The Hurewicz and the Whitehead theorems in shape theory

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(Received June 26, 1974)

Introduction. The notion of shape was originally introduced by K. Borsuk [2] for compact metric spaces. Since then shape theory has achieved a remarkable development. In particular, the notion of shape is extended to the case of arbitrary topological spaces by S. Mardešić [6]. In the present paper we shall understand the notion of shape in the sense of Mardešić [6].

In a previous paper [10] we gave a new approach to shape theory by means of the Čech systems.

Let (X, A, x_0) be a pair of pointed topological spaces. Let $\{\mathcal{U}_\lambda | \lambda \in \Lambda\}$ be the family of all locally finite normal open covers of X such that each $\mathcal{U}_\lambda$ has exactly one member containing x_0. Then we have an inverse system $\{(X_\lambda, A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}$ in the pro-category of the homotopy category of pairs of pointed CW complexes by taking the nerves of $\mathcal{U}_\lambda$ and $\mathcal{U}_\lambda \cap A$, by ordering Λ by means of refinement of covers, and by taking the homotopy classes of canonical projections. We call this inverse system the Čech system of (X, A, x_0). The Čech system of (X, A) is defined similarly by using all locally finite normal open covers of X.

According to our approach in [10], a shape morphism from (X, A, x_0) to (Y, B, y_0), where (X, A, x_0) and (Y, B, y_0) are pairs of pointed spaces, is defined to be an equivalence class of system maps from the Čech system of (X, A, x_0) to the Čech system of (Y, B, y_0). Shape morphisms between pairs of spaces are defined similarly. Our approach enables us to discuss the notions of homotopy and homology pro-groups in shape theory.

The n-th (Čech) homotopy pro-group $\pi_n(X, A, x_0)$ is defined to be a pro-group $\{\pi_n(X_\lambda, A_\lambda, x_{0\lambda}), \pi_n(p_{\lambda\lambda'}), \Lambda\}$ $(n \geqq 2)$; $\pi_1(X, A, x_0) = \{\pi_1(X_\lambda, A_\lambda, x_{0\lambda}), \pi_1(p_{\lambda\lambda'}), \Lambda\}$ is considered as a pro-object in the category of pointed sets and base-point preserving maps.

The n-th (Čech) homology pro-group $H_n(X, A)$ with coefficients in the additive group of integers is defined similarly by using the Čech system of (X, A). Since $\{\mathcal{U}_\lambda | \lambda \in \Lambda\}$ described above to define the Čech system of (X, A, x_0) is cofinal in the family of all locally finite normal open covers of X[1)], the inverse system

1) Any locally finite normal open cover $\mathcal{U}$ of X admits a locally finite cozero-set cover $\{G_\alpha | \alpha \in \Omega\}$ of X as its refinement. Let $x_0 \in G_{\alpha_0}$. Then there is a zero-set F of X such that $x_0 \in F \subset G_{\alpha_0}$. The locally finite cozero-set cover $\{G_{\alpha_0}, G_\alpha - F | \alpha \in \Omega, \alpha \neq \alpha_0\}$ of X is a refinement of $\mathcal{U}$, and G_{α_0} is the sole member containing x_0.

$\{H_n(X_\lambda, A_\lambda), H_n(p_{\lambda\lambda'}), \Lambda\}$ is isomorphic to $H_n(X, A)$ in the category of pro-groups. Hence, the set of the Hurewicz homomorphisms $\Phi_n(X_\lambda, A_\lambda, x_{0\lambda}): \pi_n(X_\lambda, A_\lambda, x_{0\lambda}) \longrightarrow H_n(X_\lambda, A_\lambda)$ for $\lambda \in \Lambda$ determines a morphism $\Phi_n(X, A, x_0): \pi_n(X, A, x_0) \longrightarrow H_n(X, A)$ in the category of pro-groups, which we called the Hurewicz morphism in [11].

A subspace A of a space X is said to be P-embedded in X if every locally finite normal open cover of A has a refinement which can be extended to a locally finite normal open cover of X. If A is P-embedded in X, $\{(A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}|(A_{\lambda'}, x_{0\lambda'})], \Lambda\}$, which is obtained from the Čech system of (X, A, x_0), is isomorphic to the Čech system of (A, x_0).

A pro-group $G=\{G_\lambda, \phi_{\lambda\lambda'}, \Lambda\}$ is a zero-object, $G=0$ in notation, if G is isomorphic to a pro-group consisting of a single trivial group, or equivalently, if for each $\lambda \in \Lambda$ there is some $\lambda' \geqq \lambda$ with $\phi_{\lambda\lambda'}=0$. In case (X, x_0) is a pointed compact metric space, $\pi_k(X, x_0)=0$ is equivalent to saying that (X, x_0) is approximatively k-connected in the sense of Borsuk [3] (cf. Lemma 2.3 below).

Let (X, A, x_0) and (Y, B, y_0) be pairs of pointed topological spaces. Then a shape morphism f from (X, A, x_0) to (Y, B, y_0) induces a morphism from $\pi_n(X, A, x_0)$ to $\pi_n(Y, B, y_0)$ in the pro-category of groups for $n \geqq 2$ and in the pro-category of pointed sets for $n=1$; this morphism is denoted by $\pi_n(f)$. Similarly a shape morphism f between pairs of spaces (X, A) and (Y, B) induces a morphism between $H_n(X, A)$ and $H_n(Y, B)$ in the pro-category of groups, which we denote by $H_n(f)$.

In the present paper we shall establish the following theorems as analogues in shape theory of the classical Hurewicz theorem and the Whitehead theorems in homotopy theory.

THEOREM A. *Let (X, A, x_0) be a pair of pointed, connected, topological spaces such that $\pi_k(X, A, x_0)=0$ for k with $1 \leqq k \leqq n$ $(n \geqq 1)$. Then $H_k(X, A)=0$ for $1 \leqq k \leqq n$. If A is P-embedded in X and $\pi_1(A, x_0)=0$, then the Hurewicz morphism $\Phi_k(X, A, x_0)$: $\pi_k(X, A, x_0) \longrightarrow H_k(X, A)$ is an isomorphism for $k=n+1$ and an epimorphism for $k=n+2$.*

THEOREM B. *Let $f: (X, x_0) \longrightarrow (Y, y_0)$ be a shape morphism of pointed connected topological spaces. For $n \geqq 2$, let us consider the following two conditions.*

(a) *$\pi_k(f)$: $\pi_k(X, x_0) \longrightarrow \pi_k(Y, y_0)$ is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k=n$.*

(b) *$H_k(f)$: $H_k(X) \longrightarrow H_k(Y)$ is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k=n$.*

Then (a) *implies* (b), *and conversely, in case $\pi_1(X, x_0)=0$ and $\pi_1(Y, y_0)=0$,* (b) *implies* (a).

THEOREM C. *Let $f: (X, x_0) \longrightarrow (Y, y_0)$ be a shape morphism of pointed connected topological spaces of finite dimension and let $n = max(1+dim X, dim Y)$. If the induced morphism $\pi_k(f)$: $\pi_k(X, x_0) \longrightarrow \pi_k(Y, y_0)$ of homotopy pro-groups is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k=n$, then f is a shape equivalence.*

Theorem A was proved in our previous paper [11] except the assertion concerning epimorphisms. Theorem B was proved in [12] for the case of f being induced by a continuous map. Theorem C has been proved hitherto for the following cases.

(i) X and Y are compact metric spaces and $n=\max(1+\dim X, \dim Y)+1$ (M. Moszyńska [13]).

(ii) X and Y are compact Hausdorff spaces with Y metrizable and n is the same as in (i) (Mardešić [7]).

(iii) f is induced by a continuous map and n is the same as in (i) (Mardešić [7]).

(iv) f is induced by a continuous map (Morita [12]).

In Mardešić [7] the problem to prove or to disprove Theorem C remained open. Our result settles this problem.

Here the covering dimension of a topological space X, $\dim X$ in notation, is defined to be the least integer such that every locally finite normal open cover of X admits a locally finite normal open cover of X of order $\leqq n+1$ as its refinement (cf. Morita [9]).

A homological version of Theorem C is obtained from Theorems B and C.

THEOREM D. *Let $f: (X, x_0) \longrightarrow (Y, y_0)$ be a shape morphism of pointed connected spaces and let $n=max(1+dimX, dimY)$. If $\pi_1(X, x_0)=0$ and $\pi_1(Y, y_0)=0$, and if the induced morphism $H_k(f): H_k(X) \longrightarrow H_k(Y)$ is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k=n$, then f is a shape equivalence.*

§1 is devoted to establishing some basic theorems on pro-categories which have applications in shape theory of topological spaces; some of them may be of interest in themselves.

§1. Some theorems on pro-categories

Let $\mathfrak{K}$ be a category. Let $X=\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system in $\mathfrak{K}$ (i.e. Λ is a directed set with a partial order $\leq$, X_λ is an object in $\mathfrak{K}$ for each $\lambda \in \Lambda$, and $p_{\lambda\lambda'}$: $X_{\lambda'} \longrightarrow X_\lambda$ is a morphism in $\mathfrak{K}$ such that $p_{\lambda\lambda}=1$, $p_{\lambda\lambda'}p_{\lambda'\lambda''}=p_{\lambda\lambda''}$ for $\lambda \leq \lambda' \leq \lambda''$). Let $Y=\{Y_\mu, q_{\mu\mu'}, M\}$ be another inverse system in $\mathfrak{K}$. A map of inverse systems, or simply a system map, from X to Y consists of a map $\phi: M \longrightarrow \Lambda$ and a collection $\{f_\mu: X_{\phi(\mu)} \longrightarrow Y_\mu | \mu \in M\}$ of morphisms in $\mathfrak{K}$ such that for every $\mu, \mu' \in M$ with $\mu \leq \mu'$ there exists some $\lambda \in \Lambda$ for which $\phi(\mu), \phi(\mu') \leq \lambda$ and $f_\mu p_{\phi(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{\phi(\mu')\lambda}$. If $g=\{\psi, g_\nu, N\}: Y \longrightarrow Z=\{Z_\nu, r_{\nu\nu'}, N\}$ is a system map, then the composite of the system maps f and g is defined to be a system map $\{\phi\psi, h_\nu, N\}: X \longrightarrow Z$, where $h_\nu = g_\nu f_{\psi(\nu)}$: $X_{\phi(\psi(\nu))} \longrightarrow Z_\nu$. The identity system map $1: X \longrightarrow X$ consists of the identity map of Λ onto Λ and the collection of identity morphisms $1_\lambda = 1: X_\lambda \longrightarrow X_\lambda$. Two system maps $f=\{\phi, f_\mu, M\}$ and $g=\{\psi, g_\mu, M\}$ both from X to Y, are called equivalent, if for any $\mu \in M$ there is some $\lambda \in \Lambda$ such that $\phi(\mu), \psi(\mu) \leq \lambda$ and $f_\mu p_{\phi(\mu)\lambda} = g_\mu p_{\psi(\mu)\lambda}$. This is an equivalence relation among all system maps from X to Y. The equivalence class containing f is denoted by $[f]$. Then $[f]=[f']$ and $[g]=[g']$ imply $[gf]=[g'f']$. Thus we have a category whose objects are inverse systems in $\mathfrak{K}$ and whose morphisms are equivalence classes of system maps. This category is called the pro-category of $\mathfrak{K}$ and denoted by pro$(\mathfrak{K})$. The pro-category defined here is somewhat less general than that of Grothendieck [4] or Artin-Mazur [1].

Let $X=\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and $Y=\{Y_\lambda, q_{\lambda\lambda'}, \Lambda\}$ be inverse systems in $\mathfrak{K}$ over the same directed set Λ. If a collection of morphisms $f_\lambda: X_\lambda \longrightarrow Y_\lambda$, $\lambda\in\Lambda$, is such that $f_\lambda p_{\lambda\lambda'} = q_{\lambda\lambda'} f_{\lambda'}$ for $\lambda\leq\lambda'$, then $\{1, f_\lambda, \Lambda\}$ defines a system map from X to Y. Such a system map is called a special system map. Special system maps are important in view of the fact that for every system map $f': X' \longrightarrow Y'$ there exist three system maps $i: X' \longrightarrow X$, $f: X \longrightarrow Y$, $j: Y' \longrightarrow Y$ such that f is a special system map, $[j][f']=[f][i]$ and $[i]$, $[j]$ are isomorphisms in pro$(\mathfrak{K})$ (cf. [1, Corallary 3.2, p. 160], [7]).

The following is a basic theorem on pro-categories.

THEOREM 1.1. *Let $X=\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ and $Y=\{Y_\lambda, q_{\lambda\lambda'}, \Lambda\}$ be inverse systems in $\mathfrak{K}$ over the same directed set Λ, and let $f=\{1, f_\lambda, \Lambda\}$ be a special system map from X to Y. Then $[f]$ is an isomorphism in pro$(\mathfrak{K})$ iff for any $\lambda\in\Lambda$ there is some $\mu\in\Lambda$ such that $\lambda\leq\mu$ and there exists $\varphi_{\lambda\mu}: Y_\mu \longrightarrow X_\lambda$ for which $\varphi_{\lambda\mu} f_\mu = p_{\lambda\mu}$ and $f_\lambda \varphi_{\lambda\mu} = q_{\lambda\mu}$.*

PROOF. To prove the "if" part of Theorem 1.1, assume the condition of Theorem 1.1 is satisfied. Let us first define a new relation $\dashv$ in Λ as follows: $\lambda\dashv\mu$ iff $\lambda\leq\mu$ and there is a morphism $\varphi_{\lambda\mu}: Y_\mu \longrightarrow X_\lambda$ such that

$$(1)\qquad \varphi_{\lambda\mu} f_\mu = p_{\lambda\mu}, \qquad f_\lambda \varphi_{\lambda\mu} = q_{\lambda\mu}.$$

Then $\lambda\dashv\mu\dashv\nu$ implies $\lambda\dashv\nu$, $\lambda\leq\lambda'\dashv\mu'\leq\mu$ implies $\lambda\dashv\mu$ and for any $\lambda\in\Lambda$ there is $\mu\in\Lambda$ with $\lambda\dashv\mu$.

If $\kappa\dashv\lambda\dashv\mu\dashv\nu$, then

$$(2)\qquad \varphi_{\kappa\lambda} q_{\lambda\nu} = \varphi_{\kappa\mu} q_{\mu\nu} = p_{\kappa\lambda} \varphi_{\lambda\mu} q_{\mu\nu},$$

since $\varphi_{\kappa\mu} q_{\mu\nu} = p_{\kappa\lambda} p_{\lambda\mu} \varphi_{\mu\nu} = p_{\kappa\lambda} \varphi_{\lambda\mu} q_{\mu\nu} = \varphi_{\kappa\lambda} q_{\lambda\mu} q_{\mu\nu}$. Moreover, if $\kappa\dashv\kappa'\dashv\mu\dashv\nu$ and $\kappa\leq\lambda\dashv\lambda'\dashv\mu\dashv\nu$, then by (1) and (2) we have

$$\varphi_{\kappa\kappa'} q_{\kappa'\nu} = \varphi_{\kappa\mu} q_{\mu\nu} = \varphi_{\kappa\lambda'} q_{\lambda'\nu} = p_{\kappa\lambda'} \varphi_{\lambda'\mu} q_{\mu\nu}$$

$$= p_{\kappa\lambda} p_{\lambda\lambda'} \varphi_{\lambda'\mu} q_{\mu\nu} = p_{\kappa\lambda} \varphi_{\lambda\lambda'} q_{\lambda'\nu},$$

that is,

$$(3)\qquad \varphi_{\kappa\kappa'} q_{\kappa'\nu} = p_{\kappa\lambda} \varphi_{\lambda\lambda'} q_{\lambda'\nu}.$$

Now, for each $\lambda\in\Lambda$ let us choose an element $\alpha(\lambda)$ of Λ so that $\lambda\dashv\alpha(\lambda)$, and define $g_\lambda: Y_{\alpha(\lambda)} \longrightarrow X_\lambda$ by $g_\lambda = \varphi_{\lambda,\alpha(\lambda)}$. Then by (3) $g=\{\alpha, g_\lambda, \Lambda\}$ defines a system map from Y to X. Now, it is easy to see that $[f][g]=1$ and $[g][f]=1$. Hence $[f]$ is an isomorphism.

The proof of the "only if" part is straightforward and is omitted.

A morphism f in a category with a zero-object is called a quasi-monomorphism (quasi-epimorphism) if $fg=0$ (resp. $gf=0$) for another morphism g implies $g=0$. In abelian categories a quasi-monomorphism (resp. quasi-epimorphism) is a monomorphism (resp. epimorphism).

THEOREM 1.2. *Let $f: X \longrightarrow Y$ be a special system map as in Theorem* 1.1. *Let $\mathfrak{K}$ be a category with a zero-object, kernels and cokernels. Then $[f]$ is a quasi-monomorphism (resp. quasi-epimorphism) in pro $(\mathfrak{K})$ iff for any $\lambda \in \Lambda$ there is some $\mu \in \Lambda$ such that $\lambda \leq \mu$ and $p_{\lambda\mu} \operatorname{Ker} f_\mu = 0$ (resp. $(\operatorname{Coker} f_\lambda) q_{\lambda\mu} = 0$). In particular, in case $\mathfrak{K}$ is the category $\mathfrak{G}$ of groups, $[f]$ is a monomorphism (resp. epimorphism) in pro $(\mathfrak{G})$ iff for any $\lambda \in \Lambda$ there is some $\mu \in \Lambda$ such that $\lambda \leq \mu$ and $p_{\lambda\mu} \operatorname{Ker} f_\mu = 0$ (resp. $\operatorname{Im} q_{\lambda\mu} \subset \operatorname{Im} f_\lambda$).*

PROOF. Let $f: X \longrightarrow Y$ be an epimorphism in pro $(\mathfrak{G})$. For $\lambda \in \Lambda$, let us put $B_\lambda = \operatorname{Im} f_\lambda$. Suppose that $B_\lambda \neq Y_\lambda$. Then there are two homomorphisms ϕ, ψ from Y_λ to some group G such that $\phi f_\lambda = \psi f_\lambda$ and $\phi(y) = \psi(y)$ iff $y \in B_\lambda$; this is obvious in case B_λ is normal, and is proved in [8, p. 38] in case B_λ is not normal. Since $[\phi][f] = [\psi][f]$, we have $[\phi] = [\psi]$, where ϕ and ψ are considered as system maps from Y to the inverse system $\{G\}$ which consists only of G. Hence there is some $\mu \in \Lambda$ such that $\lambda \leq \mu$ and $\phi q_{\lambda\mu} = \psi q_{\lambda\mu}$. Hence $\operatorname{Im} q_{\lambda\mu} \subset B_\lambda = \operatorname{Im} f_\lambda$. In case $B_\lambda = Y_\lambda$ we have $\operatorname{Im} q_{\lambda\mu} \subset \operatorname{Im} f_\lambda$ for any μ with $\lambda \leq \mu$. The proof for the other parts of Theorem 1.2 is straightforward and is omitted.

THEOREM 1.3. *In the pro-category of groups any bimorphism is an isomorphism.*

PROOF. Let $f: X \longrightarrow Y$ be a special system map such as described in Theorem 1.1 where $\mathfrak{K} = \mathfrak{G}$ in the present case. Assume $[f]$ is a bimorphism. Then, by Theorem 1.2, for any $\lambda \in \Lambda$ there are $\mu, \nu \in \Lambda$ such that $\lambda \leq \mu \leq \nu$, $p_{\lambda\mu} \operatorname{Ker} f_\mu = 0$ and $\operatorname{Im} q_{\mu\nu} \subset \operatorname{Im} f_\mu$. Let $y \in Y_\nu$. Then there is $x \in X_\mu$ with $f_\mu(x) = q_{\mu\nu}(y)$. If $f_\mu(x') = q_{\mu\nu}(y)$ for another $x' \in X_\mu$, then $p_{\lambda\mu}(x) = p_{\lambda\mu}(x')$. Hence we have a map $\psi_{\lambda\nu}: Y_\nu \longrightarrow X_\lambda$ by putting $\psi_{\lambda\nu}(y) = p_{\lambda\mu}(x)$. Then $\psi_{\lambda\nu}$ is a homomorphism and $\psi_{\lambda\nu} f_\nu = p_{\lambda\nu}$, $f_\lambda \psi_{\lambda\nu} = q_{\lambda\nu}$. Hence by Theorem 1.1 $[f]$ is an isomorphism.

In a category with a zero-object and kernels a sequence of two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called exact at Y if $gf=0$ and f' in the unique factorization $f = (\operatorname{Ker} g) f'$ of f is an epimorphism ([14, p. 123]). For a category with a zero-object and cokernels the notion of coexact is defined dually. In the categery of groups exactness implies coexactness but not vice versa. In abelian categories exactness is equivalent to coexactness (cf. [14, Lemma 13.1.4]).

THEOREM 1.4. *Let $\mathfrak{K}$ be a category with a zero-object, kernels and cokernels. Let*

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "p_{21}"] & Y_1 \arrow[r, "g_1"] \arrow[d, "q_{21}"] & Z_1 \arrow[r, "h_1"] \arrow[d, "r_{21}"] & P_1 \arrow[r, "\phi_1"] \arrow[d, "s_{21}"] & Q_1 \arrow[d, "t_{21}"] \\
X_2 \arrow[r, "f_2"] \arrow[d, "p_{32}"] & Y_2 \arrow[r, "g_2"] \arrow[d, "q_{32}"] & Z_2 \arrow[r, "h_2"] \arrow[d, "r_{32}"] & P_2 \arrow[r, "\phi_2"] \arrow[d, "s_{32}"] & Q_2 \arrow[d, "t_{32}"] \\
X_3 \arrow[r, "f_3"] & Y_3 \arrow[r, "g_3"] & Z_3 \arrow[r, "h_3"] & P_3 \arrow[r, "\phi_3"] & Q_3
\end{tikzcd}$$

be a commutative diagram in $\mathfrak{K}$ *in which each row is exact at* P_1 *and* Z_2 *and co-exact at* Y_2 *and* Y_3.

(a) *If* $s_{21}(\mathit{Ker}\ \phi_1)=0$ *and* $(\mathit{Coker}\ f_3)q_{32}=0$ *then* $r_{32}r_{21}=0$.

(b) *If* $r_{21}=0$, *then* $s_{21}(\mathit{Ker}\ \phi_1)=0$ *and* $(\mathit{Coker}\ f_2)q_{21}=0$.

PROOF. Let $k_1 : K_1 \longrightarrow P_1$ be a kernel of ϕ_1 and $m_i : Y_i \longrightarrow M_i$ a cokernel of f_i, $i=2, 3$. Then there are an epimorphism $h_1' : Z_1 \longrightarrow K_1$ and monomorphisms $g_i' : M_i \longrightarrow Z_i$, $i=2, 3$, such that $h_1=k_1h_1'$ and $g_i=g_i'm_i$. Assume that $s_{21}k_1=0$ and $m_3q_{32}=0$. Then $h_2r_{21}=0$ and $g_3q_{32}=0$. Let $l_2 : L_2 \longrightarrow Z_2$ be a kernel of h_2. Then there is an epimorphism $g_2'' : Y_2 \longrightarrow L_2$ such that $g_2=l_2g_2''$. Since $r_{32}g_2=g_3q_{32}=0$, we have $r_{32}l_2g_2''=0$. Hence $r_{32}l_2=0$. On the other hand, since $h_2r_{21}=0$, there is $r_{21}' : Z_1 \longrightarrow L_2$ such that $r_{21}=l_2r_{21}'$. Therefore $r_{32}r_{21}=0$. This proves (a).

To prove (b), assume that $r_{21}=0$. Then $s_{21}k_1h_1'=0$. Since h_1' is an epimorphism, we have $s_{21}k_1=0$. On the other hand, $g_2'm_2q_{21}=g_2q_{21}=r_{21}g_1=0$. Since g_2' is a monomorphism, we have $m_2q_{21}=0$. This proves (b).

REMARK. In case $\mathfrak{K}=\mathfrak{G}$ and the middle row is exact at Y_2, $r_{21}=0$ implies $\mathrm{Im}\ q_{21} \subset \mathrm{Im}\ f_2$.

§ 2. Some lemmas on *CW* complexes and a lemma on approximative *k*-connectedness

LEMMA 2.1. *Let* $p_{i+1,i} : (X_i, A_i, x_i) \longrightarrow (X_{i+1}, A_{i+1}, x_{i+1})$, $0 \leqq i < n$, *be continuous maps of pairs of pointed connected CW complexes such that*

$$\pi_{k+1}(p_{k+1,k})=0 : \pi_{k+1}(X_k, A_k, x_k) \longrightarrow \pi_{k+1}(X_{k+1}, A_{k+1}, x_{k+1})$$

for $0 \leqq k < n$. *Then there is a continuous map* $\phi : (X_0, X_0^n \cup A_0, x_0) \longrightarrow (X_n, A_n, x_n)$ *such that* $\phi(X_0^0)=x_n$ *and*

$$\phi j \simeq p_{n,n-1} \cdots p_{10} : (X_0, A_0, x_0) \longrightarrow (X_n, A_n, x_n),$$

where X_0^k *is the* k*-skeleton of* X_0 *and* $j : (X_0, A_0, x_0) \longrightarrow (X_0, X_0^n \cup A_0, x_0)$ *is the inclusion map.*

Moreover, if $\pi_1(p_{10}|(A_0, x_0))=0$ *and* $\pi_1(p_{10}|(X_0, x_0))=0$, *then* ϕ *can be chosen so that* $\phi(X_0^1)=x_n$.

PROOF. In what follows, maps are continuous. Assume that $\pi_1(p_{10}|(A_0, x_0))=0$ and $\pi_1(p_{10}|(X_0, x_0))=0$. Putting $L_0=X_0^0\times I\cup X_0\times 0$ and $L_k=(X_0^k\cup A_0)\times I\cup X_0\times 0$ for $1\leqq k\leqq n$, where $I=[0, 1]$, let us construct maps $\chi_k : L_k\longrightarrow X_k$, $k=0, 1, \cdots, n$ with the following properties.

(4) $\qquad \chi_0(x, 0)=x \quad \text{for } x\in X_0, \qquad \chi_0(A_0^0\times I)\subset A_0;$

(5) $\qquad \chi_1(x, 1)=x_1 \quad \text{for } x\in X_0^1, \qquad \chi_1(A_0\times I)\subset A_1;$

(6) $\qquad \chi_k|L_{k-1}=p_{k,k-1}\chi_{k-1} \quad \text{for } k\geqq 1;$

(7) $\qquad \chi_k(x, 1)\in A_k \quad \text{for } x\in X_0^k,\ k\geqq 0.$

First, let χ_0 be defined over $X_0\times 0$ by (4). For $x\in X_0^0$ let $\chi_0(x, t)$ be a path from x to x_0 so that it lies in A_0 if $x\in A_0^0$. Next, let e_λ^1 be a 1-cell in X_0 (resp. A_0), and let $h_\lambda : E^1 \longrightarrow \bar{e}_\lambda^1\subset X_0$ be its characteristic map. Define a map $\alpha_\lambda : (E^1\times 0\cup \dot{E}^1\times I, \dot{E}^1\times 1) \longrightarrow (X_0, x_0)$ (resp. (A_0, x_0)) by $\alpha_\lambda(s, t)=\chi_0(h_\lambda(s), t)$. Since $\pi_1(p_{10}|(X_0, x_0))=0$ and $\pi_1(p_{10}|(A_0, x_0))=0$, $p_{10}\alpha_\lambda$ is homotopic in X_1 (resp. A_1) relative to $\dot{E}^1\times 1$ to a constant map to x_1. This homotopy yields an extension β_λ of $p_{10}\alpha_\lambda$ over $E^1\times I$ such that $\beta_\lambda(E^1\times I)=x_1$ and $\beta_\lambda(E^1\times I)\subset A_1$ if $e_\lambda^1\in A_0$. Define first a map $\chi_1 : L_0\cup (X_0^1\times I) \longrightarrow X_1$ by (6), put $\chi_1(x, t)=\beta_\lambda(h_\lambda^{-1}(x), t)$ for $x\in e_\lambda^1$, $t\in I$ and then extend χ_1 over $L_0\cup(X_0^1\times I)\cup(A_0\times I)$ by the homotopy extension theorem so that $\chi_1(A_0\times I)\subset A_1$. Then χ_1 satisfies (5), (6) and (7).

For $k\geqq 2$, suppose that χ_{k-1} has been constructed. Let e_λ^k be a k-cell in X_0-A_0 and $h_\lambda : E^k \longrightarrow \bar{e}_\lambda^k\subset X_0$ its characteristic map. Define a map $\alpha_\lambda : (\dot{E}^k\times 0\cup \dot{E}^k\times I, E^k\times 1, s_0\times 1) \longrightarrow (X_{k-1}, A_{k-1}, x_{k-1})$ by $\alpha_\lambda(s, t)=\chi_{k-1}(h_\lambda(s), t)$ where s_0 is a point of $\dot{E}^k$ such that $h_\lambda(s_0)\in X_0^1$. Since $\pi_k(p_{k,k-1})=0$, $p_{k,k-1}\alpha_\lambda$ is homotopic relative to $\dot{E}^k\times 1$ to a map from $E^k\times 0\cup \dot{E}^k\times I$ to A_k. This homotopy yields an extension β_λ of $p_{k,k-1}\alpha_\lambda$ over $E^k\times I$ such that $\beta_\lambda(E^k\times 1)\subset A_k$. Define a map $\chi_k : L_{k-1}\cup(X_0^k\times I)$ by (6) and by $\chi_k(x, t)=\beta_\lambda(h_\lambda^{-1}(x), t)$ for $x\in e_\lambda^k$, $t\in I$. Then χ_k satisfies (7). Therefore by induction on k we can find χ_k satisfying (6) and (7) for all k with $2\leqq k\leqq n$. Here we note that $\chi_n(x, 1)=x_n$ for $x\in X_0^1$.

Finally, by the homotopy extension theorem there is a map $\theta : X_0\times I \longrightarrow X_n$ such that $\theta|L_n=\chi_n$. Let us put $\phi(x)=\theta(x, 1)$ for $x\in X_0$, Then ϕ has the desired properties. This proves the second part of Lemma 2.1.

The first part is proved similarly.

Lemma 2.1 was proved in [11] for the case of simplical complexes.

Let

$$\begin{array}{ccccccc}
(X_0, x_0) & \xrightarrow{p_{10}} & (X_1, x_1) & \xrightarrow{p_{21}} & \cdots & \xrightarrow{p_{2n,\, 2n-1}} & (X_{2n}, x_{2n}) \\
\Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f_1} & & & & \Big\downarrow{\scriptstyle f_{2n}} \\
(Y_0, y_0) & \xrightarrow{q_{10}} & (Y_1, y_1) & \xrightarrow{q_{21}} & \cdots & \xrightarrow{q_{2n,\, 2n-1}} & (Y_{2n}, y_{2n})
\end{array}$$

be a commutative diagram in the homotopy category of pointed connected CW complexes. Let $\eta_i : (X_i \times I, x_i \times I) \longrightarrow (Y_{i+1}, y_{i+1})$ be a homotopy between $f_{i+1} p_{i+1, i}$ and $q_{i+1, i} f_i : \eta_i(x, 0) = f_{i+1} p_{i+1, i}(x)$, $\eta_i(x, 1) = q_{i+1, i} f_i(x)$ for $x \in X_i$. Let Z_i be the reduced mapping cylinder of f_i, which is obtained from the disjoint union $(X_i \times I) \cup Y_i$ by identifying $(x, 1)$ with $f_i(x)$ for $x \in X_i$ and by shrinking $(x_i \times I) \cup \{y_i\}$ to a point which is denoted also by x_i; the images of (x, t) and y under this identification are denoted by $[x, t]$ and $[y]$ respectively. Let us define embeddings $\alpha_i : X_i \longrightarrow Z_i$, $\beta_i : Y_i \longrightarrow Z_i$ by putting $\alpha_i(x) = [x, 0]$, $\beta_i(y) = [y]$ and a map $\gamma_i : Z_i \longrightarrow Y_i$ by $\gamma_i[x, t] = [f_i(x)]$ and $\gamma_i[y] = y$. Then $f_i = \gamma_i \alpha_i$, $\gamma_i \beta_i = 1$ and $\beta_i \gamma_i \simeq 1$. Let $g_i : (Y_i, y_i, y_i) \longrightarrow (Z_i, X_i, x_i)$ be the composite of β_i and the inclusion map. Following [13], let us define a map $r_{i+1, i} : (Z_i, x_i) \longrightarrow (Z_{i+1}, x_{i+1})$ by

$$r_{i+1, i}([x, t]) = \begin{cases} [p_{i+1, i}(x), 2t], & x \in X_i,\ 0 \leq t \leq \dfrac{1}{2}, \\[2mm] [\eta_i(x, 2t-1)], & x \in X_i,\ \dfrac{1}{2} \leq t \leq 1, \end{cases}$$

$$r_{i+1, i}([y]) = [q_{i+1, i}(y)], \qquad y \in Y_i.$$

Then $r_{i+1, i}$ defines a map of (Z_i, X_i, x_i) into $(Z_{i+1}, X_{i+1}, x_{i+1})$ which is denoted also by the same letter $r_{i+1, i}$. Then we have the following commutative diagram in which each row is exact (the description of base-points being omitted).

$$\begin{array}{ccccccccc}
\pi_{k+1}(X_i) & \xrightarrow{\pi_{k+1}(f_i)} & \pi_{k+1}(Y_i) & \xrightarrow{\pi_{k+1}(g_i)} & \pi_{k+1}(Z_i, X_i) & \xrightarrow{\partial} & \pi_k(X_i) & \xrightarrow{\pi_k(f_i)} & \pi_k(Y_i) \\
\Big\downarrow{\scriptstyle \pi_{k+1}(p_{i+1, i})} & & \Big\downarrow{\scriptstyle \pi_{k+1}(q_{i+1, i})} & & \Big\downarrow{\scriptstyle \pi_{k+1}(r_{i+1, i})} & & \Big\downarrow & & \Big\downarrow \\
\pi_{k+1}(X_{i+1}) & \longrightarrow & \pi_{k+1}(Y_{i+1}) & \longrightarrow & \pi_{k+1}(Z_{i+1}, X_{i+1}) & \longrightarrow & \pi_k(X_{i+1}) & \longrightarrow & \pi_k(Y_{i+1})
\end{array}$$

LEMMA 2.2. *If* $\pi_k(p_{2k+1, 2k})\, Ker\, \pi_k(f_{2k}) = 0$ *and* $Im\, \pi_{k+1}(q_{2k+2, 2k+1}) \subset Im\, \pi_{k+1}(f_{2k+2})$ *for* $0 \leqq k < n$, *then there exists a continuous map* $\phi : (Z_0, Z_0^n \cup X_0, x_0) \longrightarrow (Z_{2n}, X_{2n}, x_{2n})$ *such that* $r_{2n, 2n-1} \cdots r_{10} \simeq \phi j$, *where* $j : (Z_0, X_0, x_0) \longrightarrow (Z_0, Z_0^n \cup X_0, x_0)$ *is the inclu-*

sion map and Z_0^n *is the n-skeleton of* Z_0.

If, in addition, $\pi_1(p_{10})=0$ *and* $\pi_1(q_{10})=0$, *then* ϕ *can be chosen so that* $\phi(Z_0^1)=x_{2n}$.

This lemma follows readily from Theorem 1.4 and Lemma 2.1, since in the category of groups exactness implies coexactness.

In concluding this section, we shall prove the following lemma.

LEMMA 2.3. *Let* (X, x_0) *be a pointed metric space. Let* P *be an ANR for metric spaces such that* $X \subset P$. *Then* $\pi_k(X, x_0)=0$ *iff for any open neighborhood* U *of* X *in* P *there is an open neighborhood* V *of* X *in* P *such that* $V \subset U$ *and* $\pi_k(i_{UV})=0 : \pi_k(V, x_0) \longrightarrow \pi_k(U, x_0)$, *where* $i_{UV} : V \longrightarrow U$ *is the inclusion map.*

PROOF. Let $\mathfrak{U}(X, P)$ be an inverse system which consists of open neighborhoods U of X in P and which has the inclusion maps $i_{UV} : V \longrightarrow U$ as bonding maps. By [10, Theorem 1.4] $\mathfrak{U}(X, P)$ induces an inverse system in $\mathfrak{W}_0$ which is associated with (X, x_0) in the sense of Morita [10], where $\mathfrak{W}_0$ is the homotopy category of pointed spaces having the homotopy type of a pointed CW complex. If two inverse systems in $\mathfrak{W}_0$ are associated with the same pointed space, then they are isomorphic in the pro-category of $\mathfrak{W}_0$; this is seen from [10, Theorem 2.4]. Hence we have Lemma 2.3.

Thus, for the case of a pointed compact metric space (X, x_0), $\pi_k(X, x_0)=0$ iff (X, x_0) is approximatively k-connected in the sense of Borsuk [3].

§3. Proof of Theorem A

Assume $\pi_1(A, x_0)=0$ and $\pi_1(X, A, x_0)=0$. Then by the exactness of the sequence of homotopy pro-groups (cf. [7], [13]) we have $\pi_1(X, x_0)=0$. Hence for each $\lambda \in \Lambda$ there is $\mu \in \Lambda$ which admits a sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ in Λ such that $\lambda \leq \lambda_n \leq \cdots \leq \lambda_0 \leq \mu$ and $p_{\lambda_{i+1}\lambda_i} : (X_{\lambda_i}, A_{\lambda_i}, x_{0\lambda_i}) \longrightarrow (X_{\lambda_{i+1}}, A_{\lambda_{i+1}}, x_{0\lambda_{i+1}})$, $i=0, 1, \cdots, n-1$ satisfy the conditions in Lemma 2.1 (with the subscripts i there replaced by λ_i). In such a case we write $\lambda \dashv \mu$. Then for $\lambda, \mu \in \Lambda$ with $\lambda \dashv \mu$ there exists a map $\phi_{\lambda\mu} : (X_\mu, X_\mu^n \cup A_\mu, x_{0\mu}) \longrightarrow (X_\lambda, A_\lambda, x_{0\lambda})$ such that

$$(8) \qquad p_{\lambda\mu} \simeq \phi_{\lambda\mu} j_\mu : (X_\mu, A_\mu, x_{0\mu}) \longrightarrow (X_\lambda, A_\lambda, x_{0\lambda})$$

$$(9) \qquad \phi_{\lambda\mu}(X_\mu^n \cup A_\mu) \subset A_\lambda, \ \phi_{\lambda\mu}(X_\mu^1) = x_{0\lambda},$$

where $j : (X_\mu, A_\mu, x_{0\mu}) \longrightarrow (X_\mu, X_\mu^n \cup A_\mu, x_{0\mu})$ is the inclusion map. Let us now construct the quotient space $Y_\mu = X_\mu / X_\mu^1$ and put $B_\mu = (X_\mu^n \cup A_\mu)/X_\mu^1$; let $g_\mu : (X_\mu, X_\mu^n \cup A_\mu, x_{0\mu}) \longrightarrow (Y_\mu, B_\mu, y_{0\mu})$ be the quotient map. Then there is a map $\psi_{\lambda\mu} : (Y_\mu, B_\mu, y_{0\mu}) \longrightarrow (X_\lambda, A_\lambda, x_{0\lambda})$ such that $\phi_{\lambda\mu} = \psi_{\lambda\mu} g_\mu$. It is to be noted that $\pi_k(Y_\mu, B_\mu, y_{0\mu})=0$ for $1 \leqq k \leqq n$, $\pi_1(B_\mu, y_{0\mu})=0$, and that (Y_μ, B_μ) is a pair of connected CW complexes. Thus, by the usual Hurewicz theorem (cf. [5, p. 103]) the Hurewicz homomorphism $\Phi_k(Y_\mu, B_\mu, y_{0\mu}) : \pi_k(Y_\mu, B_\mu, y_{0\mu}) \longrightarrow H_k(Y_\mu, B_\mu)$ is an isomorphism for $k=n+1$ and an epimorphism for $k=n+2$. If we put

$$\theta_{\lambda\mu}=\pi_{n+1}(\psi_{\lambda\mu})\Phi_{n+1}(Y_{\mu}, B_{\mu}, y_{0\mu})^{-1}H_{n+1}(g_{\mu}j_{\mu}):$$
$$H_{n+1}(X_{\mu}, A_{\mu}) \longrightarrow \pi_{n+1}(X_{\lambda}, A_{\lambda}, x_{0\lambda}),$$

then we have

$$\theta_{\lambda\mu}\Phi_{n+1}(X_{\mu}, A_{\mu}, x_{0\mu})=\pi_{n+1}(p_{\lambda\mu}),$$
$$\Phi_{n+1}(X_{\lambda}, A_{\lambda}, x_{0\lambda})\theta_{\lambda\mu}=H_{n+1}(p_{\lambda\mu}).$$

Therefore, by Theorem 1.1, $\{1, \Phi_{n+1}(X_{\lambda}, A_{\lambda}, x_{0\lambda}), \Lambda\}$ defines an isomorphism from $\{\pi_{n+1}(X_{\lambda}, A_{\lambda}, x_{0\lambda}), \pi_{n+1}(p_{\lambda\lambda'}), \Lambda\}$ to $\{H_{n+1}(X_{\lambda}, A_{\lambda}), H_{n+1}(p_{\lambda\lambda'}), \Lambda\}$. On the other hand, $\Phi_{n+2}(Y_{\mu}, B_{\mu}, y_{0\mu})$ is an epimorphism, and hence $\operatorname{Im} H_{n+2}(p_{\lambda\mu})\subset\operatorname{Im} H_{n+2}(\psi_{\lambda\mu})\subset \operatorname{Im}\Phi_{n+2}(X_{\lambda}, A_{\lambda}, x_{0\lambda})$. Hence by Theorem 1.2 $\Phi_{n+2}(X, A, x_0)$ is an epimorphism. Thus the second part of Theorem A is proved.

Let us prove the first part. In this case for each λ there is some $\mu\in\Lambda$ with $\lambda\leq\mu$ for which there is a map $\psi_{\lambda\mu}:(X_{\mu}, X_{\mu}^{n}\cup A_{\mu}, x_{0\mu})\longrightarrow(X_{\lambda}, A_{\lambda}, x_{0\lambda})$ such that (8) is satisfied. Since $H_k(X_{\mu}, X_{\mu}^{n}\cup A_{\mu})=0$ for $1\leqq k\leqq n$, we have $H_k(p_{\lambda\mu})=0$. Hence $H_k(X, A)=0$.

Thus, Theorem A is completely proved.

§4. Proof of Theorem B

For any system map $f:X\longrightarrow Y$ of inverse systems in a category $\mathfrak{K}$ there are system maps $i:X\longrightarrow X'$, $f':X'\longrightarrow Y'$, $j:Y\longrightarrow Y'$ such that $[j][f]=[f'][i]$, f' is a special system map and $[i], [j]$ are isomorphisms in pro$(\mathfrak{K})$. In this case each object appearing in X' (resp. Y') can be taken from those objects appearing in X (resp. Y).

Hence without loss of generality we may assume that $\{(X_{\lambda}, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}$ and $\{(Y_{\lambda}, y_{0\lambda}), [q_{\lambda\lambda'}], \Lambda\}$ are inverse systems in the homotopy category of pointed CW complexes which are isomorphic to the Čech systems of (X, x_0) and of (Y, y_0) respectively and that $f=\{1, f_{\lambda}, \Lambda\}:\{(X_{\lambda}, x_{0\lambda})\}\longrightarrow\{(Y_{\lambda}, y_{0\lambda})\}$ is a special system map (cf. §1, [10]). Now, assume (a). Let $\lambda\in\Lambda$. Then by Theorem 1.2 there are elements $\lambda_i\in\Lambda$, $i=0, 1, \cdots, 2n$, such that $\lambda=\lambda_{2n}\leq\cdots\leq\lambda_1\leq\lambda_0$ such that $\pi_k(p_{\lambda_{2k+1}, \lambda_{2k}})\operatorname{Ker}\pi_k(f_{\lambda_{2k}})=0$ and $\operatorname{Im}\pi_{k+1}(q_{\lambda_{2k+2}, \lambda_{2k+1}})\subset\operatorname{Im}\pi_{k+1}(f_{\lambda_{2k+2}})$ for $0\leqq k<n$. Hence, if we put $\mu=\lambda_0$ and $r_{\lambda\mu}=r_{\lambda, \lambda_{2n-1}}\cdots r_{\lambda_1\mu}$, by Lemma 2.1 there is a map $\psi:(Z_{\mu}, Z_{\mu}^{n}\cup X_{\mu}, x_{0\mu})\longrightarrow(Z_{\lambda}, X_{\lambda}, x_{0\lambda})$ such that $r_{\lambda\mu}\simeq\psi j$, where $j:(Z_{\mu}, X_{\mu}, x_{0\mu})\longrightarrow(Z_{\mu}, Z_{\mu}^{n}\cup X_{\mu}, x_{0\mu})$ is the inclusion map and Z_{μ} is the mapping cylinder of f_{μ}. Then we have a commutative diagram:

$$\begin{array}{ccccccccc}
H_{k+1}(X_{\mu}) & \longrightarrow & H_{k+1}(Y_{\mu}) & \longrightarrow & H_{k+1}(Z_{\mu}, X_{\mu}) & \longrightarrow & H_k(X_{\mu}) & \longrightarrow & H_k(Y_{\mu}) \\
\downarrow & & \downarrow H_{k+1}(q_{\lambda\mu}) & & \downarrow H_{k+1}(r_{\lambda\mu}) & & \downarrow H_k(p_{\lambda\mu}) & & \downarrow \\
H_{k+1}(X_{\lambda}) & \longrightarrow & H_{k+1}(Y_{\lambda}) & \longrightarrow & H_{k+1}(Z_{\lambda}, X_{\lambda}) & \longrightarrow & H_k(X_{\lambda}) & \longrightarrow & H_k(Y_{\lambda})
\end{array}$$

in which each row is exact and $H_{k+1}(r_{\lambda\mu})=0$ for $0\leqq k<n$. Hence by Theorem 1.4 we have $\operatorname{Im} H_{k+1}(q_{\lambda\mu})\subset\operatorname{Im} H_{k+1}(f_{\lambda})$ and $H_k(p_{\lambda\mu})\operatorname{Ker} H_k(f_{\mu})=0$ for $1\leqq k<n$. Hence by

Theorems 1.2 and 1.3 $H_k(f)$ is an isomorphism for $1\leqq k<n$ and an epimorphism for $k=n$. This proves (a)$\Longrightarrow$(b).

Now suppose that $\pi_1(X, x_0)=0$ and $\pi_1(Y, y_0)=0$. Let us prove the implication (b)$\Longrightarrow$(a) by induction on n. Assume that the implication is true for $n\geqq 2$ and that $H_k(f)$ is an isomorphism for $1\leqq k<n+1$ and an epimorphism for $k=n+1$. Then by the induction hypothesis $\pi_k(f)$ is an isomorphism for $1\leqq k<n$ and an epimorphism for $k=n$. For $\kappa\in\Lambda$, by Lemma 2.2 there are $\lambda\in\Lambda$ with $\kappa\leq\lambda$ and continuous maps $\phi:(Z_\lambda, X_\lambda, x_{0\lambda})\longrightarrow(P, Q, s_0)$, $\psi:(P, Q, s_0)\longrightarrow(Z_\kappa, X_\kappa, x_{0\kappa})$ such that $\psi\phi\simeq r_{\kappa\lambda}$, where $P=Z_\lambda/Z_\lambda^1$, $Q=(Z_\lambda^n\cup X_\lambda)/Z_\lambda^1$ and $r_{\kappa\lambda}$ is defined as the composite of several maps similarly as in the proof of the first part. Then we have a commutative diagram (the description of base-points being omitted).

$$\begin{tikzcd}
\pi_{n+1}(Z_\lambda, X_\lambda) \arrow[r, "\pi_{n+1}(\phi)"] \arrow[d] & \pi_{n+1}(P, Q) \arrow[r, "\pi_{n+1}(\psi)"] \arrow[d, "\Phi_{n+1}(P, Q)"] & \pi_{n+1}(Z_\kappa, X_\kappa) \arrow[d] \\
H_{n+1}(Z_\lambda, X_\lambda) \arrow[r, "H_{n+1}(\phi)"] & H_{n+1}(P, Q) \arrow[r, "H_{n+1}(\psi)"] & H_{n+1}(Z_\kappa, X_\kappa).
\end{tikzcd}$$

Since $\pi_k(P, Q)=0$ for $1\leqq k\leqq n$ and $\pi_1(Q)=0$ the Hurewicz homomorphism $\Phi_{n+1}(P, Q)$ is an isomorphism by the usual Hurewicz isomorphism theorem. If we choose $\mu, \nu\in\Lambda$ so that $\lambda\leq\mu\leq\nu$ and $\operatorname{Im} H_{n+1}(q_{\lambda\mu})\subset\operatorname{Im} H_{n+1}(f_\lambda)$, $H_n(p_{\mu\nu})\operatorname{Ker} H_n(f_\nu)=0$, then we have $H_{n+1}(r_{\lambda\mu}r_{\mu\nu})=0$. Since $\pi_{n+1}(r_{\kappa\lambda}r_{\lambda\mu}r_{\mu\nu})=\pi_{n+1}(\psi)\Phi_{n+1}(P, Q)^{-1}H_{n+1}(\phi)H_{n+1}(r_{\lambda\mu}r_{\mu\nu})\Phi_{n+1}(Z_\nu, X_\nu)$, we have $\pi_{n+1}(r_{\kappa\lambda}r_{\lambda\mu}r_{\mu\nu})=0$. Hence by Theorem 1.4 $\operatorname{Im}\pi_{n+1}(q_{\kappa\nu})\subset\operatorname{Im}\pi_{n+1}(f_\kappa)$ and $\pi_n(p_{\kappa\nu})\operatorname{Ker}\pi_n(f_\nu)=0$. Therefore, $\pi_k(f)$ is an epimorphism for $k=n+1$ and a monomorphism for $k=n$. Thus, by Theorem 1.3 (b)$\Longrightarrow$(a) holds for $n+1$.

As for the case $n=2$, for $\kappa\in\Lambda$ there exist $\lambda\in\Lambda$ with $\kappa\leq\lambda$ and two continuous maps $\phi:(Z_\lambda, X_\lambda, x_{0\lambda})\longrightarrow(P, Q, s_0)$ and $\psi:(P, Q, s_0)\longrightarrow(Z_\kappa, X_\kappa, x_{0\kappa})$ with $\psi\phi\simeq r_{\kappa\lambda}$ where $P=Z_\lambda/Z_\lambda^1$, $Q=(Z_\lambda^1\cup X_\lambda)/Z_\lambda^1$. Since $\Phi_2(P, Q, s_0)$ is an isomorphism by the usual Hurewicz isomorphism theorem, the proof may be carried out along the same line as above.

Thus, Theorem B is completely proved.

The following is a supplement to Theorem B.

THEOREM 4.1. *Let $f:(X, x_0)\longrightarrow(Y, y_0)$ be a shape morphism of connected pointed spaces. If $\pi_1(f):\pi_1(X, x_0)\longrightarrow\pi_1(Y, y_0)$ is an isomorphism (resp. an epimorphism), so is $H_1(f):H_1(X)\longrightarrow H_1(Y)$.*

PROOF. Let f be a special system map as in the first paragraph of this section. Assume $\pi_1(f)$ is an isomorphism. Then, by Theorem 1.1, for any $\lambda\in\Lambda$ there are $\mu\in\Lambda$ with $\lambda\leq\mu$ and a homomorphism $\psi_{\lambda\mu}:\pi_1(Y_\mu, y_{0\mu})\longrightarrow\pi_1(X_\lambda, x_{0\lambda})$ such that $\pi_1(p_{\lambda\mu})=\psi_{\lambda\mu}\pi_1(f_\mu)$ and $\pi_1(q_{\lambda\mu})=\pi_1(f_\lambda)\psi_{\lambda\mu}$. Since the kernel of the Hurewicz homomorphism for the first homotopy group of a connected space is its commutator subgroup, by the functorial property of the Hurewicz homomorphism we see that there is a homomorphism $\phi_{\lambda\mu}: H_1(Y_\mu) \longrightarrow H_1(X_\lambda)$ such that $H_1(p_{\lambda\mu}) = \phi_{\lambda\mu} H_1(f_\mu)$ and $H_1(q_{\lambda\mu}) = H_1(f_\lambda)\phi_{\lambda\mu}$. Hence $H_1(f)$ is an isomorphism by Theorem 1.1. The theorem for the case of epimorphisms can be proved similarly by using Theorem 1.2, although it follows also from the proof of the first part of Theorem B.

§ 5. Proof of Theorem C

As in § 4, without loss of generality we may assume that $\{(X_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}$ and $\{(Y_\lambda, y_{0\lambda}), [q_{\lambda\lambda'}], \Lambda\}$ are isomorphic to the Čech systems of (X, x_0) and of (Y, y_0) respectively in the pro-category of the homotopy category of pointed connected CW complexes, and that $f = \{1, f_\lambda, \Lambda\} : \{(X_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\} \longrightarrow \{(Y_\lambda, y_{0\lambda}), [q_{\lambda\lambda'}], \Lambda\}$ is a special system map. Moreover, by [9] we can assume that $\dim X_\lambda \leqq \dim X$, $\dim Y_\lambda \leqq \dim Y$ for each $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$. Then by Theorem 1.2 it follows from the assumption of the theorem that there is a sequence $\{\lambda_0, \cdots, \lambda_{2n}\}$ of elements of Λ such that

$$\lambda = \lambda_{2n} \leq \lambda_{2n-1} \leq \cdots \leq \lambda_1 \leq \lambda_0 \tag{10}$$

$$\pi_k(p_{\lambda_{2k+1}, \lambda_{2k}}) \operatorname{Ker} \pi_k(f_{\lambda_{2k}}) = 0, \quad 1 \leqq k < n, \tag{11}$$

$$\operatorname{Im} \pi_{k+1}(q_{\lambda_{2k+2}, \lambda_{2k+1}}) \subset \operatorname{Im} \pi_{k+1}(f_{\lambda_{2k+2}}), \quad 0 \leqq k < n. \tag{12}$$

Let us put $\mu = \lambda_0$. By Lemma 2.2 there is a continuous map $\psi : (Z_\mu, Z_\mu^n \cup X_\mu, x_{0\mu}) \longrightarrow (Z_\lambda, X_\lambda, x_{0\lambda})$ such that

$$r_{\lambda\mu} \simeq \psi j, \tag{13}$$

where $j : (Z_\mu, X_\mu, x_{0\mu}) \longrightarrow (Z_\mu, Z_\mu^n \cup X_\mu, x_{0\mu})$ is the inclusion map, Z_μ and Z_λ are the mapping cylinders of f_μ and f_λ respectively, and $r_{\lambda\mu} = r_{\lambda_{2n}, \lambda_{2n-1}} \cdots r_{\lambda_1 \lambda_0}$.

As is well known, we have $\dim Z_\mu \leqq \max(1 + \dim X_\mu, \dim Y_\mu) \leqq n$. Hence $Z_\mu^n \cup X_\mu = Z_\mu$. Therefore we have a commutative diagram below in the homotopy category of pointed CW complexes:

$$\begin{tikzcd}
(X_\mu, x_{0\mu}) \arrow[rrr, "p_{\lambda\mu}"] \arrow[dr, "\alpha_\mu"] \arrow[dd, "f_\mu"'] & & & (X_\lambda, x_{0\lambda}) \arrow[dl, "\alpha_\lambda"'] \arrow[dd, "f_\lambda"] \\
& (Z_\mu, x_{0\mu}) \arrow[r, "r_{\lambda\mu}"'] \arrow[urr, "\psi"] & (Z_\lambda, x_{0\lambda}) & \\
(Y_\mu, y_{0\mu}) \arrow[ur, "\beta_\mu"'] \arrow[rrr, "q_{\lambda\mu}"'] & & & (Y_\lambda, y_{0\lambda}) \arrow[ul, "\beta_\lambda"']
\end{tikzcd}$$

Since β_μ and β_λ are homotopy equivalences, there exists a continuous map $\phi_{\lambda\mu}$: $(Y_\mu, y_{0\mu}) \longrightarrow (X_\lambda, x_{0\lambda})$ such that

(14) $$[\phi_{\lambda\mu}][f_\mu]=[p_{\lambda\mu}], \qquad [f_\lambda][\phi_{\lambda\mu}]=[q_{\lambda\mu}].$$

Therefore, by Theorem 1.1 we see that a special system map f induces an isomorphism in the pro-categopy of the homotopy category of pointed CW complexes. Consequently $[f]$ is a shape equivalence. This completes the proof of Theorem C.

Addendum (August 6, 1974). Corresponding to another form of the classical Whitehead theorem we can prove also Theorem C′ and D′ below.

THEOREM C′. *Let $f: (X, x_0) \longrightarrow (Y, y_0)$ be a shape morphism of pointed connected topological spaces of finite dimension and let $n = max(dim\ X, dim\ Y)$. If the induced morphism $\pi_k(f): \pi_k(X, x_0) \longrightarrow \pi_k(Y, y_0)$ of homotopy pro-groups is an isomorphism for $1 \leqq k \leqq n$, then f is a shape equivalence.*

THEOREM D′. *Let f and n be the same as in Theorem C′. If $\pi_1(X, x_0)=0$ and $\pi_1(Y, y_0)=0$ and if the induced morphism $H_k(f): H_k(X) \longrightarrow H_k(Y)$ of homology pro-groups is an isomorphism for $2 \leqq k \leqq n$, then f is a shape equivalence.*

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