Topological completions and M-spaces

By

Kiiti MORITA*)

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Throughout this paper by a space we shall mean a completely regular Hausdorff space unless otherwise specified. A space is called topologically complete if it is complete with respect to its finest uniformity, whereas a space is called an absolute G_δ if it is a G_δ in its Stone-Čech compactification. The completion of a space X with respect to its finest uniformity is called the topological completion of X and will be denoted by $\mu(X)$.

The purpose of this paper is to discuss some basic properties of $\mu(X)$ and to apply them to the case of M-spaces.

In § 1, for any space X with a uniformity Φ we shall construct a space $\mu_\Phi(X)$ which is the weak completion of X with respect to Φ in the sense defined in § 1.

In § 2, we shall consider $\mu_\Phi(X)$ in case Φ is the finest uniformity of X. In this case $\mu_\Phi(X)$ is shown to coincide with the completion of X with respect to Φ and hence we write $\mu(X)$ instead of $\mu_\Phi(X)$. The space $\mu(X)$ is characterized as a space Y with properties (a) and (b):

(a) Y is a topologically complete space containing X as a dense subspace,

(b) every continuous map from X into a metric space T can be extended to a continuous map from Y into T.

For any continuous map $f\colon X\to Y$ there corresponds a continuous map $\mu(f)\colon \mu(X)\to\mu(Y)$. Thus μ defines a covariant functor from the category of all spaces into the category of all topologically complete spaces (morphisms in both categories being continuous maps).

For M-spaces which are introduced in our previous paper [11] the functor μ possesses the following remarkable properties:

(1) $\mu(X)$ is a paracompact M-space for any M-space X.

(2) If $f\colon X\to Y$ is a quasi-perfect map where X and Y are M-spaces, then $\mu(f)\colon \mu(X)\to\mu(Y)$ is a perfect map.

Thus for an M-space X $\mu(X)$ may be called the paracompactification of X. Several spaces are characterized by the property of $\mu(X)$. For example, an M-space X admits a quasi-perfect map from X onto a separable (resp. locally compact or complete) metric space if and only if $\mu(X)$ is Lindelöf (resp. locally compact or an absolute G_δ). § 4 is devoted to characterizing a space X with a paracompact M-space as $\mu(X)$.

In § 5 we are concerned with the product formula $\mu(X\times Y)=\mu X\times\mu Y$ which,

*) Visiting Andrew Mellon Professor at the University of Pittsburgh.

however, does not hold in general. The first case for which we shall assure the validity of this formula is this: X is an arbitrary space and Y is a locally compact paracompact space. If we restrict ourselves to the case of $X\times Y$ being an M-space, we can prove a more precise result that $\mu(X\times Y)=\mu X\times\mu Y$ if and only if there are quasi-perfect maps $\varphi: X\to S$, $\psi: Y\to T$ with S and T metrizable such that $\varphi\times\psi: X\times Y\to S\times T$ is also quasi-perfect.

As an application of the last result, we shall establish that

$$\dim(X\times Y)\leqq\dim X+\dim Y$$

if X is an M-space and Y is a metric space or a locally compact paracompact space. Here it should be noted that X is not necessarily normal, and hence we understand $\dim X$ in the sense of M. Katetov (that is, $\dim X$ is the covering dimension of $\beta(X)$). It seems that this is the first result which assures the validity of the product theorem in dimension theory for the case where $X\times Y$ is not assumed to be normal.

Finally a number of problems will be raised in §7.

1. Definition of $\mu_\Phi(X)$

For any space X, let Φ be a uniformity of X agreeing with the topology of X; that is, let Φ be a family of open coverings of X satisfying conditions (a) to (c) below, where for coverings $\mathcal{U}$ and $\mathcal{V}$ of X we mean by $\mathcal{U}<\mathcal{V}$ that $\mathcal{V}$ is a refinement of $\mathcal{U}$.

(a) If $\mathcal{U},\mathcal{V}\in\Phi$, then there exists $\mathcal{W}\in\Phi$ such that $\mathcal{U}<\mathcal{W}$ and $\mathcal{V}<\mathcal{W}$.

(b) If $\mathcal{U}\in\Phi$, there is $\mathcal{V}\in\Phi$ which is a star-refinement of $\mathcal{U}$ (that is, $\{\mathrm{St}(V,\mathcal{V})\mid V\in\mathcal{V}\}>\mathcal{U}$).

(c) $\{\mathrm{St}(x,\mathcal{U})\mid\mathcal{U}\in\Phi\}$ is a basis of neighborhoods at each point x of X.

Now, let $\{\Phi_\lambda\mid\lambda\in\Lambda\}$ be the totality of those normal sequences which consist of open coverings of X contained in Φ. Let $\Phi_\lambda=\{\mathcal{U}_{\lambda_i}\mid i=1,2,\cdots\}$, where $\mathcal{U}_{\lambda_i}\in\Phi$ and $\mathcal{U}_{\lambda_i}$ is a star-refinement of $\mathcal{U}_{\lambda_{i-1}}$ for $i=2,3,\cdots$. As in [11], we denote by (X,Φ_λ) the topological space obtained from X by taking $\{\mathrm{St}(x,\mathcal{U}_{\lambda_i})\mid i=1,2,\cdots\}$ as a basis of neighborhoods at each point x of X. For any subset A of X we set

$$\mathrm{Int}(A;\Phi_\lambda)=\{x\in X\mid\mathrm{St}(x,\mathcal{U}_{\lambda_i})\subset A\ \text{for some}\ i\}.$$

Then $\mathrm{Int}(A;\Phi_\lambda)$ is open in (X,Φ_λ). Let X/Φ_λ be the quotient space obtained from (X,Φ_λ) by defining those two points x and y equivalent for which $y\in\mathrm{St}(x,\mathcal{U}_{\lambda_i})$ for $i=1,2,\cdots$. Let us denote by i_λ the identity map of X viewed as a map from X onto (X,Φ_λ) and by $\tilde{\varphi}_\lambda$ the quotient map from (X,Φ_λ) onto X/Φ_λ. Let us set

$$\varphi_\lambda=\tilde{\varphi}_\lambda\circ i_\lambda: X\to X/\Phi_\lambda.$$

Since $\tilde{\varphi}_\lambda^{-1}(\tilde{\varphi}_\lambda(\mathrm{Int}(A;\Phi_\lambda)))=\mathrm{Int}(A;\Phi_\lambda)$, $\tilde{\varphi}_\lambda$ is an open continuous map and hence φ_λ is a continuous map.

Let us set

$$\mathcal{V}_{\lambda_i}=\{\mathrm{Int}(U;\Phi_\lambda)\mid U\in\mathcal{U}_{\lambda_i}\}.$$

Then $\mathcal{V}_{\lambda_i}$ is an open covering of (X, Φ_λ), since for any point x of X there exists $U \in \mathcal{U}_{\lambda_i}$ such that $\mathrm{St}(x, \mathcal{U}_{\lambda_{i+1}}) \subset U$.

We shall prove that $\mathrm{St}(A, \mathcal{U}_{\lambda_i}) \subset G$ implies $\mathrm{St}(A, \mathcal{V}_{\lambda_i}) \subset \mathrm{Int}(G; \Phi_\lambda)$. Let y be any point of $\mathrm{Int}(U; \Phi_\lambda)$ with $U \in \mathcal{U}_{\lambda_i}$ where $A \cap \mathrm{Int}(U; \Phi_\lambda) \neq \phi$. Then $\mathrm{St}(y, \mathcal{U}_{\lambda_j}) \subset U$ for some j and, since $A \cap U \supset A \cap \mathrm{Int}(U; \Phi_\lambda) \neq \phi$, we have $U \subset \mathrm{St}(A, \mathcal{U}_{\lambda_i}) \subset G$. Therefore $\mathrm{St}(y, \mathcal{U}_{\lambda_i}) \subset G$, that is, $y \in \mathrm{Int}(G, \Phi_\lambda)$.

Thus, if $\mathrm{St}(U', \mathcal{U}_{\lambda_{i+i}}) \subset U$ for $U' \in \mathcal{U}_{\lambda_{i+1}}$, $U \in \mathcal{U}_{\lambda_i}$, then $\mathrm{St}(\mathrm{Int}(U'; \Phi_\lambda), \mathcal{V}_{\lambda_{i+1}}) \subset \mathrm{Int}(U; \Phi_\lambda)$. This shows that $\mathcal{V}_{\lambda_{i+1}}$ is a star-refinement of $\mathcal{V}_{\lambda_i}$. Moreover, it is easy to see that $\mathcal{V}_{\lambda_i} > \mathcal{U}_{\lambda_i}$ while $\mathcal{U}_{\lambda_{i+1}} > \mathcal{V}_{\lambda_i}$ for $i=1, 2, \cdots$. Let us denote the normal sequence $\{\mathcal{V}_{\lambda_i} \mid i=1, 2, \cdots\}$ by Φ_λ^* and call it the normalized normal sequence associated with Φ_λ. Indeed, we have

$$(X, \Phi_\lambda^*) = (X, \Phi_\lambda), \qquad X/\Phi_\lambda^* = X/\Phi_\lambda$$

and $\{\varphi_\lambda(\mathcal{V}_{\lambda_i}) \mid i=1, 2, \cdots\}$ is a normal sequence of open coverings of X/Φ_λ which defines a uniformity of X/Φ_λ, where $\varphi_\lambda(\mathcal{V}_{\lambda_i}) = \{\varphi_\lambda(V) \mid V \in \mathcal{V}_{\lambda_i}\}$. Therefore X/Φ_λ is a metrizable space.

Here it should be noted that if order of $\mathcal{U}_{\lambda_i} \leqq n+1$ for each i, then order of $\varphi_\lambda(\mathcal{V}_{\lambda_i}) \leqq n+1$ for each i since order of $\varphi_\lambda(\mathcal{V}_{\lambda_i}) =$ order of $\mathcal{V}_{\lambda_i} \leqq$ order of $\mathcal{U}_{\lambda_i}$, and cosequently we have $\dim X/\Phi_\lambda \leqq n$ by Nagata [13, Corollary to Theorem 5. 1].

Next, we shall introduce a partial order in $\{\Phi_\lambda \mid \lambda \in \Lambda\}$. Let $\lambda, \mu \in \Lambda$. In case for each i there exists $\mathcal{U}_{\mu_j} \in \Phi_\mu$ such that $\mathcal{U}_{\mu_j} > \mathcal{U}_{\lambda_i}$, we write $\Phi_\lambda < \Phi_\mu$. Then for a countable number of elements λ_i, $i=1, 2, \cdots$ of Λ there exists an element μ of Λ such that $\Phi_{\lambda_i} < \Phi_\mu$ for $i=1, 2, \cdots$.

Suppose that $\Phi_\lambda < \Phi_\mu$. Then it is easy to see that if a subset G of X is open in (X, Φ_λ), then so is G in (X, Φ_μ) and that

$$y \in \bigcap_{j=1}^{\infty} \mathrm{St}(x, \mathcal{U}_{\mu_j}) \quad \text{implies} \quad y \in \bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathcal{U}_{\lambda_i})$$

for any two points x, y of X. Therefore there exists a canonical map

$$\varphi_\lambda^\mu : X/\Phi_\mu \to X/\Phi_\lambda .$$

φ_λ^μ is continuous as is seen from the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_\mu"] \arrow[d, equal] & (X, \Phi_\mu) \arrow[r, "\tilde{\varphi}_\mu"] \arrow[d, "i_\lambda^\mu"] & X/\Phi_\mu \arrow[d, "\varphi_\lambda^\mu"] \\ X \arrow[r, "i_\lambda"] & (X, \Phi_\lambda) \arrow[r, "\tilde{\varphi}_\lambda"] & X/\Phi_\lambda \end{tikzcd}$$

where i_λ^μ is the identity map.

Let us now utilize the normalized normal sequences $\Phi_\lambda^* = \{\mathcal{V}_{\lambda_i}\}$ and $\Phi_\mu^* = \{\mathcal{V}_{\mu i}\}$ associated with Φ_λ and Φ_μ respectively. If $\mathcal{U}_{\mu_j} > \mathcal{U}_{\lambda_{i+1}}$ then we have

$$\mathcal{V}_{\mu_j} > \mathcal{U}_{\mu_j} > \mathcal{U}_{\lambda_{i+1}} > \mathcal{V}_{\lambda_i}$$

and hence $(\varphi_\lambda^\mu)^{-1}(\varphi_\lambda(\mathcal{V}_{\lambda_i})) < \varphi_\mu(\mathcal{V}_{\mu_j})$. Thus φ_λ^μ is a uniformly continuous map from

X/Φ_μ onto X/Φ_λ.

From the above considerations it is seen that

$$\{X/\Phi_\lambda;\varphi''_\lambda\}$$

is an inverse system of metrizable spaces. Let us denote by $\mu_\Phi(X)$ the limit of this inverse system:

$$\mu_\Phi(X)=\lim_{\longleftarrow} X/\Phi_\lambda\,,$$

and by π_λ the projection from $\mu_\Phi(X)$ into X/Φ_λ. By the uniformity of $\mu_\Phi(X)$ we mean that uniformity which consists of

$$\{\pi_\lambda^{-1}\varphi_\lambda(\mathcal{V}_{\lambda_i})\,|\,\lambda\in\Lambda,\ i=1,2,\cdots\}\,.$$

For any point x of X $\{\varphi_\lambda(x)\,|\,\lambda\in\Lambda\}$ defines a point of $\mu_\Phi(X)$ and defines a map

$$\varphi\colon X\to\mu_\Phi(X)\,.$$

Since $\mathcal{V}_{\lambda_i}=\varphi^{-1}(\pi_\lambda^{-1}(\varphi_\lambda(\mathcal{V}_{\lambda_i})))$, φ is a uniformly continuous map. Moreover, if $x\neq x'$ for x, x' of X, then $\varphi_\lambda(x)\neq\varphi_\lambda(x')$ for some λ of Λ. Hence φ is one-to-one, and

$$\varphi\colon X\to\varphi(X)$$

is a uniform homeomorphism and $\varphi(X)$ is dense in $\mu_\Phi(X)$.

In case every Cauchy family $\{C_\gamma\}$ of X with the countable intersection property is non-vanishing (that is, $\cap\overline{C}_\mu\neq\phi$), we say that X is *weakly complete* with respect to Φ. Then we have

THEOREM 1.1. *The map $\varphi\colon X\to\mu_\Phi(X)$ is onto if and only if X is weakly complete with respect to Φ.*

The "if" part is easy to see. Indeed, let y be any point of $\mu_\Phi(X)$. Then $\{\varphi_\lambda^{-1}(\pi_\lambda(y))\,|\,\lambda\in\Lambda\}$ is a Cauchy family of X wity respect to Φ and has the countable intersection property; for $x\in\varphi_\lambda^{-1}(\pi_\lambda(y))$ we have

$$\varphi_\lambda^{-1}(\pi_\lambda(y))\subset\mathrm{St}\,(x,\mathcal{U}_{\lambda_i})\,,\qquad i=1,2,\cdots,$$

and for $\lambda_i\in\Lambda$, $i=1,2,\cdots$, if we take $\mu\in\Lambda$ such that $\Phi_{\lambda_i}<\Phi_\mu$, $i=1,2,\cdots$, we have

$$\varphi_\mu^{-1}(\pi_\mu(y))\subset\varphi_{\lambda_i}^{-1}(\pi_{\lambda_i}(y))\,,\qquad i=1,2,\cdots,$$

since $\varphi''_{\lambda_i}\circ\varphi_\mu=\varphi_{\lambda_i}$, $\varphi''_{\lambda_i}\circ\pi_\mu=\pi_{\lambda_i}$. Therefore, if X is weakly complete with respect to Φ, there exists a point x of X such that

$$x\in\bigcap_\lambda\varphi_\lambda^{-1}(\pi_\lambda(y))\,,$$

and we have $\varphi(x)=y$, which shows that φ is onto.

Next, let $\{C_\gamma\,|\,\gamma\in\Gamma\}$ be a Cauchy family of X with respect fo Φ which has the countable intersection property. Then for each λ and each positive integer i there exists $\gamma(\lambda_i)\in\Gamma$ such that $C_{\gamma(\lambda_i)}\subset V_{\lambda_i}$ for some $V_{\lambda_i}\in\mathcal{V}_{\lambda_i}$. From the countable intersection property of $\{C_\gamma\}$ and the agreement of $\{\varphi_\lambda(\mathcal{V}_{\lambda_i})\mid i=1,2,\cdots\}$ with the topology of X/Φ_λ it follows that $\bigcap_{i=1}^{\infty}\varphi_\lambda(C_{\gamma(\lambda_i)})$ consists of exactly one point, and this point does not depend on the choice of $C_{\gamma(\lambda_i)}$; we denote it by y_λ. Therefore, if $\Phi_\mu>\Phi_\lambda$, we have $\varphi_\lambda^\mu(y_\mu)=y_\lambda$.

Let $\mathfrak{F}=\{D_\delta\mid\delta\in\Delta\}$ be a maximal filter containing $\{C_\gamma\mid\gamma\in\Gamma\}$. Since

$$\varphi_\lambda(C_{\gamma(\lambda_i)})\subset\mathrm{St}\,(y_\lambda,\varphi_\lambda(\mathcal{V}_{\lambda_i}))$$

we have

$$\varphi_\lambda(D_\delta)\cap\mathrm{St}\,(y_\lambda,\varphi_\lambda(\mathcal{V}_{\lambda_i}))\neq\phi$$

for every δ in Δ, and hence

$$\varphi_\lambda^{-1}(\mathrm{St}\,(y_\lambda,\varphi_\lambda(\mathcal{V}_{\lambda_i}))\in\mathfrak{F}\ .$$

Since $\varphi_\lambda^{-1}(\mathrm{St}\,(y_\lambda,\varphi_\lambda(\mathcal{V}_{\lambda_i}))=\varphi^{-1}\pi_\lambda^{-1}(\mathrm{St}\,(y_\lambda,\varphi_\lambda(\mathcal{V}_{\lambda_i}))$, we see that if we denote by y the point $\{y_\lambda\}$ of $\mu_\Phi(X)$ we have $y\in\overline{\varphi(C_\gamma)}$ for every γ in Γ. This shows that if φ is onto then X is weakly complete with respect to Φ.

COROLLARY 1. 2. *If X is complete with respect to Φ, then $\varphi\colon X\to\mu_\Phi(X)$ is onto.*

Hereafter we identify X with $\varphi(X)$ and consider X as a subspace of $\mu_\Phi(X)$. The following theorems are contained in the proof of Theorem 1. 1.

THEOREM 1. 3. *$\mu_\Phi(X)$ is the weak completion of X with respect to Φ.*

THEOREM 1. 4. *If X admits a uniformity which consists of open coverings of order $\leqq n+1$ and with respect to which X is weakly complete, then X is the limit of an inverse system of metric spaces of dimension $\leqq n$. This is the case if X is paracompact and* $\dim X\leqq n$.

The notion of weak completeness is different from that of completeness; for example, every metric space is weakly complete with respect to its metric uniformity. However, there are cases for which both notions of completeness coincide. The uniformity consisting of all normal coverings of X is called the finest uniformity of X.

THEOREM 1. 5. *The following statements are equivalent.*

(a) *X is complete with respect to its finest uniformity (resp. the uniformity consisting of all countable normal coverings of X); that is, X is topologically complete (resp. realcompact).*

(b) *X is weakly complete with respect to its finest uniformity (resp. the uniformity consisting of all countable normal coverings of X).*

(c) *X is the limit of an inverse system of metric (resp. separable metric) spaces.*

(d) *X is homeomorphic to a closed subset of a product of metric (resp. separable metric) spaces.*

Proof. By Theorem 1. 1 (b) implies (c). Since a metric (resp. separable metric) space is topologically complete (resp. realcompact), (d) implies (a). Since the implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) are obvious, this completes the proof of Theorem 1. 4.

Expansions of spaces into inverse systems of metric spaces have been investigated by B. Pasynkov [5], V. Kljusin [7, 8] and some results similar to ours are obtained; in particular, the second part of Theorem 1. 4 and the equivalence of (a), (c) and (d) in Theorem 1. 5 are obtained by Pasynkov.

2. Characteristic properties of $\mu(X)$

Throughout this section, Φ is assumed to be the finest uniformity of X. In this case we write $\mu(X)$ instead of $\mu_\Phi(X)$. Thus $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ is the set of all the normal sequences of open coverings of X. For any continuous map $f: X \to Y$ there exists its extension $\beta(f): \beta(X) \to \beta(Y)$, where for a space S $\beta(S)$ means the Stone-Čech compactification of S.

LEMMA 2. 1. *$\{\beta(X/\Phi_\lambda); \beta(\varphi_\lambda^\mu)\}$ is an inverse system and its limit can be identified with $\beta(X)$.*

Proof. Let $\Psi = \{\mathcal{W}_i \mid i=1, 2, \cdots\}$ be a normal sequence of finite open coverings of $\beta(X)$. Then $\mathcal{W}_i \cap X = \{W \cap X \mid W \in \mathcal{W}_i\}$, $i=1, 2, \cdots$ determine a normal sequence of open coverings of X, which is equal to Φ_λ with some λ of Λ; $\Phi_\lambda = \{\mathcal{U}_{\lambda_i} \mid i=1, 2, \cdots\}$ and $\mathcal{U}_{\lambda_i} = \mathcal{W}_i \cap X$. As in §1, we have a canonical map θ; $\beta(X) \to \beta(X)/\Psi$ which is continuous. Then the following assertions hold:

(1) For two points x and x' of X

$$x' \in \bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathcal{W}_i) \Longleftrightarrow x' \in \bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathcal{U}_{\lambda_i}) .$$

(2) If H is open in $(\beta(X), \Psi)$, then $H \cap X$ is open in (X, Φ_λ).

(3) If G is open in (X, Φ_λ), then there is an open set H of $(\beta(X), \Psi)$ such that $G = H \cap X$.

Since (1) and (2) are obvious, we have only to prove (3). For this purpose, let x be any point of G. Since G is open in (X, Φ_λ) there exists an integer $i(x)$ such that $\mathrm{St}(x, \mathcal{U}_{\lambda_{i(x)}}) \subset G$. Let us set

$$K = \cup \{\mathrm{St}(x, \mathcal{W}_{i(x)}) \mid x \in G\} .$$

Then $K \cap X = G$. Let us set further $H = \mathrm{Int}(K; \Psi)$. Then we have $G \subset H \subset K$ and hence $H \cap X = G$. This proves (3).

From (1) to (3) it follows that

$$\theta \mid X = \eta \circ \varphi_\lambda : X \to X/\Phi_\lambda \to \beta(X)/\Psi ,$$

where $\eta: X/\Phi_\lambda \to \beta(X)/\Psi$ is the inclusion map. Hence we have $\theta = \beta(\eta) \circ \beta(\varphi_\lambda)$.

Let y and y' be any distinct points of $\beta(X)$. Then there exists a normal sequence Ψ of open coverings of $\beta(X)$ such that $\theta(y) \neq \theta(y')$. Hence $\beta(\varphi_\lambda)(y) \neq \beta(\varphi_\lambda)(y')$

for some $\lambda \in \Lambda$ as is seen from the result in the preceding paragraph.

Therefore the canonical map $\beta(X) \to \varprojlim \{\beta(X/\Phi_\lambda); \beta(\varphi_\lambda^\mu)\}$ is one-to-one and consequently a homeomorphism. This proves Lemma 2. 1.

THEOREM 2. 2. *For any space X, we have $X \subset \mu(X) \subset \beta(X)$ and*

$$\mu(X) = \cap \{\beta(\varphi_\lambda)^{-1}(X/\Phi_\lambda) \mid \lambda \in \Lambda\}$$

$$= \cap \{\beta(f)^{-1}(T) \mid f\colon X \to T \textit{ is a continusus map with } T \textit{ metrizable}\}.$$

Proof. The first equality follows directly from Lemma 2. 1. To prove the second equality, let f be an arbitrary continuous map from X into a metric space T. Then there exists $\lambda \in \Lambda$ such that $f = g \circ \varphi_\lambda$ with a suitable continuous map g: $X/\Phi \to T$; if $\{\mathcal{W}_i \mid i = 1, 2, \cdots\}$ is a normal sequence of open coverings of T which agrees with the topology of T, then $\{f^{-1}(\mathcal{W}_i) \mid i = 1, 2, \cdots\} = \Phi_\lambda$ with some λ of Λ and we have a desired map g.

Therefore we have

$$\beta(f)^{-1}(T) = \beta(\varphi_\lambda)^{-1}(\beta(g)^{-1}(T)) \supset \beta(\varphi_\lambda)^{-1}(X/\Phi_\lambda)$$

and the second equality is proved.

Now, let $f\colon X \to Y$ be any continuous map. Then for any continuous map ψ from Y into a metric space T we have

$$\mu(X) \subset \beta(\psi \circ f)^{-1}(T) = \beta(f)^{-1}(\beta(\psi)^{-1}(T))$$

and hence

$$\beta(f)(\mu(X)) \subset \beta(\psi)^{-1}(T).$$

In view of Theorem 2. 2 this shows that $\beta(f)$ carries $\mu(X)$ into $\mu(Y)$. Let us denote this map by $\mu(f)$. Then we have $\mu(gf) = \mu(g)\mu(f)$ for continuous maps $f\colon X \to Y$, $g\colon Y \to Z$ and $\mu(1_X) = 1_{\mu(X)}$.

LEMMA 2. 3. *If $X \subset Y \subset \mu(X)$, then $\mu(Y) = \mu(X)$.*

Proof. If g is a continuous map from Y into a metric space T, then $f = g \mid X$: $X \to T$ is continuous and $\beta(f) = \beta(g)$. Conversely, for any continuous map f from X into a metric space T $\mu(f)$ carries $\mu(X)$ into $\mu(T)$ and, since $T = \mu(T)$ by Corollary 1. 2, $g = \mu(f) \mid Y$ is a continuous map from Y into T such that $g \mid X = f$. By Theorem 2. 2 we have therefore $\mu(Y) = \mu(X)$.

THEOREM 2. 4. *$\mu(X)$ is characterized as a space Y with the following properties:*

(a) *Y is a topologically complete space containing X as a dense subspace.*

(b) *Any continuous map f from X into an arbitrary metric space T can be extended to a continuous map from Y into T.*

Proof. $\mu(X)$ is the limit of an inverse system of metric spaces and hence by Theorem 1. 5 $\mu(X)$ is topologically complete. Thus $\mu(X)$ satisfies (a). As for (b), the extension of f over $\mu(X)$ is given by $\mu(f): \mu(X)\to\mu(T)=T$.

Conversely, let Y be a space satisfying (a) and (b). From (b) it follows that $\beta(Y)=\beta(X)$, and hence $X\subset Y\subset\beta(X)$. For any continuous map f from X into a metric space T there exists a continuous map $g: Y\to T$ such that $f=g\,|\,X$. Since $\beta(f)=\beta(g): \beta(X)\to\beta(T)$ and $\beta(g)\,|\,Y=g$, we have $\beta(f)(Y)\subset T$ and hence $Y\subset\beta(f)^{-1}(T)$. Therefore we have $Y\subset\mu(X)$ by Theorem 2. 2. Now, Lemma 2. 3 shows that $\mu(Y)=\mu(X)$. On the other hand, $\mu(Y)=Y$ since Y is topologically complete. Thus we have $Y=\mu(X)$.

THEOREM 2. 5. *$\mu(X)$ is characterized as a topologically complete space Y which is the smallest with respect to properties* (a) *and* (b) *below*:

(a) *Y contains X as a dense subspace,*

(b) *every bounded real-valued continuous function on X can be extended to a continuous function over Y.*

Proof. Let Y be a topologically complete space Y satisfying (a) and (b). Then $X\subset Y\subset\beta(Y)=\beta(X)$. Let f be the inclusion map: $X\to Y$. Then $\beta(f): \beta(X)\to\beta(Y)$ is the identity. Since $\mu(f)=\beta(f)\,|\,\mu(X): \mu(X)\to\mu(Y)$ and $\mu(Y)=Y$, we have $\mu(X)\subset Y$.

THEOREM 2. 6. *$\mu(X)$ is homeomorphic to the completion of X with respect to its finest uniformity.*

Proof. Let Y be the completion of X with respect to its finest uniformity. Let f be any continuous map from X into a metric space T. Then f is a uniformly continuous map if we consider X and T as uniform spaces with the finest uniformity. Since T is complete with respect to its finest uniformity, f can be extended to a continuous map from Y into T. Hence by Theorem 2. 4 we have Theorem 2. 6.

The above considerations except Theorem 2. 5 apply equally well to $\mu_\Phi(X)$ for the case where Φ consists of all normal coverings of X with cardinality $\leqq m$ (m: an infinite cardinal number) if we require T to be a metric space with weight $\leqq m$. In particular, in case $m=\aleph_0$ $\mu_\Phi(X)$ is the Hewitt realcompactification of X.

In concluding this section, we state the following theorem; its proof is simple and is left to the reader.

THEOREM 2. 7. *If X is the topological sum of X_λ, $\lambda\in\Lambda$, then $\mu(X)$ is the topological sum of $\mu(X_\lambda)$, $\lambda\in\Lambda$.*

3. $\mu(X)$ for an M-space X

We shall first prove

THEOREM 3. 1. *$\mu(X)$ is compact if and only if X is pseudocompact.*

Proof. Suppose that X is pseudocompact. Then for any $\varphi_\lambda: X\to X/\Phi_\lambda$ we see that X/Φ_λ is compact and hence $\beta(\varphi_\lambda)^{-1}(X/\Phi_\lambda)=\beta(X)$. Therefore by Theorem 2. 2

we have $\mu(X)=\beta(X)$.

Conversely, suppose that $\mu(X)$ is compact. Then we have $\mu(X)=\beta(X)$. Let f: $X\rightarrow T$ be any continuous map from X onto a metric space T. Then from the property of $\mu(X)$ it follows that $\mu(f)$ carries $\mu(X)$ into T. Since $\mu(f)$ is clearly onto, T is compact. Therefore X is pseudocompact.

Suggested by Theorem 3. 1, we shall say that X is *pseudo-paracompact*, if $\mu(X)$ is paracompact. As is seen from Theorem 5. 1 below, the product of a pseudo-paracompact space with a locally compact paracompact space is pseudo-paracompact. As an example of pseudo-paracompact spaces we can mention M-spaces.

In a previous paper [11] we have called a space X an M-space in case there is a normal sequence $\{\mathcal{U}_i\}$ of open coverings of X satisfying condition (M) below:

(M) If $\{K_i\}$ is a decreasing sequence of non-empty closed sets of X such that $K_i\subset \mathrm{St}(x, \mathcal{U}_i)$ for each i and for some point x of X, then $\cap K_i\neq\phi$.

For paracompact spaces M-spaces coincide with p-spaces in the sense of Arhangel'skii [1].

Let X be an M-space and let $\Phi'=\{\Phi_\lambda \mid \lambda\in\Lambda'\}$ be the totality of all normal sequences of X satisfying condition (M). Then $\{\Phi_\lambda \mid \lambda\in\Lambda'\}$ is a cofinal subset of $\{\Phi_\lambda \mid \lambda\in\Lambda\}$ by using the notations in § 2, and hence $\mu(X)$ is the limit of the inverse subsystem $\{X/\Phi_\lambda \mid \lambda\in\Lambda'\}$ of $\{X/\Phi_\lambda \mid \lambda\in\Lambda\}$. As was shown in [11], φ_λ: $X\rightarrow X/\Phi_\lambda$ is a quasi-perfect map for each λ of Λ'. In case $\lambda, \mu\in\Lambda'$ and $\Phi_\lambda<\Phi_\mu$ we have $\varphi_\lambda=\varphi_\lambda^\mu\circ\varphi_\mu$ and hence φ_λ^μ: $X/\Phi_\mu\rightarrow X/\Phi_\lambda$ is a perfect map. Therefore

$$\beta(\varphi_\lambda)^{-1}(X/\Phi_\lambda)=\beta(\varphi_\mu)^{-1}(X/\Phi_\mu)$$

since $X/\Phi_\mu=\beta(\varphi_\lambda^\mu)^{-1}(X/\Phi_\lambda)$, and consequently by Theorem 2. 2 we have

$$\mu(X)=\beta(\varphi_\lambda)^{-1}(X/\Phi_\lambda) \qquad \text{for each} \quad \lambda\in\Lambda' .$$

Since a quasi-perfect map from X onto a metric space T coincides with φ_λ: $X\rightarrow X/\Phi_\lambda$ for some λ of Λ', we obtain the following theorem.

THEOREM 3. 2. *Suppose that X is an M-space. Then we have*

$$\mu(X)=\beta(f)^{-1}(T)$$

for any quasi-perfect map f from X onto a metric space T, and $\mu(f)$: $\mu(X)\rightarrow T$ is a perfect map. Moreover, $\mu(X)$ is a paracompact M-space.

The following lemma is useful.

LEMMA 3. 3. *Let f be a continuous map from an M-space X into a metric space T. Then the following assertions hold for $\mu(f)$: $\mu(X)\rightarrow T$.*

(a) *f is onto if and only if $\mu(f)$ is onto.*

(b) *f is closed if and only if $\mu(f)$ is closed.*

(c) *f is quasi-perfect if and only if $\mu(f)$ is perfect.*

Proof. There exists some $\lambda\in\Lambda'$ such that $f=h\circ\varphi_\lambda$ for some continuous map h: $X/\Phi_\lambda\rightarrow T$. From the commutative diagram

$$\begin{tikzcd} \mu(X) \arrow[r, "\mu(\varphi_\lambda)"] & X/\Phi_\lambda \arrow[d, "h"] \\ X \arrow[u] \arrow[ur, "\varphi_\lambda"] \arrow[r, "f"'] & T \end{tikzcd}$$

we see that $\mu(f)=h\circ\mu(\varphi_\lambda)$.

To prove (a) which holds for a non-M-space X also, suppose that $\mu(f)$ is onto. Then h must be onto and hence $f=h\circ\varphi_\lambda$ is onto. This proves the non-trivial part of (a).

Next, it is easy to see that f is closed (resp. quasi-perfect) if and only if h is closed (resp. perfect) and the latter holds if and only if $\mu(f)$ is closed (resp. perfect). This proves (b) and (c).

THEOREM 3. 4. *Let f be a quasi-perfect map from an M-space X onto an M-space Y. Then $\mu(f)$: $\mu(X)\to\mu(Y)$ is a perfect map.*

Proof. Let ψ be any quasi-perfect map from Y onto a metric space T. Then $\psi\circ f$: $X\to T$ is a quasi-perfect map and by Theorem 3. 2 we have

$$\mu(X)=\beta(f)^{-1}(\beta(\psi)^{-1}(T))=\beta(f)^{-1}(\mu(Y)) .$$

This shows that $\mu(f)$ is perfect since $\mu(f)=\beta(f)\,|\,\mu(X)$.

REMARK. As was shown in [12], the image of an M-space under a perfect map is not necessarily an M-space and Theorem 3. 4 does not hold if we do not assume Y to be an M-space.

Some properties of an M-space are expressed by those of $\mu(X)$.

THEOREM 3. 5. *Let X be an M-space. Then there exists a quasi-perfect map from X onto a separable (resp. locally compact or complete) metric space if and only if $\mu(X)$ is Lindelöf (resp. locally compact or an absolute G_δ).*

This theorem is easy to see in view of Lemma 3. 3.

Theorem 3. 5, together with Theorem 3. 4, gives rise to the following theorem, since the properties of a space such as " Lindelöf ", " locally compact " or " absolute G_δ " are preserved under perfect maps.

THEOREM 3. 6. *Let f be a quasi-perfect map from an M-space X onto an M-space Y. If X admits a quasi-perfect map from X onto a separable (resp. locally compact or complete) metric space, so does Y.*

4. Spaces whose topological completions are *M*-spaces

In this section we shall first prove the following theorem.

THEOREM 4. 1. *A space X is a paracompact M-space if and only if X is the limit of an inverse system $\{T_\lambda\,;\psi_\lambda^\mu\}$ of metric spaces T_λ with bonding maps ψ_λ^μ perfect.*

The "only if" part is proved in the paragraph preceding Theorem 3.2 and was proved by Kljusin [7]. The "if" part is a direct consequence of the following theorem.

THEOREM 4.2. *Let* $\{X_\lambda; \varphi_\lambda^\mu \mid \lambda, \mu \in \Lambda\}$ *be an inverse system such that* φ_λ^μ *is a perfect map for each pair* λ, μ *with* $\lambda < \mu$, *and let* X *be its limit. Then the projection* $\varphi_\lambda: X \to X_\lambda$ *is a perfect map.*

Proof. For any point x_λ of X_λ we have

$$\varphi_\lambda^{-1}(x_\lambda) = \varprojlim \{(\varphi_\lambda^\mu)^{-1}(x_\lambda); \varphi_\mu^\nu \mid \mu > \lambda,\ \nu > \mu > \lambda\}.$$

Since $(\varphi_\lambda^\mu)^{-1}(x_\lambda)$ is compact and non-empty, $\varphi_\lambda^{-1}(x_\lambda)$ is compact and non-empty, and hence φ_λ is onto.

Let F be any closed subset of X. Suppose that there is a point x_λ of X_λ such that $x_\lambda \notin \varphi_\lambda(F)$. Then $\varphi_\lambda^{-1}(x_\lambda) \cap F = \phi$ and, since $\varphi_\lambda^{-1}(x_\lambda)$ is compact, there exist a finite number of elements μ_j, $j=1, \cdots, m$ of Λ such that

$$\varphi_\lambda^{-1}(x_\lambda) \subset \bigcup_{j=1}^m \varphi_{\mu_j}^{-1}(W_j), \qquad \varphi_{\mu_j}^{-1}(W_j) \cap F = \phi$$

with some open subsets W_j of X_{μ_j} for $j=1, \cdots, m$. Here we can assume that $\mu_j > \lambda$, $j=1, \cdots, m$. Now, take $\mu \in \Lambda$ so that $\mu > \mu_j$, $j=1, \cdots, m$, and set

$$W = \bigcup_{j=1}^m (\varphi_{\mu_j}^\mu)^{-1}(W_j).$$

Then W is an open set of X_μ and we have

$$\varphi_\lambda^{-1}(x_\lambda) \subset \varphi_\mu^{-1}(W), \qquad \varphi_\mu^{-1}(W) \cap F = \phi.$$

Since $\varphi_\lambda = \varphi_\lambda^\mu \circ \varphi_\mu$, we have $(\varphi_\lambda^\mu)^{-1}(x_\lambda) \subset W$. Hence if we set

$$V = X_\lambda - \varphi_\lambda^\mu(X_\mu - W),$$

then $x_\lambda \in V$ and $(\varphi_\lambda^\mu)^{-1}(V) \subset W$, and V is open since φ_λ^μ is a closed map. Thus we have

$$\varphi_\lambda^{-1}(V) \cap F \subset \varphi_\mu^{-1}(W) \cap F = \phi.$$

This shows that $x_\lambda \notin \overline{\varphi_\lambda(F)}$. Therefore $\varphi_\lambda(F)$ is closed. This completes the proof of Theorem 4.2.

COROLLARY 4.3. *Under the same assumptions as in Theorem* 4.2, *if each* X_λ *is a* (*paracompact*) *M-space, so is* X.

After the first draft of this paper had been completed, T. Isiwata [5] generalized the notion of M-spaces as follows: A space X is called an M'-space if there is a normal sequence $\{\mathcal{U}_i \mid i=1, 2, \cdots\}$ of open coverings of X satisfying condition (M') which is obtained from condition (M) in § 3 by restricting each K_i there to be a zero-set (of a real-valued continuous function over X). Analogously as in the case of M-spaces a space X is an M'-space if and only if there is a continuous map f from X onto a metric space T such that $f(F)$ is closed for each zero-set F of X and $f^{-1}(t)$ is relatively pseudocompact (that is, every real-valued continuous function over X is bounded on $f^{-1}(t)$). He calls a map with such a property as f an SZ-map. In case X is an M'-space, the set Λ' of $\lambda \in \Lambda$ such that $\varphi_\lambda: X \to X/\Phi_\lambda$, with notations in § 2, is an SZ-map is cofinal in Λ and φ''_λ is a perfect map if $\Phi_\lambda < \Phi_\mu$ and $\lambda, \mu \in \Lambda'$. Thus by Theorem 4. 2 $\mu(X)$ is a paracompact M-space. This is seen also from Theorem 2. 5 and Isiwata [5, I; Theorem 2. 5].

THEOREM 4. 4. *$\mu(X)$ is a paracompact M-space if and only if X is an M'-space.*

We have only to prove the "only if" part. Suppose that $\mu(X)$ is a paracompact M-space. Then, by using notations in § 2, $\mu(\varphi_\lambda): \mu(X) \to X/\Phi_\lambda$ is a perfect map for each λ of a cofinal subset Λ' of Λ, and hence if $\Phi_\lambda < \Phi_\mu$ for $\lambda, \mu \in \Lambda'$, the map $\varphi''_\lambda: X/\Phi_\mu \to X/\Phi_\lambda$ is a perfect map. Let g be an arbitrary real-valued continuous function over X. For a given λ of Λ' there exists $\mu \in \Lambda'$ such that $\Phi_\lambda < \Phi_\mu$ and $g = h \circ \varphi_\mu$ with some real-valued continuous function h. If $F = g^{-1}(0)$, then $\varphi_\mu(F) = h^{-1}(0)$ is closed and, since φ''_λ is perfect, $\varphi_\lambda(F) = \varphi''_\lambda \varphi_\mu(F)$ is closed. Moreover, g is bounded on $\varphi_\lambda^{-1}(y)$ for $y \in X/\Phi_\lambda$, since $g(\varphi_\lambda^{-1}(y)) = g(\varphi_\mu^{-1}((\varphi''_\lambda)^{-1}(y))) = h((\varphi''_\lambda)^{-1}(y))$. This shows that X is an M'-space.

It should be noted that Theorem 3. 5 remains true for an M'-space if we replace "a quasi-perfect map" by "an SZ-map".

5. Conditions for $\mu(X \times Y) = \mu X \times \mu Y$

We shall first prove the following theorem.

THEOREM 5. 1. *Let Y be a locally compact paracompact space or a locally compact, topologically complete space. Then we have*

$$\mu(X \times Y) = \mu X \times \mu Y (= \mu X \times Y)$$

for any space X.

Proof. (i) Suppose that Y is compact. Let f be any continuous map from $X \times Y$ into a metric space T. For each point x of X let us define a map $\Psi(x)$ from Y into T by

$$[\Psi(x)](y) = f(x, y) .$$

Then $\Psi(x)$ is a continuous maps from Y into T. The set of all continuous maps from Y into T becomes a metric space if we define a distance between two continuous maps α and β from Y into T by

$$\rho(\alpha, \beta) = \sup \{\rho_T(\alpha(y), \beta(y)) \mid y \in Y\} ,$$

where ρ_T is a metric in T. This space is denoted by T^Y. Then we have a map

$$\Psi : X \to T^Y .$$

The map Ψ is continuous. Because for any point x of X, any positive number ε and any point y of Y there exist a neighborhood $U_y(x)$ of x and a neighborhood $V(y)$ of y such that

$$\rho_T(f(x, y), f(x', y')) < \frac{\varepsilon}{2} \qquad \text{for} \quad x' \in U_y(x),\ y' \in V(y) .$$

Then there exist a finite number of points y_i, $i=1, \cdots, m$ of Y such that

$$Y = \bigcup_{i=1}^{m} V(y_i) .$$

If we set

$$U(x) = \bigcap_{i=1}^{m} U_{y_i}(x) ,$$

then

$$\rho_T(f(x, y), f(x', y)) < \varepsilon \qquad \text{for} \quad x' \in U(x),\ y \in Y ,$$

and hence Ψ is continuous.

Since T^Y is a metric space, there is an extension $\mu(\Psi) : \mu(X) \to T^Y$ of Ψ. Let us set

$$g(\xi, y) = [\mu(\Psi)(\xi)](y) \qquad \text{for} \quad \xi \in \mu(X),\ y \in Y .$$

To prove the continuity of the map $g : \mu X \times Y \to T$, suppose that $\xi \in \mu(X)$, $y \in Y$ and $\varepsilon > 0$. Then there exists a neighborhood $U(\xi)$ of ξ such that

$$\rho(\mu(\Psi)(\xi'), \mu(\Psi)(\xi)) < \frac{\varepsilon}{2} \qquad \text{for} \quad \xi' \in U(\xi) .$$

Take a neighborhood $V(y)$ of y so that

$$\rho_T(\mu(\Psi)(\xi)](y), [\mu(\Psi)(\xi)](y')) < \frac{\varepsilon}{2} \qquad \text{for} \quad y' \in V(y) .$$

Then we have

$$\rho_T(g(\xi, y), g(\xi', y')) < \varepsilon \qquad \text{for} \quad \xi' \in U(\xi),\ y' \in V(y) .$$

This shows that $g : \mu X \times Y \to T$ is a continuous map which is an extension of $f : X \times Y \to T$.

Since $\mu(X) \times Y$ is topologically complete, by Theorem 2. 4 we have $\mu(X \times Y) = \mu X \times Y$.

(ii) In case Y is not compact, there exists an open covering $\{B_\gamma \mid \gamma \in \Gamma\}$ of Y such that $\bar{B}_\gamma$ is compact for each γ. Let $f : X \times Y \to T$ be any continuous map, where T is a metric space. As has been shown above, $f \mid X \times \bar{B}_\gamma$ can be extended to a continuous map g_γ: $\mu(X)\times\bar{B}_\gamma\to T$. Since $g_\gamma(x, y)=g_{\gamma'}(x, y)$ for $x\in X$, $y\in B_\gamma\cap B_{\gamma'}$, we have also $g_\gamma(\xi, y)=g_{\gamma'}(\xi, y)$ for $\xi\in\mu X$, $y\in B_\gamma\cap B_{\gamma'}$. Therefore, if we define a map g: $\mu(X)\times Y\to T$ by

$$g(\xi, y)=g_\gamma(\xi, y) \qquad \text{for} \quad \xi\in\mu(X),\ y\in B_\gamma,$$

then g is well defined and since $g\,|\,\mu X\times B_\gamma$ is continuous for each γ, g is continuous. On the other hand, $\mu X\times Y$ is topologically complete. Therefore by Theorem 2. 4 we have $\mu(X\times Y)=\mu X\times Y=\mu X\times\mu Y$. This completes the proof of Theorem 5. 1.

As is well known, there are pseudocompact spaces P and Q such that $P\times Q$ is not pseudocompact. In this case, by Theorem 3. 1, $\mu(P)$ and $\mu(Q)$ are compact but $\mu(P\times Q)$ is not. Therefore, $\mu(X\times Y)=\mu(X)\times\mu(Y)$ does not hold in general. If we restrict our consideration to M-spaces, we obtain the following theorem.

THEOREM 5. 2. *Let $X\times Y$ be an M-space. Then the following statements are equivalent.*

(a) $\mu(X\times Y)=\mu X\times\mu Y$.

(b) *There exist quasi-perfect maps φ: $X\to S$ and ψ: $Y\to T$ with S, T metrizable such that $\varphi\times\psi$: $X\times Y\to S\times T$ is quasi-perfect.*

(c) *If K (resp. L) is any countably compact closed subset of X (resp. Y), then $K\times L$ is countably compact.*

Proof. (a)$\Rightarrow$(b). Let φ: $X\to S$ and ψ: $Y\to T$ be any quasi-perfect maps, where S and T are metric spaces. Then $\mu(\varphi)$: $\mu(X)\to S$ and $\mu(\psi)$: $\mu(Y)\to T$ are perfect maps, and hence

$$\mu(\varphi)\times\mu(\psi):\ \mu(X)\times\mu(Y)\to S\times T$$

is also perfect. If *(a)* holds, by Lemma 3. 3 $\varphi\times\psi$ is a quasi-perfect map. This proves (b).

(b)$\Rightarrow$(c). Suppose that there are quasi-perfect maps φ: $X\to S$, ψ: $Y\to T$ with S and T metrizable such that $\varphi\times\psi$: $X\times Y\to S\times T$ is quasi-perfect.

Let K (resp. L) be any countably compact closed subset of X (resp. Y). If we set $K_0=\varphi^{-1}\varphi(K)$ and $L_0=\psi^{-1}\psi(L)$, it follows from the quasi-perfectness of $\varphi\times\psi$ that $K_0\times L_0$ is countably compact, since $\varphi(K)\times\psi(L)$ is compact. Hence $K\times L$ is countably compact.

(c)$\Rightarrow$(a). Assume (c). Let f be any continuous map from $X\times Y$ onto a metric space R. Let K (resp. L) be any countably compact closed subset of X (resp. Y). Then, by (c) and a theorem of Glicksberg (cf. [3]), $f\,|\,K\times L$ is extended to a continuous map $g_{K\times\beta(L)}$: $K\times\beta(L)\to R$. If we take another countably compact closed set K' of X, then we have a continuous map $g_{K'\times\beta(L)}$: $K'\times\beta(L)\to R$. Then two maps $g_{K\times\beta(L)}$ and $g_{K'\times\beta(L)}$ coincide with each other over $(K\cap K')\times L$ and hence over $(K\cap K')\times\beta(L)$. Any point x of X is contained in some countably compact closed set. Hence we have a single-valued map

$$g_{X\times\beta(L)}:\ X\times\beta(L)\to R$$

which coincides with $g_{K\times\beta(L)}$ over $K\times\beta(L)$. If C is a countably compact subset of

$X\times\beta(L)$, then

$$C\subset K\times\beta(L)$$

for some countably compact closed subset K of X. Since $g_{X\times\beta(L)}$ is continuous over $K\times\beta(L)$, $g_{X\times\beta(L)}$ is continuous over C. According to Nagata [14, Theorem 1], every M-space is a quasi-k-space. Since $X\times\beta(L)$ is an M-space we can conclude that $g_{X\times\beta(L)}$ is a continuous map. Applying Theorem 5. 1 to the present case we see that $g_{X\times\beta(L)}$ is extended to a continuous map

$$g_{\mu(X)\times\beta(L)}:\ \mu(X)\times\beta(L)\to R\,.$$

Thus $f|X\times L$ is extended to a continuous map g_L: $\mu X\times L\to R$.

If L' is another countably compact closed subset of Y, then $f\,|\,X\times L'$ is extended to a continuous map $g_{L'}$: $\mu(X)\times L'\to R$. Then two maps g_L and $g_{L'}$ coincide with each other over $X\times(L\cap L')$ and hence over $\mu(X)\times(L\cap L')$. Therefore we have a singlevalued map g: $\mu(X)\times Y\to R$, which coincides with g_L over $\mu(X)\times L$. Since $\mu(X)\times Y$ is an M-space and hence a quasi-k-space, we can conclude by the same argument as given before that g is a continuous map.

Similarly as above, the map g: $\mu(X)\times Y\to R$ is extended to a continuous map h: $\mu(X)\times\mu(Y)\to R$. Thus any continuous map f: $X\times Y\to R$ is extended to a continuous map h: $\mu(X)\times\mu(Y)\to R$. On the other hand, $\mu(X)\times\mu(Y)$ is a paracompact M-space by Morita [11, Theorem 6. 4] and hence topologically complete. Therefore we have $\mu(X\times Y)=\mu X\times\mu Y$ by Theorem 2. 4. This proves (a).

REMARK. Recently Y. Tanaka has proved that the product map $\varphi\times\psi$: $X\times Y\to S\times T$, where φ: $X\to S$ and ψ: $Y\to T$ are quasi-perfect maps and S, T are first countable spaces, is a quasi-perfect map if and only if the product of any countably compact closed subsets of X and Y is countably compact.

In applications of Theorem 5. 2 the following theorem is useful.

THEOREM 5. 3. *For an M-space X the following statements are equivalent.*

(a) *$X\times Y$ is an M-space for any M-space Y.*

(b) *Every quasi-perfect map φ from X onto a metric space S has the property that $\varphi\times\psi$: $X\times Y\to S\times T$ is quasi-perfect for any quasi-perfect map ψ from any space Y onto a metric space T.*

(c) *Every countably compact closed subset A of X has the property that $A\times Y$ is countably compact for any countably compact space Y.*

(d) *Every normal sequence $\{\mathcal{U}_i\}$ of open coverings of X satisfying condition* (M) *in* § 3 *has the property* (CM).

(CM): *For any discrete subsequence N of a sequence $\{x_i\}$ such that $x_i\in \mathrm{St}\,(x, \mathcal{U}_i)$ for each i and some point x of X, and for any non-empty subset S of $K-X$ where K is any compactification of X, the subspace $N\cup S$ of K is not countably compact.*

Proof. (b)$\Rightarrow$(a) is obvious. (a)$\Longleftrightarrow$(d) is proved by T. Isiwata [6]. To prove (d)$\Rightarrow$(c), let A be any countably compact closed subset of X, and let φ be a quasi-perfect map from X onto a metric space S. Then $\varphi(A)$ is compact. Let T be a space obtained from S by contracting $\varphi(A)$ to a point t_0. Then the quotient map $\psi: S\to T$ is perfect and T is metrizable. Hence $f=\psi\circ\varphi: X\to T$ is quasi-perfect and $A\subset f^{-1}(t_0)$. It is sufficient to prove that if (d) holds, then $B=f^{-1}(t_0)$ has the property that $B\times Y$ is countably compact for any countably compact space Y.

Let $\{\mathcal{W}_i\}$ be a normal sequence of open coverings of T such that $\{\mathrm{St}\,(t, \mathcal{W}_i)\}$ is a basis of neighborhoods at each point t of T. Let us set $\mathcal{U}_i=f^{-1}(\mathcal{W}_i)$, $i=1, 2, \cdots$. Then $\{\mathcal{U}_i\}$ is a normal sequence of open coverings of X satisfying condition (M). Let K be any compactification of X and L the closure of B in K. Then L is a compactification of B and $L-B\subset K-X$ since B is closed in X.

Let N be any countable discrete subset of B. Then we have $N\subset B\subset \mathrm{St}\,(b, \mathcal{U}_i)$ for each i and each point b of B. Hence, if (d) holds, $N\cup S$ is not countably compact for any subset S of $L-A$. Therefore, by a theorem of Frolik [2] $B\times Y$ is countably compact for any countably compact space Y. This proves that (d) implies (c).

Finally, assume (c). Let $\{\mathcal{U}_i\}$ (resp. $\{\mathcal{V}_i\}$) be a normal sequence of open coverings of X (resp. Y) satisfying condition (M). Let us set $\mathcal{W}_i=\{U\times V\,|\,U\in\mathcal{U}_i, V\in\mathcal{V}_i\}$. Then $\{\mathcal{W}_i\}$ is a normal sequence of open coverings of $X\times Y$ (e.g. cf. [4]). Let (x_0, y_0) be any point of $X\times Y$, and suppose that

$$(x_i, y_i)\in \mathrm{St}\,((x_0, y_0), \mathcal{W}_i) \qquad \text{for} \quad i=1, 2, \cdots.$$

Then the closure A of $\{x_i\}$ in X and the closure B of $\{y_i\}$ in Y are countably compact. Hence by (c) $A\times B$ is countably compact, and consequently $\{(x_i, y_i)\}$ has an accumulation point in $X\times Y$. Thus (b) holds.

REMARK. If X is an M-space satisfying one of the conditions: (i) X satisfies the first axiom of countability, (ii) X is locally compact, (iii) X is paracompact, then X satisfies condition (a) of Theorem 5. 3. T. Ishii, M. Tsuda and S. Kunugi [4] have given a condition for an M-space which implies (a) and is implied by any one of conditions (i) to (iii) above. However, as has been shown by T. Isiwata, their condition is not necessary for (a) to hold.

6. The dimension of product spaces

For a space X we define the covering dimension of X, $\dim X$, as the smallest integer n with the property that every finite normal open covering of X admits a finite normal open covering of order $\leqq n+1$ as a refinement. This definition is due to M. Katetov (cf. [3]) and it coincides with the covering dimension in the usual sense if X is normal. Since $\dim X=\dim \beta(X)$, the lemma below is a direct consequence of Theorem 2. 2.

LEMMA 6. 1. $\dim X=\dim \mu(X)$ *for any space* X.

THEOREM 6. 2. *If X is a pseudo-paracompact space and Y is a locally compact paracompact space, then* $\dim (X\times Y)\leqq \dim X+\dim Y$.

Proof. Since $\mu(X)$ is paracompact, we have

$$\dim(\mu(X)\times Y)\leqq\dim\mu(X)+\dim Y=\dim X+\dim Y$$

by Morita [10]. On the other hand, by Theorem 5. 1 we have

$$\dim(X\times Y)=\dim\mu(X\times Y)=\dim(\mu(X)\times Y).$$

This proves Theorem 6. 2.

As another case for which the product theorem on dimension holds we can state the following theorem.

THEOREM 6. 3. *If X is an M-space and Y is a metric space, then* $\dim(X\times Y)\leqq\dim X+\dim Y$.

Proof. In this case we have $\mu(X\times Y)=\mu X\times Y$ by Theorem 5. 2 and Remark following Theorem 5. 3. Since $\mu(X)$ is a paracompact M-space, so is $\mu(X)\times Y$. Therefore, by Kodama [9, Theorem 4] we have

$$\dim(\mu(X)\times Y)\leqq\dim\mu(X)+\dim Y=\dim X+\dim Y.$$

By the same argument as in the proof of Theorem 6. 2 we obtain the desired inequality.

By virtue of Morita [10, Theorem 7] the following theorem is proved similarly.

THEOREM 6. 4. *If X is pseudo-paracompact and Y is a polytope of finite dimension, then* $\dim(X\times Y)=\dim X+\dim Y$.

7. Unsolved problems

Let us list some of unsolved problems relating to our results in this paper.

I. Find a characterization of a space X such that $\mu(X)$ has a give property (P).

As examples of (P) we mention: "paracompact", "locally compact", and "Lindelöf".

II. Find a necessary and sufficient condition on X and Y for $\mu(X\times Y)=\mu X\times\mu Y$ to hold.

As far as $X\times Y$ is an M-space, Problem II is settled by Theorem 5. 2. However, the following problem is open.

III. If $X\times Y$ is an M-space, then does the formula $\mu(X\times Y)=\mu X\times\mu Y$ hold?

This problem is equivalent to Problem IV below.

IV. If X and Y are countably compact spaces and if $X\times Y$ is an M-space, is $X\times Y$ countably compact?

Suppose that the answer of III is yes. If X and Y are countably compact spaces and $X\times Y$ is an M-space, then $\mu(X\times Y)=\mu X\times\mu Y$ and $\mu(X)$, $\mu(Y)$ are compact by Theorem 3. 1, and hence by applying Theorem 3. 1 again we see that $X\times Y$ is countably compact. Thus IV is answered affirmatively.

Conversely, suppose that the answer of IV is yes. Let K (resp. L) be any countably compact closed subset of X (resp. Y). Then $K\times L$ is a closed subset of the M-space $X\times Y$ and so is itself an M-space, and hence $K\times L$ is countably compact, and consequently we have $\mu(X\times Y)=\mu X\times\mu Y$ by Theorem 5. 2.

V. Does the product theorem $\dim(X\times Y)\leqq\dim X+\dim Y$ hold for any spaces X and Y?

Try the cases (a) X is arbitrary and Y is locally compact and paracompact, and (b) $X\times Y$ is an M-space. The problem raised by J. Nagata: "Is $\dim(X\times Y)\leqq\dim X+\dim Y$ true for paracompact M-spaces X and Y?" is related to case (b) intimately as is seen from the proof of Theorem 6. 3.

Added in proof (October 20, 1970). Recently Problem IV in §7 has been answered in the negative by A. K. Steiner.

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