisfying \( \int_{-l_0}^{l_0} f_{S'}(s) ds = 1 - \alpha \) for the as.m.p.d.f. \( f_{S'}(s) \) of \( S' \).
If $\xi$ is known, we have

$$P\{|M_n - \theta| \leq d\} = P\{n^{1/(\gamma+1)}|M_n - \theta|/\xi \leq dn^{1/(\gamma+1)}/\xi\} \\ \approx \int_{-dn^{1/(\gamma+1)}/\xi}^{dn^{1/(\gamma+1)}/\xi} f_S(s)ds,$$

where “$\approx$” means that the distribution of $n^{1/(\gamma+1)}|M_n - \theta|/\xi$ is approximated by the asymptotic distribution. The optimal size of sample required for attaining the preassigned coverage probability $1 - \alpha$ is the smallest positive integer $\geq (l_0\xi/d)^{\gamma+1} =: n^{**}$ (say). Define a stopping rule as

$$\tau_3 := \inf \left\{ n \geq n_0 \mid \frac{R_n}{n^{1/(\gamma+1)}} \leq \frac{2ad}{l_0} \right\},$$

where $n_0(\geq 2)$ is an initial size of samples. Then the next theorem follows.

**Theorem 2.** For the sequential estimation procedure $(\tau_3, [M_{\tau_3} - d, M_{\tau_3} + d])$, the following hold.

(i) $\lim_{d \to 0+} P\{|M_{\tau_3} - \theta| \leq d\} = 1 - \alpha$ (asymptotic consistency).

(ii) $\tau_3/n^{**} \xrightarrow{a.s.} 1 \ (d \to 0+)$.

(iii) $E(\tau_3)/n^{**} \to 1 \ (d \to 0+)$ (asymptotic efficiency).

**Proof.** The proof for (i) is similar to the one of Theorem 1 (i). (ii) follows from $(\tau_3/n^{**})^{1/(\gamma+1)} \xrightarrow{a.s.} 1$ as $d \to 0+$.

(iii) From (ii), by Fatou’s lemma,

$$\liminf_{d \to 0+} \frac{E(\tau_3)}{n^{**}} \geq E \left( \liminf_{d \to 0+} \frac{\tau_3}{n^{**}} \right) = 1. \quad (3.4)$$

On the other hand, since $0 \leq R_n \leq 2a\xi$ with probability 1 for any $n \in \mathbb{N}$, we have $0 \leq (R_n l_0/(2ad))^{\gamma+1} \leq (2a\xi l_0/(2ad))^{\gamma+1} = (l_0\xi/d)^{\gamma+1}$ with probability 1 for any $n \in \mathbb{N}$. So, $0 \leq (R_n l_0/(2ad))^{\gamma+1} \leq n$ with probability 1 for $n$ satisfying $n \geq (l_0\xi/d)^{\gamma+1} + 1$. Therefore, since $\tau_3 = \inf \{ n \geq n_0 \mid (R_n l_0/(2ad))^{\gamma+1} \leq n \}$, we have $\tau_3 \leq (l_0\xi/d)^{\gamma+1} + 1$. Then, using the definition of $n^{**}$, we have

$$\frac{E(\tau_3)}{n^{**}} \leq \left\{ \left(\frac{l_0\xi}{d}\right)^{\gamma+1} + 1 \right\} \left(\frac{l_0\xi}{d}\right)^{-(\gamma+1)} = 1 + \left(\frac{d}{l_0\xi}\right)^{\gamma+1},$$
hence
\[
\limsup_{d \to 0+} \frac{E(\tau_3)}{n^{**}} \leq 1. \tag{3.5}
\]
Combining (3.4) and (3.5), we obtain (iii).

From Theorem 2 and Theorem of Chow and Robbins (1965), \( \tau_3 \approx (l_0\xi/d)^{\gamma+1} \) and \( \tau_{CR} \approx u^2\alpha/2\sigma^2/d^2 \) as \( d \to 0+ \). Therefore,
\[
\tau_3/\tau_{CR} \begin{cases} 
= o(1) & (0 < \gamma < 1), \\
= O(1) & (\gamma = 1), \\
\to \infty & (\gamma > 1)
\end{cases}
\]
as \( d \to 0+ \). Therefore, \( \tau_3 \) is asymptotically better than \( \tau_{CR} \) in the sense of the average size of sample if \( 0 < \gamma < 1 \).

In this paper, we considered the cases when the values at the endpoints of the support of the p.d.f. are positive simultaneously, or tend to 0 at the same speed. In the meantime, if the either value at the endpoints of the support of the p.d.f. is positive, or tend to 0 at a different speed, then the coefficients of \( n^\gamma(X_1 - a - \theta) \) and \( n^\delta(X_n - b - \theta) \) converging to nontrivial random variables are different and estimation by using the midrange \( M_n \) is inappropriate.

4. NUMERICAL EXAMPLE

In this section we examine the coverage probability of the procedure \([M_{n_2} - d, M_{n_2} + d]\) by simulation based on 100000 repetitions. Suppose that \( X_1, X_2, \ldots, X_n, \ldots \) is a sequence of i.i.d. random variables with the p.d.f. \((1/\xi)f_0((x - \theta)/\xi)\), where \( \theta \in \mathbb{R} \), \( \xi > 0 \) and \( f_0(\cdot) \) is a trapezoid-shape p.d.f. given by
\[
f_0(x) = \begin{cases} 
(1/2 - c)x + 1/2 & (x \in (-1, 1)), \\
0 & \text{(otherwise)}
\end{cases}
\]
with \( 0 < c < 1 \). Note that, \( f_0 \) is the p.d.f. of the uniform distribution over \((-1, 1)\) and an asymmetric p.d.f. over \((-1, 1)\) for \( c = 0.5 \) and a sufficiently small \( c > 0 \), respectively. Since \( M_{n_2} \) is location equivariant, we may assume \( \theta = 0 \) without loss of generality.

When \( \alpha = 0.10 \), \( d = 0.01(0.01)0.05 \), \( \xi = 1(1)5 \) and \( n_0 = 5 \), Tables 1 and 2 show the values of coverage probabilities of the sequential estimation
procedure \((\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])\) for \(c = 0.1\) and \(c = 0.5\), respectively. The result suggests that the estimation procedure is consistent for this case.

Table 1. Coverage probabilities of \([M_{\tau_2} - d, M_{\tau_2} + d]\) for \(c = 0.1\)

<table>
<thead>
<tr>
<th>(\xi \ ) (d)</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.90637</td>
<td>0.91545</td>
<td>0.92348</td>
<td>0.93092</td>
<td>0.93758</td>
</tr>
<tr>
<td>2</td>
<td>0.89830</td>
<td>0.90544</td>
<td>0.90960</td>
<td>0.91424</td>
<td>0.92017</td>
</tr>
<tr>
<td>3</td>
<td>0.90123</td>
<td>0.90313</td>
<td>0.90713</td>
<td>0.90832</td>
<td>0.91030</td>
</tr>
<tr>
<td>4</td>
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<td>0.90117</td>
<td>0.90333</td>
<td>0.90615</td>
<td>0.90804</td>
</tr>
<tr>
<td>5</td>
<td>0.89817</td>
<td>0.89952</td>
<td>0.90318</td>
<td>0.90421</td>
<td>0.90561</td>
</tr>
</tbody>
</table>

Table 2. Coverage probabilities of \([M_{\tau_2} - d, M_{\tau_2} + d]\) for \(c = 0.5\)

<table>
<thead>
<tr>
<th>(\xi \ ) (d)</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.91183</td>
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</tr>
<tr>
<td>2</td>
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<td>0.90131</td>
<td>0.90330</td>
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<td>0.91176</td>
</tr>
<tr>
<td>3</td>
<td>0.89849</td>
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<td>0.90235</td>
<td>0.90525</td>
</tr>
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<td>0.89729</td>
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<tr>
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<td>0.8998</td>
<td>0.89906</td>
<td>0.89862</td>
<td>0.90054</td>
</tr>
</tbody>
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