On expansions of Tychonoff spaces into inverse systems of polyhedra

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Introduction

In a previous paper [8] we have established that the shape category of topological spaces is category-equivalent to a full subcategory of the pro-category of the homotopy category of CW complexes, and that such a category-equivalence can be obtained by assigning to each topological space X an inverse system in the homotopy category of CW complexes which is associated with X in the sense of our paper [8]. As such an inverse system we have the Čech system of X; it consists of the nerves of locally finite normal open covers of X.

On the other hand, in defining the notion of shape, inverse systems of ANR's for metric spaces are utilized by S. Mardešić and J. Segal [3] for the case of compact Hausdorff spaces and by R. H. Fox [2] for the case of metric spaces. The inverse systems with X as their inverse limit, which are used by these authors, induce the inverse systems in the homotopy category of spaces which are associated with X.

In view of these results it is meaningful to find a condition under which an inverse system of spaces with a given space X as its inverse limit induces an inverse system associated with X in the homotopy category of spaces. As such a condition we have introduced the notion of proper inverse systems in our previous paper [8].

In this paper we shall establish that a Tychonoff space X admits a proper inverse system of polyhedra with X as its inverse limit if and only if $\mu(X)=X$, where $\mu(X)$ is the completion of X with respect to the finest uniformity of X. This result will be obtained by making use of a recent result of P. Bacon [1].

Finally, it will be shown that zero-dimensional spaces X and Y have the same shape if and only if $\mu(X)$ is homeomorphic to $\mu(Y)$.

Throughout this paper we shall mean by a cover of a space a locally finite normal open cover, and by a polyhedron a simplicial complex with the weak topology.

§ 1. Proper inverse systems

Let $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system of topological spaces over a directed set Λ, and let X be its inverse limit; let $p_\lambda : X \to X_\lambda$ be the projection for each $\lambda \in \Lambda$.

DEFINITION 1.1. $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is said to be proper if for every cover $\mathfrak{G}$ of X and for every cover $\mathfrak{H}$ of X_λ with a given $\lambda \in \Lambda$ there are $\mu \in \Lambda$ and a cover $\mathfrak{U}$ of X_μ such that $\lambda \leq \mu$, $\mathfrak{U}$ refines $p_{\lambda\mu}^{-1}(\mathfrak{H})$, $p_\mu^{-1}(\mathfrak{U})$ refines $\mathfrak{G}$ and $\mathfrak{U}$ is proper with respect to the map p_μ. Here a cover $\mathfrak{V}$ of a space Y is defined to be proper with respect to a continuous map $f : X \to Y$ if f^{-1} induces an isomorphism between the nerves of $\mathfrak{V}$ and $f^{-1}(\mathfrak{V})$.

In a previous paper [8] we have proved the following theorems.

THEOREM 1.2. *If each X_λ is a compact Hausdorff space, then $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is proper.*

THEOREM 1.3. *If X is a metric space which is contained in another metric space P and if $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ consists of all open neighborhoods of X in P and of the inclusion maps, then $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is proper.*

THEOREM 1.4. *Let $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be a proper inverse system. Then* (i) *for any continuous map f from X to a polyhedron Q there are some $\lambda \in \Lambda$ and a continuous map $f_\lambda : X_\lambda \to Q$ such that $f \simeq f_\lambda p_\lambda : X \to Q$, and* (ii) *for any two continuous map $f_\lambda, g_\lambda : X_\lambda \to Q$ with Q a polyhedron such that $f_\lambda p_\lambda \simeq g_\lambda p_\lambda : X \to Q$ there is $\lambda' \geq \lambda$ with $f_\lambda p_{\lambda\lambda'} \simeq g_\lambda p_{\lambda\lambda'} : X_{\lambda'} \to Q$; that is, the inverse system $\{X_\lambda, [p_{\lambda\lambda'}], \Lambda\}$ in the homtopy category of spaces is associated with X in the sense of* [8].

Theorem 1.3 will be generalized in § 5.

§ 2. Level inverses ystems

Having read the first version of [8], P. Bacon kindly informed me that the definition of proper inverse systems there had a defect which made the proof of [8, Theorem 1.9] (=Theorem 1.4 above) invalid. In the second version of [8] I modified the definition slightly (as in § 1 above) so that the proof may be valid without any alteration. After this I received a copy of [1], in which Bacon defined level inverse systems and proved actually the assertion that a level inverse system of paracompact Hausdorff spaces induces an inverse system in the homotopy category of spaces which is associated with its inverse limit in the sense of [8] (cf. Theorem 1.4 above); in particular, his proof for the property (i) of Theorem 1.4 above is carried out along the same line as our proof described in [8] by showing essentially the assertion that a level inverse system of paracompact Hausdorff spaces is proper (the defect mentioned above does not concern the proof for the property (i) of Theorem 1.4 above). The latter assertion (and hence the former by Theorem 1.4) holds without the assumption that a level inverse system consists of paracompact Hausdorff spaces, as will be shown in Theorem 3.1 below.

Let $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system of spaces and let X be its inverse limit; let $p_\lambda : X \to X_\lambda$ be the projection for $\lambda \in \Lambda$.

DEFINITION 2.1. $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is said to be level if (a) for any cover $\mathfrak{G}$ of X there exist $\lambda \in \Lambda$ and a cover $\mathfrak{U}$ of X_λ such that $p_\lambda^{-1}(\mathfrak{U})$ refines $\mathfrak{G}$ and (b) for any $\lambda \in \Lambda$ and any open set V of X_λ with $V \supset p_\lambda(X)$ there is $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subset V$.

Let X be a topological space. Let $\{\mathfrak{U}_\alpha | \alpha \in \Omega\}$ be the set of all locally finite cozero-set covers of X. Then we can associate with each $\mathfrak{U}_\alpha$ a partition of unity $\{\phi_{\alpha,j} | j \in J_\alpha\}$ such that $\mathfrak{U}_\alpha$ consists of cozero-sets $\{x \in X | \phi_{\alpha,j}(x) > 0\}$ with $j \in J_\alpha$. Let Γ be the set of all non-empty finite subsets of Ω. For $\gamma = \{\alpha_1, \cdots, \alpha_n\} \in \Gamma$ let us put $\mathfrak{U}_\gamma = \bigwedge_{i=1}^{n} \mathfrak{U}_{\alpha_i}$ and define a canonical map ϕ_γ from X into the nerve $N(\mathfrak{U}_\gamma)$ of $\mathfrak{U}_\gamma$ by

$$\phi_\gamma(x) = \sum \left(\prod_{i=1}^{n} \phi_{\alpha_i, j_i}(x) \right) v((\alpha_1, j_1), \cdots, (\alpha_n, j_n))$$

where $v((\alpha_1, j_1), \cdots, (\alpha_n, j_n))$ denotes the vertex of the nerve of $\bigwedge_{i=1}^{n} \mathfrak{U}_{\alpha_i}$ corresponding to the set $\bigcap_{i=1}^{n} \{x \in X | \phi_{\alpha_i, j_i}(x) > 0\}$, and $\sum$ ranges over all such $((\alpha_1, j_1), \cdots, (\alpha_n, j_n))$. For γ, $\gamma' \in \Gamma$ we define $\gamma \leq \gamma'$ by $\gamma \subset \gamma'$, and for $\gamma \leq \gamma'$ we shall define a simplicial map $\phi_{\gamma\gamma'}$ from $N(\mathfrak{U}_{\gamma'})$ to $N(\mathfrak{U}_\gamma)$ by assigning to each vertex $v((\alpha_1, j_1), \cdots, (\alpha_m, j_m))$ the vertex $v((\alpha_{k_1}, j_{k_1}), \cdots, (\alpha_{k_n}, j_{k_n}))$, where we assume that $\gamma' = \{\alpha_1, \cdots, \alpha_m\}$ and $\gamma = \{\alpha_{k_1}, \cdots, \alpha_{k_n}\}$ with $1 \leqq k_1 < k_2 < \cdots < k_n \leqq m$. Then we have

$$\phi_\gamma = \phi_{\gamma\gamma'} \phi_{\gamma'} \qquad \text{for} \qquad \gamma \leq \gamma'.$$

Now, let (γ, U) be a pair such that U is an open set of $N(\mathfrak{U}_\gamma)$ containing $\phi_\gamma(X)$ and let us define a partial order $(\gamma, U) \leq (\gamma', U')$ by requiring that $\gamma \leq \gamma'$, $\phi_{\gamma\gamma'}(U') \subset U$. Let Λ be the set of all such pairs (γ, U). For $\lambda = (\gamma, U)$ let us put $X_\lambda = U$ and define $p_\lambda : X \to X_\lambda$ by $p_\lambda(x) = \phi_\gamma(x)$ for $x \in X$. For $\lambda = (\gamma, U) \leq \lambda' = (\gamma', U')$ let us define $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ by $p_{\lambda\lambda'}(x) = \phi_{\gamma\gamma'}(x)$, $x \in X_{\lambda'}$. Then it is easy to see that inverse system $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ satisfies the conditions (a) and (b) in Definition 2.1. In the present case X is not necessarily the inverse limit of $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ but for each $\lambda \in \Lambda$ there is a continuous map $p_\lambda : X \to X_\lambda$ such that $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$ for $\lambda \leq \lambda'$.

In such a case Bacon [1] say that *the inverse system* $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *has* X *as its complement.* Since an open subspace of a polyhedron is itself a polyhedron, Bacon [1] proved the following theorem by the above construction.

THEOREM 2.2. *Any topological space* X *admits an inverse system of polyhedra with* X *as is complement.*

§ 3. Main theorems

We are now in a position to state our main theorems.

THEOREM 3.1. *Every level inverse system of topological spaces is proper.*

THEOREM 3.2. *A Tychonoff space* X *admits a proper inverse system of polyhedra with* X *as its inverse limit iff* X *is topologically complete.*

Since any polyhedron is paracompact Hausdorff[1)] and any paracompact Hausdorff space is topologically complete, the inverse limit of an inverse system of polyhedra is topologically complete. Hence, in view of Theorems 2.2 and 3.1, Theorem 3.2 is a direct consequence of Theorem 3.3 below which is proved by Bacon [1] for the case of X being paracompact Hausdorff.

THEOREM 3.3 *If X is a Tychonoff space which is topologically complete and $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is an inverse system of T_1-spaces with X as its complement, then X is the inverse limit of $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$.*

PROOF. Let $\{g_\lambda : Y \to X_\lambda | \lambda \in \Lambda\}$ be a set of continuous maps such that

$$g_\lambda = p_{\lambda\lambda'} g_{\lambda'} \qquad \text{for} \qquad \lambda \leq \lambda'.$$

Let us put

$$C(y) = \{p_\lambda^{-1} g_\lambda(y) | \lambda \in \Lambda\}.$$

First, let us note that $p_\lambda^{-1} g_\lambda(y) \neq \emptyset$ for each λ; because, otherwise we would have $p_{\lambda\mu}(X_\mu) \cap g_\lambda(y) = \emptyset$ for some $\mu \in \Lambda$ with $\lambda \leq \mu$ by virtue of condition (b) but this is contradictory to the fact that $p_{\lambda\mu} g_\mu(y) = g_\lambda(y)$. Secondly, $C(y)$ has the finite intersection property since we have $p_{\lambda'}^{-1} g_{\lambda'}(y) \subset p_\lambda^{-1} g_\lambda(y)$ for $\lambda \leq \lambda'$. Finally, for any cover $\mathfrak{G}$ of X there exist $\lambda \in \Lambda$ and a cover $\mathfrak{U}$ of X_λ such that $p_\lambda^{-1}(\mathfrak{U})$ refines $\mathfrak{G}$. If $g_\lambda(y) \in U \in \mathfrak{U}$, then $p_\lambda^{-1} g_\lambda(y) \subset p_\lambda^{-1}(U)$. Since $p_\lambda^{-1}(U) \subset G$ for some $G \in \mathfrak{G}$ we have $p_\lambda^{-1} g_\lambda(y) \subset G \in \mathfrak{G}$. Thus, $C(y)$ is a Cauchy family with respect to the finest uniformity of X. Since X is complete with respect to this uniformity, $\cap \{p_\lambda^{-1} g_\lambda(y) | \lambda \in \Lambda\}$ consists of a single point, which shall be denoted by $g(y)$.

Let G be any open set of X containing $g(y)$. Then there are $\lambda \in \Lambda$ and a cover $\mathfrak{H}$ of X such that $\mathrm{St}\,(p_\lambda^{-1} g_\lambda(y), \mathfrak{H}) \subset G$. From condition (a) we see that there are $\mu \in \Lambda$ with $\lambda \leq \mu$ and a cover $\mathfrak{V}$ of X_μ such that $p_\mu^{-1}(\mathfrak{V})$ refines $\mathfrak{H}$. Hence we have

$$\text{(1)} \qquad \mathrm{St}\,(p_\mu^{-1} g_\mu(y), p_\mu^{-1}(\mathfrak{V})) \subset G.$$

On the other hand, by the continuity of g_μ we can find an open neighborhood W of y such that

$$\text{(2)} \qquad g_\mu(W) \subset \mathrm{St}\,(g_\mu(y), \mathfrak{V}).$$

Since $p_\mu^{-1}(\mathrm{St}\,(g_\mu(y), \mathfrak{V})) = \mathrm{St}\,(p_\mu^{-1} g_\mu(y) p_\mu^{-1}(\mathfrak{V}))$, we have from (1) and (2)

$$\text{(3)} \qquad p_\mu^{-1} g_\mu(W) \subset G.$$

Since (3) shows that $g(W) \subset G$, the continuity of g is proved hereby.

1) More generally, any CW complex is paracompact and Hausdorff. This result was proved first by Morita [5] (a simpler proof can be found in [6, § 3]). In [5] I wrote that the result was proved earlier by H. Miyazaki, the paracompactness of CW complexes, Tohoku Math. J. 4 (1952), 309-313. This quotation, however, was wrong. Because, as was pointed out by topologists in Osaka, his proof was incorrect.

On the other hand, we have

$$g_\lambda = p_\lambda g \quad \text{for} \quad \lambda \in \Lambda,$$

since $p_\lambda g(y) \subset \cap \{p_\lambda(p_\mu^{-1} g_\mu(y)) | \mu \in \Lambda\} \subset p_\lambda p_\lambda^{-1} g_\lambda(y) = g_\lambda(y)$. Moreover, if $h: Y \to X$ is another continuous map such that $g_\lambda = p_\lambda h$ for each $\lambda \in \Lambda$, then $h(y) \in \cap \{p_\lambda^{-1} g_\lambda(y) | \lambda \in \Lambda\} = g(y)$ and hence we have $h = g$. This completes the proof of Theorem 3.3.

THEOREM 3.4. *Let $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system of topologically complete Tychonoff spaces with a Tychonoff space X as its complement. Then the completion of X with respect to its finest uniformity, $\mu(X)$ in notation, is the inverse limit of $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$.*

PROOF. For an open set G of X let us put

$$G^* = \mu X - Cl_{\mu X}(X - G).$$

Then for any open set U of μX we have

$$U \subset (U \cap X)^* \subset Cl_{\mu X}(U \cap X). \tag{4}$$

Let μ be the covariant functor from the category of Tychonoff spaces and continuous maps to its full subcategory of topologically complete spaces which assigns to each space X the completion of X with respect to the finest uniformity of X. Then we have continuous maps $\mu(p_\lambda): \mu X \to X_\lambda$ for $\lambda \in \Lambda$ and $\mu(p_\lambda) = p_{\lambda\lambda'} \mu(p_{\lambda'})$ for $\lambda \leq \lambda'$.

Let $\mathfrak{G}$ be any cover of μX, and $\mathfrak{H}$ a star-refinement of $\mathfrak{G}$. By condition (a) there are $\lambda \in \Lambda$ and a cover $\mathfrak{U}$ of X_λ such that $\mathfrak{H} \cap X$ is refined by $p_\lambda^{-1}(\mathfrak{U})$. Let U be any set belonging to $\mathfrak{U}$. Then there is some $H \in \mathfrak{H}$ such that $X \cap \mu(p_\lambda)^{-1}(U) \subset X \cap H$. Hence by (4) we have

$$\mu(p_\lambda)^{-1}(U) \subset \mathrm{Cl}_{\mu X}(X \cap H) \subset \mathrm{St}(H, \mathfrak{H}). \tag{5}$$

Since there is some $G \in \mathfrak{G}$ such that $\mathrm{St}(H, \mathfrak{H}) \subset G$, we have $\mu(p_\lambda)^{-1}(U) \subset G$. This shows that $\mu(p_\lambda)^{-1}(\mathfrak{U})$ refines $\mathfrak{G}$. Thus condition (a) is satisfied for $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and $\mu(p_\lambda): \mu X \to X_\lambda$ with $\lambda \in \Lambda$.

If $\mu(p_\lambda)(\mu X) \subset U$ for an open set U in X_λ, then $p_\lambda(X) \subset U$ and hence $p_{\lambda\mu}(X_\mu) \subset U$ for some $\mu \in \Lambda$ with $\lambda \leq \mu$.

Therefore, the inverse system $\{X_\lambda p_{\lambda\lambda'}, \Lambda\}$ has $\mu(X)$ as its complement. Since μX is topologically complete, $\mu(X)$ is the inverse limit of $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ by Theorem 3.3.

§ 4. Proof of Theorem 3.1

Before proceeding to the proof of Theorem 3.1 we shall need two lemmas.

LEMMA 4.1. (Morita and Hoshina [9]). *Let X be a topological space. Let $\{F_\lambda | \lambda \in \Lambda\}$ and $\{G_\lambda | \lambda \in \Lambda\}$ be locally finite collections of zero-sets and of cozero-sets in X respectively such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$. Then $\cup \{F_\lambda | \lambda \in \Lambda\}$ is a zero-set in X.*

LEMMA 4.2. *Let* $\{F_\lambda|\lambda\in\Lambda\}$ *and* $\{G_\lambda|\lambda\in\Lambda\}$ *be the same as in Lemma 4.1. Then there is a collection* $\{V_\lambda|\lambda\in\Lambda\}$ *of cozero-sets in* X *such that* (i) $F_\lambda\subset V_\lambda\subset\mathrm{Cl}\ V_\lambda\subset G_\lambda$ *for each* $\lambda\in\Lambda$ *and* (ii) $\{\mathrm{Cl}\ V_\lambda|\lambda\in\Lambda\}$ *is similar to* $\{F_\lambda|\lambda\in\Lambda\}$.

PROOF. Let us well-order the index set Λ and assume that $\Lambda=\{\alpha|0\leqq\alpha<\xi\}$ for some ordinal ξ. Let Γ be the set of all finite sets $\gamma=\{\alpha_1,\cdots,\alpha_n\}$ with $\alpha_i<\xi$, $i=1,2,\cdots,n$, such that $\bigcap_{i=1}^{n} F_{\alpha_i}\neq\emptyset$ but $F_0\cap(\bigcap_{i=1}^{n} F_{\alpha_i})=\emptyset$.

Then $\cap\{F_\alpha|\alpha\in\gamma\}$ is a zero-set for each $\gamma\in\Gamma$. On the other hand, $\{\bigcap_{\alpha\in\gamma} G_\alpha|\gamma\in\Gamma\}$ is locally finite and each member of this collection is a cozero-set. Let us put

$$S_0=\cup\{\bigcap_{\alpha\in\gamma} F_\alpha|\gamma\in\Gamma\}.$$

Then S_0 is a zero-set of X by Lemma 4.1. Since

$$F_0\cap(S_0\cup(X-G_0))=\emptyset,$$

there is a continuous map $\phi_0: X\to[0,1]$ such that

$$\phi_0(x)=\begin{cases}0, & \text{for}\quad x\in F_0\\ 1, & \text{for}\quad x\in(X-G_0)\cup S.\end{cases}$$

Let us put

$$V_0=\left\{x\in X\,|\phi_0(x)<\frac{1}{2}\right\},$$

$$K_0=\left\{x\in X\,|\phi_0(x)\leq\frac{1}{2}\right\}.$$

Then V_0 is a cozero-set, K_0 is a zero-set and $F_0\subset V_0\subset\mathrm{Cl}\ V_0\subset K_0\subset G_0$. Since $K_0\cap(S_0\cup(X-G_0))=\emptyset$, $\{F_\alpha|0\leq\alpha<\xi\}$ and $\{K_0, F_\alpha|0<\alpha<\xi\}$ are similar.

By transfinite induction we can construct cozero-sets V_α, zero-sets K_α for each $\alpha<\xi$ such that

$$(\mathrm{a})_\alpha \qquad F_\alpha\subset V_\alpha\subset\mathrm{Cl}\ V_\alpha\subset K_\alpha\subset G_\alpha,$$

$$(\mathrm{b})_\alpha \qquad \{K_\beta, F_\gamma|\beta<\alpha, \alpha\leqq\gamma<\xi\} \text{ is similar to } \{K_\beta, F_\gamma|\beta\leqq\alpha, \alpha<\gamma<\xi\}.$$

This construction is carried out by the same argument as in the case $\alpha=0$. (Cf. the argument in the proof of [4, Theorem 1.3]).

Then $\{\mathrm{Cl}\ V_\alpha|\alpha<\xi\}$ is similar to $\{F_\alpha|\alpha<\xi\}$. This proves Lemma 4.2.

Now we are able to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system with X as its complement. Then, for each $\lambda\in\Lambda$ there is a continuous map $p_\lambda: X\to X_\lambda$ such that $p_\lambda=p_{\lambda\lambda'}\cdot p_{\lambda'}$ for $\lambda\leq\lambda'$.

Let $\mathfrak{G}$ be any cover of X and $\mathfrak{H}$ any cover of X_λ with $\lambda\in\Lambda$. Then by condition (a) in Definition 2.1 there are $\nu\in\Lambda$ with $\lambda\leq\nu$ and a cover $\mathfrak{U}$ of X_ν such that $\mathfrak{U}$

refines $p_{\lambda\nu}^{-1}(\mathfrak{H})$ and $p_{\nu}^{-1}(\mathfrak{U})$ refines $\mathfrak{G}$.

Here we may assume that $\mathfrak{U}=\{U_\alpha|\alpha\in\Omega\}$ is a locally finite cozero-set cover of X_ν. Then there is a locally finite zero-set cover $\mathfrak{F}=\{F_\alpha|\alpha\in\Omega\}$ such that $F_\alpha\subset U_\alpha$ for each $\alpha\in\Omega$.

Let Γ be the set of all finite subsets γ of Ω such that

$$\cap\{F_\alpha|\alpha\in\gamma\}\neq\emptyset,\quad [\cap\{F_\alpha|\alpha\in\gamma\}]\cap p_\nu(X)=\emptyset.$$

Since $\{\bigcap_{\alpha\in\gamma} F_\alpha|\gamma\in\Gamma\}$ is locally finite, the union F of all the sets in this collection is closed. Hence by condition (b) in Definition 2.1 there exists $\mu\in\Lambda$ with $\nu\leq\mu$ such that $p_{\nu\mu}(X_\mu)\cap F=\emptyset$, that is, $p_{\nu\mu}^{-1}(F)=\emptyset$.

Let us put

$$K_\alpha=p_{\nu\mu}^{-1}(F_\alpha),\quad L_\alpha=p_{\nu\mu}^{-1}(U_\alpha)\quad \text{for}\quad \alpha\in\Omega.$$

Then for any finite subset γ of Ω

$$(6)\qquad \cap\{K_\alpha|\alpha\in\gamma\}\neq\emptyset \Longrightarrow \cap\{p_\mu^{-1}(K_\alpha)|\alpha\in\gamma\}\neq\emptyset.$$

Because, if $\cap\{p_\mu^{-1}(K_\alpha)|\alpha\in\gamma\}=\emptyset$, then $p_\mu(X)\cap[\cap\{K_\alpha|\alpha\in\gamma\}]=\emptyset$ and hence $p_\nu(X)\cap[\cap\{F_\alpha|\alpha\in\gamma\}]=\emptyset$, and hence we would have $\gamma\in\Gamma$, but this is a contradiction since $\cap\{K_\alpha|\alpha\in\gamma\}=\emptyset$ for $\gamma\in\Gamma$.

Now let us observe that $\{K_\alpha\}$ and $\{L_\alpha\}$ are locally finite collections of zero-sets and of cozero-sets in X_μ respectively such that $K_\alpha\subset L_\alpha$ for each α. Hence by Lemma 4.2 there is a locally finite collection $\mathfrak{V}=\{V_\alpha|\alpha\in\Omega\}$ of cozero-sets in X_μ such that $\mathfrak{V}$ is similar to $\mathfrak{K}=\{K_\alpha|\alpha\in\Omega\}$. This collection $\mathfrak{V}$ is a cover of X_μ which refines $p_{\nu\mu}^{-1}(\mathfrak{U})$. Since $\mathfrak{U}$ refines $p_{\lambda\nu}^{-1}(\mathfrak{H})$ and $p_\nu^{-1}(\mathfrak{U})$ refines $\mathfrak{G}$, $\mathfrak{V}$ refines $p_{\lambda\mu}^{-1}(\mathfrak{H})$ and $p_\mu^{-1}(\mathfrak{V})$ refines $\mathfrak{G}$.

Suppose that $\cap\{U_\alpha|\alpha\in\gamma\}\neq\emptyset$ for a finite subset γ of Ω. Then $\cap\{K_\alpha|\alpha\in\gamma\}\neq\emptyset$ and hence $\cap\{p_\mu^{-1}(K_\alpha)|\alpha\in\gamma\}\neq\emptyset$ by (6). Therefore, we have $\cap\{p_\mu^{-1}(V_\alpha)|\alpha\in\gamma\}\neq\emptyset$. On the other hand, $\cap\{p_\mu^{-1}(V_\alpha)|\alpha\in\gamma\}\neq\emptyset$ implies $\cap\{V_\alpha|\alpha\in\gamma\}\neq\emptyset$ for a finite subset γ of Ω.

Therefore $\mathfrak{V}$ is a proper cover of X_μ with respect to the map p_μ. This completes the proof of Theorem 3.1.

§ 5. Applications

A subset A of a space X is called P-embedded (resp. $P^{\mathfrak{m}}$-embedded) in X if every cover of A (resp. every cover of A of cardinality $\leqq\mathfrak{m}$) has a refinement which can be extended to a cover of X (cf. the convention for cover at the end of the introduction). Then as an immediate consequence of Theorem 3.1 we have the following theorem.

THEOREM 5.1. *Let A be a subset of a topological space X. Let $\mathfrak{U}(A, X)$ be the inverse system which consists of open neighborhoods of A in X and which has the inclusion maps between them as bonding maps. Suppose that either A is P-embedded in X or X is hereditarily paracompact Hausdorff. Then $\mathfrak{U}(A, X)$ is proper.*

As a corollary to Theorem 5.1 we have the following theorem by virtue of Theorem 1.4 and [8, Theorem 3.2].

THEOREM 5.2. *Under the same assumption as in Theorem 5.1, we have an isomorphism*:

$$H^n(A;G)\cong\varinjlim\{H^n(U;G)\mid U\in\mathfrak{U}(A;X)\}$$

where $H^n(S;G)$ denotes the n-th Čech cohomology group of a space S with coefficients in an abelian group G which is defined by using locally finite normal open covers of S.

The proof of Theorem 5.1 yields also the following theorem, which may be regarded as a substitute for [2, Lemma, 5.5].

THEOREM 5.3. *Let A be a subset of a topological space X. Let $\mathfrak{m}$ be an infinite cardinal number. If either A is $P^{\mathfrak{m}}$-embedded in X or every subspace of X is $\mathfrak{m}$-paracompact and normal, then for any normal open cover $\mathfrak{G}$ of A of cardinality $\leqq\mathfrak{m}$ there exist an open neighborhood U of A in X and a locally finite normal open cover $\mathfrak{V}$ of U of cardinality $\leqq\mathfrak{m}$ such that $\mathfrak{V}\cap A$ refines $\mathfrak{G}$ and $\mathfrak{V}$ is a proper cover of U with respect to the inclusion map from A to U.*

§ 6. The shape of zero-dimensional spaces

Let X be a Tychonoff space such that $\dim X=0$ in the sense of [7]. Then the set $\{\mathfrak{U}_\lambda|\lambda\in\Lambda\}$ of all locally finite normal open covers of X of order 1 is cofinal (with respect to the partial order by refinement) in the set of all locally finite normal open covers of X. Let X_λ be the nerve of $\mathfrak{U}_\lambda$; let $p_\lambda: X\to X_\lambda$ be a canonical map and $p_{\lambda\lambda'}: X_{\lambda'}\to X_\lambda$ a canonical projection for $\lambda\leq\lambda'$, where by $\lambda\leq\lambda'$ we mean that $\mathfrak{U}_{\lambda'}$ is a refinement of $\mathfrak{U}_\lambda$. If $\lambda\leq\mu\leq\nu$, then $p_{\lambda\mu}p_{\mu\nu}\simeq p_{\lambda\nu}$ and $p_{\lambda\mu}p_\mu\simeq p_\lambda$, but, since each X_λ is discrete, we have actually $p_\lambda=p_{\lambda\mu}p_\mu$ and $p_{\lambda\mu}p_{\lambda\nu}=p_{\lambda\nu}$. Thus, $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is an inverse system of discrete spaces with X as its complement; in the present case each p_λ is surjective. Hence by Theorem 3.4 we have

$$\mu(X)=\varprojlim\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}.$$

Let Y be another Tychonoff space with $\dim Y=0$, and let us construct an inverse system $\{Y_\mu, q_{\mu\mu'}, M\}$ of discrete spaces for Y, which corresponds to $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ above. Then $\{Y_\mu, p_{\mu\mu'}, M\}$ has $\mu(Y)$ as its inverse limit and $\{Y_\mu, [q_{\mu\mu'}], M\}$ is isomorphic to the Čech system of Y in the homotopy category $\mathfrak{W}$ of polyhedra.

According to our approach to shape theory in [8], a shape morphism from X to Y is an equivalence class of system maps from the Čech system of X to the Čech system of Y. In the present case, any system map from $\{X_\lambda, [p_{\lambda\lambda'}], \Lambda\}$ to $\{Y_\mu, [q_{\mu\mu'}], M\}$ is obtained from a system map from $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ to $\{Y_\mu, q_{\mu\mu'}, M\}$. Thus, any shape morphism from X to Y is induced by a continuous map from $\mu(X)$ to $\mu(Y)$. Therefore if X and Y are of the shape, then $\mu(X)$ and $\mu(Y)$ are homeomorphic. Conversely, if $\mu(X)$ and $\mu(Y)$ are homeomorphic, then $\mu(X)$ and

$\mu(Y)$ are of the same shape, and hence by [8, Theorem 5.2] X and Y are of the same shape. Thus, we have by [8, Theorem 5.1]

THEOREM 6.1. *Let X and Y be topological spaces such that* $\dim X=0$ *and* $\dim Y=0$ *in the sense of* [7]. *Then X and Y are of the same shape if and only if $\mu\tau(X)$ is homeomorphic to $\mu\tau(Y)$, where τ is the Tychonoff functor which is a reflector from the category of topological spaces to its full subcategory of Tychonoff spaces* (*cf.* [7, § 1]).

For the special case of X and Y being paracompact Hausdorff, Theorem 6.1 reduces to a theorem proved by Kozlowski and Segal [10].

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