Countably-compactifiable spaces[1)]

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1. Throughout this paper by a space we shall mean a completely regular T_1-space, and by N the set of positive integers.

For a space X we shall call a space S a *countably-compactification* of X if

1) S is countably compact and contains X as a dense subset,

2) every countably compact closed subset of X is closed also in S.

In case X has a countably-compactification we shall say that X is *countably-compactifiable.*

Countably compact spaces and paracompact spaces, or more generally, isocompact spaces in the sense of Bacon [1] (that is, every countably compact closed subset is compact) are trivial examples of countably-compactifiable spaces. As a non-trivial example we can mention an M-space which admits a quasi-perfect map onto a locally compact metric space. Moreover, the product of a countably-compactifiable space with a product of paracompact spaces is countably-compactifiable. However, it is to be noted that a locally compact space is not always countably-compactifiable.

In this paper we shall establish a number of basic results concerning countably-compactifiable spaces.

As an application, we shall prove that a space X is a countably-compactifiable M-space if and only if X is homeomorphic to a closed subspace of the product $C\times T$ of a countably compact space C with a metric space T. Thus Nagata's problem concerning embeddability of M-spaces is reduced to the problem: "Is every M-space countably-compactifiable?" Our result above shows that an M-space which is either paracompact or carried onto a locally compact metric space by a quasi-perfect map is countably-compactifiable.

2. Generalizing the notion of weak normality in the sense of Dugundji [2], we shall say that a space is weakly normal if each pair of disjoint closed subsets, one of which is countable and discrete, have disjoint neighborhoods. Weak normality is actually weaker than normality.

PROPOSITION 2.1. *A countably paracompact space is weakly normal.*

This proposition was observed already by Isiwata [6]. Indeed, let $A=\{a_n|n\in N\}$ and B be disjoint closed sets in a countably paracompact space X. Then for each

1) Some of the results of this paper were announced in [11].

n of N there is an open set U_n such that $a_n \in U_n$, Cl $U_n \cap B=\phi$. Since $\{X-A, U_n | n \in N\}$ is a countable open covering of X, there is a locally finite open covering $\{V_0, V_n | n \in N\}$ such that $V_0 \subset X-A$ and $V_n \subset U_n$. Let us put $V=\cup\{V_n | n \in N\}$. Then $A \subset V$ and Cl $V \cap B=\phi$. This proves Proposition 2.1.

In discussing the properties of weakly normal spaces the following lemma is useful.

LEMMA 2.2. *Let $D=\{d_n | n \in N\}$ be a countable discrete set in a space X. Then the following conditions are equivalent.*

(a) *D and any disjoint closed set have disjoint neighborhoods.*

(b) *D and any disjoint closed set are completely separated.*

(c) *D is C-embedded in X.*

(d) *There is $f \in C(X)$ such that $\lim_{n\to\infty} f(d_n)=\infty$.*

(e) *There is a locally finite collection $\{U_n | n \in N\}$ of open sets in X such that $d_n \in U_n$ for $n \in N$.*

(f) *There is a discrete collection $\{U_n | n \in N\}$ of open sets in X such that $d_n \in U_n$ for $n \in N$.*

The equivalence of (a), (b) and (c) is observed in Gillman and Jerison [4, p. 51], and the equivalence of (c), (d), (e) and (f) is easy to see.

Recently Hansard [5] has called a space X well-separated if every countably infinite discrete closed subset D satisfies condition (e) of Lemma 2.2, and hence well-separatedness is equivalent to weak normality. However, in this paper we shall say, more generally, that a space X is well-separated, if every infinite discrete closed set is not relatively pseudocompact. Indeed, in view of Lemma 2.3 below weakly normal spaces are well-separated and the theorems 10 to 14 except 11 in [5] remain true for well-separated spaces in our sense.

LEMMA 2.3. *A discrete closed set A of a space X is not relatively pseudocompact if and only if A contains a countably infinite discrete closed set D satisfying any one of conditions stated in Lemma 2.2.*

After having read the first draft of the present paper, T. Isiwata introduced originally the notion of ss-discrete property: a space X has the ss-discrete property if for every discrete closed subset $\{d_n | n \in N\}$ of X and for every collection $\{U_n\}$ of open sets of X with $d_n \in U_n$ for $n \in N$ there are a subsequence $\{n_i | i \in N\}$ of N and a locally finite collection $\{V_{n_i}\}$ of open sets of X such that $x_{n_i} \in V_{n_i} \subset U_{n_i}$ for each $i \in N$. From Lemma 2.3 above it follows that well-separated spaces are precisely the spaces having the ss-discrete property. Our definition of well-separatedness, however, has a merit that it leads naturally to the following characterization.

PROPOSITION 2.4. *A space X is well-separated if and only if the closure of every relatively pseudocompact subset of X is countably compact.*

As in our previous paper [9], let us denote by $\mu(X)$ the completion of X with respect to its finest uniformity. Then $X \subset \mu(X) \subset \nu(X) \subset \beta(X)$ where $\nu(X)$ is the

Hewitt realcompactification of X. In case $\mu(X)=X$, X is said to be topologically complete. The following lemma is useful.

LEMMA 2.5. *Let A be a subset of a space X. Then the following conditions are equivalent.*

(a) *A is relatively pseudocompact,*

(b) $\mathrm{Cl}_{\beta(X)}A\subset\mu(X)$,

(c) $\mathrm{Cl}_{\beta(X)}A\subset\nu(X)$.

Proof. Since (b)$\Rightarrow$(c)$\Rightarrow$(a) is obvious, we have only to prove the implication (a)$\Rightarrow$(b). For this purpose, suppose that (a) holds. $f: X\to T$ be any continuous map of X into a metric space T. Then $f(A)$ is relatively pseudocompact and hence $\mathrm{Cl}\, f(A)$ is compact. Therefore we have $\mathrm{Cl}_{\beta(X)}A\subset\beta(f)^{-1}(\mathrm{Cl}\, f(A))\subset\beta(f)^{-1}(T)$, where $\beta(f)$: $\beta(X)\to\beta(T)$ is the extension of f. By Morita [9, Theorem 2.2] this shows that (b) holds.

An application of Lemma 2.5 is the following proposition due to Dykes [3].

PROPOSITION 2.6. *If a space X is topologically complete, then the closure of a relatively pseudocompact subset of X is compact.*

Proof. Let A be a relatively pseudocompact set. Then by Lemma 2.5 we have

$$\mathrm{Cl}_{\beta(X)}A\subset\mu(X)=X.$$

Hence $\mathrm{Cl}_X A=\mathrm{Cl}_{\beta(X)}A$ and so $\mathrm{Cl}_X A$ is compact.

The following proposition is a direct consequence of Proposition 2.4.

PROPOSITION 2.7. *A space X is isocompact and well-separated if and only if the closure of every relatively pseudocompact subset of X is compact.*

3. In this section we shall give a sufficient condition for a space to be countably-compactifiable.

THEOREM 3.1. *Let X be a well-separated space. Then the following statements are valid.*

a) *If $\beta(X)-\mu(X)$ is countably compact then $X\cup(\beta(X)-\mu(X))$ is countably compact.*

b) *If $X\subset S\subset X\cup(\beta(X)-\mu(X))$ and S is countably compact, then S is a countably-compactification of X.*

Proof. Suppose that D is a countably infinite discrete closed set in X. Then D is not relatively pseudocompact and hence by Lemma 2.5 we have $\mathrm{Cl}_{\beta(X)}D\cap(\beta(X)-\mu(X))\neq\phi$. Therefore $X\cup(\beta(X)-\mu(X))$ is countably compact if $\beta(X)-\mu(X)$ is countably compact.

On the other hand, let C be any countably compact closed subset of X. Then by Lemma 2.5 $\mathrm{Cl}_{\beta(X)}C\subset\mu(X)$. Hence $\mathrm{Cl}_S C\subset X$. This shows that S is a countably-compactification of X.

REMARK. If X is well-separated and

$$\mu(X)=\cup\{\mathrm{Cl}_{\beta(X)}C|C \text{ ranges over all countably compact closed subsets of } X\},$$

in particular, if X is an M-space, then X is countably-compactifiable if and only if there is a countably compact space S such that $X\subset S\subset X\cup(\beta(X)-\mu(X))$.

COROLLARY 3.2. *Let X be an M-space such that there is a quasi-perfect map from X onto a locally compact metric space. Then X is countably-compactifiable.*

Proof. In this case $\mu(X)$ is locally compact by [9, Theorem 3.5]. Since X is an M-space, X is countably paracompact and hence it is weakly normal by Proposition 2.1. Thus Theorem 3.1 is applicable to the present case.

Theorem 3.1 fails to be valid if we drop the assumption that X is well-separated.

EXAMPLE 3.3. Let ω_0 be the first infinite ordinal and ω_1 the first uncountable ordinal. Let us put

$$X=W(\omega_0+1)\times W(\omega_1+1)-(\omega_0, \omega_1)$$

where $W(\alpha)=\{\beta|\beta<\alpha\}$ has the order topology for an ordinal α. Then X is a locally compact, pseudocompact space which is not countably compact and $\beta(X)=\mu(X)=W(\omega_0+1)\times W(\omega_1+1)$. Let $A=\{(\omega_0, \alpha)|\alpha<\omega_1\}$. Then A is a countably compact closed subset of X but $(\omega_0, \omega_1)\in\mathrm{Cl}_{\beta(X)}A$. This shows that $\beta(X)$ is not a countably-compactification of X in our sense. Hence X is not countably-compactifiable, as will be seen from the following proposition.

PROPOSITION 3.4. *If a space X is countably-compactifiable, then there is a countably-compactification S of X such that $X\subset S\subset\beta(X)$.*

Proof. Let R be a countably compactification of X. Let $f:X\to R$ be the inclusion map and $\beta(f):\beta(X)\to\beta(R)$ the extension of f. Let us put $S=\beta(f)^{-1}(R)$. Then $X\subset S\subset\beta(X)$. Since $\beta(S)=\beta(X)$, $g=\beta(f)|S:S\to R$ is a perfect map and hence S is countably compact. Let A be a countably compact closed subset of X. Then A is closed in R and hence $g^{-1}(A)$ is closed in S. On the other hand, $g^{-1}(A)=A$, since $\beta(R)$ is a compactification of X, and hence $\beta(f)(\beta(X)-X)=\beta(R)-X$. Thus S is a countably-compactification of X.

4. In this section we shall discuss a number of operations under which the property of being countably-compactifiable is preserved.

PROPOSITION 4.1. *If a space X is countably-compactifiable, so is every closed subspace of X.*

PROPOSITION 4.2. *Let $f: X\to Y$ be a perfect map. If Y is countably-compactifiable, so is X.*

Proof. By Proposition 3.4 there is a countably-compactification R of Y such that $Y\subset R\subset\beta(Y)$. Let us put

$$S=\beta(f)^{-1}(R),\ g=\beta(f)|S:S\longrightarrow R$$

where $\beta(f)\colon\beta(X)\rightarrow\beta(Y)$ is the extension of f. Then g is a perfect map, and hence S is countably compact.

Let A be a countably compact closed subset of X, and let us put $B=f(A)$. Then B is a countably compact closed subset of Y. Hence B is closed in R, and so $g^{-1}(B)$ is closed also in S. Since f is a perfect map, $\beta(f)^{-1}(\beta(Y)-Y)=\beta(X)-X$, we have $g^{-1}(B)=f^{-1}(f(A))$. Since A is closed in $f^{-1}(f(A))$, we see that A is closed in S. Thus S is a countably-compactification of X.

PROPOSITION 4.3. *Let X and Y be well-separated spaces which have countably-compactifications R and S respectively. If $R\times S$ is countably-compactifiable, so is $X\times Y$.*

Proof. Let us denote by g and h the projections from $X\times Y$ onto X and Y respectively. Let A be a countably compact closed subset of $X\times Y$. Then $\mathrm{Cl}_X g(A)$ and $\mathrm{Cl}_Y h(A)$ are countably compact by Proposition 2.4 and hence $\mathrm{Cl}_X g(A)\times\mathrm{Cl}_Y h(A)$ is closed in $R\times S$. Thus A is closed in $R\times S$. Hence if T is a countably-compactification of $R\times S$ then A is closed also in T and so T is a countably compactification of $X\times Y$.

PROPOSITION 4.4. Let Y_λ be a well-separated isocompact space for $\lambda\in\Lambda$. If a space X is countably-compactifiable, so is $X\times\Pi Y_\lambda$.

Proof. Let R be a countably-compactification of X and S_λ a compactification of Y_λ. Then S_λ is a countably-compactification of Y_λ. Let A be a countably compact closed subset of $X\times\Pi Y_\lambda$ and let g and h_λ be projections from $X\times\Pi Y_\lambda$ onto X and Y_λ respectively. Then $\mathrm{Cl}\, h_\lambda(A)$ is compact by Proposition 2.6. On the other hand, $g(A)$ is closed in X since the projection from $X\times\Pi\,\mathrm{Cl}\, h_\lambda(A)$ onto X is a closed map. Therefore $g(A)\times\Pi\,\mathrm{Cl}\, h_\lambda(A)$ is closed in $R\times\Pi S_\lambda$ and consequently A is closed in $R\times\Pi S_\lambda$. Since $R\times\Pi S_\lambda$ is countably compact, $R\times\Pi S_\lambda$ is a countably-compactification of $X\times\Pi Y_\lambda$.

5. As is well-known, for a locally countably compact space X which is not countably compact we can construct a countably compact space S by adding a new point p_∞ to X; as neighborhoods of p_∞ we take the sets of the form $p_\infty\cup(X-C)$ with a countably compact closed subset C of X. The space S, however, is not necessarily a countably-compactification of X because S is not necessarily completely regular. In case S is a countably-compactification, we shall call S a one-point countably-compactification of X.

THEOREM 5.1. *A space X admits a one-point countably-compactification if and only if for any countably compact closed subset A of X there is a real-valued continuous function f over X such that*

$$f=0 \text{ on } A \text{ and } f=1 \text{ on } X-B$$

for some countably compact closed subset B of X.

Proof. The "only if" part is obvious. To prove the "if" part assume that the condition in the theorem is satisfied. In this case, if we put $G=\{x|f(x)<1/2\}$ for f satisfying the condition of the theorem, then $A\subset G\subset \mathrm{Cl}\, G\subset B$. Thus every countably compact closed subset of X is contained in an open set whose closure is countably compact. In particular, X is locally countably compact. Let us put $S=p_\infty \cup X$ where the topology of S is that described at the beginning of this section. Let $U(p_\infty)=p_\infty\cup(X-A)$ be any neighborhood of p_∞ where A is a countably compact closed set of X. Then there are $f\in C(X)$ and a countably compact closed subset B of X such that $f=0$ on A and $f=1$ on $X-B$. If we put $f(p_\infty)=1$, then f is countinuous over X as well as over $p_\infty\cup(X-B)$ and hence f is continuous over S.

Next, let $U(x_0)$ be any neighborhood of a point x_0 of X. Then there is an open set G of X such that $x_0\in G$, $\mathrm{Cl}\, G\subset U(x_0)$ and $\mathrm{Cl}\, G$ is countably compact. Then there is $f\in C(X)$ such that $f(x_0)=1$ and $f=0$ on $X-G$. If we put $f(p_\infty)=0$, then f is continuous over X and over $p_\infty\cup(X-\mathrm{Cl}\, G)$. Thus S is completely regular.

COROLLARY 5.2. *Let S be a one-point countably-compactification of a space X. If X is normal, so is S.*

It is pointed out by M. Atsuji that if X is a normal space such that every countably compact closed set of X is contained in an open set with a countably compact closure, then X admits a one-point countably-compactification which is normal. This, combined with Theorem 5.1, yields Corollary 5.2. The following is a direct proof of Corollary 5.2. Let A and B be disjoint closed subset of S. Let us consider first the case that $A\cup B\subset X$. Then there is an open set G of X such that $A\cup B\subset G$ and $\mathrm{Cl}\, G$ is countably compact. Then there is $f\in C(X)$ such that $f=0$ on A, $f=1$ on B, $f=2$ on $X-G$. If we put $f(p_\infty)=2$, then f is continuous over S. In case $p_\infty\in B$ then there is an open set G of X such that $A\subset G$, $\mathrm{Cl}\, G\cap(B-p_\infty)=\phi$ and $\mathrm{Cl}\, G$ is countably compact. If we put $H=p_\infty\cup(X-\mathrm{Cl}\, G)$, then $B\subset H$, $G\cap H=\phi$. Thus S is normal.

As for M-spaces we have

THEOREM 5.3. *An M-space X admits a one-point countably-compactification if and only if there is a quasi-perfect map from X onto a locally compact metric space.*

Proof. Let $\phi: X\to T$ be a quasi-perfect map from X onto a metric space T. Then it follows from Theorem 5.1 that X admits a one-point countably-compactification if and only if T is locally compact.

It should be noted that a locally compact space does not not necessarily admit a one-point countably-compactification even if it is countably-compactifiable.

EXAMPLE 5.4. Let ω_0 (resp. ω_1) be the first infinite (resp. uncountable) ordinal and let us put

$$P = W(\omega_0+1)\times W(\omega_1+1)\times W(\omega_1+1)$$

In P we consider the following closed subsets:

$$C=\{(n, \omega_1, \omega_1)|n\leq\omega_0\},$$

$$A(m, \alpha, \omega_1)=\{(2m, \alpha, \omega_1), (2m+1, \alpha, \omega_1)\},$$

$$B(n, \omega_1, \beta)=\{(2n+1, \omega_1, \beta), (2n+2, \omega_1, \beta)\}$$

where $m<\omega_0$, $n<\omega_0$, $\alpha<\omega_1$, $\beta<\omega_1$.

By contracting each of these sets into a single point we have a quotient space Q and a quotient map $\phi: P\to Q$. Then Q is a compact Hausdorff space. Let us put

$$q_0=\phi(C),$$

$$K=\phi(C\cup(\omega_0\times W(\omega_1+1)\times W(\omega_1+1))),$$

$$Y=Q-K.$$

Then K is compact and $K-q_0\cong W(\omega_1+1)\times W(\omega_1+1)-(\omega_1, \omega_1)$. The space Y is the same as constructed in our previous paper [8] and $\mu(Y)=Y\cup q_0$. Thus Y is locally compact, weakly normal and $\beta(Y)-\mu(Y)$ is countably compact; Y is dense in Q and $\beta(Y)-\mu(Y)$ is the preimage of $K-q_0$ under the perfect map $\beta(\psi)$ where ψ: $\mu(Y)\to Q$ is the inclusion map. Therefore Y is countably-compactifiable by Theorem 3.1, but Y does not admit a one-point countably-compactification.

6. Now we are in a position to prove the following theorem.

THEOREM 6.1. *A space X is a countably-compactifiable M-space if and only if X is homeomorphic to a closed subspace of the product space $C\times T$ for a countably compact space C and a metric space T.*

Proof of Theorem 6.1. We first note that $C\times T$ in this theorem is an M-space; this is seen from the fact that the projection from $C\times T$ onto T is quasi-perfect. Since a metric space is paracompact, the "if" part follows immediately from Propositions 4.1 and 4.4.

To prove the "only if" part, suppose that X is an M-space with a countably-compactification S. Since X is an M-space, there is a quasi-perfect map f from X onto a metric space T. Let us put

$$A=\{(x, f(x))\in S\times T|x\in X\}.$$

Then A is a closed subset of $S\times T$.

To see this, let $(s_0, t_0)\in S\times T-A$. Then $s_0\notin f^{-1}(t_0)$. Since $f^{-1}(t_0)$ is a countably compact closed subset of X, $f^{-1}(t_0)$ is closed also in S since S is a countably-compactification of X. Since S is regular, there is an open neighborhood $W(s_0)$ of s_0

such that $\mathrm{Cl}_S W(s_0) \cap f^{-1}(t_0) = \phi$. Hence it follows that $f^{-1}(t_0) \subset X - \mathrm{Cl}_S W(s_0)$. Since f is a closed map, there is an open neighborhood $V(t_0)$ of t_0 in T such that $f^{-1}(V(t_0)) \subset X - \mathrm{Cl}_S W(s_0)$.

If there were a point

$$(s, t) \in [W(s_0) \times V(t_0)] \cap A,$$

we would have

$$s \in f^{-1}(t) \subset f^{-1}(V(t_0)) \subset X - W(s_0),$$

which contradicts the assumption $s \in W(s_0)$. Hence $[W(s_0) \times V(t_0)] \cap A = \phi$. This shows that A is closed in $S \times T$.

Since X is homeomorphic to A, the proof of Theorem 6.1 is completed.

COROLLARY 6.2. *A space X is a paracompact M-space if and only if X is homeomorphic to a closed subspace of the product of a compact space with a metric space.*

Proof. Let X be a paracompact M-space such that $f: X \to T$ is a perfect map and that T is a metric space. Then any compactification S of X is a countably-compactification of X and the above proof of Theorem 6.1 shows that X is homeomorphic to a closed subspace of $S \times T$. Thus the "only if" part is proved. The "if" part is obvious.

COROLLARY 6.3. *A space X is mapped onto a locally compact metric space by a quasi-perfect map if and only if X is homeomorphic to a closed subset of the product of a locally compact metric space with a countably compact space.*

Proof. In view of our proof of Theorem 6.1, the "only if" part is a direct consequence of Corollary 3.2. The "if" part is obvious.

Corollary 6.2 is due to J. Nagata [12].

Nagata's problem concerning embeddability of M-spaces is now reduced to the following problem by Theorem 6.1.

PROBLEM 6.4. Is every M-space countably-compactifiable?

For countably-compactifiable spaces there are many unsolved problems. For example, we have

PROBLEM 6.5. Is every normal space countably-compactifiable?

A. K. Steiner [13] proved that there is a product of two countably compact spaces which is an M-space but not countably compact. This product space is countably-compactifiable by Corollary 3.2.

PROBLEM 6.6. Is the product of two countably compact (or countably-compactifiable) spaces countably-compactifiable?

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