A theorem on Frobenius extensions

By

Kiiti MORITA

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§ 1. Introduction

Throughout this note A and B are assumed to be rings with an identity element, and all modules will be assumed unitary. By a subring of A we shall always mean one containing the identity element of A. For any right (resp. left) A-module V the A-endomorphism ring of V will be denoted by $\mathrm{End}_A(V_A)$ (resp. $\mathrm{End}_A({}_AV)$); we consider $\mathrm{End}_A(V_A)$ as a left operator domain of V, and $[\mathrm{End}_A({}_AV)]^0$, the opposite ring of $\mathrm{End}_A({}_AV)$, as a right operator domain of V.

The notion of Frobenius extensions was first introduced by F. Kasch [1], and later generalized by Kasch himself [2] and by T. Nakayama and T. Tsuzuku [7]. In case B is a subring of A, A is said to be a Frobenius extension of B if

(1) $\quad A_B$ is finitely generated and projective,

(2) $\quad {}_BA_A \cong {}_B[\mathrm{Hom}_B({}_AA_B, B_B)]_A$.

The purpose of this note is to establish the following theorem.

THEOREM 1. 1. *If A is a Frobenius extension of B, then A is also a Frobenius extension of L for any subring L of A such that*

$$B_0 \subset L \subset B'.$$

Here B' and B_0 are defined as follows:

(3) *B' is the second commutator ring of A_B (that is, $B' = [\mathrm{End}_C({}_CA)]^0$ where $C = \mathrm{End}_B(A_B)$); we consider B' as a subring of A containing B.*

(4) *B_0 is the subring of B which is generated by the identity element of B and by all the elements of the form $f(a)$ where $a \in A$, $f \in \mathrm{Hom}_B(A_B, B_B)$.*

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In a previous paper [4] we have introduced the notion of adjoint pairs of functors and discussed Frobenius extensions from this point of view. Let ${}_A\mathfrak{M}$ (resp. ${}_B\mathfrak{M}$) be the category of all left A-modules (resp. B-modules). In case S: ${}_A\mathfrak{M}\rightarrow{}_B\mathfrak{M}$ and T: ${}_B\mathfrak{M}\rightarrow{}_A\mathfrak{M}$ are covariant additive functors, we call $\{S, T\}$ a strongly adjoint pair of functors if S is a left adjoint of T and a right adjoint of T, that is, there exist natural isomorphisms

$$\mathrm{Hom}_B(S(X), Y)\cong\mathrm{Hom}_A(X, T(Y)), \qquad \mathrm{Hom}_A(T(Y), X)\cong\mathrm{Hom}_B(Y, S(X))$$

where $X\in{}_A\mathfrak{M}$ and $Y\in{}_B\mathfrak{M}$. We have proved in [4] that $\{S, T\}$ is a strongly adjoint pair if and only if there are natural equivalences

$$S(X)\cong{}_B[\mathrm{Hom}_B({}_AV_B, B_B)]_A\otimes X,\ T(Y)\cong{}_AV_B\otimes Y,$$

with an A-B-bimodule ${}_AV_B$ satisfying conditions (5) and (6) below:

(5) ${}_AV$ and V_B are finitely generated and projective,

(6) ${}_B[\mathrm{Hom}_A({}_AV_B, {}_AA)]_A\cong{}_B[\mathrm{Hom}_B({}_AV_B, B_B)]_A$.

We shall say that an A-B-bimodule ${}_AV_B$ is Frobenius if ${}_AV_B$ satisfies conditions (5) and (6). Then A is a Frobenius extension of B if and only if the A-B-bimodule ${}_AA_B$ is Frobenius.

Thus our Theorem 1. 1 is an immediate consequence of Theorem 1. 2 below.

THEOREM 1. 2. *Let ${}_AV_B$ be an A-B-bimodule such that ${}_AV$ and V_B are faithful. If ${}_AV_B$ is Frobenius, then the K-L-bimodule ${}_KV_L$ is also Frobenius for any subring K of A' and for any subring L of B' such that*

$$A_0\subset K\subset A', \qquad B_0\subset L\subset B'.$$

Here A', B', A_0 and B_0 are defined as follows (A and B are considered respectively as subrings of A' and B'):

(7) $A'=\mathrm{End}_D(V_D)$ *where* $D=[\mathrm{End}_A({}_AV)]^0$.

(8) $B'=[\mathrm{End}_C({}_CV)]^0$ *where* $C=\mathrm{End}_B(V_B)$.

(9) *A_0 (resp. B_0) is the subring of A (resp. B) generated by the identity element of A (resp. B) and by all the elements of the form $f(v)$ for $v\in V$, $f\in\mathrm{Hom}_A({}_AV, {}_AA)$ (resp. $g(v)$ for $v\in V$, $g\in\mathrm{Hom}_B(V_B, B_B)$).*

The notion of quasi-Frobenius extensions was introduced by B. Müller [6] and by A. Rosenberg and S. Chase independently.

In [4] we have defined the notion of similarity for modules as follows: an A-B-bimodule ${}_AW_B$ is similar to another A-B-bimodule ${}_AW'_B$ if each of ${}_AW_B$ and ${}_AW'_B$

is isomorphic to a direct summand of a finite direct sum of copies of the other; in this case we write ${}_A W_B \sim {}_A W'_B$. If we replace conditions (2) and (6) respectively by (2)′ and (6)′ below

$$(2)' \qquad {}_B A_A \sim {}_B[\mathrm{Hom}_B({}_A A_B, B_B)]_A,$$

$$(6)' \qquad {}_B[\mathrm{Hom}_A({}_A V_B, {}_A A)]_A \sim {}_B[\mathrm{Hom}_B({}_A V_B, B_B)]_A,$$

we have the notion of quasi-Frobenius extensions and quasi-Frobenius bimodules: A is a quasi-Frobenius extension of its subring B if (1) and (2)′ are satisfied, and an A-B-bimodule ${}_A V_B$ is quasi-Frobenius if (5) and (6)′ are satisfied. The notion of quasi-Frobenius bimodules is related to quasi-strongly adjoint pairs of functors introduced in [4].

Correponding to Theorems 1. 1 and 1. 2 we have the following theorems.

THEOREM 1. 3. *If A is a quasi-Frobenius extension of B, then A is also a quasi-Frobenius extension of L for any subring L of A such that*

$$B_0 \subset L \subset B',$$

where B' and B_0 are the same as described in Theorem 1. 1.

THEOREM 1. 4. *Let ${}_A V_B$ be an A-B-bimodule such that ${}_A V$ and V_B are faithful. If ${}_A V_B$ is quasi-Frobenius, then the K-L-bimodule ${}_K V_L$ is also quasi-Frobenius for any subring K of A' and for any subring L of B' such that*

$$A_0 \subset K \subset A', \qquad B_0 \subset L \subset B',$$

where A', A_0, B' and B_0 are the same as described in Theorem 1. 2.

Since A is a quasi-Frobenius extension of its subring B if and only if the A-B-bimodule ${}_A A_B$ is quasi-Frobenius, Theorem 1. 3 is a special case of Theorem 1. 4.

Our proofs of Theorems 1. 2 and 1. 4 are based on three theorems; one is the endomorphism ring theorem for adjoint pairs of functors which was established in our previous paper [5], and the others are theorems concerning modules which may be of interest by themselves.

§ 2. Theorems on modules

Let V_B be a right B-bimodule; we set

$$(10) \qquad C = \mathrm{End}_B(V_B).$$

Let us set further

(11) $$ {}_BU_C = {}_B[\mathrm{Hom}_B({}_CV_B, B_B)]_C\,, $$

(12) $$ \omega_B(u, v) = u(v) \qquad \text{for} \quad u \in U,\ v \in V\,. $$

Then ω_B: ${}_BU \times V_B \to {}_BB_B$ is a B-bilinear form in the sense that $\omega_B(u, v)$ is additive with respect to $u \in U$, $v \in V$, and that

$$ \omega_B(bu, v) = b\omega_B(u, v), \qquad \omega_B(u, vb) = \omega_B(u, v)b $$

for $b \in B$. Moreover, we have

$$ \omega_B(uc, v) = \omega_B(u, cv), \qquad \text{for} \quad c \in C\,. $$

The correspondence $v' \to v\omega_B(u, v')$ for $v' \in V$, with $u \in U$, $v \in V$ fixed, defines a B-endomorphism of V which shall be denoted by $\omega_C(v, u)$; $\omega_C(v, u)$ is an element of C and we have

(13) $$ v\omega_B(u, v') = \omega_C(v, u)v', \qquad \text{for} \quad u \in U,\ v,\ v' \in V\,. $$

Then ω_C: ${}_CV \times U_C \to {}_CC_C$ is a C-bilinear form and

$$ \omega_C(vb, u) = \omega_C(v, bu) \qquad \text{for} \quad b \in B,\ u \in U,\ v \in V\,. $$

Next, let us set

(14) $$ {}_BU'_C = {}_B[\mathrm{Hom}_C({}_CV_B, {}_CC)]_C\,, $$

(15) $$ \omega'_C(v, u') = u'(v), \qquad \text{for} \quad u' \in U',\ v \in V\,. $$

Then ω'_C: ${}_CV \times U'_C \to {}_CC_C$ is a C-bilinear form. To $u \in U$ we assign an element $\Phi(u)$ of U' defined by $[\Phi(u)](v) = \omega_C(v, u)$, $v \in V$; then we have

(16) $$ \omega'_C(v, \Phi(u)) = \omega_C(v, u), \qquad \text{for} \quad u \in U,\ v \in V\,. $$

The map Φ: ${}_BU_C \to {}_BU'_C$ is a B-C-homomorphism.

If Φ is a B-C-isomorphism, we shall say that V_B has the centralizer-dual property.

Now, assume that V_B is faithful. Then Φ is a B-C-monomorphism as is proved in [5, §1] and B is considered as a subring of B' where

(17) $$ B' = [\mathrm{End}_C({}_CV)]^0\,. $$

For $u' \in U'$, $v \in V$, the correspondence $v' \to \omega'_C(v', u')v$ defines a C-endomorphism of V, which shall be denoted by $\omega'_{B'}(u', v)$; $\omega'_{B'}(u', v)$ is an element of B' and we have

(18) $$\omega'_C(v', u')v = v'\omega'_{B'}(u', v), \qquad \text{for} \quad u' \in U',\ v,\ v' \in V.$$

The map $\omega'_{B'}$: ${}_BU' \times V_B \to {}_BB_B$ is a B-bilinear form. From (13), (16) and (18) we get

(19) $$\omega'_{B'}(\Phi(u), v) = \omega_B(u, v), \qquad \text{for} \quad u \in U,\ v \in V.$$

In a previous paper [5] we have proved that if $B' = B$ then V_B has the centralizer-dual property. We shall now prove the following theorem.

THEOREM 2. 1. *Let V_B be a faithful right B-module which has the centralizer-dual property. Then V_L has also the centralizer-dual property for any subring L of B' such that $B \subset L$, or more generally, such that $B_0 \subset L$ and $C = \mathrm{End}_L(V_L)$, where B_0 is the subring of B which is generated by the identity element of B and by all the elements $f(v)$ for $v \in V$, $f \in \mathrm{Hom}_B(V_B, B_B)$.*

Proof. Since Φ: ${}_BU_C \to {}_BU'_C$ is an isomorphism, from (19) it follows that

(20) $$\{\omega'_{B'}(u', v) \mid u' \in U',\ v \in V\} \subset B_0 .$$

Let L be any subring of B' such that $B_0 \subset L$ and $C = \mathrm{End}_L(V_L)$. Let us set

(21) $${}_LW_C = {}_L[\mathrm{Hom}_L({}_CV_L, L_L)]_C ,$$

(22) $$\tau_L(w, v) = w(v), \qquad \text{for} \quad w \in W,\ v \in V.$$

Then τ_L: ${}_LW \times V_L \to {}_LL_L$ is an L-bilinear form. Let τ_C: ${}_CV \times W_C \to {}_CC_C$ be defined by τ_L as in (13):

(13)′ $$v\tau_L(w, v') = \tau_C(v, w)v', \qquad \text{for} \quad w \in W,\ v,\ v' \in V.$$

Then we define

$$\Psi:\ {}_LW_C \to {}_LU'_C = {}_L[\mathrm{Hom}_C({}_CV_L, {}_CC)]_C$$

by $[\Psi(w)](v) = \tau_C(v, w)$, for $w \in W$, $v \in V$, and we have

(16)′ $$\omega'_C(v, \Psi(w)) = \tau_C(v, w), \qquad \text{for} \quad w \in W,\ v \in V.$$

(19)′ $$\omega'_{B'}(\Psi(w), v) = \tau_L(w, v), \qquad \text{for} \quad w \in W,\ v \in V.$$

Let u' be any element of U'. Then the correspondence $v \to \omega'_{B'}(u', v)$ defines an element of W since $\omega'_{B'}(u', v) \in B_0 \subset L$; this element shall be denoted by $\Psi'(u')$. Then Ψ': $U' \to W$ is an L-C-homomorphism and we have

(23) $$\omega'_{B'}(u', v) = \tau_L(\Psi'(u'), v), \qquad \text{for} \quad u' \in U',\ v \in V.$$

Hence by (19)′ we get $\omega'_{B'}(u', v)=\omega'_{B'}(\Psi\Psi'(u'), v)$, and consequently by (15) and (18) we have $u'=\Psi\Psi'(u')$. This shows that Ψ is an epimorphism. Since Ψ is a monomorphism, Ψ is an L-C-isomorphism.

THEOREM 2. 2. *Let V_B be a faithful right B-module which is finitely generated and projective. If L is a subring of B' containing B_0 where B_0 is the ring defined in Theorem* 2. 1, *then V_L is finitely generated and projective, and $C=\mathrm{End}_L(V_L)$.*

Proof. There exist dual sets of generators $\{u_i\in {}_BU|i=1, \cdots, n\}$ and $\{v_i\in V_B|i=1, \cdots, n\}$ of ${}_BU$ and V_B with respect to ω_B (cf. [4, § 2]); we have

$$v=\sum_{i=1}^{n} v_i\omega_B(u_i, v), \qquad \text{for} \quad v\in V.$$

By (13) we have

$$\sum_{i=1}^{n} \omega_C(v_i, u_i)=1\,.$$

On the other hand, (16) shows that $\omega_C(v_i, u_i)=\omega'_C(v_i, \Phi(u_i))$. Hence by (18) we have

$$v=\sum_{i=1}^{n} v_i\omega'_{B'}(\Phi(u_i), v)\,. \tag{24}$$

Since $\omega'_{B'}(\Phi(u_i), v)=\omega_B(u_i, v)\in B_0\subset L$, (24) shows that V_L is finitely generated and projective.

Let φ be any L-endomorphism of V. Let us set

$$c=\sum_{i=1}^{n} \omega'_C(\varphi(v_i),\ \Phi(u_i))\,.$$

Then we have

$$cv=\sum_{i=1}^{n} \varphi(v_i)\omega'_{B'}(\Phi(u_i), v)=\varphi\left(\sum_{i=1}^{n} v_i\omega'_{B'}(\Phi(u_i), v_i)\right)=\varphi(v)\,.$$

Thus Theorem 2. 2 is proved.

§ 3. Proofs of Theorems 1. 2 and 1. 4

By investigating our proof of [5, Theorem 5. 1] we see that the following theorem was actually established in [5, Theorem 5. 1].

THEOREM 3. 1. *Let ${}_AV_B$ be an A-B-bimodule such that ${}_AV$ and V_B are finitely generated and projective. Let us set $C=\mathrm{End}_B(V_B)$. Then ${}_AV_B$ is Frobenius (resp.*

quasi-Frobenius) if and only if the ring-homomorphism $\varphi: A \to C$, *defined by* $\varphi(a)v = av$ *for* $v \in V$, *is Frobenius (resp. quasi-Frobenius) and* V_B *has the centralizer-dual property.*

Here a ring-homomorphism $\varphi: A \to C$ is said to be *Frobenius* (resp. *quasi-Frobenius*) if

(1) C_A is finitely generated and projective,

(2) ${}_A C_C \cong$ (resp. $\sim$)${}_A[\mathrm{Hom}({}_C C_A, A_A)]_C$.

In case ${}_A V$ is faithful, φ is the inclusion map and hence "φ is Frobenius (resp. quasi-Frobenius)" means "C is a Frobenius (resp. quasi-Frobenius) extension of A".

Now we are in a position to prove our main theorems.

Proofs of Theorems 1. 2 *and* 1. 4. Let ${}_A V_B$ be a Frobenius (resp. quasi-Frobenius) A-B-bimodule such that ${}_A V$ and V_B are faithful. Let L be any subring of B' containing B_0. Then by Theorem 2. 2 V_L is finitely generated and projective, and $C = \mathrm{End}_L(V_L)$ where $C = \mathrm{End}_B(V_B)$. According to Theorem 3. 1 the right B-module V_B has the centralizer-dual property, and hence by Theorem 2. 1 V_L has also the centralizer-dual property. By applying Theorem 3. 1 to the A-L-bimodule ${}_A V_L$, it is seen that ${}_A V_L$ is Frobenius (resp. quasi-Frobenius). If K is any subring of A' containing A_0, it is proved similarly that the K-L-bimodule ${}_K V_L$ is Frobenius (resp. quasi-Frobenius), since the right-left dual of Theorem 3. 1 holds. This completes our proof of Theorems 1. 2 and 1. 4.

§ 4. Remarks

Let ${}_A V_B$ be an A-B-bimodule, and let us set

$$C = \mathrm{End}_B(V_B), \qquad D = [\mathrm{End}_A({}_A V)]^0.$$

Then there are ring-homomorphisms

$$\varphi: A \to C, \qquad \psi: B \to D$$

defined respectively by

$$\varphi(a)v = av, \ v\psi(b) = vb \ \text{for} \ v \in V, \ a \in A, \ b \in B.$$

We set

$$E = \mathrm{Im}\ \varphi, \qquad F = \mathrm{Im}\ \psi.$$

Then V is an E-F-bimodule such that ${}_E V$ and V_F are faithful.

PROPOSITION 4. 1. *If ${}_AV_B$ is Frobenius (resp. quasi-Frobenius), then ${}_EV_F$ is also Frobenius (resp. quasi-Frobenius).*

Proof. We have

$$ {}_B[\mathrm{Hom}_E({}_EV_B, {}_EE)]_E \cong {}_B[\mathrm{Hom}_A({}_AV_B, {}_AE)]_E $$

$$ \cong {}_B[\mathrm{Hom}_A({}_AV_B, {}_AA)]_A \otimes {}_AE_E $$

$$ \cong (\text{resp. } \sim) {}_B[\mathrm{Hom}_B({}_AV_B, B_B)]_A \otimes {}_AE_E $$

$$ \cong {}_B[\mathrm{Hom}_B({}_EV_B, B_B)]_E . $$

On the other hand, ${}_EV$ is finitely generated and projective. Therefore the E-B-bimodule ${}_EV_B$ is Frobenius (resp. quasi-Frobenius). By the same argument we see that ${}_EV_F$ is Frobenius (resp. quasi-Frobenius). This proves Proposition 4. 1.

In Theorems 1. 2 and 1. 4 we have assumed that ${}_AV$ and V_B are faithful. For a Frobenius (resp. quasi-Frobenius) A-B-bimodule ${}_AV_B$ for which ${}_AV$ and V_B are not necessarily faithful, Theorems 1. 2 and 1. 4 can be applied to the E-F-bimodule ${}_EV_F$ by virtue of Proposition 4. 1.

Secondly, we give an example of a Frobenius extension A of B in which $B' \neq B$, $B_0 \neq B$.

EXAMPLE 4. 2. Let K be a field and B a subalgebra of the full matrix algebra $(K)_4$ such that B has a K-basis consisting of

$$ e_1 = c_{11} + c_{44}, \qquad e_2 = c_{22}, \qquad e_3 = c_{33},\ c_{21},\ c_{31},\ c_{41},\ c_{42},\ c_{43}, $$

where c_{ik} are matrix units in $(K)_4$. Let us set ${}_KV_B = {}_K[e_1B]_B$. Then ${}_KV_B$ is Frobenius (and B is a QF-3 algebra), and the rings B' and B_0 in Theorem 1. 2 have K-bases $\{e_1, e_2, e_3, c_{21}, c_{31}, c_{41}, c_{42}, c_{43}, c_{23}, c_{32}\}$ and $\{e_1, e_2+e_3, c_{21}, c_{31}, c_{41}, c_{42}, c_{43}\}$, and hence $B' \neq B$, $B_0 \neq B$. Let us set $A = (K)_4 = [\mathrm{End}_K({}_KV)]^0$. Then

$$ A_B \cong \sum_{i=1}^{4} \oplus V_B . $$

Hence by the left-right dual of Theorem 3. 1 A is a Frobenius extension of B and the rings B' and B_0 in Theorem 1. 1 coincides with B' and B_0 described above.

As is indicated in the above consideration, the following result obtained in [3, § 17] is a special case of our Theorem 1. 2: Let B be a QF-3 algebra over a field K such that Be and eB are minimal faithful modules for some idempotent e of B. Then a subring L of B' containing B_0 is also a QF-3 algebra where B' is the second commutator algebra of $[eB]_B$ and B_0 is the subalgebra of B generated by the identity element of B and by the elements of BeB.

Finally, we note that the following characterization for the centralizer-dual property is obtained as a direct consequence of Theorem 2. 1 and [5, Theorem 1. 1].

THEOREM 4. 3. *Let V_B be a faithful right B-module and B' the second commutator ring of V_B. Then V_B has the centralizer-dual property if and only if $f(v)\in B$ for $v\in V$, $f\in \mathrm{Hom}_{B'}(V_{B'}, B'_{B'})$.*

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