Dynamical systems which realize given random bi-sequences of points on their orbits

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Abstract. A dynamical system consists of a phase space of possible states, together with an evolution rule that determines all future states and all past states given a state at any particular moment. In this paper, we show that for any countable random infinite bi-sequences of states of some phase space, there exists an evolution rule in C^0-topology which realizes precisely the given sequences of states on their orbits and satisfies some regular conditions on the times to realize the states.

[1]AMS Subject Classification: Primary 54F11; 54H20; 58FE60, Secondary 58F11; 28A32; 57Q40.

[2]Key words and phrases: Dynamical system, orbits, transitive homeomorphism, chaotic, good measure, measure-preserving homeomorphism, flow.

1 Introduction

A dynamical system consists of a phase space of possible states, together with an evolution rule that determines all future states and all past states given a state at any particular moment. In this paper, we consider some kinds of chaotic properties of dynamical systems. We show that in the world admitting C^0-topology, for any countable random infinite bi-sequences of states of some phase space there exists an evolution rule which realizes precisely the given bi-sequences of states on their orbits and satisfies some regular conditions on the times to realize the states. In other words, for any countable random infinite itineraries, by making a slight modification on our dynamical system, we have a new dynamical system in C^0-topology which realizes the given infinite itineraries and satisfies some regular conditions on the times of itineraries. The ideas of this paper depend on works of Oxtoby-Ulam [7] and Bennett [2]. We need the following terminology and concepts. Let $\mathbb{N} = \{1, 2, ..., \}$ be the set of all positive integers and $\mathbb{Z} = \{0, \pm 1, \pm 2, ..., \}$ the set of all integers. Also let $\mathbb{R}$ be the set of all real numbers and $I = [0, 1]$ the unit interval.

In this paper, we suppose that $f : X \to X$ is a homeomorphism of a compact metric space (X, d), where d is a metric on X. We put $Supp(f) = \{x \in X |\ f(x) \neq x\}$. A point $x \in X$ is a *periodic point* of f if there exists a positive integer $n \in \mathbb{N}$ such that $f^n(x) = x$. A point $x \in X$ is *recurrent* under f if for any neighborhood U of x there exists a positive integer $n \in \mathbb{N}$ such that $f^n(x) \in U$. The orbit of a point $x \in X$ under f, denoted by $Orb(f; x)$, is the set $\{f^n(x)|\ n \in \mathbb{Z}\}$. If x is not a periodic point of f, we consider the infinite bi-sequence (=ordered orbit) $(f^n(x)|\ n \in \mathbb{Z}) = (...., f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), ...)$ of x under f. If x is a periodic point of f with period n, we also consider the finite sequence (=ordered orbit) $(f^i(x)|\ 0 \leq i \leq n-1)$ of x under f. For any $x \in X$ and $i, j \in \mathbb{Z}$ with $i \leq j$, we put $Orb(f; x)_{[i,j]} = \{f^n(x)|\ i \leq n \leq j\}$. Suppose that $Orb(f; x)$ is not a periodic orbit and $y \in Orb(f; x)$. In this case, we put $Time_f(x \to y) = n$, where n is the (unique) integer satisfying $f^n(x) = y$ $(n \in \mathbb{Z})$.

Let $\varphi : X \times \mathbb{R} \to X$ be a *flow*, i.e., φ is a map (=continuous function) such that

1. $\varphi(x, 0) = x$ and
2. $\varphi(x, s + t) = \varphi(\varphi(x, s), t)$ for any $x \in X$ and any $s, t \in \mathbb{R}$.

A point $x \in X$ is a *periodic point* of φ if there exists a positive number $t \in \mathbb{R}$ such that $\varphi(x, t) = x$. The orbit of a point $x \in X$ under φ, denoted by $Orb(\varphi; x)$, is the set $\{\varphi(x, t)|\ t \in \mathbb{R}\}$. If x is not a periodic point of φ, we consider the ordered orbit $(\varphi(x, t))|\ t \in \mathbb{R})$ of x under φ. If x is a periodic point of φ with period $t_0 > 0$, we consider the ordered orbit $(\varphi(x, t))|\ 0 \leq t < t_0)$ of x under φ. If x is not a periodic point and $y \in Orb(\varphi, x)$, we put $Time_\varphi(x \to y) = t$ if $\varphi(x, t) = y$.

Let $\Lambda = \mathbb{Z}$ or $\Lambda = \{0, 1, 2, ..., s\}$ $(s < \infty)$. A sequence $\mathcal{S} = (a_n | n \in \Lambda)$ of points of X is said to be *realized* by a homeomorphism f if $\mathcal{S}$ is a subsequence of the ordered orbit of a_0 under f. Similarly $\mathcal{S} = (a_n | n \in \Lambda)$ is said to be *realized* by a flow φ if $\mathcal{S}$ is a subsequence of the ordered orbit of a_0 under φ. A sequence $(x_n | n \in \Lambda)$ of points of X is a *pseudo η-orbit* $(\eta > 0)$ of f if $d(f(x_n), x_{n+1}) < \eta$ for any $n, n+1 \in \Lambda$. Let $a, b \in X$. A finite sequence $(x_n | 0 \leq n \leq s)$ is a *pseudo η-orbit* of f from a to b if $x_0 = a, x_s = b$ and $d(f(x_n), x_{n+1}) < \eta$ for any $0 \leq n \leq s - 1$.

Let $(k_n|\ n \in \mathbb{Z})$ be an arbitrary increasing bi-sequence of integers with $k_0 = 0$ i.e., $k_n < k_{n+1}$ for $n \in \mathbb{Z}$ and let $\mathcal{S}_i = (a^i_n | n \in \mathbb{Z})$ $(i \in \mathbb{N})$ be infinite bi-sequences of distinct points of X. Then the (countable) family $\{\mathcal{S}_i | i \in \mathbb{N}\}$ is said to be *chaotic* for $(k_n|\ n \in \mathbb{Z})$ if the following conditions are satisfied;

1. $\mathcal{S}_i$ and $\mathcal{S}_j$ $(i \neq j)$ are mutually disjoint,
2. the set $\{a^i_0 | i \in \mathbb{N}\}$ is dense in X,
3. the sets $\{a^i_{k_n} | n \in \mathbb{N}\}$ and $\{a^i_{k_{-n}} | n \in \mathbb{N}\}$ are dense in X for each i,
4. if $i, j \in \mathbb{N}$ and $i \neq j$, then $\mathcal{S}_i$ and $\mathcal{S}_j$ are Li-Yorke pair with respect to $(k_n|\ n \in \mathbb{Z})$ and the diameter $\delta(X)$ of X, that is,

$$\liminf_{n \to \pm\infty} d(a^i_{k_n}, a^j_{k_n}) = 0,$$

$$\limsup_{n \to \pm\infty} d(a^i_{k_n}, a^j_{k_n}) = \delta(X).$$

It is easy to see that if X has no isolated point (i.e., X is perfect), then we have many kinds of chaotic families $\{\mathcal{S}_i | i \in \mathbb{N}\}$.

An m-dimensional compact connected polyhedron X is said to be *regularly connected* if the set

$$Int(X) = \{x \in X|\ x \text{ has an open neighborhood which is homeomorphic to } \mathbb{R}^m\}$$

is a connected dense open subset of X. Put $\partial(X) = X - Int(X)$.

The theory of Menger manifolds was founded by Anderson and Bestvina (see [1] and [3]) and has been studied by many authors. We also study Menger manifolds from the viewpoint of dynamical systems. Anderson and Bestvina gave characterizations of Menger manifolds as follows: For a compactum M, M is a k-dimensional Menger manifold if and only if (1) $\dim M = k$, (2) M is locally $(k-1)$-connected, (3) M has the disjoint k-cell property, i.e., for any $\epsilon > 0$ and any maps $f, g : I^k \to M$, there are maps $f', g' : I^k \to M$ such that $d(f, f') < \epsilon$, $d(g, g') < \epsilon$ and $f'(I^k) \cap g'(I^k) = \phi$. Note that every 0-dimensional Menger manifold is a Cantor set, and every 1-dimensional Menger connected manifold is a Menger curve. If X is a Menger manifold, we put $Int(X) = X$ and $\partial(X) = \phi$.

Let μ be a probability measure on a compact metric space (X, d) which is nonatomic, locally positive; such a measure is called a *good measure*. Put

$$M(X; good) = \{\mu|\ \mu \text{ is a good measure on } X\}.$$

If X is a regularly connected polyhedron, we consider the following subset of measures:

$$M_\partial(X; good) = \{\mu \in M(X; good)|\ \mu(\partial X) = 0\}.$$

Let $H(X, \mu)$ be the set of all μ-measure preserving homeomorphisms of X with metric

$$\rho(f, g) = d(f, g) + d(f^{-1}, g^{-1}),$$

where $d(f,g) = \sup\{d(f(x), g(x)) | x \in X\}$. Also, put

$$H_\partial(X, \mu) = \{f \in H(X, \mu) |\ f|\partial X = Id\}.$$

Note that $H(X, \mu)$ and $H_\partial(X, \mu)$ are complete metric spaces (see [7]). Note that if X is a regularly connected polyhedron and $\mu, \mu' \in M_\partial(X; good)$, then there is a homeomorphism $h : X \to X$ such that $h_*\mu = \mu'$ (see [7, Corollary 1]). Also, note that if X is a k-dimensional Menger manifold ($k \geq 1$) and $\mu, \mu' \in M(X; good)$, then there is a homeomorphism $h : X \to X$ such that $h_*\mu = \mu'$ (see [5, Theorem 3.1]).

2 Homeomorphisms which realize precisely the given sequences of points on their orbits

In this section, we consider the case of discrete dynamical systems. A metric d on a space X is a *convex metric* if for any $x, y \in X$ there is a point z of X such that $d(x, z) = d(z, y) = (1/2)d(x, y)$. It is well-known that a continuum (compact metric connected space) X is locally connected (=Peano continuum) if and only if X admits a convex metric d on X. First, we need the following lemmas (cf. [7, Lemma 14]).

Lemma 2.1. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 1$ or a Menger k-dimensional manifold with $k \geq 1$ and d is a convex metric on X. Let μ be a good measure on X and $h \in H(X, \mu)$. For any $\delta > 0$, there is a natural number N such that for any $a, b \in X$ and any $n \geq N$, there is a pseudo δ-orbit $x_0, x_1, ..., x_n$ of h from a to b.*

Proof. For a subset A of X, let $U(A, \delta)$ be the δ-neighborhood of A in X. Put $U_1 = U(h(a), \delta)$. By induction on i, we define $U_{i+1} = U(h(U_i), \delta)$. Since $h \in H(X, \mu)$, by [7, Lemma 14] and [5] there is a natural number N such that $U_N = X$. Let $a, b \in X$ and n any natural number with $n \geq N$. We choose the point $y \in X$ such that $b = h^{n-N}(y)$. Since $U_N = X$, there is a pseudo δ-orbit $x_0, x_1, ..., x_N$ of h from a to y. Then the sequence $x_0, x_1, ..., x_N(= y), x_{N+1}(= h(y)), x_{N+2}(= h^2(y)), ..., x_n(= h^{n-N}(y) = b)$ is a pseudo δ-orbit $x_0, x_1, ..., x_n$ of h from a to b.

The following lemma follows from [7, Lemma 12] and [5, Proposition 4.16]. We omit the proof.

Lemma 2.2. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 2$ or a Menger k-dimensional manifold ($k \geq 1$). Let μ be a good measure on X. Suppose that U is a connected open set of $Int(X)$ with $a, b \in U$. Then there exists $h \in H(X, \mu)$ such that $h(a) = b$ and $Supp(h) \subset U$.*

The following lemma is a slight modification of [7, Lemma 13]. For completeness, we give the proof.

Lemma 2.3. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 2$ or a Menger k-dimensional manifold ($k \geq 1$) and d is a convex metric on X. Let F be a* *finite subset of X and μ a good measure on X. Suppose that p_i, q_i $(i = 1, 2, ..., l)$ are points of $Int(X) - F$ such that $\{p_i, q_i\} \cap \{p_j, q_j\} = \phi$ $(i \neq j)$ with $d(p_i, q_i) < \delta$. Then there exists $h \in H(X, \mu)$ such that $h(p_i) = q_i$ for each i, $d(h, Id) < \delta$ and $Supp(h) \cap (F \cup \partial X) = \phi$.*

Proof. We shall prove the case that X is a regularly connected polyhedron of dimension 2. Since F is a finite set and $Int(X)$ is 2-dimensional manifold, we can choose arcs L_i from p_i to q_i such that the length $l(L_i)$ of L_i is less than δ, $L_i \cap L_j$ is an at most one point set for $i \neq j$ and

$$F \cap L_i = \phi, L_i \cap \{p_j, q_j |\ j = 1, 2, ..., l\} = \{p_i, q_i\}.$$

Note that if X is the other cases, we can find arcs L_i from p_i to q_i such that the length $l(L_i)$ of L_i is less than δ, $F \cap L_i = \phi$ and $L_i \cap L_j = \phi$ $(i \neq j)$. Let k be a sufficiently large natural number. For each i, we can take $k+1$ points $p_i = p_{i,0} < p_{i,1} < .... < p_{i,k} = q_i$ on L_i such that the length $l(L_{i,j})$ of $L_{i,j}$ is less than δ/k and for each $1 \leq j \leq k$, the family $\{L_{i,j} |\ i = 1, 2, ..., l\}$ are disjoint, where $L_{i,j}$ is the sub arc from $p_{i,j-1}$ to $p_{i,j}$ in L_i (see the proof of [7, Lemma 13]). Take a sufficiently small neighborhood $U_{i,j}$ of $L_{i,j}$ for each i, j such that $\delta(U_{i,j}) < \delta/k$, $U_{i,j} \cap F = \phi$ and for each $j = 1, 2, .., k$, $\{U_{i,j} |\ i = 1, 2, ..., l\}$ are disjoint. For each $j = 1, 2, ..., k$, we can choose $h_j \in H(X, \mu)$ such that $h_j(p_{i,j-1}) = p_{i,j}$ for each i and $h_j | X - \cup_{i=1}^{l} U_{i,j} = Id$. Put $h = h_k \circ .... \circ h_1$. Since $d(h_j, Id) < \delta/k$, we see $d(h, Id) < \delta$. Hence h is a desired homeomorphism.

The main result of this section is the following theorem. This theorem means that in the world admitting C^0-topology, random infinite sequences of any prophecies will come true by making a slight change. From now on, we may assume that X admits a convex metric d if X is a Peano continuum.

Theorem 2.4. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 2$ or a Menger k-dimensional manifold $(k \geq 1)$, and μ is a good measure on X. Let $h \in H(X, \mu)$, $\epsilon > 0$ and let $(k_n |\ n \in \mathbb{Z})$ be an arbitrary increasing bi-sequence of integers with $k_0 = 0$. Suppose that $\mathcal{S}_i = (a_n^i | n \in \Lambda_i)$ $(i \in \mathbb{N})$ are arbitrary infinite bi-sequences or finite sequences of distinct points of $Int(X)$ and $\mathcal{S}_i$, $\mathcal{S}_j$ $(i \neq j)$ are mutually disjoint. Then there is $f \in H(X, \mu)$ satisfying the following conditions:*

1. *$d(f, h) < \epsilon$ and $f | \partial(X) = h | \partial(X)$.*
2. *$\mathcal{S}_i$ is realized by f for each $i \in \mathbb{N}$. Moreover if $\mathcal{S}_i = (a_n^i |\ 0 \leq n \leq s_i)$ is a finite sequence, then a_0^i is a periodic point of f and $\mathcal{S}_i$ is realized by f on the periodic ordered orbit of a_0^i.*
3. *If $\mathcal{S}_i$ and $\mathcal{S}_j$ are infinite bi-sequences, then there is $n(i, j) \in \mathbb{N}$ such that if $n \in \mathbb{Z}$ and $|n| \geq n(i, j)$, then $Time_f(a_0^i \to a_n^i) = Time_f(a_0^j \to a_n^j)$.*
4. *If $\mathcal{S}_i$ is an infinite bi-sequence, then $(Time_f(a_0^i \to a_n^i) |\ n \in \mathbb{Z})$ is a bi-subsequence of $(k_n |\ n \in \mathbb{Z})$.*

Proof. We may assume that $\mathcal{S}_{2i-1} = (a^i_n|\ n \in \mathbb{Z})$ is an infinite bi-sequence and $\mathcal{S}_{2i} = (b^i_n|\ 0 \le n \le s_i)$ is a finite sequence for each $i \in \mathbb{N}$. We consider the set $S = \cup_{i=1}^{\infty} S_i$, where $S_{2i-1} = \{a^i_n|\ n \in \mathbb{Z}\}$ and $S_{2i} = \{b^i_n|\ 0 \le n \le s_i\}$. Also, put $S_{2i-1,n} = \{a^i_j|\ -n \le j \le n\}$ $(n \in \mathbb{N})$. By induction on n, we will construct a sequence $(h_n)_{n\in\mathbb{N}}$ of homeomorphisms of X and a bi-subsequence of $(l_n|\ n \in \mathbb{Z})$ of $(k_n|\ n \in \mathbb{Z})$ with $l_0 = 0$ such that for each $n \in \mathbb{N}$, the following conditions are satisfied:

1. $h_n \in H_\partial(X, \mu)$.
2. $d(h_n \circ h_{n-1} \circ ...h_1 \circ h, h_{n-1} \circ h_{n-2} \circ ...h_1 \circ h) < \epsilon/3^n$ and $d((h_n \circ h_{n-1} \circ ... \circ h_1 \circ h)^{-1}, (h_{n-1} \circ h_{n-2} \circ ...h_1 \circ h)^{-1}) < \epsilon/3^n$.
3. For each $1 \le i \le n$, the finite subsequence $(a^i_{-i}, a^i_{-i+1}, ..., a^i_{i-1}, a^i_i)$ of $\mathcal{S}_{2i-1}$ is realized by $h_i \circ h_{i-1} \circ ...h_1 \circ h$. Moreover
$$Time_{(h_i \circ h_{i-1} \circ ... h_1)}(a^i_0 \to a^i_i) = l_i,$$
$$Time_{(h_i \circ h_{i-1} \circ ... h_1)}(a^i_0 \to a^i_{-i}) = l_{-i}$$
and $(Time_{(h_i \circ h_{i-1} \circ ... h_1 \circ h)}(a^i_0 \to a^i_j)|-i \le j \le i)$ is a finite subsequence of $(k_n|\ i \in \mathbb{Z})$.
4. For each $1 \le i \le n$, the point b^i_0 is a periodic point of $h_i \circ h_{i-1} \circ ...h_1 \circ h$ and the sequence $\mathcal{S}_{2i} = (b^i_0, b^i_1, b^i_2, ..., b^i_{s_i})$ is realized by $h_i \circ h_{i-1} \circ ...h_1 \circ h$.
5. If $1 \le i \le j < n$, then for $j < s \le n$
$$Supp(h_s) \cap Orb((h_j \circ h_{j-1} \circ ...h_1 \circ h); a^i_0)_{[l_{-j}, l_j]} = \phi.$$
6. If $1 \le i < n$, then for $i < s \le n$
$$Supp(h_s) \cap Orb((h_i \circ h_{i-1} \circ ...h_1 \circ h); b^i_0) = \phi.$$
7. If $1 \le i \le j \le n$, then $(h_n \circ h_{n-1} \circ ...h_1 \circ h)^{l_j}(a^i_0) = a^i_j$ and $(h_n \circ h_{n-1} \circ ...h_1 \circ h)^{l_{-j}}(a^i_0) = a^i_{-j}$.
8. For each $1 \le i \le n$,
$$Orb(h_n \circ h_{n-1} \circ ...h_1 \circ h; a^i_0)_{[l_{-n}, l_n]} - S_{2i-1,n} \subset X - S$$
and
$$Orb(h_n \circ h_{n-1} \circ ...h_1 \circ h; b^i_0) - S_{2i} \subset X - S.$$

Let $n = 1$. Suppose that $\delta > 0$ is a sufficiently small positive number. By Lemma 2.1, we can choose $l_{-1}, l_1 \in \mathbb{Z}$ and a pseudo δ-orbit
$$a^1_{-1} = x(l_{-1}), x(l_{-1} + 1), ..., x(-1), x(0), x(1), ..., x(l_1 - 1), x(l_1) = a^1_1$$
of h from a^1_{-1} to a^1_1 in $Int(X)$ such that $a^1_0 = x(0)$. We may assume that
$$\{x(j)|\ l_{-1} \le j \le l_1\} \cap S = S_{1,1}$$

and l_{-1}, l_1 are elements of the sequence $(k_n|\ n \in \mathbb{Z})$ such that $l_{-1} < 0 < l_1$. Also, we may assume that there is a pseudo δ-orbit

$$b_0^1 = z(0), z(1), ..., z(l_1 - 1), z(l_1) = b_0^1$$

of h from b_0^1 to b_0^1 in $Int(X)$ such that $(b_0^1, b_1^1, ..., b_{s_1}^1)$ is a subsequence of the sequence

$$z(0), z(1), ..., z(l_1 - 1)).$$

Also, we may assume that

$$\{z(j)|\ 0 \leq j \leq l_1 - 1\} \cap (S \cup \{x(j)|\ l_{-1} \leq j \leq l_1\}) = S_2.$$

For the sake of simplicity, we may assume h satisfies that $h(x(j)) \neq x(j)$ and $h(z(j)) \neq z(j)$ for each j (see Lemma 2.2); if necessary, we replace h with the composition $h' \circ h$ of h and h', where $h' \in H_\partial(X, \mu)$ and $d(h', Id)$ is sufficiently small. Then

$$\{h(x(j)), x(j+1)\} \cap \{h(x(j')), x(j'+1)\} = \phi,$$

$$\{h(z(j)), z(j+1)\} \cap \{h(z(j')), z(j'+1)\} = \phi\ (j \neq j').$$

By Lemma 2.3, there is a homeomorphism $h_1 \in H_\partial(X, \mu)$ such that $d(h_1, Id) < \delta$ and $h_1(h(x(j))) = x(j+1)$ and $h_1(h(z(j))) = z(j+1)$ for each j. If δ is sufficiently small, then we may assume that $d(h_1 \circ h, h) < \epsilon/3$ and $d((h_1 \circ h)^{-1}, h^{-1}) < \epsilon/3$. Note that the sequences (a_{-1}^1, a_0^1, a_1^1) and $(b_0^1, b_1^1, ..., b_{s_1}^1)$ are realized by $h_1 \circ h$ and $Time_{(h_1 \circ h)}(a_0^1 \to a_{\pm 1}^1) = l_{\pm 1}$.

Assume that $h_1, h_2, ..., h_n$ and $l_{\pm 1}, l_{\pm 2}, ..., l_{\pm n}$ have been defined for certain n and they satisfy the conditions 1-8. We define h_{n+1} and $l_{\pm(n+1)}$ as follows.

Let $\delta > 0$ be a sufficiently small positive number. Choose integers $l_{n+1}, l_{-(n+1)} \in \mathbb{Z}$ and a pseudo δ-orbit

$$a_{-(n+1)}^{n+1} = x(l_{-(n+1)}), x(l_{-(n+1)} + 1), ..., x(-1), x(0), x(1), ..., x(l_{(n+1)} - 1), x(l_{(n+1)}) = a_{n+1}^{n+1}$$

of $h_n \circ h_{n-1} \circ ... h_1 \circ h$ from $a_{-(n+1)}^{n+1}$ to a_{n+1}^{n+1} such that the points x_j are distinct points of $Int(X)$, $a_0^{n+1} = x(0)$, and $(a_i^{n+1}| - (n+1) \leq i \leq n+1)$ is a subsequence of the sequence

$$x(l_{-(n+1)}), x(l_{-(n+1)} + 1), ..., x(-1), x(0), x(1), ..., x(l_{(n+1)} - 1), x(l_{n+1}).$$

Also, by Lemma 2.1, for each $1 \leq i \leq n$, we may choose a pseudo δ-orbit

$$a_n^i = y^i(l_n), y^i(l_n + 1), ..., y^i(l_{n+1}) = a_{n+1}^i$$

of $h_n \circ h_{n-1} \circ ... h_1 \circ h$ from a_n^i to a_{n+1}^i and a pseudo δ-orbit

$$a_{-(n+1)}^i = y^i(l_{-(n+1)}), y^i(l_{-(n+1)} + 1), ..., y^i(l_{-n}) = a_{-n}^i$$

of $h_n \circ h_{n-1} \circ ... h_1 \circ h$ from $a_{-(n+1)}^i$ to a_{-n}^i. Also, we may assume that there is a pseudo δ-orbit

$$b_0^{n+1} = z(0), z(1), ..., z(l_{(n+1)} - 1), z(l_{(n+1)}) = b_0^{n+1}$$

of $h_n \circ h_{n-1} \circ ... h_1 \circ h$ from b_0^{n+1} to b_0^{n+1} such that the pointa z_j are distinct points of $Int(X)$, $(b_0^{n+1}, b_1^{n+1}, b_2^{n+1}, ..., b_{s_{n+1}}^{n+1})$ is a subsequence of $z(0), z(1), ..., z(l_{(n+1)} - 1))$. Moreover, we may assume that A, B_i $(i = 1, 2, ..., n), C$ and D are mutually disjoint, where

$$A = \{x(j) |\ l_{-(n+1)} \leq j \leq l_{n+1}\},$$

$$B_i = \{y^i(j) |\ l_{-(n+1)} \leq j \leq l_{-n} - 1\} \cup \{y^i(j) |\ l_n + 1 \leq j \leq l_{n+1}\},$$
$$C = \{z(j) |\ 0 \leq j \leq l_{n+1}\},$$

$$D = \bigcup_{i=1}^{n} (\{(h_n \circ h_{n-1} \circ ... h_1 \circ h)^j (a_0^i) |\ l_{-n} \leq j \leq l_n\} \cup \bigcup_{i=1}^{n} (\{(h_n \circ h_{n-1} \circ ... h_1 \circ h)^j (b_0^i) |\ 0 \leq j \leq l_i\}).$$

Also we may assume that $l_{-(n+1)}, l_{n+1}$ are elements of the sequence $(k_n |\ n \in \mathbb{Z})$ and the finite sequence

$$(Time_{(h_{n+1} \circ h_n \circ ... h_1 \circ h)}(a_0^{n+1} \to a_j^n) |\ -(n+1) \leq j \leq n+1)$$

is a subsequence of $(k_n |\ i \in \mathbb{Z})$. By the same argument as above, we may assume that $x(j), y^i(j), z(j)$ are not fixed points of $h_n \circ h_{n-1} \circ ... h_1 \circ h$. By Lemma 2.3, there is a homeomorphism $h_{n+1} \in H_\partial(X, \mu)$ such that $h_{n+1}|D = Id$, $d(h_{n+1}, Id) < \delta$ and

$$h_{n+1}(h_n \circ h_{n-1} \circ ... h_1 \circ h(x(j)) = x(j+1),$$

$$h_{n+1}(h_n \circ h_{n-1} \circ ... h_1 \circ h(y^i(j))) = y^i(j+1),$$
$$h_{n+1}(h_n \circ h_{n-1} \circ ... h_1 \circ h(z(j))) = z(j+1)$$

for each i, j. If δ is sufficiently small, then we may assume that

$$d(h_{n+1} \circ h_n \circ h_{n-1} \circ ... h_1 \circ h, h_n \circ h_{n-1} \circ ... h_1 \circ h) < \epsilon/3^{n+1},$$

$$d((h_{n+1} \circ h_n \circ h_{n-1} \circ ... h_1 \circ h)^{-1}, (h_n \circ h_{n-1} \circ ... h_1 \circ h)^{-1}) < \epsilon/3^{n+1}.$$

Also, we may assume that the condition 8 is satisfied for $h_{n+1} \circ h_n \circ h_{n-1} \circ ... h_1 \circ h$.

By using the sequence $(h_n)_{n \in \mathbb{N}}$ of homeomorphisms of X, we put

$$f = \lim_{n \to \infty} h_n \circ h_{n-1} \circ ... h_1 \circ h.$$

Note that if $i, j \leq n$, then

$$f(a_j^i) = h_n \circ h_{n-1} \circ ... h_1 \circ h(a_j^i),$$

$$f(b_j^i) = h_n \circ h_{n-1} \circ ... h_1 \circ h(b_j^i).$$

Then we can see that f is a desired homeomorphism.

Let $f : X \to X$ be a map of a compact metric space (X, d). Then f is *chaotic in the sense of Devaney* if f satisfies the following conditions;

1. f has sensitive dependence on initial conditions, i.e., there is a positive number $\tau > 0$ such that for any $x \in X$ and any neighborhood U of x in X, there is a point $y \in U$ such that $d(f^n(x), f^n(y)) \geq \tau$ for some positive integer $n \in \mathbb{N}$,

2. f is topologically transitive, i.e., the (positive) orbit $\{f^n(x) |\ n \in \mathbb{N}\}$ is dense in X for some point $x \in X$,

3. the set of all periodic points is dense in X.

A subset S of X is a *scrambled set* of f if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$
$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau.$$

If there is an uncountable scrambled set S of f, we say that f is *chaotic in the sense of Li-Yorke*. A map $f : X \to X$ is *everywhere-chaotic* (in the sense of Li-Yorke) if the following conditions are satisfied;

1. there is $\tau > 0$ such that if U and V are any nonempty open subsets of X and N is any natural number, then there is a natural number $n \geq N$ such that $d(f^n(x), f^n(y)) \geq \tau$ for some $x \in U$, $y \in V$, and

2. for any nonempty open subsets U, V of X and any $\epsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \epsilon$ for some $x \in U$, $y \in V$.

Suppose that X is a regularly connected polyhedron of dimension $m \geq 1$. A space homeomorphic to I^m is an *m-cell*. A 0-dimensional compactum D in $Int(X)$ is *flat* if for any neighborhood V of D in X, there is a closed neighborhood U of D in X such that $U \subset V$ and $U = B_1 \cup \cdots \cup B_p$, where B_i $(i = 1, 2, ..., p)$ are mutually disjoint k-cells. By Generalized Schoenflies theorem, we see that if C and C' are flat Cantor sets in $Int(X)$, then any homeomorphism $f : C \cup \partial X \to C' \cup \partial X$ can be extended to a homeomorphism $\overline{f} : X \to X$ (e.g., see the proof of [6, p. 93, Theorem 7]). Also, note that any closed subset of a flat 0-dimensional compactum is also flat.

Theorem 2.5. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 2$ and E is a dense F_σ-set of X such that E is a countable union of flat Cantor sets in $Int(X)$. Let μ be a good measure on X with $\mu(E) = 1$. Suppose that $h \in H(X, \mu)$, $\epsilon > 0$ and $(k_n |\ n \in \mathbb{Z})$ is an arbitrary increasing bi-sequence of integers with $k_0 = 0$. Then there is a homeomorphism $f : X \to X$ satisfying the following conditions:*

1. *$d(f, h) < \epsilon$ and $f|\partial(X) = h|\partial(X)$.*

2. *f and f^{-1} are chaotic in the sense of Devaney and chaotic in the sense of Li-Yorke such that the set E is a scrambled set of f. Moreover, if $a, b \in E$ and $a \neq b$, then*

   (a) the sets $\{f^{k_n}(a)|n \in \mathbb{N}\}$ and $\{f^{k_{-n}}(a)|n \in \mathbb{N}\}$ are dense in X,

   (b) $\liminf_{n\to\pm\infty} d(f^{k_n}(a), f^{k_n}(b)) = 0$ and $\limsup_{n\to\pm\infty} d(f^{k_n}(a), f^{k_n}(b)) = \delta(X)$.

To prove the above theorem, we need the following notions: Let X be a space and R be any subset of X^m $(m \geq 2)$. A subset $F \subset X$ is said to be *independent in R* if for any different m points $x_1, \cdots, x_m$ of F (i.e., $x_i \neq x_j$ for $i \neq j$), we have $(x_1, x_2, \cdots, x_m) \in X^m - R$. A countable union of nowhere dense sets is called a set *of the first category*.

Proposition 2.6. [4, Proposition 2.3] *Suppose that X is a regularly connected polyhedron of dimension ≥ 1 and $R \subset X^m$ $(m \geq 2)$. If X has no isolated point and R is of the first category, then there is a subset S of X such that $S = \bigcup_{n=1}^{\infty} C_n$, where C_n are flat Cantor sets in X, S is independent in R, and $\mathrm{Cl}(S) = X$.*

By modifying the proof of [4, Theorem 2.6], we can prove the following.

Proposition 2.7. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 1$. Let E and S be sets which are countable unions of flat Cantor sets of $Int(X)$. Then for any $\delta > 0$ there is a homeomorphism $u : X \to X$ such that $u(E) = S$ and $d(u, Id) < \delta$.*

Proof of Theorem 2.5. Let $\{\mathcal{S}_i | i \in \mathbb{N}\}$ be a countable family which is chaotic for $(k_n | \, n \in \mathbb{Z})$. By Theorem 2.4, there is $g \in H(X, \mu)$ such that $d(g, h) < \epsilon/2$ and g satisfies the conditions as in Theorem 2.4. Then g and g^{-1} are everywhere-chaotic. Also we may assume that g and g^{-1} are chaotic in the sense of Devaney. We shall show that the set

$$T(g) = \{x \in X | \; Cl(\{f^{k_n}(x) | \; n \in \mathbb{N}\}) = X = Cl(\{f^{k_{-n}}(x) | \; n \in \mathbb{N}\})\}$$

is a dense G_δ-set of X. Let $\{U_i\}_{i \in \mathbb{N}}$ be an open countable base of X. For each $i, j \in \mathbb{N}$, consider the sets

$$T^+_{i,j} = \{x \in X | \; g^{k_n}(x) \in (X - U_i) \text{ for } n \geq j\},$$
$$T^-_{i,j} = \{x \in X | \; g^{k_{-n}}(x) \in (X - U_i) \text{ for } n \geq j\}.$$

Then

$$T(g) = X - \bigcup_{i,j \in \mathbb{N}} (T^+_{i,j} \cup T^-_{i,j}).$$

Note that each $T^{\pm}_{i,j}$ is a closed and nowhere dense set of X and hence we see that $T(g)$ is a dense G_δ-set of X. Put

$$R_0 = ((X - T(g)) \times X) \cup (X \times (X - T(g)).$$

Then R_0 is of the first category in X^2.

Next, we consider the following sets:

$$R^+_1 = \{(x, y) \in X^2 | \; \limsup_{n \to \infty} d(g^{k_n}(x), g^{k_n}(y) < \delta(X)\},$$

$$R^+_2 = \{(x, y) \in X^2 | \; \liminf_{i \to \infty} d(g^{k_i}(x), g^{k_i}(y) > 0\}.$$

Let $\{\epsilon_i\}$ be a decreasing sequence of positive numbers with $\lim_{i \to \infty} \epsilon_i = 0$. Then $R^+_1 = \bigcup_{i=1}^{\infty} T_i$, where

$$T_i = \{(x, y) \in X^2 | \, d(g^{k_n}(x), g^{k_n}(y)) \leq \delta(X) - \epsilon_i \text{ for every } n \geq i\}.$$

Also, $R^+_2 = \bigcup_{i=1}^{\infty} W_i$, where

$$W_i = \{(x, y) \in X^2 | \, d(g^{k_n}(x), g^{k_n}(y)) \geq \epsilon_i \text{ for every } n \geq i\}.$$

Since T_i and $W_i \subset X^2$ are closed, R_1^+ and R_2^+ are of the first category in X^2. Put

$$R_1^- = \{(x,y) \in X^2 | \limsup_{n \to -\infty} d(g^{k_n}(x), g^{k_n}(y) < \delta(X)\},$$

$$R_2^- = \{(x,y) \in X^2 | \liminf_{n \to -\infty} d(g^{k_n}(x), g^{k_n}(y) > 0\}.$$

Then $R = R_0 \cup R_1^+ \cup R_2^+ \cup R_1^- \cup R_2^-$ is of the first category. By Proposition 2.6, there is a subset S of X such that $S = \bigcup_{n=1}^{\infty} C_n$, where C_n are flat Cantor sets in $Int(X)$, S is independent in R and $\mathrm{Cl}(S) = X$. By Proposition 2.7, there is a homeomorphism $u : X \to X$ such that $u(E) = S$ and $d(u, Id)$ is sufficiently small. Put $f = u^{-1} \circ g \circ u$. Then $f : X \to X$ is topologically conjugate to g, $d(f,g) < \epsilon/2$ and E is the scrambled set E of f. We see that f is a desired homeomorphism.

3 Flows which realize precisely the given sequences of points on their orbits

In this section, we consider the case of continuous dynamical systems. For any $t \in \mathbb{R}$, we define the integer $< t > \in \mathbb{Z}$ by $< t > = [t + 1/2]$, where $[x]$ is the greatest integer that is less than or equal to $x \in \mathbb{R}$. Note that if $t \in \mathbb{R} - (\mathbb{Z} + 1/2)$, then the integer $< t > \in \mathbb{Z}$ satisfies $|t - < t > | < 1/2$. The main result of this section is the following theorem.

Theorem 3.1. *Suppose that X is a regularly connected polyhedron of dimension $m \geq 3$. Let $(k_n | \ n \in \mathbb{Z})$ be an arbitrary increasing bi-sequence of integers with $k_0 = 0$. Suppose that $\mathcal{S}_i = (a_n^i | n \in \Lambda_i)$ $(i \in \mathbb{N})$ are any infinite bi-sequences or finite sequences of (distinct) points which are contained in some polyhedral m-cell C of $Int(X)$ and $\mathcal{S}_i$, $\mathcal{S}_j$ $(i \neq j)$ are mutually disjoint. Then there exist $\mu \in M_\partial(X; good)$ and a μ-measure preserving flow $\varphi : X \times \mathbb{R} \to X$ satisfying the following conditions:*

1. *Each $\mathcal{S}_i$ $(i \in \mathbb{N})$ is realized by φ. Moreover if $\mathcal{S}_i = (a_n^i | \ 0 \leq n \leq s_i)$ is a finite sequence, then a_0^i is a periodic point of φ and $\mathcal{S}_i$ is realized by φ on the periodic ordered orbit of a_0^i.*

2. *If $\mathcal{S}_i$ and $\mathcal{S}_j$ are infinite bi-sequences, then there is $n(i,j) \in \mathbb{N}$ such that if $n \in \mathbb{Z}$ with $|n| \geq n(i,j)$, then*

$$< Time_\varphi(a_0^i \to a_n^i) > = < Time_\varphi(a_0^j \to a_n^j) > .$$

3. *If $\mathcal{S}_i$ is an infinite bi-sequence, then the bi-sequence $(< Time_\varphi(a_0^i \to a_n^i) > | \ n \in \mathbb{Z})$ is a subsequence of $(k_n | \ n \in \mathbb{Z})$.*

Proof. We use the methods of [7]. By [7, Lemma 1], X is a continuous image of an m-cell Z under a map which is a homeomorphism up to the boundary and which is a simplicial map of a certain subdivision of Z onto X. Hence we may assume that X is the m-dimensional unit cube and C is an m-dimensional cube in the interior $Int(X)$ of X.

Let $B = I^{m-1}$ be the $(m-1)$-dimensional unit cube and B_1 an $(m-1)$-dimensional cube in the interior of B. Also, let Q be the m-dimensional tube, that is, the product space of B with $= [0,1]$ where points $(b,0)$ and $(b,1)$ are identified and $p : B \times I \to Q$ denotes the quotient map. By the proof of [7, Theorem 3], there is an onto map $q : Q \to X$ such that $q|Int(Q)$ is an embedding and $q(\partial Q)$ is an $(m-1)$-dimensional subpolyheron of X. Hence we may assume that $X = Q$ and C is a subset of Q such that $C \subset p(B_1 \times [0,1/2])$. Choose a countable subset D of the interior $Int(B_1)$ of B_1 with $Cl(D) = B_1$. Let S be the set which is the union of all $\mathcal{S}_i$. By modifying the proof of Bennett's theorem [2], we have a homeomorphism $h : Q \to Q$ such that $h|\partial Q = Id$, $h|S : S \to D \times [0,1/2]$ is an embedding satisfying that $h(S) \cap p(\{d\} \times I)$ is an empty set or a one point set for each each $d \in D$. Consequently, we may assume that S is contained in $p(D \times [0,1/2])$ and for each each $d \in D$, $S \cap p(\{d\} \times [0,1/2])$ is an empty set or a one point set. Let d^i_n be the point of D such that $a^i_n \in p(\{d^i_n\} \times [0,1/2])$ for each $i \in \mathbb{N}, n \in \Lambda_i$. We consider the corresponding sequences $\mathcal{D}_i = (d^i_n | n \in \Lambda_i)$ $(i \in \mathbb{N})$ of the sequences $\mathcal{S}_i$. We define a measure ν in B by $\nu(A) = \int_A 1/f(p)dp$, where $f : Int(B) \to \mathbb{R}$ is a map (=continuous function) such that $\int_B 1/f(p)dp = 1$ and $f(p) > 0$ for $p \in B - \partial B$, $f(B_1) = 1$ and $f(p)$ tends to infinity at the boundary ∂B (see the proof of [7, Theorem 3]). By Theorem 2.4, we have $g \in H_\partial(B, \nu)$ satisfying the conditions of Theorem 2.4 with respect to $h = Id$, the the sequences $\mathcal{D}_i = (d^i_n | n \in \Lambda_i)$ $(i \in \mathbb{N})$ and $(k_n|\ n \in \mathbb{Z})$. Then there is an isotopy h_t of B, $0 \le t \le 1$, such that $h_t = Id$ $(0 \le t \le 1/2)$, $h_1 = g$, $h_t|\partial(B) = Id$. Define a map $\phi : B \times I \to Q$ by $\phi(x,t) = h_t(x)$ for $0 \le t \le 1$. Consider the mapping torus Q_1 of the map $g : B \to B$, i.e., Q_1 is obtained from $B \times I$ by identifying points $(x,1)$ and $(g(x),0)$ for $x \in B$. Then there is the natural homeomorphism $h : Q_1 \to Q$ such that $h([x,t]) = h_t(x)$. Hence we may assume that $Q = Q_1$. By the proof of [7,Theorem 3], we can define a flow φ upward along streamlines perpendicular to B, taking the velocity at any point to be $1/f(x)$, where x is the last intersection of the streamline with B. Then the flow φ preserves m-dimensional Lebesgue measure in Q_1 (see the proof of [7, Theorem 3]). Note that the velocity at any point x on streamlines perpendicular to B_1 is $1/f(x) = 1$. By the construction of φ, each $\mathcal{S}_i$ $(i \in \mathbb{N})$ is realized by the flow φ. Also, the m-dimensional Lebesgue measure in Q_1 induces a good measure μ on X by the map $q : Q \to X$. Let $\mathcal{S}_i$ and $\mathcal{S}_j$ be infinite bi-sequences. Since $|Time_g(d^i_0 \to d^i_n) - Time_\varphi(a^i_0 \to a^i_n)| < 1/2$, we see that

$$< Time_\varphi(a^i_0 \to a^i_n) >= Time_g(d^i_0 \to d^i_n) \in \mathbb{Z}.$$

Note that $Time_g(d^i_0 \to d^i_n) = Time_g(d^j_0 \to d^j_n)$ for $|n| \ge n(i,j)$. Hence we see that if $\mathcal{S}_i$ and $\mathcal{S}_j$ are infinite bi-sequences, then for $n \in \mathbb{Z}$ with $|n| \ge n(i,j)$,

$$< Time_\varphi(a^i_0 \to a^i_n) >=< Time_\varphi(a^j_0 \to a^j_n) > .$$

We can see that μ and φ satisfy the desired conditions of Theorem 3.1.

By a modification of the proof of Theorem 3.1, we can prove the following theorem. We omit the proof.

Theorem 3.2. *Suppose that X is a regularly connected polyhedron of dimension $m \ge 3$. If $\mathcal{S}_i$ $(i \in \mathbb{N})$ are any infinite bi-sequences or finite sequences of distinct points of $Int(X)$*

and $\mathcal{S}_i$, $\mathcal{S}_j$ $(i \neq j)$ are mutually disjoint, then there exist $\mu \in M_\partial(X; good)$ and a μ-measure preserving flow $\varphi : X \times \mathbb{R} \to X$ such that for each $i \in \mathbb{N}$, $\mathcal{S}_i$ is realized by the flow φ, and moreover if $\mathcal{S}_i$ is a finite sequence, then $\mathcal{S}_i$ is realized by φ on the periodic ordered orbit of φ.

Note that if a separable metric space S is a countable set and perfect, then S is homeomorphic to the set $\mathbb{Q}$ of all rational numbers. If $f : X \to X$ is a transitive homeomorphism of a perfect compact metric space X and $Orb(x, f)$ is dense in X, then $Orb(x, f)$ is homeomorphic to the set $\mathbb{Q}$.

Theorem 3.3. *Suppose that X is a regularly connected polyhedron of dimension $n \geq 2$ or a Menger k-dimensional manifold with $k \geq 1$. Let μ be a good measure on X, $h \in H(X, \mu)$ and $\epsilon > 0$. Suppose that S_i $(i \in \mathbb{N})$ is a dense countable subset or a finite set of $Int(X)$ such that the family $\{S_i |\ i \in \mathbb{N}\}$ are mutually disjoint. Then there is $f \in H(X, \mu)$ satisfying the following conditions:*

1. *$d(f, h) < \epsilon$ and $f|\partial(X) = h|\partial(X)$.*
2. *If S_i is an infinite set, then S_i coincides with a dense orbit of f, i.e., $S_i = Orb(a_i, f)$ for $a_i \in S_i$, and if S_i is a finite set, then S_i is a subset of a periodic orbit of f.*

Proof. Let $\mathcal{T}_i$ $(i \in \mathbb{N})$ be infinite bi-sequences of points of $Int(X)$ such that $\{\mathcal{T}_i | i \in \mathbb{N}\}$ is a chaotic family for $\mathbb{Z}$. Also, we can choose $\{\mathcal{P}_i | i \in \mathbb{N}\}$ which is a family of finite sequences of points of $Int(X)$ such that $\lim_{i\to\infty} P_i = X$ and P_i $(i \in \mathbb{N})$ are mutually disjoint, where P_i is the set induced by the sequence $\mathcal{P}_i$. By Theorem 2.4, there is $g \in H(X, \mu)$ such that $d(h, g) < \epsilon/2$, $g|\partial(X) = h|\partial(X)$ and $\mathcal{T}_i$ and $\mathcal{P}_i$ are realized by g. Moreover, we may assume that $\mathcal{P}_i$ is realized on a periodic orbit of g. Hence we can choose a countable family $\{T'_i |\ i \in \mathbb{N}\}$ of mutually disjoint dense orbits of g and a countable family $\{P'_i |\ i \in \mathbb{N}\}$ of mutually disjoint periodic orbits of g such that $\lim_{i\to\infty} P'_i = X$. By modifying the proof of Bennett [2], we can prove that there is $u \in H_\partial(X, \mu)$ satisfying the following conditions; if S_i is an infinite set, then $u(S_i) = T'_i$ and if S_i is a finite set, then $u(S_i) \subset P'_{j_i}$ for some j_i. Put $f = u^{-1} \circ g \circ u$. Then f is a desired homeomorphism.

Finally, we have the following problem.

Problem 3.4. *Are any versions of the results of this paper true in the smooth category?*

Acknowledgement: The author thanks the referee for helpful comments on the problem above.

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